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SUMS OF ALMOST EQUAL PRIME SQUARES

HONGZE LI¹ AND JIE WU

ABSTRACT. In this short note, we prove that almost all integers N satisfying $N \equiv 3 \pmod{24}$ and $5 \nmid N$ or $N \equiv 4 \pmod{24}$ is the sum of three or four almost equal prime squares, respectively: $N = p_1^2 + \cdots + p_j^2$ with $|p_i - (N/j)^{1/2}| \leq N^{1/2-9/80+\varepsilon}$ for $j = 3$ or 4 and $1 \leq i \leq j$.

1. INTRODUCTION

Motivated by Lagrange's theorem, it is natural to conjecture that all large integers subject to a natural congruence condition are the sum of four squares of prime numbers. Using the Hardy-Littlewood method, Hua [5] has shown that an analogous result holds for sums of five squares of primes. On the other hand, he has also proved that almost all integers n with $n \equiv 4 \pmod{24}$ are the sum of three squares of prime numbers. Define

$$\begin{aligned}\mathcal{A}_3 &:= \{N \in \mathbb{N} : N \equiv 3 \pmod{24}, 5 \nmid N\}, \\ \mathcal{A}_4 &:= \{N \in \mathbb{N} : N \equiv 4 \pmod{24}\},\end{aligned}$$

and denote by $\mathcal{E}_j(z)$ the set of integers $N \in \mathcal{A}_j \cap [z/2, z]$ such that $N \neq p_1^2 + \cdots + p_j^2$. Hua proved that $|\mathcal{E}_3(z)| \ll_A z/(\log z)^A$ for some positive constant A . The study on size of $\mathcal{E}_j(z)$ has received attention of many authors such as Schwarz [15], Liu & Liu [7], Wooley [18], Liu [6], Liu, Wooley & Yu [9]. The best record is due to Harman & A. V. Kumchev [4]: $|\mathcal{E}_3(z)| \ll z^{5/14+\varepsilon}$ and $|\mathcal{E}_4(z)| \ll z^{6/7+\varepsilon}$ for any $\varepsilon > 0$.

In this short note, we investigate this problem in form of short intervals:

$$(1.1) \quad N = p_1^2 + \cdots + p_j^2 \quad \text{with} \quad |p_i - (N/j)^{1/2}| \leq N^{1/2-\delta} \quad (1 \leq i \leq j),$$

where $\delta > 0$ is a constant, which is hoped to be "large" as soon as possible.

In the case of $j = 3$ or 4 , our result is as follows.

Theorem 1. *Let $j = 3$ or 4 . For any fixed $\varepsilon > 0$, the equation (1.1) with $\delta = \frac{9}{80} - \varepsilon$ is solvable for almost all integers $N \in \mathcal{A}_j$.*

Following Liu & Zhan [11], we shall use the circle method to prove Theorems 1 and 2. Our improves essentially come from an estimate for exponential sums over prime numbers of Liu, Lü & Zhan [8] (see Lemma 2.1 below) and a mean value theorem of Choi & Kumchev [3] (see Lemma 2.2 below). However in order to exploit these we need to introduce some new arguments in Liu & Zhan's method.

2. OUTLINE AND PRELIMINARY LEMMAS

Throughout this paper, the letter p , with or without subscript, denotes a prime number and ε an arbitrarily small positive number. Let $j = 3$ or 4 and N be a sufficiently large integer. Define

$$(2.1) \quad x = x_j := (N/j)^{1/2}, \quad y = y_j := N^{1/2-9/80+4\varepsilon}$$

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and

$$(2.2) \quad P = P_j := N^{24\varepsilon}, \quad Q = Q_j := N^{-6\varepsilon}y^2.$$

The circle method, in the form we require here, begins with the observation that

$$(2.3) \quad \mathcal{R}_j(N) := \sum_{\substack{x-y \leq p_1, \dots, p_j \leq x+y \\ p_1^2 + \dots + p_j^2 = N}} (\log p_1) \cdots (\log p_j) = \int_{1/Q}^{1+1/Q} S(\alpha)^j e(-\alpha N) d\alpha,$$

where $e(t) := e^{2\pi it}$ and

$$(2.4) \quad S(\alpha) := \sum_{x-y \leq p \leq x+y} (\log p) e(\alpha p^2).$$

Clearly in order to prove our theorems 1 and 2, it is sufficient to show that $\mathcal{R}_j(N) > 0$ for almost all integers $N \in \mathcal{A}_j$ if $j = 3, 4$.

By Dirichlet's lemma ([17], Lemma 2.1), each $\alpha \in [1/Q, 1 + 1/Q]$ can be written as

$$(2.5) \quad \alpha = a/q + \beta \quad \text{with} \quad |\beta| \leq 1/(qQ)$$

for some integers a and q with $1 \leq a \leq q \leq Q$ and $(a, q) = 1$. We denote by $I(a, q)$ the set of α satisfying (2.5), and define the major arcs $\mathfrak{M} = \mathfrak{M}_j$ and the minor arcs $\mathfrak{m} = \mathfrak{m}_j$ as follows:

$$(2.6) \quad \mathfrak{M} := \bigcup_{1 \leq q \leq P} \bigcup_{\substack{1 \leq a \leq q \\ (a, q) = 1}} I(a, q) \quad \text{and} \quad \mathfrak{m} := [1/Q, 1 + 1/Q] \setminus \mathfrak{M}.$$

Thus we can write

$$(2.7) \quad \mathcal{R}_j(N) = \int_{\mathfrak{M}} S(\alpha)^j e(-\alpha N) d\alpha + \int_{\mathfrak{m}} S(\alpha)^j e(-\alpha N) d\alpha =: \mathcal{R}_j(N; \mathfrak{M}) + \mathcal{R}_j(N; \mathfrak{m}).$$

We shall establish an asymptotic formula for $\mathcal{R}_j(N; \mathfrak{M})$ in Section 3 and treat $\mathcal{R}_j(N; \mathfrak{m})$ in Section 4. As indicated in the introduction, the new tools that we need are an estimate for exponential sums over prime numbers of Liu, Lü & Zhan [8] and a mean value theorem of Choi & Kumchev [3], which are stated as follows.

Lemma 2.1. ([8], Theorem 1.1) *Let $k \in \mathbb{N}$, $2 \leq y \leq x$ and $\alpha = a/q + \beta$ be a real number with $1 \leq a \leq q$ and $(a, q) = 1$. Define*

$$\Xi := |\beta|x^k + (x/y)^2.$$

Then for any fixed $\varepsilon > 0$, we have

$$\sum_{x < p \leq x+y} \Lambda(n) e(\alpha n^k) \ll (qx)^\varepsilon \{y(q\Xi/x)^{1/2} + (qx)^{1/2} \Xi^{1/6} + x^{3/10} y^{1/2} + x^{4/5} \Xi^{-1/6} + x(q\Xi)^{-1/2}\},$$

where the implied constant depends on ε and k only.

Lemma 2.2. ([3], Theorem 1.1) *Let $\ell \in \mathbb{N}$, $R \geq 1$, $T \geq 1$, $X \geq 1$ and $\kappa := 1/\log X$. Then there is an absolute positive constant c such that*

$$\sum_{\substack{r \sim R \\ \ell | r}} \sum_{\chi \pmod{r}}^* \int_{-T}^T \left| \sum_{X \leq n \leq 2X} \frac{\Lambda(n) \chi(n)}{n^{\kappa+i\tau}} \right| d\tau \ll (\ell^{-1} R^2 T X^{11/20} + X) (\log RTX)^c,$$

where $\sum_{\chi \pmod{r}}^$ means summation over the primitive characters modulo r . The implied constant is absolute.*

In Choi & Kumchev's original statement (in a more general form), there is no factor $n^{-\kappa}$. Since $n \mapsto n^{-\kappa}$ is completely multiplicative and $n^{-\kappa} \asymp 1$ for $X \leq n \leq 2X$, their proof rests available with some trivial modification.

Next we bound $S(\alpha)$ on the minor arcs \mathfrak{m} by combining Lemma 2.1 with Liu & Zhan's estimate for short exponential sums over prime numbers ([10], Theorem 2): Let $1 \leq a \leq q \leq uv$ with $(a, q) = 1$ and $u, v \geq 1$ and let $\alpha \in \mathbb{R}$ such that $|\alpha - a/q| < 1/q^2$. Then for any $\varepsilon > 0$ we have

$$(2.8) \quad \sum_{u \leq n \leq u+v} \Lambda(n) e(\alpha n^2) \ll_{\varepsilon} v^{1+\varepsilon} (q^{-1/4} + u^{1/8} v^{-1/4} + u^{1/3} v^{-1/2} + (qu)^{1/4} v^{-3/4}),$$

where $\Lambda(n)$ is von Mangoldt's function and the implied constant depends on ε only.

Proposition 2.1. *With the previous notation, we have*

$$(2.9) \quad \max_{\alpha \in \mathfrak{m}_j} |S(\alpha)| \ll_{\varepsilon} N^{-2\varepsilon} y \quad (j = 3, 4).$$

The implied constant depends on ε only.

Proof. Let

$$(2.10) \quad Q' = Q'_j := N^{-1/2-10\varepsilon} y^3.$$

By Dirichlet's lemma, each $\alpha \in \mathfrak{m}$ can be written as

$$\alpha = a/q + \beta \quad \text{with} \quad 1 \leq a \leq q \leq Q', \quad (a, q) = 1 \quad \text{and} \quad |\beta| \leq 1/(qQ').$$

We discuss three possibilities according to the size of q :

(i) If $P \leq q \leq Q'$, we can use (2.8) with $(u, v) = (x - y, 2y)$ to write

$$(2.11) \quad |S(\alpha)| \ll_{\varepsilon} N^{-2\varepsilon} y.$$

(ii) If $q \leq P$, we must have $1/(qQ) < |\alpha - a/q| \leq 1/(qQ')$. We shall apply Lemma 2.1 with $k = 2$. Since $Q^{-1} \geq y^{-2}$, we have

$$NQ^{-1} \ll q\Xi \asymp q|\beta|N \ll NQ'^{-1}.$$

Thus we have, for $j = 3, 4$,

$$\begin{aligned} |S(\alpha)| &\ll_{\varepsilon} N^{\varepsilon/10} \{ N^{-1/4} y (q\Xi)^{1/2} + N^{1/4} q^{1/3} (q\Xi)^{1/6} + N^{3/20} y^{1/2} + N^{2/5} \Xi^{-1/6} + N^{1/2} (q\Xi)^{-1/2} \} \\ &\ll_{\varepsilon} N^{\varepsilon/10} \{ N^{1/4} Q'^{-1/2} y + N^{5/12} P^{1/3} Q'^{-1/6} + N^{3/20} y^{1/2} + N^{2/5} (N^{-1} P Q)^{1/6} + Q^{1/2} \} \\ &\ll_{\varepsilon} N^{\varepsilon/10} \{ N^{1/2+10\varepsilon} y^{-1/2} + N^{3/20} y^{1/2} + N^{7/30+3\varepsilon} y^{1/3} + N^{-3\varepsilon} y \} \\ &\ll_{\varepsilon} N^{-2\varepsilon} y, \end{aligned}$$

provided $y \geq N^{1/2-3/20+8\varepsilon}$. □

In order to exploit Choi & Kumchev's mean value theorem effectively, we need to prove a preliminary lemma.

Lemma 2.3. *Let χ be a Dirichlet character modulo r . Let $Q \geq r$, $2 \leq X < Y \leq 2X$, $T_0 := (\log(Y/X))^{-1}$, $T_1 := (\log(Y/X))^{-2}$, $T_2 := 8\pi X^2/(rQ)$, $T_3 := X^4$ and $\kappa := (\log X)^{-1}$. Define*

$$(2.12) \quad F(s, \chi) := \sum_{X \leq n \leq 2X} \Lambda(n) \chi(n) n^{-s}.$$

Then we have

$$(2.13) \quad \begin{aligned} \max_{|\beta| \leq 1/(rQ)} \left| \sum_{X \leq n \leq Y} \Lambda(n) \chi(n) e(\beta n^2) \right| &\ll \log \left(\frac{Y}{X} \right) \int_{|\tau| \leq T_1} |F(\kappa + i\tau, \chi)| d\tau \\ &+ \int_{T_1 < |\tau| \leq T_2} \frac{|F(\kappa + i\tau, \chi)|}{|\tau|^{1/2}} d\tau \\ &+ \int_{T_2 < |\tau| \leq T_3} \frac{|F(\kappa + i\tau, \chi)|}{|\tau|} d\tau + 1 \end{aligned}$$

and

$$(2.14) \quad \sum_{X \leq n \leq Y} \Lambda(n) \chi(n) \ll \log \left(\frac{Y}{X} \right) \int_{|\tau| \leq T_0} |F(\kappa + i\tau, \chi)| d\tau \\ + \int_{T_0 < |\tau| \leq T_3} \frac{|F(\kappa + i\tau, \chi)|}{|\tau|} d\tau + 1.$$

The implied constants are absolute.

Proof. By Perron's formula ([16], Corollary II.2.1), for any $t \in [X, 2X]$ we can write

$$\sum_{X \leq n \leq t} \Lambda(n) \chi(n) = \frac{1}{2\pi i} \int_{\kappa - iT_3}^{\kappa + iT_3} F(s, \chi) \frac{t^s - X^s}{s} ds + O\left(\frac{\log(T_3 X)}{T_3}\right).$$

From this, a simple partial summation gives

$$(2.15) \quad \sum_{X \leq n \leq Y} \Lambda(n) \chi(n) e(\beta n^2) = \int_X^Y e(\beta t^2) d\left(\sum_{X \leq n \leq t} \Lambda(n) \chi(n) \right) \\ = \frac{1}{2\pi i} \int_{\kappa - iT_3}^{\kappa + iT_3} F(s, \chi) V(s, \beta) ds + O(1),$$

where

$$V(s, \beta) := \int_X^Y t^{s-1} e(\beta t^2) dt.$$

First for all $\beta \in \mathbb{R}$, we have trivially

$$(2.16) \quad |V(\kappa + i\tau, \beta)| \leq \int_X^Y t^{\kappa-1} dt \ll \log(Y/X).$$

On the other hand, the change of variables $u = t^2$ and the second mean value formula ([16], Theorem I.0.3) imply

$$V(s, \beta) = \frac{1}{2} \int_{X^2}^{Y^2} u^{\kappa/2-1} e(\beta u + (\tau/4\pi) \log u) du \\ = \frac{X^{\kappa-2}}{2} \int_{X^2}^{\xi} e(\beta u + (\tau/4\pi) \log u) du + \frac{Y^{\kappa-2}}{2} \int_{\xi}^{Y^2} e(\beta u + (\tau/4\pi) \log u) du$$

for some $\xi \in [X^2, Y^2]$. We estimate the last two integrals by using Theorem I.6.2 [16] if $T_2 < |\tau| \leq T_3$ and Theorem I.6.3 [16] if $T_1 < |\tau| \leq T_2$ and use (2.16) for $|\tau| \leq T_1$. We obtain

$$\max_{|\beta| \leq 1/(rQ)} |V(s, \beta)| \ll \begin{cases} \log(Y/X) & \text{if } |\tau| \leq T_1, \\ |\tau|^{-1/2} & \text{if } T_1 < |\tau| \leq T_2, \\ |\tau|^{-1} & \text{if } T_2 < |\tau| \leq T_3. \end{cases}$$

Now the inequality (2.13) follows from (2.15) by splitting the integral into three parts according to $|\tau| \leq T_1$ or $T_1 < |\tau| \leq T_2$ or $T_2 < |\tau| \leq T_3$ and by using the preceding estimates.

Similarly there is a real number $\xi \in [X, Y]$ such that

$$(2.17) \quad V(\kappa + i\tau, 0) = X^{\kappa-1} \int_X^{\xi} t^{i\tau} dt + Y^{\kappa-1} \int_{\xi}^Y t^{i\tau} dt \ll (|\tau| + 1)^{-1}.$$

Now the inequality (2.14) follows from (2.15) with $\beta = 0$ by splitting the integral into two parts according to $|\tau| \leq T_0$ or $T_0 < |\tau| \leq T_3$ and by using (2.17) and (2.16) with $\beta = 0$. This completes the proof. \square

Next we shall prove three estimates (see (2.21), (2.22) and (2.23) below), which play an important role in Liu's iterative procedure [6]. Define

$$(2.18) \quad S_0(\beta) := \sum_{x-y \leq n \leq x+y} e(\beta n^2),$$

$$(2.19) \quad W_\chi(\beta) := \sum_{x-y \leq p \leq x+y} (\log p) \chi(p) e(\beta p^2) - \delta_\chi S_0(\beta)$$

and $\delta_\chi = 1$ or 0 according as χ is principal or not. We also set $\mathcal{L} := \log N$,

$$(2.20) \quad W_\chi^\# := \max_{|\beta| \leq 1/(rQ)} |W_\chi(\beta)| \quad \text{and} \quad \|W_\chi\|_2 := \left(\int_{-1/(rQ)}^{1/(rQ)} |W_\chi(\beta)|^2 d\beta \right)^{1/2}.$$

Proposition 2.2. *Let $d \geq 1$ and $j = 3, 4$. Let $(x, y) = (x_j, y_j)$ and $(P, Q) = (P_j, Q_j)$ be defined as in (2.1) and (2.2), respectively. Then there is an absolute positive constant c such that for any $\varepsilon > 0$ we have*

$$(2.21) \quad \sum_{r \leq P} [d, r]^{-(j-2)/2+\varepsilon} \sum_{\chi \pmod{r}}^* W_\chi^\# \ll_\varepsilon d^{-(j-2)/2+\varepsilon} y \mathcal{L}^c,$$

$$(2.22) \quad \sum_{r \leq P} [d, r]^{-(j-2)/2+\varepsilon} \sum_{\chi \pmod{r}}^* \|W_\chi\|_2 \ll_\varepsilon d^{-(j-2)/2+\varepsilon} N^{-1/4} y^{1/2} \mathcal{L}^c.$$

Further if $d = 1$, the first estimate can be improved to

$$(2.23) \quad \sum_{r \leq P} r^{-(j-2)/2+\varepsilon} \sum_{\chi \pmod{r}}^* W_\chi^\# \ll_A y \mathcal{L}^{-A}$$

for any fixed $A > 0$.

Proof. Introducing

$$(2.24) \quad \widetilde{W}_\chi(\beta) := \sum_{x-y \leq n \leq x+y} \Lambda(n) \chi(n) e(\beta n^2) - \delta_\chi S_0(\beta),$$

we have, for all $\beta \in \mathbb{R}$,

$$(2.25) \quad |\widetilde{W}_\chi(\beta) - W_\chi(\beta)| \leq 2 \sum_{\substack{x-y \leq p^\nu \leq x+y \\ \nu \geq 2}} \log p \ll \sum_{\substack{x-y \leq p^\nu \leq x+y \\ \nu \geq 2}} \mathcal{L} \ll \mathcal{L}^2,$$

where we have used the fact that $\sum_{\substack{x-y \leq p^\nu \leq x+y \\ \nu \geq 2}} 1 \leq \mathcal{L}$. Thus

$$W_\chi^\# \leq \widetilde{W}_\chi^\# + O(\mathcal{L}^2).$$

The contribute of $O(\mathcal{L}^2)$ to (2.21) is, writing $[d, r] = dr/\ell$ and $\ell = (d, r)$,

$$\begin{aligned} &\ll \mathcal{L}^2 \sum_{\ell|d, \ell \leq P} \sum_{r \leq P, \ell|r} (dr/\ell)^{-(j-2)/2+\varepsilon} r \\ &\ll d^{-(j-2)/2+\varepsilon} \mathcal{L}^2 P^{(9-j)/4+\varepsilon} \\ &\ll d^{-(j-2)/2+\varepsilon} y, \end{aligned}$$

since $P^{(9-j)/4+\varepsilon} \ll_\varepsilon y$ in view of our choice of P (see (2.2)).

Therefore in order to prove (2.21), it is enough to show

$$(2.26) \quad \sum_{r \sim R} [d, r]^{-(j-2)/2+\varepsilon} \sum_{\chi \pmod{r}}^* \widetilde{W}_\chi^\# \ll d^{-(j-2)/2+\varepsilon} y \mathcal{L}^c$$

for any $R \leq P$, where $r \sim R$ means that $R \leq r < 2R$.

If $R = 1$ and $r \sim R$, we have $\chi = \chi_0^*(\bmod 1)$ (the primitive character modulo 1). Thus

$$\widetilde{W}_\chi^\sharp \leq \sum_{x-y \leq n \leq x+y} 2\mathcal{L} \ll y\mathcal{L}.$$

This will contribute $O(d^{-(j-2)/2+\varepsilon}y\mathcal{L})$, which is acceptable.

For $2 \leq R \leq P$ and $r \sim R$, we have $\delta_\chi = 0$. Thus we can apply (2.13) to write

$$(2.27) \quad \begin{aligned} \widetilde{W}_\chi^\sharp &\ll \frac{y}{x} \int_{|\tau| \leq T_1} |F(\kappa + i\tau, \chi)| \, d\tau + \int_{T_1 < |\tau| \leq T_2} \frac{|F(\kappa + i\tau, \chi)|}{|\tau|^{1/2}} \, d\tau \\ &+ \int_{T_2 < |\tau| \leq T} \frac{|F(\kappa + i\tau, \chi)|}{|\tau|} \, d\tau + 1, \end{aligned}$$

where $T_1 \asymp (x/y)^2$, $T_2 \asymp x^2/(RQ)$ and $T \asymp x^4$.

By Lemma 2.2, the contribution of the first term on the right-hand side of (2.27) to (2.21) is

$$(2.28) \quad \begin{aligned} &\ll d^{-(j-2)/2+\varepsilon} x^{-1} y \sum_{\ell|d, \ell \leq 2R} (R/\ell)^{-(j-2)/2+\varepsilon} (\ell^{-1} R^2 T_1 x^{11/20} + x) \\ &\ll d^{-(j-2)/2+\varepsilon} y (P^{(9-j)/4+\varepsilon} N^{31/40} y^{-2} + 1) \mathcal{L}^c \\ &\ll d^{-(j-2)/2+\varepsilon} y \mathcal{L}^c \end{aligned}$$

in view of our choice of (P, y) (see (2.1) and (2.2)).

Introducing

$$M(\ell, R, T', x) := \sum_{r \sim R, \ell|r} \sum_{\chi \pmod{r}}^* \int_{T'}^{2T'} |F(\kappa + i\tau, \chi)| \, d\tau,$$

the contribution of the second term on the right-hand side of (2.27) to (2.21) is

$$(2.29) \quad \begin{aligned} &\ll d^{-(j-2)/2+\varepsilon} \mathcal{L}^c \sum_{\ell|d, \ell \leq R} (R/\ell)^{-(j-2)/2+\varepsilon} \max_{T_1 \leq T' \leq T_2} (T'^{-1/2} M(\ell, R, T', x)) \\ &\ll d^{-(j-2)/2+\varepsilon} \mathcal{L}^c \sum_{\ell|d, \ell \leq R} (R/\ell)^{-(j-2)/2+\varepsilon} (\ell^{-1} R^2 T_2^{1/2} x^{11/20} + T_1^{-1/2} x) \mathcal{L}^c \\ &\ll d^{-(j-2)/2+\varepsilon} y (P^{(7-j)/4+\varepsilon} Q^{-1/2} N^{31/40} y^{-1} + 1) \mathcal{L}^c \\ &\ll d^{-(j-2)/2+\varepsilon} y \mathcal{L}^c, \end{aligned}$$

in view of our choice of (P, Q, y) (see (2.1) and (2.2)).

Similarly the contribution of the third term on the right-hand side of (2.27) to (2.21) is

$$(2.30) \quad \begin{aligned} &\ll d^{-(j-2)/2+\varepsilon} \mathcal{L}^c \sum_{\ell|d, \ell \leq R} (R/\ell)^{-(j-2)/2+\varepsilon} \max_{T_2 \leq T' \leq T} (T'^{-1} M(\ell, R, T', x)) \\ &\ll d^{-(j-2)/2+\varepsilon} \mathcal{L}^c \sum_{\ell|d, \ell \leq R} (R/\ell)^{-(j-2)/2+\varepsilon} (\ell^{-1} R^2 x^{11/20} + T_2^{-1} x) \mathcal{L}^c \\ &\ll d^{-(j-2)/2+\varepsilon} y (P^{(9-j)/4+\varepsilon} N^{11/40} y^{-1} + PQ(xy)^{-1}) \mathcal{L}^c \\ &\ll d^{-(j-2)/2+\varepsilon} y \mathcal{L}^c, \end{aligned}$$

in view of our choice of (P, Q, y) (see (2.1) and (2.2)).

Finally the contribution of the last term on the right-hand side of (2.27) to (2.21) is

$$(2.31) \quad \ll d^{-(j-2)/2+\varepsilon} \sum_{\ell|d, \ell \leq R} (R/\ell)^{-(j-2)/2+\varepsilon} \ll d^{-(j-2)/2+\varepsilon} \ll d^{-(j-2)/2+\varepsilon} y.$$

Now the inequality (2.26) follows from (2.28), (2.29), (2.30) and (2.31). This proves (2.21).

The proof of (2.22) is rather similar. Therefore we shall only point out different places. First the inequality (2.25) implies

$$\|W_\chi\|_2 \ll \|\widetilde{W}_\chi\|_2 + \mathcal{L}^2(r/Q)^{1/2}.$$

The contribution of $O(\mathcal{L}^2(r/Q)^{1/2})$ to (2.22) is

$$\begin{aligned} &\ll \mathcal{L}^2 Q^{-1/2} \sum_{\ell|d, \ell \leq P} \sum_{r \leq P, \ell|r} (dr/\ell)^{-(j-2)/2+\varepsilon} r^{1/2} \\ &\ll d^{-(j-2)/2+\varepsilon} \mathcal{L}^2 P^{1/2+\varepsilon} Q^{-1/2} \\ &\ll d^{-(j-2)/2+\varepsilon} N^{-1/4} y^{1/2}, \end{aligned}$$

since $P^{1+\varepsilon} N^{1/2} \ll_\varepsilon Qy$ in view of our choice of (P, Q, y) (see (2.1) and (2.2)). Thus in order to prove (2.22), it suffices to show that

$$(2.32) \quad \sum_{r \sim R} [d, r]^{-(j-2)/2+\varepsilon} \sum_{\chi \pmod{r}}^* \|\widetilde{W}_\chi\|_2 \ll d^{-(j-2)/2+\varepsilon} N^{-1/4} y^{1/2} \mathcal{L}^c$$

for any $R \leq P$. For this, by Lemma 1.9 of [14] we write, for $r \sim R$,

$$\begin{aligned} \|\widetilde{W}_\chi\|_2 &\ll \frac{1}{RQ} \left(\int_{-\infty}^{\infty} \left| \sum_{\substack{v-RQ/3 < n^2 \leq v+RQ/3 \\ x-y \leq n \leq x+y}} (\Lambda(n)\chi(n) - \delta_\chi) \right|^2 dv \right)^{1/2} \\ &\ll \frac{1}{RQ} \left(\int_{(x-y)^2-RQ}^{(x+y)^2+RQ/3} \left| \sum_{X \leq n \leq Y} (\Lambda(n)\chi(n) - \delta_\chi) \right|^2 dv \right)^{1/2}, \end{aligned}$$

where $X := \max\{(v-RQ/3)^{1/2}, x-y\}$ and $Y := \min\{(v+RQ/3)^{1/2}, x+y\}$.

If $R = 1$, we have

$$\begin{aligned} \left| \sum_{X \leq n \leq Y} (\Lambda(n)\chi(n) - \delta_\chi) \right| &= \left| \sum_{Y < n \leq X} (\Lambda(n) - 1) \right| \leq 2(X-Y)\mathcal{L} \\ &\ll \{(v+Q/3)^{1/2} - (v-Q/3)^{1/2}\} \mathcal{L} \\ &\ll Qv^{-1/2} \mathcal{L} \ll N^{-1/2} Q \mathcal{L}, \end{aligned}$$

which implies, in view of $Q < xy$,

$$(2.33) \quad \begin{aligned} d^{-(j-2)/2+\varepsilon} \|\widetilde{W}_{\chi_0^*}\|_2 &\ll d^{-(j-2)/2+\varepsilon} Q^{-1} ((N^{-1/2} Q \mathcal{L})^2 (xy+Q))^{1/2} \\ &\ll d^{-(j-2)/2+\varepsilon} N^{-1/4} y^{1/2} \mathcal{L}. \end{aligned}$$

For $R \geq 2$ and $r \sim R$, we have $\delta_\chi = 0$. Thus we can apply (2.14) of Lemma 2.3 to write

$$(2.34) \quad \|\widetilde{W}_\chi\|_2 \ll \left(\frac{y}{x^3}\right)^{1/2} \int_{|\tau| \leq T_0} |F(\kappa+i\tau, \chi)| d\tau + \frac{(xy)^{1/2}}{RQ} \int_{T_0 < |\tau| \leq T} \frac{|F(\kappa+i\tau, \chi)|}{|\tau|} d\tau + \frac{(xy)^{1/2}}{RQ},$$

since $T_0^{-1} = \log(Y/X) \asymp RQv^{-1} \asymp RQx^{-2}$ and $(x+y)^2 + RQ/3 - (x-y)^2 + RQ/3 \asymp xy$.

As before, the contribution of the first term on the left-hand side of (2.34) to (2.32) is

$$(2.35) \quad \begin{aligned} &\ll d^{-(j-2)/2+\varepsilon} (x^{-3}y)^{1/2} \sum_{\ell|d, \ell \leq 2R} (R/\ell)^{-(j-2)/2+\varepsilon} (\ell^{-1} R^2 T_0 x^{11/20} + x) \\ &\ll d^{-(j-2)/2+\varepsilon} N^{-1/4} y^{1/2} (P^{(5-j)/4+\varepsilon} Q^{-1} N^{31/40} + 1) \mathcal{L}^c \\ &\ll d^{-(j-2)/2+\varepsilon} N^{-1/4} y^{1/2} \mathcal{L}^c \end{aligned}$$

in view of our choice of (P, Q) ; the contribution of the second term on the left-hand side of (2.34) to (2.32) is

$$\begin{aligned}
&\ll d^{-(j-2)/2+\varepsilon} (xy)^{1/2} (RQ)^{-1} \mathcal{L}^c \sum_{\ell|d, \ell \leq R} (R/\ell)^{-(j-2)/2+\varepsilon} \max_{T_0 \leq T' \leq T} (T'^{-1} M(\ell, R, T', x)) \\
(2.36) \quad &\ll d^{-(j-2)/2+\varepsilon} (xy)^{1/2} (RQ)^{-1} \mathcal{L}^c \sum_{\ell|d, \ell \leq R} (R/\ell)^{-(j-2)/2+\varepsilon} (\ell^{-1} R^2 x^{11/20} + T_0^{-1} x) \mathcal{L}^c \\
&\ll d^{-(j-2)/2+\varepsilon} N^{-1/4} y^{1/2} (P^{(5-j)/4+\varepsilon} Q^{-1} N^{31/40} + 1) \mathcal{L}^c \\
&\ll d^{-(j-2)/2+\varepsilon} N^{-1/4} y^{1/2} \mathcal{L}^c,
\end{aligned}$$

the contribution of the last term on the left-hand side of (2.34) to (2.32) is

$$\begin{aligned}
(2.37) \quad &\ll d^{-(j-2)/2+\varepsilon} Q^{-1} (xy)^{1/2} \sum_{\ell|d, \ell \leq 2R} \sum_{r \sim R, \ell|r} (r/\ell)^{-(j-2)/2+\varepsilon} \\
&\ll d^{-(j-2)/2+\varepsilon} N^{-1/4} y^{1/2} R^{(5-j)/4+\varepsilon} Q^{-1} x \\
&\ll d^{-(j-2)/2+\varepsilon} N^{-1/4} y^{1/2} \mathcal{L}^c,
\end{aligned}$$

since $R^{(5-j)/4+\varepsilon} x \leq P^{(5-j)/4+\varepsilon} N^{1/2} \leq Q$.

Now the estimate (2.32) follows from (2.33), (2.36), (2.35) and (2.37). This proves (2.22).

The estimate (2.23) can be proved in the same way as Lemma 2.3 of [13] and we omit detail. This completes the proof of Lemma 2.2. \square

3. ASYMPTOTIC FORMULA FOR $\mathcal{R}_j(N; \mathfrak{M})$

The aim of this section is to treat the integral $\mathcal{R}_j(N; \mathfrak{M})$ over the major arcs \mathfrak{M} .

Proposition 3.1. *Let $j = 3, 4$. Then for $N \in \mathcal{A}_j$ with $N \rightarrow \infty$, we have*

$$(3.1) \quad \int_{\mathfrak{M}} S(\alpha)^j e(-\alpha N) d\alpha \sim C_j \mathfrak{S}_j(N) N^{-1/2} y^{j-1},$$

where C_j are some positive constants, $\phi(q)$ is the Euler function and

$$\mathfrak{S}_j(N) := \sum_{q=1}^{\infty} \frac{1}{\phi(q)^j} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left(\sum_{h=1}^q e^{2\pi i a h^2 / q} \right)^j e^{-2\pi i a N / q}.$$

Proof. Since $q \leq P < x - y$, we have $(p, q) = 1$ for all $p \in (x - y, x + y]$. By using the orthogonality relation, we can write

$$\begin{aligned}
S(a/q + \beta) &= \sum_{1 \leq h \leq q} e^{2\pi i a h^2 / q} \sum_{\substack{x-y \leq p \leq x+y \\ p \equiv h \pmod{q}, (p,q)=1}} (\log p) e(\beta p^2) \\
&= \frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \sum_{1 \leq h \leq q} \bar{\chi}(h) e^{2\pi i a h^2 / q} \sum_{x-y \leq p \leq x+y} \chi(p) (\log p) e(\beta p^2).
\end{aligned}$$

Introducing notation

$$(3.2) \quad C(\chi, a) := \sum_{1 \leq h \leq q} \bar{\chi}(h) e^{2\pi i a h^2 / q} \quad \text{and} \quad C(q, a) := C(\chi_0, a),$$

where χ_0 is the principal character modulo q , the preceding relation can be written as

$$(3.3) \quad S(a/q + \beta) = \frac{C(q, a)}{\phi(q)} S_0(\beta) + \frac{1}{\phi(q)} \sum_{\chi \pmod{q}} C(\chi, a) W_\chi(\beta),$$

where $S_0(\beta)$ and $W_\chi(\beta)$ are defined as in (2.18) and (2.19), respectively. In view of our choice of P and Q , we have $2P < Q$. Thus the intervals $I(a, q)$ are mutually disjoint and we can write, by using (3.3),

$$(3.4) \quad \begin{aligned} \int_{\mathfrak{M}} S(\alpha)^j e(-\alpha N) d\alpha &= \sum_{1 \leq q \leq P} \sum_{\substack{1 \leq a \leq q \\ (a, q) = 1}} e^{-2\pi i a N/q} \int_{-1/(qQ)}^{1/(qQ)} S(a/q + \beta)^j e(-\beta N) d\beta \\ &= \sum_{0 \leq k \leq j} C_k^j I_k, \end{aligned}$$

where

$$\begin{aligned} I_k &:= \sum_{1 \leq q \leq P} \frac{1}{\phi(q)^j} \sum_{\substack{1 \leq a \leq q \\ (a, q) = 1}} C(q, a)^{j-k} e^{-2\pi i a N/q} \times \\ &\quad \times \int_{-1/(qQ)}^{1/(qQ)} S_0(\alpha)^{j-k} \left(\sum_{\chi \pmod{q}} C(\chi, a) W_\chi(\beta) \right)^k e(-\beta N) d\beta. \end{aligned}$$

We shall see that I_0 contributes the main term and the others I_j are as error terms.

By the standard major arcs techniques, we can prove

$$(3.5) \quad I_0 = C_j \mathfrak{S}_j(N) y^{j-1} N^{-1/2} \{1 + o(1)\}.$$

It remains to control the I_k ($1 \leq k \leq j$). We shall only treat I_j , the others can be treated similarly (even more easily). We can write

$$I_j = \sum_{1 \leq q \leq P} \sum_{\chi_1 \pmod{q}} \cdots \sum_{\chi_j \pmod{q}} B_j(N, q; \chi_1, \dots, \chi_j) J_j(N, q; \chi_1, \dots, \chi_j),$$

where

$$\begin{aligned} B_j(N, q; \chi_1, \dots, \chi_j) &:= \frac{1}{\phi(q)^j} \sum_{\substack{a=1 \\ (a, q) = 1}}^q C(\chi_1, a) \cdots C(\chi_j, a) e^{-2\pi i a N/q}, \\ J_j(N, q; \chi_1, \dots, \chi_j) &:= \int_{-1/(qQ)}^{1/(qQ)} W_{\chi_1}(\beta) \cdots W_{\chi_j}(\beta) e(-\beta N) d\beta. \end{aligned}$$

Suppose that $\chi_k^* \pmod{r_k}$ with $r_k \mid q$ is the primitive character inducing χ_k . Thus we can write $\chi_k = \chi_0 \chi_k^*$. It is easy to see that $W_{\chi_k}(\beta) = W_{\chi_k^*}(\beta)$. By Cauchy's inequality, it follows that

$$(3.6) \quad |J_j(N, q; \chi_1, \dots, \chi_j)| \leq W_{\chi_1^*}^\sharp \cdots W_{\chi_{j-2}^*}^\sharp W_{\chi_{j-1}^*} W_{\chi_j^*},$$

where W_χ^\sharp and $\|W_\chi\|_2$ are defined as in (2.20) with $r := [r_1, \dots, r_j]$. From (3.6) and the inequality (see [12] for $j = 3$ and [1] for $j = 5$. The general case can be treated in the same way.)

$$\sum_{q \leq z, r \mid q} |B_j(N, q; \chi_1^* \chi_0, \dots, \chi_j^* \chi_0)| \ll_\varepsilon r^{-(j-2)/2+\varepsilon} (\log z)^c,$$

we deduce

$$\begin{aligned} I_j &\ll \mathcal{L}^c \sum_{r_1 \leq P} \sum_{\chi_1 \pmod{r_1}}^* W_{\chi_1}^\sharp \cdots \sum_{r_{j-2} \leq P} \sum_{\chi_{j-2} \pmod{r_{j-2}}}^* W_{\chi_{j-2}}^\sharp \times \\ &\quad \times \sum_{r_{j-1} \leq P} \sum_{\chi_{j-1} \pmod{r_{j-1}}}^* W_{\chi_{j-1}} \sum_{r_j \leq P} [r_1, \dots, r_j]^{-(j-2)/2+\varepsilon} \sum_{\chi_j \pmod{r_j}}^* W_{\chi_j}. \end{aligned}$$

By noticing that $[r_1, \dots, r_j] = [[r_1, \dots, r_{j-1}], r_j]$, we use consecutively (2.22) (2 times), (2.21) ($j-3$ times) and (2.23) (1 time) of Proposition 2.2 to write

$$\begin{aligned}
I_j &\ll N^{-1/4} y^{1/2} \mathcal{L}^c \sum_{r_1 \leq P} \sum_{\chi_1 \pmod{r_1}}^* W_{\chi_1}^\# \cdots \sum_{r_{j-2} \leq P} \sum_{\chi_{j-2} \pmod{r_{j-2}}}^* W_{\chi_{j-2}}^\# \times \\
&\quad \times \sum_{r_{j-1} \leq P} [r_1, \dots, r_{j-1}]^{-(j-2)/2+\varepsilon} \sum_{\chi_{j-1} \pmod{r_{j-1}}}^* W_{\chi_{j-1}} \\
(3.7) \quad &\ll N^{-1/2} y \mathcal{L}^c \sum_{r_1 \leq P} \sum_{\chi_1 \pmod{r_1}}^* W_{\chi_1}^\# \cdots \sum_{r_{j-2} \leq P} [r_1, \dots, r_{j-2}]^{-(j-2)/2+\varepsilon} \sum_{\chi_{j-2} \pmod{r_{j-2}}}^* W_{\chi_{j-2}}^\# \\
&\ll N^{-1/2} y^{j-2} \mathcal{L}^c \sum_{r_1 \leq P} r_1^{-(j-2)/2+\varepsilon} \sum_{\chi_1 \pmod{r_1}}^* W_{\chi_1}^\# \\
&\ll N^{-1/2} y^{j-1} \mathcal{L}^{-A}
\end{aligned}$$

for any fixed $A > 0$.

Now the required asymptotic formula follows from (3.4), (3.5) and (3.7). \square

4. PROOF OF THEOREM 1 (THE MINOR ARCS)

We first prove a preliminary lemma, which can be regarded as generalisation of Hua's lemma ([17], Lemma 2.5) in the case of short intervals.

Lemma 4.1. *Let $X \geq Y \geq 2$, $k \in \mathbb{N}$ and*

$$S_k^*(\alpha) := \sum_{X-Y \leq n \leq X+Y} e(\alpha n^k)$$

Then for any $\varepsilon > 0$ and $1 \leq j \leq k$, we have

$$\int_0^1 |S_k^*(\alpha)|^{2^j} d\alpha \ll_\varepsilon X^\varepsilon Y^{2^j-j}.$$

Proof. We prove only the case of $j = k = 2$. The general case can be treated by recurrence. We first write

$$\begin{aligned}
|S_2^*(\alpha)|^2 &= \sum_{X-Y \leq m \leq X+Y} \sum_{X-Y \leq n \leq X+Y} e(\alpha(m^2 - n^2)) \quad (m = n + h) \\
&= \sum_{-2Y \leq h \leq 2Y} \sum_{X-Y \leq n \leq X+Y} e(\alpha h(2n + h)) \\
&= \sum_d a_d e(\alpha h(2n + h))
\end{aligned}$$

with

$$a_d := \sum_{-2Y \leq h \leq 2Y} \sum_{\substack{X-Y \leq n \leq X+Y \\ h(2n+h)=d}} 1.$$

Clearly $a_0 \leq 4Y$ and $a_d \leq \tau(|d|) \ll_\varepsilon |d|^{\varepsilon/2} \ll_\varepsilon (XY)^{\varepsilon/2} \ll_\varepsilon X^\varepsilon$ ($d \neq 0$) where $\tau(|d|)$ is the divisor function. On the other hand, we can write

$$|S_2^*(\alpha)|^2 = \sum_{X-Y \leq m \leq X+Y} \sum_{X-Y \leq n \leq X+Y} e(-\alpha(m^2 - n^2)) = \sum_d b_d e(-\alpha d),$$

where

$$b_d := \sum_{X-Y \leq m \leq X+Y} \sum_{\substack{X-Y \leq n \leq X+Y \\ m^2 - n^2 = d}} 1.$$

Clearly $b_0 \leq 2Y$ and

$$\sum_d b_d = |S^*(0)|^2 \leq (2Y)^2.$$

Thus

$$\begin{aligned} \int_0^1 |S_2^*(\alpha)|^4 d\alpha &= \int_0^1 \left(a_0 + \sum_{d_1 \neq 0} a_{d_1} e(\alpha d_1) \right) \left(b_0 + \sum_{d_2 \neq 0} b_{d_2} e(-\alpha d_2) \right) d\alpha \\ &= a_0 b_0 + \sum_{d \neq 0} a_d b_d \ll Y^2 + X^\varepsilon Y^2 \ll_\varepsilon X^\varepsilon Y^2. \end{aligned}$$

This completes the proof. \square

Next we shall apply the device introduced by Wooley [18] to prove Theorem 1. Let $j = 3$ or 4 and denote by $\mathcal{E}_j^*(z)$ the set of integers $N \in \mathcal{A}_j \cap [z/2, z]$ such that

$$N \neq p_1^2 + \cdots + p_j^2 \quad \text{with} \quad |p_i - (N/j)^{1/2}| \leq N^{1/2-9/100+\varepsilon} \quad (1 \leq i \leq j).$$

Introduce the generating function

$$Z(\alpha) := \sum_{N \in \mathcal{E}_j^*(z)} e(-\alpha N).$$

Clearly we have

$$\int_0^1 S(\alpha)^j Z(\alpha) d\alpha = 0.$$

By using Proposition 3.1 with $j = 3, 4$, it follows that

$$\begin{aligned} \left| \int_{\mathfrak{m}} S(\alpha)^j Z(\alpha) d\alpha \right| &= \left| \int_{\mathfrak{M}} S(\alpha)^j Z(\alpha) d\alpha \right| \\ &= \sum_{N \in \mathcal{E}_j^*(z)} \int_{\mathfrak{M}} S(\alpha)^j e(-\alpha N) d\alpha \\ &\gg |\mathcal{E}_j^*(z)| N^{-1/2} y^{j-1}. \end{aligned}$$

From this and (2.9), we deduce that

$$\begin{aligned} |\mathcal{E}_j^*(z)| &\ll N^{1/2} y^{-j+1} \int_{\mathfrak{m}} |S(\alpha)^j Z(\alpha)| d\alpha \\ &\ll N^{1/2-2(j-2)\varepsilon} y^{-1} \int_0^1 |S(\alpha)^2 Z(\alpha)| d\alpha \\ &\ll N^{1/2-2(j-2)\varepsilon} y^{-1} \left(\int_0^1 |Z(\alpha)|^2 d\alpha \right)^{1/2} \left(\int_0^1 |S(\alpha)|^4 d\alpha \right)^{1/2}. \end{aligned}$$

Clearly

$$\int_0^1 |Z(\alpha)|^2 d\alpha = |\mathcal{E}_j^*(z)|$$

and Lemma 4.1 implies

$$\int_0^1 |S(\alpha)|^4 d\alpha \ll \int_0^1 |S_2^*(\alpha)|^4 d\alpha \ll N^\varepsilon y^2.$$

Thus

$$|\mathcal{E}_j^*(z)| \ll N^{1/2-(2j-5)\varepsilon} |\mathcal{E}_j^*(z)|^{1/2},$$

which is equivalent to $|\mathcal{E}_j^*(z)| \ll z^{1-(4j-10)\varepsilon}$. This completes the proof of Theorem 1. \square

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