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Hypothesis H and the prime number theorem for automorphic representations

JIE WU (Nancy) & YANGBO YE ¹ (Iowa)

Dedicated to Jean-Marc Deshouillers on the occasion of his sixtieth birthday

Abstract. For any unitary cuspidal representations π_n of $GL_n(\mathbb{Q}_{\mathbb{A}})$, $n = 2, 3, 4$, respectively, consider two automorphic representations Π and Π' of $GL_6(\mathbb{Q}_{\mathbb{A}})$, where $\Pi_p \cong \wedge^2 \pi_{4,p}$ for $p \neq 2, 3$ and $\pi_{4,p}$ not supercuspidal, and $\Pi' = \pi_2 \boxtimes \pi_3$. First, Hypothesis H for Π and Π' is proved. Then contributions from prime powers are removed from the prime number theorem for cuspidal representations π and π' of $GL_m(\mathbb{Q}_{\mathbb{A}})$ and $GL_{m'}(\mathbb{Q}_{\mathbb{A}})$, respectively. The resulting prime number theorem is unconditional when $m, m' \leq 4$ and is under Hypothesis H otherwise.

Keywords. Hypothesis H, functoriality, prime number theorem

§ 1. Introduction

Recent developments in functoriality by the Langlands-Shahidi method have many profound applications in prime distribution. To name a few, we recall a recent proof of Hypothesis H for any cuspidal representation of $GL_4(\mathbb{Q}_{\mathbb{A}})$ and for $\text{Sym}^4(\pi)$ by Kim [2], where π is an automorphic cuspidal representation of $GL_2(\mathbb{Q}_{\mathbb{A}})$. Here Hypothesis H predicts the convergence of a certain Dirichlet series associated with $(L'/L)'(s, \pi \times \tilde{\pi})$ taken over prime powers.

More precisely, let $\pi = \otimes_p \pi_p$ be a unitary automorphic cuspidal representation of $GL_m(\mathbb{Q}_{\mathbb{A}})$. Or more generally, let π be an automorphic representation irreducibly induced from unitary cuspidal representations, i.e., $\pi = \text{Ind } \sigma_1 \otimes \cdots \otimes \sigma_k$, where σ_j is a cuspidal representation of $GL_{m_j}(\mathbb{Q}_{\mathbb{A}})$, with $m_1 + \cdots + m_k = m$. The local component π_p with $p < \infty$ can be parameterized by the Satake parameters $\text{diag}[\alpha_{\pi}(p, 1), \dots, \alpha_{\pi}(p, m)]$. For $\nu \geq 1$ define

$$(1.1) \quad a_{\pi}(p^{\nu}) = \sum_{j=1}^m \alpha_{\pi}(p, j)^{\nu}.$$

Let $\tilde{\pi}$ be the contragredient representation of π , and $L(s, \pi \times \tilde{\pi})$ the Rankin-Selberg L -function. Then for $\Re s > 1$, we have (see [10], **RS 1**)

$$(1.2) \quad \left(\frac{L'}{L}\right)'(s, \pi \times \tilde{\pi}) = \sum_{n=1}^{\infty} \frac{(\log n) \Lambda(n) |a_{\pi}(n)|^2}{n^s}.$$

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Here $\Lambda(n) = \log p$ if $n = p^\nu$ and $\Lambda(n) = 0$ otherwise, so that the series in (1.2) is taken over primes and prime powers.

Hypothesis H. (Rudnick and Sarnak [10]) *For any fixed $\nu \geq 2$,*

$$\sum_p \frac{(\log p)^2 |a_\pi(p^\nu)|^2}{p^\nu} < \infty.$$

Hypothesis H is trivial for $m = 1$. When $m = 2$ it follows from bounds toward the Ramanujan conjecture $|\alpha_\pi(p, j)| \leq p^\theta$ with $\theta = 7/64$ (see [9]), another result based on the Langlands-Shahidi method proved by Kim and Sarnak in [1]. For $m = 3$, Hypothesis H follows from the Rankin-Selberg theory [10]. The GL_4 case was proved by Kim [2] based on his proof of the (weak) functoriality of the exterior square $\wedge^2 \pi$ from a cuspidal representation π of $GL_4(\mathbb{Q}_\mathbb{A})$ (see [1]). Beyond GL_4 , the only known special case for Hypothesis H is the symmetric fourth power $\text{Sym}^4(\pi)$ of a cuspidal representation π of $GL_2(\mathbb{Q}_\mathbb{A})$, which is an automorphic representation of $GL_5(\mathbb{Q}_\mathbb{A})$.

The first goal of the present paper is to prove Hypothesis H for two types of automorphic representations of $GL_6(\mathbb{Q}_\mathbb{A})$.

Theorem 1. *Let π be a cuspidal representation of $GL_4(\mathbb{Q}_\mathbb{A})$. Denote by T the set of places consisting of $p = 2, 3$ and those p at which π_p is supercuspidal. Let Π be the automorphic representation of $GL_6(\mathbb{Q}_\mathbb{A})$ such that $\Pi_p \cong \wedge^2 \pi_p$ if $p \notin T$, according to [1]. Then Hypothesis H holds for Π .*

Theorem 2. *Let π_1 (resp. π_2) be a cuspidal representation of $GL_2(\mathbb{Q}_\mathbb{A})$ (resp. $GL_3(\mathbb{Q}_\mathbb{A})$). Let Π' be the automorphic representation of $GL_6(\mathbb{Q}_\mathbb{A})$ equal to $\pi \boxtimes \pi_2$ according to [3]. Then Hypothesis H holds for Π' .*

As an application, one can use Hypothesis H to deduce the following Mertens' theorem for automorphic representations, or the so-called Selberg orthogonality conjecture, from unconditional results on similar sums taken over primes and prime powers:

$$(1.3) \quad \sum_{p \leq x} \frac{|a_\pi(p)|^2}{p} = \log \log x + O(1);$$

$$(1.4) \quad \sum_{p \leq x} \frac{a_\pi(p) \bar{a}_{\pi'}(p)}{p} = O(1),$$

when $\pi \not\cong \pi'$. Here (1.3) was proved by Rudnick and Sarnak [10], while (1.4) was proved by Liu, Wang and Ye ([6], [4]). Results in (1.3) and (1.4) played crucial roles in the n -level correlation of nontrivial zeros of automorphic L -functions and random matrix theory ([10], [5], [7]).

Another application of Hypothesis H is on the prime number theorem for automorphic representations. For any self-dual cuspidal representation π of $GL_m(\mathbb{Q}_\mathbb{A})$, Liu, Wang and Ye [4] showed that there is a constant $c > 0$ such that

$$(1.5) \quad \sum_{n \leq x} \Lambda(n) |a_\pi(n)|^2 = x + O(x \exp(-c\sqrt{\log x})).$$

In [8], Liu and Ye proved that

$$(1.6) \quad \sum_{n \leq x} \Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n) = \begin{cases} \frac{x^{1+i\tau_0}}{1+i\tau_0} + O(x \exp(-c\sqrt{\log x})) & \text{if } \pi' \cong \pi \otimes |\det|^{i\tau_0} \text{ for some } \tau_0 \in \mathbb{R}; \\ O(x \exp(-c\sqrt{\log x})) & \text{if } \pi' \not\cong \pi \otimes |\det|^{i\tau} \text{ for any } \tau \in \mathbb{R}, \end{cases}$$

where π and π' are cuspidal representations of $GL_m(\mathbb{Q}_\mathbb{A})$ and $GL_{m'}(\mathbb{Q}_\mathbb{A})$, respectively, such that at least one of them is self-dual.

The second goal of the present paper is to use Hypothesis H to remove terms on prime powers from the left side of (1.6) and deduce a prime number theorem over primes.

Theorem 3. *Let π and π' be as above. (i) If $m, m' \leq 4$, then*

$$(1.7) \quad \sum_{p \leq x} (\log p) a_\pi(p) \bar{a}_{\pi'}(p) = \begin{cases} \frac{x^{1+i\tau_0}}{1+i\tau_0} + O(x \exp(-c\sqrt{\log x})) & \text{if } \pi' \cong \pi \otimes |\det|^{i\tau_0} \text{ for some } \tau_0 \in \mathbb{R}, \\ O(x \exp(-c\sqrt{\log x})) & \text{if } \pi' \not\cong \pi \otimes |\det|^{i\tau} \text{ for any } \tau \in \mathbb{R}. \end{cases}$$

(ii) If $\max(m, m') \geq 5$, (1.7) is true under Hypothesis H with error terms replaced by $O(x/\log x)$.

We remark that (i) is an unconditional result.

§ 2. Proof of Theorems 1 and 2

Lemma 2.1. *Let π be a unitary cuspidal representation for $GL_m(\mathbb{Q}_\mathbb{A})$, or an automorphic representation irreducibly induced from unitary cuspidal representations. Then for any $\nu_0 \geq (m^2 + 1)/2 + 1$, $\varepsilon > 0$, and integer $\ell \geq 0$,*

$$(2.1) \quad \sum_{\nu \geq \nu_0, p^\nu \leq x} (\log p) |a_\pi(p^\nu)|^2 \ll x^{1-2/(m^2+1)+1/\nu_0} \log x,$$

$$(2.2) \quad \sum_p \frac{(\log p)^\ell |a_\pi(p)|^2}{p^{1+\varepsilon}} < \infty.$$

Proof. From (1.1) and the bound toward the Ramanujan conjecture ([10])

$$(2.3) \quad |\alpha_\pi(p, j)| \leq p^{1/2-1/(m^2+1)} \quad (j = 1, \dots, m),$$

we know that

$$|a_\pi(p^\nu)|^2 \leq m^2 p^{\{1-2/(m^2+1)\}\nu}.$$

Then

$$\begin{aligned} \sum_{\nu \geq \nu_0, p^\nu \leq x} (\log p) |a_\pi(p^\nu)|^2 &\leq m^2 \sum_{\nu_0 \leq \nu \leq 2 \log x} \sum_{p \leq x^{1/\nu}} (\log p) p^{\{1-2/(m^2+1)\}\nu} \\ &\ll_m x^{1-2/(m^2+1)+1/\nu_0} \log x. \end{aligned}$$

Eqn. (2.2) follows from the fact that the ℓ th-derivation of $\log L(s, \pi \times \tilde{\pi})$ converges absolutely for $\Re s > 1$. \square

Lemma 2.2. *Let π' (resp. π'') be a unitary cuspidal representation, or an automorphic representation irreducibly induced from unitary cuspidal representations, for $GL_{m'}(\mathbb{Q}_{\mathbb{A}})$ (resp. $GL_{m''}(\mathbb{Q}_{\mathbb{A}})$). Let $\nu \geq 2$ be an integer and \mathcal{P} a set of prime numbers. If there are fixed constants $\delta' \in (0, 1]$ and $\delta'' \in (0, \frac{1}{2}]$ such that*

$$(2.4) \quad |a_{\pi'}(p^\nu)|^2 \ll_{\nu} |a_{\pi''}(p)|^2 p^{(1-\delta')(\nu-1)} + p^{(1/2-\delta'')\nu}$$

for all $p \in \mathcal{P}$, then for any $\varepsilon > 0$ we have

$$(2.5) \quad \sum_{p^\nu \leq x, p \in \mathcal{P}} (\log p) |a_{\pi'}(p^\nu)|^2 \ll_{\nu, \varepsilon} x^{1-\delta}$$

with $\delta := \min\{\delta'/(2+\delta') - \varepsilon, \delta''\}$.

Proof. By (2.4) and the Rankin-Selberg theory, for any $\eta > 0$ we can write

$$\begin{aligned} \sum_{\substack{p^\nu \leq x \\ p \in \mathcal{P}}} (\log p) |a_{\pi'}(p^\nu)|^2 &\ll_{\nu} \sum_{\substack{p^\nu \leq x \\ p \in \mathcal{P}}} (\log p) |a_{\pi''}(p)|^2 p^{(1-\delta')(\nu-1)} + x^{1/2+1/\nu-\delta''} \\ &\ll_{\nu} x^{\eta} \sum_{\substack{p^\nu \leq x^{\eta} \\ p \in \mathcal{P}}} \frac{(\log p) |a_{\pi''}(p)|^2}{p^{1+\delta'(\nu-1)}} + x \sum_{\substack{x^{\eta} < p^\nu \leq x \\ p \in \mathcal{P}}} \frac{(\log p) |a_{\pi''}(p)|^2}{p^{1+\delta'(\nu-1)}} + x^{1-\delta''}. \end{aligned}$$

By (2.2) with $\pi = \pi''$ and $\ell = 1$, it follows that

$$\sum_{\substack{p^\nu \leq x^{\eta} \\ p \in \mathcal{P}}} \frac{(\log p) |a_{\pi''}(p)|^2}{p^{1+\delta'(\nu-1)}} \ll 1$$

and

$$\begin{aligned} \sum_{\substack{x^{\eta} < p^\nu \leq x \\ p \in \mathcal{P}}} \frac{(\log p) |a_{\pi''}(p)|^2}{p^{1+\delta'(\nu-1)}} &\leq \frac{1}{(x^{\eta}/\nu)^{\delta'(\nu-1)-\varepsilon}} \sum_{\substack{x^{\eta} < p^\nu \leq x \\ p \in \mathcal{P}}} \frac{(\log p) |a_{\pi''}(p)|^2}{p^{1+\varepsilon}} \\ &\leq x^{-\eta[\delta'(\nu-1)-\varepsilon]/\nu}. \end{aligned}$$

Inserting these two estimates into the preceding inequality, we find

$$\sum_{\substack{p^\nu \leq x \\ p \in \mathcal{P}}} (\log p) |a_{\pi'}(p^\nu)|^2 \ll_{\nu, \varepsilon} x^{\eta} + x^{1-\eta[\delta'(\nu-1)-\varepsilon]/\nu} + x^{1-\delta''}.$$

Taking $\eta = \nu/\{(1+\delta')\nu - \delta'\} + \varepsilon$, we obtain

$$\begin{aligned} \sum_{p^\nu \leq x, p \in \mathcal{P}} (\log p) |a_{\pi'}(p^\nu)|^2 &\ll_{\nu, \varepsilon} x^{\nu/\{(1+\delta')\nu - \delta'\} + \varepsilon} + x^{1-\delta''} \\ &\ll_{\nu, \varepsilon} x^{1-\delta'/(2+\delta') + \varepsilon} + x^{1-\delta''} \\ &\ll_{\nu, \varepsilon} x^{1-\delta}. \end{aligned}$$

In the second inequality, we have used the fact that $\nu \geq 2$. □

Remark. In proving Hypothesis H, an inequality of the form of (2.4) plays a crucial role. Lemma 2.2 has more flexibility as π'' is allowed to be different from π' .

Lemma 2.3. *Let Π'' be either Π or Π' as in Theorems 1 and 2. Then for any $\varepsilon > 0$, we have*

$$(2.6) \quad \sum_{\nu \geq 2, p^\nu \leq x} (\log p) |a_{\Pi''}(p^\nu)|^2 \ll_\varepsilon x^{1-1/38+\varepsilon}.$$

Proof. In view of (2.1) with the choice of $m = 6$ and $\nu_0 = [37 \times 38/39] + 1$, it suffices to show that for any fixed $\varepsilon > 0$ and $\nu \geq 2$ we have

$$(2.7) \quad \sum_{p^\nu \leq x} (\log p) |a_\Pi(p^\nu)|^2 \ll_{\nu, \varepsilon} x^{1-1/38+\varepsilon}, \quad \sum_{p^\nu \leq x} (\log p) |a_{\Pi'}(p^\nu)|^2 \ll_{\nu, \varepsilon} x^{1-1/38+\varepsilon}.$$

First let us consider the case of Π . Let $\pi = \otimes \pi_p$ be a cuspidal automorphic representation for $GL_4(\mathbb{A}_\mathbb{Q})$. Recall that Π is irreducibly induced from unitary cuspidal representations. Let S_0 be the set of places where Π_p is tempered. Then

$$(2.8) \quad \sum_{p \in S_0} (\log p)^2 |a_\Pi(p^\nu)|^2 < \infty.$$

Eqn. (2.8) is also true if we replace S_0 by T , which is given in Theorem 1, because at most two terms for $p = 2, 3$ will then be added to (2.8).

If $p \notin S_0 \cup T$, the Satake parameters of π_p are in one of the following forms:

$$(2.9) \quad \begin{aligned} S_1 &: \text{diag}[u_1 p^a, u_2 p^a, u_1 p^{-a}, u_2 p^{-a}], \quad \text{where } 0 < a \leq \frac{1}{2} - \frac{1}{17}, \\ S_2 &: \text{diag}[u_1 p^a, u_2, u_3, u_1 p^{-a}], \quad \text{where } 0 < a \leq \frac{1}{2} - \frac{1}{17}, \\ S_3 &: \text{diag}[u_1 p^{a_1}, u_2 p^{a_2}, u_1 p^{-a_1}, u_2 p^{-a_2}], \quad \text{where } 0 < a_2 < a_1 \leq \frac{1}{2} - \frac{1}{17}, \end{aligned}$$

where u_1, u_2, u_3 are complex numbers of absolute value 1 and we have suppressed their dependence on p for the simplicity of notation. As in [1], the corresponding Satake parameters of $\Pi_p \simeq \wedge^2 \pi_p$ are as follows:

$$\begin{aligned} S_1 &: \text{diag}[u_1 u_2 p^{2a}, u_1 u_2, u_1^2, u_2^2, u_1 u_2, u_1 u_2 p^{-2a}], \\ S_2 &: \text{diag}[u_1 u_2 p^a, u_1 u_3 p^a, u_1^2, u_2 u_3, u_1 u_2 p^{-a}, u_1 u_3 p^{-a}], \\ S_3 &: \text{diag}[u_1 u_2 p^{a_1+a_2}, u_1 u_2 p^{a_1-a_2}, u_1^2, u_2^2, u_1 u_2 p^{-(a_1-a_2)}, u_1 u_2 p^{-(a_1+a_2)}]. \end{aligned}$$

Since Π is a automorphic representation for $GL_6(\mathbb{A}_\mathbb{Q})$ which is irreducibly induced from unitary cuspidal, (2.3) gives

$$(2.10) \quad \begin{cases} 0 < 2a \leq \frac{1}{2} - \frac{1}{37} & \text{if } p \in S_1, \\ 0 < a \leq \frac{1}{2} - \frac{1}{17} & \text{if } p \in S_2, \\ 0 < a_2 < a_1 \leq \frac{1}{2} - \frac{1}{17} \quad \text{and} \quad a_1 + a_2 \leq \frac{1}{2} - \frac{1}{37} & \text{if } p \in S_3. \end{cases}$$

If $p \in S_1$, then

$$\begin{aligned} |a_\Pi(p^\nu)| &= |(u_1 u_2)^\nu (p^{2a\nu} + p^{-2a\nu} + 2) + u_1^{2\nu} + u_2^{2\nu}| \leq p^{2a\nu} + 5, \\ |a_\Pi(p)| &= |u_1 u_2 (p^{2a} + p^{-2a} + 2) + u_1^2 + u_2^2| \geq p^{2a}. \end{aligned}$$

From these and (2.3) with $m = 6$, we deduce that

$$|a_\Pi(p^\nu)|^2 \leq (|a_\Pi(p)|^\nu + 5)^2 \ll_\nu |a_\Pi(p)|^{2\nu} + 1 \ll_\nu |a_\Pi(p)|^2 p^{(1-2/37)(\nu-1)} + 1,$$

where the implied constants are all independent of p .

Similarly if $p \in S_2$, then

$$\begin{aligned} |a_\Pi(p^\nu)| &= |u_1^\nu(u_2^\nu + u_3^\nu)(p^{a\nu} + p^{-a\nu}) + u_1^{2\nu} + (u_2u_3)^\nu| \leq 2p^{a\nu} + 4, \\ |a_\pi(p)| &= |u_1(p^a + p^{-a}) + u_2 + u_3| \geq p^a - 2. \end{aligned}$$

These and (2.3) with $m = 4$ imply

$$(2.11) \quad |a_\Pi(p^\nu)|^2 \leq \{2(|a_\pi(p)| + 2)^\nu + 4\}^2 \ll_\nu |a_\pi(p)|^{2\nu} + 1 \ll_\nu |a_\pi(p)|^2 p^{(1-2/17)(\nu-1)} + 1.$$

Finally if $p \in S_3$, then

$$|a_\Pi(p^\nu)| \leq 2p^{(a_1+a_2)\nu} + 4, \quad |a_\Pi(p)| \geq p^{a_1+a_2} - 1,$$

from which we deduce, as before,

$$(2.12) \quad |a_\Pi(p^\nu)|^2 \leq \{2(|a_\Pi(p)| + 1)^\nu + 4\}^2 \ll_\nu |a_\Pi(p)|^{2\nu} + 1 \ll_\nu |a_\Pi(p)|^2 p^{(1-2/37)(\nu-1)} + 1.$$

Now we apply Lemma 2.2 with the choice of parameters

$$(\pi', \pi'', \delta', \delta'') = \begin{cases} (\Pi, \Pi, \frac{2}{37}, \frac{1}{2}) & \text{if } \mathcal{P} = S_1 \text{ or } S_3 \\ (\Pi, \pi, \frac{2}{17}, \frac{1}{2}) & \text{if } \mathcal{P} = S_2 \end{cases}$$

to write

$$(2.13) \quad \sum_{p^\nu \leq x, p \in S_j} (\log p) |a_\Pi(p^\nu)|^2 \ll_\nu \begin{cases} x^{1-1/38+\varepsilon} & \text{if } j = 1, 3. \\ x^{1-1/19+\varepsilon} & \text{if } j = 2, \end{cases}$$

Now the required estimate (2.7) for Π follows from (2.10) and (2.13).

Next let us turn to the case of Π' . Let $\pi_1 = \otimes_p \pi_{1,p}$ (resp. $\pi_2 = \otimes_p \pi_{2,p}$) be a cuspidal representation of $GL_2(\mathbb{Q}_\mathbb{A})$ (resp. $GL_3(\mathbb{Q}_\mathbb{A})$). We may just consider those p such that at least one of $\pi_{1,p}$ and $\pi_{2,p}$ is not tempered. Then the Satake parameters of $\pi_{1,p}$ and $\pi_{2,p}$ are as follows:

$$\begin{aligned} \pi_{1,p} &: \text{diag}[u_1 p^a, u_1 p^{-a}], \text{ where } 0 \leq a \leq \frac{7}{64}, \\ \pi_{2,p} &: \text{diag}[u_2 p^b, u_3, u_2 p^{-b}], \text{ where } 0 \leq b \leq \frac{1}{2} - \frac{1}{10}, \end{aligned}$$

where u_1, u_2, u_3 are complex numbers of absolute value 1. Thus the Satake parameters of $\Pi'_p = \pi_{1,p} \boxtimes \pi_{2,p}$ are:

$$\text{diag}[u_1 u_2 p^{a+b}, u_1 u_2 p^{b-a}, u_1 u_3 p^a, u_1 u_3 p^{-a}, u_1 u_2 p^{-(b-a)}, u_1 u_2 p^{-(a+b)}]$$

with

$$(2.14) \quad 0 < a + b \leq \frac{1}{2} - \frac{1}{37}.$$

Then

$$(2.15) \quad |a_{\Pi'}(p^\nu)| = |(u_1 u_2)^\nu (p^{(a+b)\nu} + p^{(a-b)\nu} + p^{(b-a)\nu} + p^{-(a+b)\nu}) + (u_1 u_3)^\nu (p^{a\nu} + p^{-a\nu})|.$$

From (2.15) we can see that

$$(2.16) \quad |a_{\Pi'}(p^\nu)| \leq 6p^{(a+b)\nu}, \quad |a_{\Pi'}(p)| \geq p^{a+b} - p^a.$$

Thus in view of (2.14), (2.16) and the fact that $a \leq \frac{7}{64}$, we can deduce

$$(2.17) \quad \begin{aligned} |a_{\Pi'}(p^\nu)|^2 &\ll (|a_{\Pi'}(p)| + p^a)^{2\nu} \ll_\nu |a_{\Pi'}(p)|^{2\nu} + p^{2a\nu} \\ &\ll_\nu |a_{\Pi'}(p)|^2 p^{(1-2/37)(\nu-1)} + p^{(1/2-9/32)\nu}. \end{aligned}$$

Applying Lemma 2.2 with $\pi' = \pi'' = \Pi'$, $\delta' = \frac{2}{37}$ and $\delta'' = \frac{9}{32}$, we now conclude that

$$\sum_{p^\nu \leq x} (\log p) |a_{\Pi'}(p^\nu)|^2 \ll x^{1-1/38+\varepsilon}.$$

This completes the proof. □

The proof of Theorems 1 and 2.

Let Π'' be either Π or Π' . We can write

$$\begin{aligned} \sum_{p^\nu > x, \nu \geq 2} \frac{(\log p)^2 |a_{\Pi''}(p^\nu)|^2}{p^\nu} &= \sum_{j \geq 0} \sum_{2^j x < p^\nu \leq 2^{j+1} x, \nu \geq 2} \frac{(\log p)^2 |a_{\Pi''}(p^\nu)|^2}{p^\nu} \\ &\leq \sum_{j \geq 0} \frac{\log(2^{j+1} x)}{2^j x} \sum_{2^j x < p^\nu \leq 2^{j+1} x, \nu \geq 2} (\log p) |a_{\Pi''}(p^\nu)|^2. \end{aligned}$$

Using Lemma 2.3, we have

$$\begin{aligned} \sum_{p^\nu > x, \nu \geq 2} \frac{(\log p)^2 |a_{\Pi''}(p^\nu)|^2}{p^\nu} &\ll \sum_{j \geq 0} \frac{\log(2^{j+1} x)}{2^j x} (2^{j+1} x)^{1-1/38+\varepsilon} \\ &\ll \sum_{j \geq 0} \frac{\log(2^{j+1} x)}{(2^{j+1} x)^{1/38-\varepsilon}} \\ &\ll x^{-1/38+2\varepsilon}. \end{aligned}$$

This implies the required result. □

§ 3. Proof of Theorem 3

Theorem 3 follows immediately from (1.6) and the following lemma.

Lemma 3.1. *Let π be a unitary automorphic cuspidal representation for $GL_m(\mathbb{Q}_\mathbb{A})$.*

(i) *For each $m \in \{1, \dots, 4\}$, there is a constant $\delta_m > 0$ such that*

$$\sum_{p^\nu \leq x, \nu \geq 2} (\log p) |a_\pi(p^\nu)|^2 \ll x^{1-\delta_m}.$$

(ii) *If $m \geq 5$, under Hypothesis H we have*

$$\sum_{p^\nu \leq x, \nu \geq 2} (\log p) |a_\pi(p^\nu)|^2 \ll x / \log x.$$

Proof. In view of (2.1) of Lemma 2.1 with a suitable choice of ν_0 , it suffices to show, for fixed $\nu \geq 2$, that (i)

$$(3.1) \quad \sum_{p^\nu \leq x} (\log p) |a_\pi(p^\nu)|^2 \ll_\nu x^{1-\delta_m},$$

if $m \leq 4$, and (ii)

$$(3.2) \quad \sum_{p^\nu \leq x} (\log p) |a_\pi(p^\nu)|^2 \ll_\nu x / \log x$$

if $m \geq 5$ under Hypothesis H.

First we prove (3.2):

$$\begin{aligned} \sum_{p^\nu \leq x} (\log p) |a_\pi(p^\nu)|^2 &= \sum_{p^\nu \leq x^{1/2}} (\log p) |a_\pi(p^\nu)|^2 + \sum_{x^{1/2} < p^\nu \leq x} (\log p) |a_\pi(p^\nu)|^2 \\ &\leq x^{1/2} \sum_{p^\nu \leq x^{1/2}} \frac{(\log p)^2 |a_\pi(p^\nu)|^2}{p^\nu} + \frac{2x}{\log x} \sum_{x^{1/2} < p^\nu \leq x} \frac{(\log p)^2 |a_\pi(p^\nu)|^2}{p^\nu}, \end{aligned}$$

which is $\ll x / \log x$ under Hypothesis H.

Next we prove (3.1) for $m = 4$, since other cases are easier. As before it suffices to consider the sum on the left side of (3.1) taken over $p \neq 2, 3$ with π_p being not tempered. Then for such a p , $\Pi_p \cong \wedge^2 \pi_p$. There are then three possibilities.

If $p \in S_1$ as in (2.9), using Π_p we get $0 < 2a \leq \frac{1}{2} - \frac{1}{37}$ as in (2.10). Then

$$|a_\pi(p^\nu)|^2 = |(u_1^\nu + u_2^\nu)(p^{a\nu} + p^{-a\nu})|^2 \leq 16p^{(1/2-1/37)\nu}.$$

From this, we deduce that

$$(3.3) \quad \sum_{p^\nu \leq x, p \in S_1} (\log p) |a_\pi(p^\nu)|^2 \ll \sum_{p^\nu \leq x, p \in S_1} (\log p) p^{(1/2-1/37)\nu} \ll x^{1-1/37}.$$

If $p \in S_2$, we have

$$|a_\pi(p^\nu)| = |u_1^\nu(p^{a\nu} + p^{-a\nu}) + u_2^\nu + u_3^\nu| \leq p^{a\nu} + 3,$$

$$|a_\pi(p)| = |u_1(p^a + p^{-a}) + u_2 + u_3| \geq p^a - 2$$

with $0 < a \leq 1/2 - 1/17$. Then

$$(3.4) \quad \begin{aligned} |a_\pi(p^\nu)|^2 &\leq \{(|a_\pi(p)| + 2)^\nu + 3\}^2 \ll_\nu |a_\pi(p)|^{2\nu} + 1 \\ &\ll_\nu |a_\pi(p)|^2 p^{(1-2/17)(\nu-1)} + 1. \end{aligned}$$

Similarly if $p \in S_3$, then

$$|a_\pi(p^\nu)| = |u_1^\nu(p^{a_1\nu} + p^{-a_1\nu}) + u_2^\nu(p^{a_2\nu} + p^{-a_2\nu})| \leq 2p^{a_1\nu} + 2,$$

$$|a_\pi(p)| = |u_1(p^{a_1} + p^{-a_1}) + u_2(p^{a_2} + p^{-a_2})| \geq p^{a_1} - 2p^{a_2}.$$

From this, (2.3) with $m = 4$ and the last inequality of (2.10), we deduce that

$$(3.5) \quad \begin{aligned} |a_\pi(p^\nu)|^2 &\leq \{2(|a_\pi(p)| + 2p^{a_2})^\nu + 2\}^2 \ll_\nu |a_\pi(p)|^{2\nu} + p^{2a_2\nu} \\ &\ll_\nu |a_\pi(p)|^2 p^{(1-2/17)(\nu-1)} + p^{(1/2-1/37)\nu}. \end{aligned}$$

As before, we can apply Lemma 2.2 with the choice of parameters

$$(\pi', \pi'', \delta', \delta'') = \begin{cases} (\pi, \pi, \frac{2}{17}, \frac{1}{2}) & \text{if } \mathcal{P} = S_2 \\ (\pi, \pi, \frac{2}{17}, \frac{1}{37}) & \text{if } \mathcal{P} = S_3 \end{cases}$$

to write

$$(3.6) \quad \sum_{p^\nu \leq x, p \in S_j} (\log p) |a_\pi(p^\nu)|^2 \ll_\nu x^{1-1/37} \quad (j = 2, 3).$$

Now the required result follows from (3.3) and (3.6). \square

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