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► **To cite this version:**

Yuk Kam Lau, Jie Wu. On the least quadratic non-residue. *International Journal of Number Theory*, 2008, 4 (3), pp.423-435. hal-00097136

HAL Id: hal-00097136

<https://hal.science/hal-00097136v1>

Submitted on 21 Sep 2006

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On the least quadratic non-residue

Y.-K. LAU & J. WU

Abstract. We prove that for almost all real primitive characters χ_d of modulus $|d|$, the least positive integer n_{χ_d} at which χ_d takes a value not equal to 0 and 1 satisfies $n_{\chi_d} \ll \log |d|$, and give a quite precise estimate on the size of the exceptional set. Also, we generalize Burgess' bound for $n_{\chi_{p'}}$ (with p' being a prime up to \pm sign) to composite modulus $|d|$ and improve Garaev's upper bound for the least quadratic non-residue in Pajtechĭ-Šapiro's sequence.

§ 1. Introduction

Let $q \geq 2$ be an integer and χ a non principal Dirichlet character modulo q . Here the evaluation of the least integer n_χ among all positive integers n for which $\chi(n) \neq 0, 1$ is referred as Linnik's problem. In case χ coincides with the Legendre symbol, n_χ is a least quadratic non-residue. Concerning the size of n_χ , Pólya-Vinogradov's inequality

$$(1.1) \quad \max_{x \geq 1} \left| \sum_{n \leq x} \chi(n) \right| \ll q^{1/2} \log q$$

implies trivially $n_\chi \ll q^{1/2} \log q$. But for prime q , Vinogradov [24] proved the better bound

$$(1.2) \quad n_\chi \ll q^{1/(2\sqrt{\varepsilon})} (\log q)^2$$

by combining a simple argument with (1.1). He also conjectured that $n_\chi \ll_\varepsilon q^\varepsilon$ for all integers $q \geq 2$ and any $\varepsilon > 0$. Under the Generalized Riemann Hypothesis (GRH), Linnik [18] settled this conjecture, and later Ankeny [1] gave a sharper estimate

$$(1.3) \quad n_\chi \ll (\log q)^2$$

(still assuming GRH). Burgess ([3], [4], [5]) wrote a series of important papers on sharpening (1.1). His well known estimate on character sums is as follows: For any $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that

$$(1.4) \quad \left| \sum_{n \leq x} \chi(n) \right| \ll_\varepsilon x q^{-\delta(\varepsilon)}$$

provided $x \geq q^{1/3+\varepsilon}$. The last condition can be improved to $x \geq q^{1/4+\varepsilon}$ if q is cubefree. When q is prime, he deduced, via (1.4) and Vinogradov's argument,

$$(1.5) \quad n_\chi \ll_\varepsilon q^{1/(4\sqrt{\varepsilon})+\varepsilon}.$$

Since Burgess' estimate (1.4) on character sums holds for composite modulus, one expects a bound analogous to (1.5) for n_χ in general cases, but this seems not available in literature. Our first result is to propose such a generalisation, by modifying Vinogradov's argument.

2000 *Mathematics Subject Classification*: 11K31

Key words and phrases: Special sequences

Theorem 1. *Let ε be an arbitrarily small positive number. For all integers $q \geq 2$ and χ non principal characters (mod q), we have*

$$n_\chi \ll_\varepsilon \begin{cases} q^{1/(4\sqrt{\varepsilon})+\varepsilon} & \text{if } q \text{ is cubefree,} \\ q^{1/(3\sqrt{\varepsilon})+\varepsilon} & \text{otherwise.} \end{cases}$$

The proof of Theorem 1 will be given in the Section 2.

Let us now focus on real primitive characters. Denote \mathcal{D} (resp. $\mathcal{D}(Q)$) to be the set of fundamental discriminants d (resp. with $|d| \leq Q$), that is, the set of non-zero integers d which are products of coprime factors of the form $-4, 8, -8, p'$ where $p' := (-1)^{(p-1)/2}p$ (p odd prime). Also, we write \mathcal{K} (resp. $\mathcal{K}(Q)$) for the set of real primitive characters (resp. with modulus $q \leq Q$). Then there is a bijection between \mathcal{D} and \mathcal{K} given by

$$d \mapsto \chi_d(\cdot) = \left(\frac{d}{\cdot}\right)_K$$

where $\left(\frac{d}{\cdot}\right)_K$ is the Kronecker symbol. Note that the modulus of χ_d equals $|d|$ and

$$(1.6) \quad |\mathcal{D}(Q)| = |\mathcal{K}(Q)| = \frac{6}{\pi^2}Q + O(Q^{1/2}).$$

In the opposite direction of (1.2), Frilender [12], Salié [23] and Chowla & Turán (see [10]) independently shew that there are infinitely many primes p for which

$$(1.7) \quad n_{\chi_{p'}} \gg \log p,$$

or in other words, $n_{\chi_{p'}} = \Omega(\log p)$. Under GRH, Montgomery [20] gave a stronger result $n_{\chi_{p'}} = \Omega(\log p \log_2 p)$, where \log_k denotes the k -fold iterated logarithm. Without any assumption Graham & Ringrose [14] obtained $n_{\chi_{p'}} = \Omega(\log p \log_3 p)$. In view of these results, it is natural to wonder what is the size of the majority of $n_{\chi_{p'}}$, or more generally n_{χ_d} . Indeed the density of large $n_{\chi_{p'}}$'s like those in (1.7) or bigger is low, which can be seen from an Erdős' result [11],

$$(1.8) \quad \lim_{x \rightarrow \infty} \frac{1}{\pi(x)} \sum_{p \leq x} n_{\chi_{p'}} = \text{constant}$$

where $\pi(x)$ denotes the number of primes up to x . This result is extended and refined by Elliott in [7] and [8]. Using (1.8) or its refinement in [7], it follows, for any fixed constant $\delta > 0$, we see

$$(1.9) \quad \sum_{p \leq x, n_{\chi_{p'}} \geq \delta \log p} 1 \ll_\delta \frac{x}{(\log x)^2}.$$

In [6], Duke & Kowalski indicated: Let $\alpha > 1$ be given. Denote by $N(Q, \alpha)$ the number of primitive characters χ (not necessarily real) of modulus $q \leq Q$ such that $\chi(n) = 1$ for all $n \leq (\log Q)^\alpha$ and $(n, q) = 1$. Then one has

$$N(Q, \alpha) \ll_\varepsilon Q^{2/\alpha+\varepsilon}$$

for all $\varepsilon > 0$. This follows that

$$|\{ |d| \leq Q : n_{\chi_d} \geq (\log Q)^\alpha \}| \ll_\varepsilon Q^{2/\alpha+\varepsilon}.$$

However, in view of (1.6) this result is non-trivial only when $\alpha > 2$ and it tells that $n_{\chi_d} \geq (\log |d|)^{2+\varepsilon}$ for almost all fundamental discriminants d . Very recently Baier [2] improved $2 + \varepsilon$ to $1 + \varepsilon$ by using the large sieve inequality of Heath-Brown [15] for real primitive characters. Still it is unable to cover the case $\alpha = 1$ or to provide information on the sparsity of the primes p with $n_{\chi_{p'}} \gg \log p$ as in (1.9).

Our second result is to supplement the case $\alpha = 1$, using the large sieve inequality of Elliott-Montgomery-Vaughan (see [9] and [21]). We obtain an almost all result, which is strong enough to yield a more tight estimate on the low density of exceptional non-residues than in (1.9).

Theorem 2. For $2 \leq P \leq Q$, define

$$\mathcal{E}(Q, P) := \{d \in \mathcal{D}(Q) : \chi_d(p) = 1 \text{ for } P < p \leq 2P \text{ and } p \nmid |d|\}.$$

Then there are two absolute positive constants C and c such that

$$(1.10) \quad |\mathcal{E}(Q, P)| \ll Q e^{-c(\log Q)/\log_2 Q}$$

holds uniformly for $Q \geq 10$ and $C \log Q \leq P \leq (\log Q)^2$. In particular we have

$$(1.11) \quad n_{\chi_d} \ll \log |d|$$

for all but except $O(Q e^{-c(\log Q)/\log_2 Q})$ characters $\chi_d \in \mathcal{K}(Q)$.

Sections 3 and e are devoted to the proof of Theorem 2.

The next Theorem 3 (essentially due to Graham & Ringrose [14]) shows that the upper bound for exceptional real primitive characters set is optimal. Graham & Ringrose considered a problem of the quasi-random graphs (Paley graphs) which leads to study the lower bound for the sum of the right-hand side of (6.5) below. This will also be the essential part of our proof of Theorem 3. We shall provide the salient points along the line of arguments in [14] to prove Theorem 3, see Sections 5 and 6.

Theorem 3. For any fixed constant $\delta > 0$, there are a sequence of positive real numbers $\{Q_n\}_{n=1}^\infty$ with $Q_n \rightarrow \infty$ and a positive constant c such that

$$(1.12) \quad \sum_{\substack{Q_n^{1/2} < p \leq Q_n \\ n_{\chi_p} \geq \delta \log p}} 1 \gg_\delta Q_n e^{-c(\log Q_n)/\log_2 Q_n}.$$

Further if we assume that both $\mathbf{L}_1(s, P_y)$ and $\mathbf{L}_4(s, P_y)$ defined in (5.3) below have no exceptional zeros in the region (5.4), then (1.12) holds for all $Q \geq 10$.

Finally we consider the least quadratic non-residue problem in Pajtechÿ-Šapiro's sequence $\{[n^c]\}_{n=1}^\infty$, where $c > 1$ is a constant and $[t]$ denotes the integral part of $t \in \mathbb{R}$. Denote by $n_{\chi_{p'}, c}$ the least positive integer n such that $[n^c]$ is a quadratic non-residue (mod p). Garaev [13] proved that for $1 < c < \frac{12}{11}$ and any $\varepsilon > 0$, one has

$$(1.13) \quad n_{\chi_{p'}, c} \ll_{c, \varepsilon} p^{3/(8(3-2c)\sqrt{c})+\varepsilon}$$

for all primes p . He pointed out also that by the method of exponential pairs the range of c and the exponent of p can be improved to $1 < c < \frac{12}{11} + 0.00257 \dots$ and $1/(8(1-\theta_2 c)\sqrt{c})$, respectively, where $\theta_2 = 0.66451 \dots$. Here we propose a further improvement by applying a recent result of Robert & Sargos [22], and give an almost result based on Theorem 2.

Theorem 4. *Let $1 < c < \frac{32}{29}$. Then for all primes p and any $\varepsilon > 0$, we have*

$$n_{\chi_{p',c}} \ll_{c,\varepsilon} p^{9/((64-40c)\sqrt{\varepsilon})+\varepsilon}.$$

For all but except $O(Qe^{-c(\log Q)/\log_2 Q})$ primes p with $p \leq Q$, we have

$$n_{\chi_{p',c}} \ll_{c,\varepsilon} (\log p)^{9/(16-10c)+\varepsilon}.$$

We prove Theorem 4 in Section 7.

Our range of c is larger than $\frac{12}{11} + 0.00257 \dots$ ($\frac{32}{29} = \frac{12}{11} + 0.01253 \dots$) and our exponent is definitely better than (1.13) but is smaller than $1/(8(1-\theta_2c)\sqrt{\varepsilon})$ only when $c > 1/(9\theta_2 - 5) = 1.019794 \dots$. It is possible to give a slightly better result with Huxley's estimates for exponential sums [16, § 18.5]. We can also generalize Theorem 4 to composite modulus $|d|$ as in Theorem 1, but with smaller range of c and larger exponent of $|d|$.

§ 2. Vinogradov's argument and proof of Theorem 1

Without loss of generality we assume $n_\chi \geq q^{1/(4\sqrt{\varepsilon})}$ (otherwise there is nothing to prove). Let x be a number specified later but satisfy

$$q > x \geq \begin{cases} q^{1/4+\varepsilon} & \text{if } q \text{ is cubefree,} \\ q^{1/3+\varepsilon} & \text{otherwise.} \end{cases}$$

By Burgess' well known estimate (1.4) on character sums, for any $\varepsilon > 0$ there are two positive constants C_ε and $\delta(\varepsilon) > 0$ such that

$$(2.1) \quad \begin{aligned} C_\varepsilon x q^{-\delta(\varepsilon)} &\geq \left| \sum_{n \leq x} \chi(n) \right| \\ &\geq \sum_{\substack{n \leq x \\ (n,q)=1}} 1 - 2 \sum_{\substack{n \leq x \\ (n,q)=1, \chi(n) \neq 1}} 1 \\ &\geq \sum_{\substack{n \leq x \\ (n,q)=1}} 1 - 2 \sum_{n_\chi < p \leq x} \sum_{\substack{m \leq x/p \\ (m,q)=1}} 1. \end{aligned}$$

As usual we denote by $\varphi(n)$ the Euler function, $\mu(n)$ the Möbius function and $\omega(n)$ the number of distinct prime factors of n . With the Möbius inversion formula, we have, for some $|\theta| \leq 1$,

$$(2.2) \quad \sum_{\substack{n \leq x \\ (n,q)=1}} 1 = \sum_{d|q} \mu(d) \sum_{m \leq x/d} 1 = \frac{\varphi(q)}{q} x + \theta 2^{\omega(q)}.$$

To estimate the last double sum on the right-hand side of (2.1), we divide the sum over p into two parts according as $n_\chi < p \leq x/2^{\omega(q)}$ or $x/2^{\omega(q)} < p \leq x$. By (2.2), the first part contributes at most

$$(2.3) \quad \begin{aligned} &\sum_{n_\chi < p \leq x/2^{\omega(q)}} \left(\frac{\varphi(q)}{q} \frac{x}{p} + 2^{\omega(q)} \right) \\ &\leq \frac{\varphi(q)}{q} x \left\{ \log \left(\frac{\log x}{\log n_\chi} \right) + O \left(e^{-\sqrt{\log n_\chi}} \right) \right\} + \frac{(1+\varepsilon)x}{\log(x2^{-\omega(q)})} \\ &\leq \frac{\varphi(q)}{q} x \log \left(\frac{\log x}{\log n_\chi} \right) + (1+2\varepsilon) \frac{x}{\log x}. \end{aligned}$$

Note that $2^{\omega(q)} \ll x^\varepsilon$ and $n_\chi \geq q^{1/(4\sqrt{\varepsilon})}$. For the second part, we interchange the summations and apply the Rankin trick,

$$\begin{aligned} \sum_{x/2^{\omega(q)} < p \leq x} \sum_{\substack{m \leq x/p \\ (m,q)=1}} 1 &\leq \sum_{\substack{1 \leq m \leq 2^{\omega(q)} \\ (m,q)=1}} \sum_{p \leq x/m} 1 \\ &\ll \frac{x}{\log x} \sum_{\substack{1 \leq m \leq 2^{\omega(q)} \\ (m,q)=1}} \frac{1}{m} \\ &\leq \frac{x}{\log x} \prod_{\substack{p \leq 2^{\omega(q)} \\ (p,q)=1}} \left(1 - \frac{1}{p}\right)^{-1} \\ &= \frac{\varphi(q)}{q} \frac{x}{\log x} \prod_{\substack{p > 2^{\omega(q)} \\ p|q}} \left(1 - \frac{1}{p}\right)^{-1} \times \prod_{p \leq 2^{\omega(q)}} \left(1 - \frac{1}{p}\right)^{-1}. \end{aligned}$$

In virtue of the simple estimates

$$\begin{aligned} \prod_{\substack{p > 2^{\omega(q)} \\ p|q}} \left(1 - \frac{1}{p}\right)^{-1} &\ll \exp \left\{ \sum_{\substack{p > 2^{\omega(q)} \\ p|q}} \frac{1}{p} \right\} \ll \exp \left\{ \frac{\omega(q)}{2^{\omega(q)}} \right\} \ll 1, \\ \prod_{p \leq 2^{\omega(q)}} \left(1 - \frac{1}{p}\right)^{-1} &\ll \exp \left\{ \sum_{p \leq 2^{\omega(q)}} \frac{1}{p} \right\} \ll \omega(q), \end{aligned}$$

it follows immediately that

$$(2.4) \quad \sum_{x/2^{\omega(q)} < p \leq x} \sum_{\substack{m \leq x/p \\ (m,q)=1}} 1 \ll \frac{\varphi(q)}{q} x \frac{\omega(q)}{\log x}.$$

Inserting (2.2), (2.3) and (2.4) into (2.1), we conclude

$$C_\varepsilon x q^{-\delta(\varepsilon)} \geq \frac{\varphi(q)}{q} x \left\{ 1 - 2 \log \left(\frac{\log x}{\log n_\chi} \right) \right\} - 2^{\omega(q)} - (1 + 2\varepsilon) \frac{x}{\log x} - C_\varepsilon \frac{\varphi(q)}{q} x \frac{\omega(q)}{\log x}.$$

From this we deduce that

$$\begin{aligned} \log \left(\frac{\log x}{\log n_\chi} \right) &\geq \frac{1}{2} - \frac{C_\varepsilon q^{1-\delta(\varepsilon)}}{2 \varphi(q)} - \frac{(1/2 + \varepsilon)q}{\varphi(q) \log x} - \frac{C_\varepsilon \omega(q)}{2 \log x} \\ &\geq \frac{1}{2} - C_\varepsilon \left(\frac{q}{\varphi(q) \log x} + \frac{\omega(q)}{\log x} \right) \end{aligned}$$

provided $q \geq q_0(\varepsilon)$. Since $q/\varphi(q) \log x + \omega(q)/\log x \ll (\log_2 q)^{-1}$, the preceding inequality implies

$$n_\chi \ll x^{1/\sqrt{\varepsilon}} \exp \left\{ O \left(\frac{q}{\varphi(q)} + \omega(q) \right) \right\},$$

which gives the required result, by taking

$$x = \begin{cases} q^{1/4+\varepsilon} & \text{if } q \text{ is cubefree,} \\ q^{1/3+\varepsilon} & \text{otherwise.} \end{cases}$$

This completes the proof of Theorem 1. \square

§ 3. A large sieve inequality of Montgomery-Vaughan

Our key tool for proving Theorem 2 is a large sieve inequality of Montgomery & Vaughan in [21, page 1050] following from [21, Lemma 2]. Here we state a slightly refined version (see Lemma 1 below). Their original statement absorbs the factors $(6/\log P)^j$ and $\{6/(\log P)^2\}^j$ in the implied constant. We reproduce here their proof with a minuscule modification.

Lemma 1. *We have*

$$(3.1) \quad \sum_{d \in \mathcal{D}(Q)} \left| \sum_{P < p \leq 2P} \frac{\chi_d(p)}{p} \right|^{2j} \ll Q \left(\frac{6j}{P \log P} \right)^j + \left(\frac{6P}{(\log P)^2} \right)^j$$

uniformly for $2 \leq P \leq Q$ and $j \geq 1$. The implied constant is absolute.

Proof. Since $\chi_d(n)$ is completely multiplicative on n , we can write

$$\left(\sum_{P < p \leq 2P} \frac{\chi_d(p)}{p} \right)^j = \sum_{P^j < m \leq (2P)^j} \frac{a_j(m)}{m} \chi_d(m),$$

where

$$a_j(m) := |\{(p_1, \dots, p_j) : p_1 \cdots p_j = m, P < p_i \leq 2P\}|.$$

By Lemma 2 of [21] with the choice of parameters

$$X = P^j, \quad Y = (2P)^j \quad \text{and} \quad a_m = a_j(m)/m,$$

it follows that as $a_j(m_1)a_j(m_2) \leq a_{2j}(n^2)$ for $n^2 = m_1m_2$,

$$(3.2) \quad \sum_{d \in \mathcal{D}(Q)} \left| \sum_{P < p \leq 2P} \frac{\chi_d(p)}{p} \right|^{2j} \ll Q \sum_{P^j < n \leq (2P)^j} \frac{a_{2j}(n^2)}{n^2} + \left(\sum_{P < p \leq 2P} \frac{1}{p^{1/2}} \right)^{2j}.$$

Writing $n = p_1^{\nu_1} \cdots p_i^{\nu_i}$ with $\nu_1 + \cdots + \nu_i = j$, we have

$$\begin{aligned} a_{2j}(n^2) &= \frac{(2j)!}{(2\nu_1)! \cdots (2\nu_i)!} \\ &= \frac{(2j)!}{j!} \frac{\nu_1!}{(2\nu_1)!} \cdots \frac{\nu_i!}{(2\nu_i)!} a_j(n). \end{aligned}$$

From this, it is easy to see $a_{2j}(n^2) \leq j^j a_j(n)$, and thus

$$\begin{aligned} \sum_{P^j < n \leq (2P)^j} \frac{a_{2j}(n^2)}{n^2} &\leq j^j \sum_{P^j < n \leq (2P)^j} \frac{a_j(n)}{n^2} \\ &= \left(j \sum_{P < p \leq 2P} \frac{1}{p^2} \right)^j \\ &\leq \left(\frac{6j}{P \log P} \right)^j. \end{aligned}$$

Inserting it into (3.2) and using the estimate

$$\sum_{P < p \leq 2P} \frac{1}{p^{1/2}} \leq \frac{6P^{1/2}}{\log P},$$

we obtain the required result (3.1). □

§ 4. Proof of Theorem 2

Define

$$\mathcal{E}^*(Q, P) := \{d \in \mathcal{D}(Q) : Q^{1/2} \leq |d| \leq Q \text{ and } \chi_d(p) = 1 \ (P < p \leq 2P, p \nmid |d|)\}.$$

Let $C \log Q \leq P \leq (\log Q)^2$. For $d \in \mathcal{E}^*(Q, P)$, we invoke the prime number theorem to deduce

$$\begin{aligned} \sum_{P < p \leq 2P} \frac{\chi_d(p)}{p} &= \sum_{P < p \leq 2P} \frac{1}{p} - \sum_{P < p \leq 2P, p \parallel |d|} \frac{1}{p} \\ &\geq \frac{\log 2 + o(1)}{\log P} - \frac{\{1 + o(1)\} \log Q}{P \log_2 Q} \\ &\geq \frac{\log 2 - 2/C + o(1)}{\log P} \\ &> \frac{1}{2 \log P}, \end{aligned}$$

provided C is sufficiently large. It is apparent from (3.1) that

$$\begin{aligned} \frac{|\mathcal{E}^*(Q, P)|}{(2 \log P)^{2j}} &\leq \sum_{d \in \mathcal{D}(Q)} \left| \sum_{P < p \leq 2P} \frac{\chi_d(p)}{p} \right|^{2j} \\ &\ll Q \left(\frac{6j}{P \log P} \right)^j + \left(\frac{6P}{(\log P)^2} \right)^j. \end{aligned}$$

Hence we obtain

$$|\mathcal{E}^*(Q, P)| \ll Q (12j \log P / P)^j + (12P)^j$$

uniformly for $C \log Q \leq P \leq (\log Q)^2$ and $j \geq 1$. Taking

$$j = \left\lceil \frac{\log Q}{48 \log P} \right\rceil + 1,$$

a simple calculation shows that

$$|\mathcal{E}^*(Q, P)| \ll Q e^{-c(\log Q) / \log_2 Q}$$

with $c = (\log 2)/48$. This implies (1.10).

Finally let

$$\mathcal{E}^*(Q) := \{d \in \mathcal{D}(Q) : d \leq Q^{1/2}\} \cup \mathcal{E}^*(Q, C \log Q).$$

Then by (1.10), we have

$$|\mathcal{E}^*(Q)| \ll Q e^{-c(\log Q) / \log_2 Q},$$

and for any $d \in \mathcal{D}(Q) \setminus \mathcal{E}^*(Q)$ there is a prime number $p \asymp \log Q \asymp \log |d|$ such that $\chi_d(p) \neq 1$, which implies (1.11). The proof is complete.

§ 5. Graham-Ringrose's method

In this section, we shall state and extend the main results of ([14], Theorems 2, 3 and 4) for our purposes. For characters of certain moduli, Graham & Ringrose [14] obtained a wide zero-free region and good zero density estimates for the corresponding Dirichlet L -functions. The main ingredient of their method is an q -analogue of van der Corput's result, which can be stated as follows: *Suppose that $q = 2^\nu r$, where $0 \leq \nu \leq 3$ and r is an odd squarefree integer, and that χ is a non-principal character mod q . Let p be the largest prime factor of q . Suppose that k is a non-negative integer, and $K = 2^k$. Finally, assume that $N \leq M$. Then*

$$(5.1) \quad \sum_{M < n \leq M+N} \chi(n) \ll M^{1 - \frac{k+3}{8K-2}} p^{\frac{k^2+3k+4}{32K-8}} q^{\frac{1}{8K-2}} d(q)^{\frac{32k^2+11k+8}{16K-4}} (\log q)^{\frac{k+3}{8K-2}} \sigma_{-1}(q),$$

where $\sigma_a(q) := \sum_{d|q} d^a$ and $d(q) := \sigma_0(q)$. The implied constant is absolute.

Recall that for any odd prime p ,

$$\chi_8(p) = \left(\frac{2}{p}\right), \quad \chi_{q'}(p) = \left(\frac{q}{p}\right)_K = \left(\frac{q}{p}\right) \quad (q \text{ odd prime, } q' := (-1)^{(q-1)/2}q)$$

by definition. For squarefree $m \geq 2$, the character $\chi_m := \prod_{p|m} \chi_{p'}$ for odd m or $\chi_m := \chi_8 \chi_{m'}$ for $m = 2m'$ is a real primitive of modulus m or $4m$, respectively. By convention, we set $\chi_1 \equiv 1$. Moreover, if χ_4 is the real primitive character mod 4, i.e. $\chi_4(n) = (-1)^{(n-1)/2}$ for odd n , then $\chi_{4m} := \chi_4 \chi_m$ is also a real primitive character of modulus $4m$.

Let

$$(5.2) \quad P_y := \prod_{p \leq y} p = e^{\{1+o(1)\}y} \quad (y \rightarrow \infty),$$

and define for $\ell = 1$ or 4,

$$(5.3) \quad \mathbf{L}_\ell(s, P_y) := \prod_{m|P_y} L(s, \chi_{\ell m}),$$

where $L(s, \chi_{\ell m})$ is the Dirichlet L -function associated to $\chi_{\ell m}$. Denote by $N_\ell(\alpha)$ the number of zeros of $\mathbf{L}_\ell(s, P_y)$ in the rectangle

$$\alpha \leq \sigma \leq 1 \quad \text{and} \quad |\tau| \leq \log P_y.$$

Here and in the sequel we implicitly define the real numbers σ and τ by the relation $s = \sigma + i\tau$.

The next lemmas 2, 3 and 4 are trivial extensions of Theorems 2, 3 and 4 of [14], respectively.

Lemma 2. *Let $y \geq 100$. Then there is an absolute positive constant C_1 such that the L -function $\prod_{\ell=1,4} \mathbf{L}_\ell(s, P_y)$ has at most one zero in the region*

$$(5.4) \quad \sigma \geq 1 - \frac{C_1(\log_2 P_y)^{1/2}}{\log P_y} \quad \text{and} \quad |\tau| \leq \log P_y.$$

The exceptional zero, if exists, is real.

Proof. As the crucial estimate (5.1) holds for all non-principal primitive characters of modulus $q = 2^\nu r \geq 2$ with $0 \leq \nu \leq 3$ and r being odd squarefree. Consider the case $\nu = 0$ or 3, and $\nu = 2$ or 3, respectively. We see that (5.1) applies to χ_m and χ_{4m} for any $m|P_y$. It follows that [14, Lemma 6.1] is valid for these characters. Proceeding with the same argument, we have [14, Lemma 6.2] for our L -function $\prod_{\ell=1,4} \mathbf{L}_\ell(s, P_y)$ in place of $\mathbf{L}(s, P_y)$ there. Then the same proof of [14, Theorem 2] will give the desired result. (Note that the value of ϕ suffers a negligible change when P_y is replaced by $4P_y$ or $8P_y$.) The exceptional zero must be real, for otherwise, its conjugate is another zero in the specified region. \square

Lemma 3. *Let C_1 be as in Lemma 2. There is a sequence of positive real numbers $\{y_n\}_{n=1}^{\infty}$ with $y_n \rightarrow \infty$ such that both $\mathbf{L}_1(s, P_{y_n})$ and $\mathbf{L}_4(s, P_{y_n})$ have no zeros in the region*

$$(5.5) \quad \sigma \geq 1 - \eta(y_n) \quad \text{and} \quad |\tau| \leq \log P_{y_n},$$

where

$$\eta(y) := \frac{C_1(\log_2 P_y)^{1/2}}{2 \log P_y}.$$

Proof. Similar to [14, Theorem 3], our proof is also based on an interesting argument attributed to Maier [19]. Suppose that for some y , the product $\mathbf{L}_1(s, P_y)\mathbf{L}_4(s, P_y)$ has an exceptional zero in the region (5.4). That is, it has a real zero $\beta > 1 - 2\eta(y)$. In view of (5.2), we can take $y_n \geq y$ such that

$$(5.6) \quad \eta(y_n) < 1 - \beta < 2\eta(y_n).$$

By Lemma 2, β is the only exceptional zero of $\prod_{\ell=1,4} \mathbf{L}_\ell(s, P_{y_n})$ in the region

$$\sigma > 1 - 2\eta(y_n) \quad \text{and} \quad |\tau| \leq \log P_{y_n}.$$

Together with the first inequality in (5.6), this forces $\prod_{\ell=1,4} \mathbf{L}_\ell(s, P_{y_n})$ to have no zero in the region (5.5). It follows that we can find a sequence of positive real numbers $\{y_n\}_{n=1}^{\infty}$ with $y_n \rightarrow \infty$ such that both $\mathbf{L}_1(s, P_{y_n})$ and $\mathbf{L}_4(s, P_{y_n})$ have no zero in this region. \square

Lemma 4. *Let $\ell = 1$ or 4 and $y \geq 100$. Then there is an absolute constant C_2 such that*

$$(5.7) \quad N_\ell(\alpha) \ll \begin{cases} \exp \left\{ \frac{C_2(1-\alpha) \log P_y}{\sqrt{\log_2 P_y}} + \frac{\log_3 P_y}{2} \right\} & \text{if } \alpha \geq 1 - \eta_1(y), \\ \exp \left\{ \frac{C_2(1-\alpha) \log P_y}{\log(1/(1-\alpha))} \right\} & \text{if } \alpha < 1 - \eta_1(y), \end{cases}$$

where

$$k_0(y) := \lceil (\log_2 P_y)^{1/2} \rceil \quad \text{and} \quad \eta_1(y) := \frac{k_0(y)}{2(2^{k_0(y)} - 2)}.$$

Proof. The case of $\ell = 1$ has been done in [14, Sections 7 and 8] and $N_4(\alpha)$ can be treated in the same way by applying (5.1) to our χ_{4m} . \square

§ 6. Proof of Theorem 3

In this section, we denote by p and q prime numbers. Define

$$\mathbb{P}_y := \{p : p \equiv 1 \pmod{4} \text{ and } \chi_p(q) = 1 \text{ for all } q \leq y\}.$$

Clearly we have $n_{\chi_p} > y$ for any $p \in \mathbb{P}_y$. We shall first show that the set \mathbb{P}_y is not too small for suitable y .

Proposition. Let $\delta > 0$ be a fixed small constant and $y(x)$ be an strictly increasing function defined on $[120, \infty)$ satisfying

$$(6.1) \quad (\log x)e^{-\delta(\log_2 x)^{1/2}} \leq y(x) \leq \delta(\log x) \log_3 x.$$

Then there are a positive constant $c = c(\delta)$ and a sequence of positive real numbers $\{x_n\}_{n=1}^{\infty}$ with $x_n \rightarrow \infty$ such that

$$(6.2) \quad \sum_{\substack{x_n^{1/2} < p \leq x_n \log x_n \\ p \in \mathbb{P}_{y(x_n)}}} 1 \gg x_n e^{-cy(x_n)/\log y(x_n)}.$$

Further if we assume that both $\mathbf{L}_1(s, P_y)$ and $\mathbf{L}_4(s, P_y)$ have no zeros in the region (5.4) for all $y \geq 100$, then there is a positive constant c such that for all $x \geq 100$ we have

$$(6.3) \quad \sum_{\substack{x^{1/2} < p \leq x \log x \\ p \in \mathbb{P}_y}} 1 \gg x e^{-cy(x)/\log y(x)}.$$

Proof. First let $10 \leq y \leq x^{1/2}$. As usual, $\pi(y)$ denotes the number of prime numbers $\leq y$. Clearly we have

$$(6.4) \quad 2^{-\pi(y)-1} (1 + \chi_4(p)) \prod_{q \leq y} (1 + \chi_p(q)) = \begin{cases} 1 & \text{if } p \in \mathbb{P}_y, \\ 0 & \text{if } p \notin \mathbb{P}_y. \end{cases}$$

When p and q are odd primes with $p \equiv 1 \pmod{4}$, i.e. $\chi_4(p) = 1$, we infer by quadratic reciprocity law that

$$\chi_p(q) = \left(\frac{p}{q}\right) = \left(\frac{q}{p}\right) = \chi_{q'}(p) \quad (q' := (-1)^{(q-1)/2}q).$$

Note also for odd prime p ,

$$\chi_p(2) = \left(\frac{p}{2}\right)_K = \left(\frac{2}{p}\right) = \chi_8(p).$$

Thus we can replace $\chi_p(q)$ by $\left(\frac{q}{p}\right)$ in (6.4) to write

$$\sum_{\substack{x^{1/2} < p \leq x \log x \\ p \in \mathbb{P}_y}} 1 = \frac{1}{2^{\pi(y)+1}} \sum_{x^{1/2} < p \leq x \log x} (1 + \chi_4(p)) \prod_{q \leq y} \left(1 + \left(\frac{q}{p}\right)\right).$$

It is more convenient to introduce the weight factor $(\log p) (e^{-p/(2x)} - e^{-p/x})$ to the summands,

$$\begin{aligned} \sum_{\substack{x^{1/2} < p \leq x \log x \\ p \in \mathbb{P}_y}} 1 &\geq \frac{1}{2^{\pi(y)+2} \log x} \sum_{x^{1/2} < p \leq x \log x} (\log p) (e^{-p/(2x)} - e^{-p/x}) \times \\ &\quad \times (1 + \chi_4(p)) \prod_{q \leq y} \left(1 + \left(\frac{q}{p}\right)\right). \end{aligned}$$

We want to relax the range of the sum over p . To this end, we observe that by the prime number theorem and integration by parts,

$$\begin{aligned} &\frac{1}{2^{\pi(y)} \log x} \sum_{x \log x < p \leq x^2} (\log p) (e^{-p/(2x)} - e^{-p/x}) (1 + \chi_4(p)) \prod_{q \leq y} \left(1 + \left(\frac{q}{p}\right)\right) \\ &\ll \sum_{x \log x < p \leq x^2} (e^{-p/(2x)} - e^{-p/x}) \\ &\ll x^{1/2} / \log x. \end{aligned}$$

Combining this with the preceding inequality, we obtain

$$(6.5) \quad \sum_{\substack{x^{1/2} < p \leq x \log x \\ p \in \mathbb{P}_y}} 1 \geq \frac{1}{(\log x)2^{\pi(y)+2}} \sum_{m|P_y} (S_x(m) + S_x(4m)) + O\left(\frac{x^{1/2}}{\log x}\right),$$

where $\ell = 1$ or 4 , and

$$S_x(\ell m) := \sum_{x^{1/2} < p \leq x^2} (\log p)(e^{-p/(2x)} - e^{-p/x})\chi_{\ell m}(p).$$

By the Perron formula, we can write

$$(6.6) \quad S_x(\ell m) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} -\frac{L'}{L}(s, \chi_{\ell m})(2^s - 1)\Gamma(s)x^s ds + O(x^{1/2} \log x).$$

We shift the line of integration to $\sigma = -\frac{3}{4}$. The function $(2^s - 1)\Gamma(s)x^s$ has no pole in the strip $-\frac{3}{4} \leq \sigma \leq 2$ since the pole of $\Gamma(s)$ at $s = 0$ is canceled by the zero of $(2^s - 1)$. Thus the only poles of the integrand in (6.6) occur at $s = 1$ if $\ell m = 1$ (note that $L(s, \chi_1)$ is the Riemann ζ -function), or at the zeros $\rho(\ell m) = \beta(\ell m) + i\gamma(\ell m)$ of $L(s, \chi_{\ell m})$. It follows that

$$S_x(\ell m) = \delta_{\ell m, 1}x - \sum_{\rho(\ell m)} (2^{\rho(\ell m)} - 1)\Gamma(\rho(\ell m))x^{\rho(\ell m)} + O(x^{1/2} \log x),$$

where $\delta_{j,1} = 1$ if $j = 1$ and 0 otherwise, and the sum is over all zeros with $0 \leq \beta(\ell m) < 1$.

We write $N(T, \chi_{\ell m})$ for the number of zeros of $L(s, \chi_{\ell m})$ in the rectangle $0 < \beta(\ell m) < 1$ and $|\gamma| \leq T$. Then we have the classical bound

$$(6.7) \quad N(T, \chi_{\ell m}) \ll T \log(Tm),$$

which implies, for any $\alpha \in (0, 1)$,

$$(6.8) \quad N_\ell(\alpha) \leq \sum_{m|P_y} N(\log P_y, \chi_{\ell m}) \ll 2^{\pi(y)}y^2.$$

On the other hand, by means of $(2^s - 1)\Gamma(s)x^s \ll x^\sigma |\tau| e^{-(\pi/2)|\tau|}$, the contribution of the zeros with $|\gamma(\ell m)| \geq \log P_y$ to $S_x(\ell m)$ is $\ll 1$. Let ε be an arbitrarily small positive number. The zeros with $\beta(\ell m) \leq 1 - \varepsilon$ and $|\gamma(\ell m)| \leq \log P_y$ contribute

$$\ll x^{1-\varepsilon} N(\log P_y, \chi_{\ell m}) \ll x^{1-\varepsilon} (\log P_y)^2 \ll x^{1-\varepsilon} y^2.$$

Combining these with (6.5), we conclude

$$(6.9) \quad \sum_{\substack{x^{1/2} < p \leq x \log x \\ p \in \mathbb{P}_y}} 1 \geq \frac{x}{(\log x)2^{\pi(y)+2}} + O\left(x^{1-\varepsilon} 2^{\pi(y)}y^2 + \frac{T_1(x, y) + T_4(x, y)}{(\log x)2^{\pi(y)}}\right)$$

uniformly for $x \geq 10$ and $1 \leq y \leq x^{1/2}$, where

$$\begin{aligned} T_\ell(x, y) &:= \sum_{m|P_y} \sum_{\substack{\rho(\ell m) \\ \beta(\ell m) \geq 1-\varepsilon, |\gamma(\ell m)| \leq \log P_y}} x^{\beta(\ell m)} \\ &= - \int_{1-\varepsilon}^1 x^\alpha dN_\ell(\alpha). \end{aligned}$$

It remains to estimate $T_\ell(x, y)$. From now on we take $y = y(x)$. By integration by parts and by using (6.8), we can deduce

$$(6.10) \quad T_\ell(x, y) \ll x^{1-\varepsilon} 2^{\pi(y)} y^2 + x(\log x) I_\ell,$$

where

$$I_\ell := \int_0^\varepsilon x^{-\beta} N_\ell(1-\beta) d\beta.$$

Let $\eta = \eta(y)$ and $\eta_1 = \eta_1(y)$ be defined as in Lemmas 3 and 4, respectively. Set $\eta_2 := 2y(x)/(\log x) \log y$. It is easy to verify that $0 < \eta < \eta_1 < \eta_2 < \varepsilon$. (The inequality $\eta_1 < \eta_2$ governs the lower bound of $y(x)$ in (6.1).) Thus we can divide the interval $[0, \varepsilon]$ into four subintervals $[0, \eta]$, $[\eta, \eta_1]$, $[\eta_1, \eta_2]$ and $[\eta_2, \varepsilon]$, and denote by $I_{\ell,0}$, $I_{\ell,1}$, $I_{\ell,2}$ and $I_{\ell,3}$ the corresponding contribution to I_ℓ . Plainly we have

$$\frac{1}{2} \log_3 P_y \leq \frac{\eta}{4} \log x, \quad \frac{C_2 \log P_y}{\sqrt{\log_2 P_y}} \leq \frac{1}{4} \log x, \quad \frac{C_2 \log P_y}{\log(1/\eta_2)} \leq \frac{1}{2} \log x, \quad \frac{y}{\log y} = \frac{\eta_2}{2} \log x.$$

(The third inequality governs the upper bound of $y(x)$ in (6.1).) From Lemma 4 and (6.8), we deduce that

$$\begin{aligned} I_{\ell,1} &\ll \int_\eta^{\eta_1} \exp \left\{ -\beta \log x + \frac{C_2 \beta \log P_y}{\sqrt{\log_2 P_y}} + \frac{1}{2} \log_3 P_y \right\} d\beta \ll \frac{x^{-\eta/2}}{\log x}, \\ I_{\ell,2} &\ll \int_{\eta_1}^{\eta_2} \exp \left\{ -\beta \log x + \frac{C_2 \beta \log P_y}{\log(1/\beta)} \right\} d\beta \ll \frac{x^{-\eta_1/2}}{\log x}, \\ I_{\ell,3} &\ll \int_{\eta_2}^\varepsilon \exp \left\{ -\beta \log x + \frac{y}{\log y} \right\} d\beta \ll \frac{x^{-\eta_2/2}}{\log x}. \end{aligned}$$

Hence, all of them satisfy

$$I_{\ell,i} = o((\log x)^{-1}) \quad (i = 1, 2, 3).$$

If we assume that both $\mathbf{L}_1(s, P_y)$ and $\mathbf{L}_4(s, P_y)$ have no zeros in the region (5.4) for all $y \geq 100$, then $I_{\ell,0} = 0$. Otherwise we use Lemma 3 to assure the existence of $\{y_n\}_{n=1}^\infty$ such that $I_{\ell,0} = 0$.

With (6.10), our conclusion is

$$T_\ell(x_n, y_n) = o\left(\frac{x_n}{(\log x_n) 2^{\pi(y_n)}}\right) \quad (n \rightarrow \infty),$$

or

$$T_\ell(x, y) = o\left(\frac{x}{(\log x) 2^{\pi(y)}}\right) \quad (x \rightarrow \infty)$$

under the assumption that both $\mathbf{L}_1(s, P_y)$ and $\mathbf{L}_4(s, P_y)$ have no exceptional zeros. Clearly this and (6.9) imply the required result. This completes the proof of Proposition. \square

Now we are ready to prove Theorem 3.

Taking $Q_n = x_n \log x_n$ and $y(x) = 100\delta \log x$ in Proposition and noticing that $p \in \mathbb{P}_y \Rightarrow n_{\chi_p} \geq y$, we have

$$\sum_{\substack{(Q_n/\log Q_n)^{1/2} < p \leq Q_n \\ n_{\chi_p} \geq 100\delta \log Q_n}} 1 \gg Q_n e^{-c_1(\log Q_n)/\log_2 Q_n}.$$

It implies the first assertion of Theorem 2, and the second one can be treated similarly. This concludes Theorem 2. \square

§ 7. Proof of Theorem 4

Let $1 < c < \frac{32}{29}$ and ε be an arbitrary but sufficiently small positive constant. The upshot is to show

$$(7.1) \quad n_{\chi_{p'}, c} \ll n_{\chi_{p'}}^{9/(16-10c)+\varepsilon}$$

whenever $n_{\chi_{p'}} \geq N_0(c, \varepsilon)$ for some suitably large constant $N_0(c, \varepsilon)$ depending only on c and ε . Once (7.1) is established, the required results follow from Burgess' upper bound (1.5) or (1.11).

To prove (7.1), we make use of the observation that the integer $mn_{\chi_{p'}}$ is quadratic non-residue for any integer $m < n_{\chi_{p'}}$. Now, we want to find a positive $M (< \frac{1}{2}n_{\chi_{p'}})$ as small as possible such that

$$(7.2) \quad [n^c] = mn_{\chi_{p'}}$$

for some integers $m \in (M, 2M]$ and $n > 1$. This implies

$$(7.3) \quad n_{\chi_{p'}, c} \ll (Mn_{\chi_{p'}})^{1/c}$$

which leads to (7.1) with a suitable estimate on M .

Apparently, (7.2) is equivalent to

$$(7.4) \quad (mn_{\chi_{p'}})^{1/c} \leq n < (mn_{\chi_{p'}} + 1)^{1/c}.$$

Denote by $\{x\}$ the fractional part of x . Then (7.4) holds if

$$(7.5) \quad 0 < \{(mn_{\chi_{p'}} + 1)^{1/c}\} \leq (2^{1/c-2}/c)(Mn_{\chi_{p'}})^{1/c-1} =: \Delta < 1 \quad (c > 1),$$

since

$$(mn_{\chi_{p'}} + 1)^{1/c} - (mn_{\chi_{p'}})^{1/c} \geq (1/c)(2Mn_{\chi_{p'}})^{1/c-1}.$$

Let $\delta_\Delta(t)$ be the periodic function of period 1 such that $\delta_\Delta(t) = 1$ if $t \in (0, \Delta]$ and $= 0$ if $t \in (\Delta, 1]$. Then (7.5) will follow from

$$(7.6) \quad \sum_{M < m \leq 2M} \delta_\Delta((mn_{\chi_{p'}} + 1)^{1/c}) > 0.$$

Introducing the function $\psi(t) := \frac{1}{2} - \{t\}$, we can express

$$\delta_\Delta(t) = \Delta + \psi(\Delta - t) - \psi(-t).$$

Thus we have

$$\sum_{M < m \leq 2M} \delta_\Delta((mn_{\chi_{p'}} + 1)^{1/c}) = \Delta M + R,$$

where

$$R := \sum_{M < m \leq 2M} (\psi(\Delta - (mn_{\chi_{p'}} + 1)^{1/c}) - \psi(-(mn_{\chi_{p'}} + 1)^{1/c})).$$

Consider respectively

$$f(t) = \Delta - ((M+t)n_{\chi_{p'}} + 1)^{1/c}, \quad f(t) = -((M+t)n_{\chi_{p'}} + 1)^{1/c}.$$

Then the treatment of R is reduced to the sum $\sum_{M < m \leq 2M} \psi(f(m))$, which can be handled using a recent result in [22] via third derivative of $f(t)$. Applying Theorem 2 of [22], we obtain

$$R \ll_{c,\varepsilon} \left\{ M(M^{1/c-3}n_{\chi_{p'}}^{1/c})^{3/19} + M^{3/4} + (M^{1/c-3}n_{\chi_{p'}}^{1/c})^{-1/3} \right\} M^{\varepsilon^2}.$$

Thus (7.6) will hold provided

$$M^{1-\varepsilon} \geq n_{\chi_{p'}}^{(19c-16)/(16-10c)}.$$

Taking $M = n_{\chi_{p'}}^{(19c-16)/(16-10c)+\varepsilon}$, it follows that

$$R \leq C_0(c, \varepsilon) n_{\chi_{p'}}^{\varepsilon(10c-16)/19c} M^{\varepsilon^2} \Delta M$$

for $n_{\chi_{p'}} \geq N_1(c, \varepsilon)$ where $C_0(c, \varepsilon)$ and $N_1(c, \varepsilon)$ are absolute constants depending only on c and ε . The hypothesis $1 < c < \frac{32}{29}$ yields that $M < \frac{1}{2}n_{\chi_{p'}}$ for all sufficiently large $n_{\chi_{p'}}$. Furthermore, this hypothesis assures that the exponent of $n_{\chi_{p'}}$ is negative and hence R is suppressed by ΔM for all large $n_{\chi_{p'}}$. Consequently, we derive (7.6) for $n_{\chi_{p'}} \geq N_2(c, \varepsilon)$, and therefore (7.1) by inserting the value of M into (7.3). The proof of Theorem 4 is thus complete. \square

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