# On a conjecture of Montgomery-Vaughan on extreme values of automorphic L-functions at 1

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J.-Y. Liu, E. Royer & J. Wu

**Abstract**. In this paper, we prove a weaker form of a conjecture of Montgomery-Vaughan on extreme values of automorphic $L$-functions at 1.

**Contents**



## § 1. Introduction

The automorphic $L$-functions constitute a powerful tool for studying arithmetic, algebraic or geometric objects. For squarefree integer $N$ and even integer $k$, denote by $\mathrm{H}_k^*(N)$ the set of all newforms of level $N$ and of weight $k$. It is known that

$$|\mathrm{H}_k^*(N)| = \frac{k-1}{12}\varphi(N) + O\big((kN)^{2/3}\big), \tag{1.1}$$

where $\varphi(N)$ is the Euler function and the implied constant is absolute. Let $m \geq 1$ be an integer and let $L(s, \mathrm{sym}^m f)$ be the $m$th symmetric power $L$-function of $f \in \mathrm{H}_k^*(N)$ normalised so that the critical strip is given by $0 < \Re e\, s < 1$. The values of these functions at the edge of the critical strip contain information of great interest. For example, Serre [18] showed that the Sato-Tate conjecture is equivalent to $L(1+i\tau, \mathrm{sym}^m f) \neq 0$ for all $m \in \mathbb{N}$ and $\tau \in \mathbb{R}$. The distribution of the values $L(1, \mathrm{sym}^m f)$ has received attention of many authors, including Goldfeld, Hoffstein & Lieman [2], Hoffstein & Lockhart [7], Luo [12], Royer [14, 15], Royer & Wu [16, 17], Cogdell & Michel [1], Habsieger & Royer [5] and Lau & Wu [10, 11]. In particular, Lau & Wu (Theorems 2 and 3 of [10], and Corollary 2 of [11]) proved the following results:

(i) For every fixed integer $m \geq 1$, there are four positive constants $A_m^{\pm}$ and $B_m^{\pm}$ such that for any newform $f \in \mathrm{H}_k^*(1)$, under the Great Riemann Hypothesis (GRH) for $L(s, \mathrm{sym}^m f)$, we

have, for $k \to \infty$,

$$\{1+o(1)\}(2B_m^- \log_2 k)^{-A_m^-} \le L(1, \mathrm{sym}^m f) \le \{1+o(1)\}(2B_m^+ \log_2 k)^{A_m^+}. \tag{1.2}$$

Here (and in the sequel) $\log_j$ denotes the $j$-fold iterated logarithm. For most values of $m$, the constants $A_m^\pm$ and $B_m^\pm$ can be explicitly evaluated, for example,

$$\begin{cases} A_m^+ = m+1, & B_m^+ = e^\gamma & (m \in \mathbb{N}), \\ A_m^- = m+1, & B_m^- = e^\gamma \zeta(2)^{-1} & (\text{odd } m), \\ A_2^- = 1, & B_2^- = e^\gamma \zeta(2)^{-2}, & \\ A_4^- = \frac{5}{4}, & B_4^- = e^\gamma B_4'^-, & \end{cases}$$

where $\zeta(s)$ is the Riemann zeta-function, $\gamma$ denotes the Euler constant and $B_4'^-$ is a positive constant given by a rather complicated Euler product (cf. [10], (1.16)).

(ii) In the opposite direction, it was shown unconditionally that for $m \in \{1,2,3,4\}$ there are newforms $f_m^\pm \in \mathrm{H}_k^*(1)$ such that for $k \to \infty$,

$$\begin{cases} L(1, \mathrm{sym}^m f_m^+) \ge \{1+o(1)\}(B_m^+ \log_2 k)^{A_m^+}, \\ L(1, \mathrm{sym}^m f_m^-) \le \{1+o(1)\}(B_m^- \log_2 k)^{-A_m^-}. \end{cases} \tag{1.3}$$

(iii) In the aim of removing GRH and closing up the gap coming from the factor 2 in (1.2) (comparing it with (1.3)), an almost all result was established. Let $\varepsilon > 0$ be an arbitrarily small positive number, $m \in \{1,2,3,4\}$ and $2 \mid k$. Then there is a subset $\mathrm{E}_k^*$ of $\mathrm{H}_k^*(1)$ such that $|\mathrm{E}_k^*| \ll \mathrm{H}_k^*(1) e^{-(\log k)^{1/2-\varepsilon}}$ and for each $f \in \mathrm{H}_k^*(1) \setminus \mathrm{E}_k^*$, we have, for $k \to \infty$,

$$\{1+O(\varepsilon_k)\}(B_m^- \log_2 k)^{-A_m^-} \le L(1, \mathrm{sym}^m f) \le \{1+O(\varepsilon_k)\}(B_m^+ \log_2 k)^{A_m^+}, \tag{1.4}$$

where $\varepsilon_k := (\log k)^{-\varepsilon}$ and the implied constants depend on $\varepsilon$ only.

By comparing (1.3) with (1.4), the extreme values of $L(1, \mathrm{sym}^m f)$ seem to be given by (1.3). Clearly it is interesting to investigate further the size of exceptional set $\mathrm{E}_k^*$. In the case of quadratic characters $L$-functions, Montgomery & Vaughan [13] proposed, based on a probabilistic model, three conjectures on the size of exceptional set. The first one has been proved recently by Granville & Soundararajan [4]. As Cogdell & Michel indicated in [1], it would be interesting to try to get, as close as possible, the analogues of the conjectures of Montgomery-Vaughan for automorphic $L$-functions. The analogue of Montgomery-Vaughan's first conjecture for the automorphic symmetric power $L$-functions can be stated as follows.

**Conjecture.** *Let $m \ge 1$ be a fixed integer and*

$$F_k(t, \mathrm{sym}^m) := |\mathrm{H}_k^*(1)|^{-1} \sum_{f \in \mathrm{H}_k^*(1),\ L(1, \mathrm{sym}^m f) \ge (B_m^+ t)^{A_m^+}} 1,$$

$$G_k(t, \mathrm{sym}^m) := |\mathrm{H}_k^*(1)|^{-1} \sum_{f \in \mathrm{H}_k^*(1),\ L(1, \mathrm{sym}^m f) \le (B_m^- t)^{-A_m^-}} 1.$$

*Then there are positive constants $c_i = c_i(m)$ $(i = 1, 2)$ such that for $k \to \infty$,*

$$\begin{cases} e^{-c_1 (\log k)/\log_2 k} \ll F_k(\log_2 k, \mathrm{sym}^m) \ll e^{-c_2 (\log k)/\log_2 k}, \\ e^{-c_1 (\log k)/\log_2 k} \ll G_k(\log_2 k, \mathrm{sym}^m) \ll e^{-c_2 (\log k)/\log_2 k}. \end{cases} \tag{1.5}$$

The aim of this paper is to prove a weaker form of this conjecture for $m = 1$. In this case, we write, for simplification of notation,

$$L(s, f) = L(s, \mathrm{sym}^1 f), \qquad F_k(t) = F_k(t, \mathrm{sym}^1), \qquad G_k(t) = G_k(t, \mathrm{sym}^1).$$

In view of the trace formula of Petersson ([8], Theorem 3.6), it is more convenient to consider the weighted arithmetic distribution function. As usual, denote by

$$\omega_f := \frac{\Gamma(k-1)}{(4\pi)^{k-1}\|f\|}$$

the harmonic weight in modular forms theory and define the weighted arithmetic distribution functions

$$\widetilde{F}_k(t) := \Big(\sum_{f\in \mathrm{H}_k^*(1)} \omega_f\Big)^{-1} \sum_{f\in \mathrm{H}_k^*(1),\ L(1,f)\ge (e^\gamma t)^2} \omega_f,$$

$$\widetilde{G}_k(t) := \Big(\sum_{f\in \mathrm{H}_k^*(1)} \omega_f\Big)^{-1} \sum_{f\in \mathrm{H}_k^*(1),\ L(1,f)\le (6\pi^{-2}e^\gamma t)^{-2}} \omega_f.$$

By using (1.1), the classical estimate

$$\sum_{f\in \mathrm{H}_k^*(1)} \omega_f = 1 + O\big(k^{-5/6}\big) \tag{1.6}$$

and the bound of Goldfeld, Hoffstein & Lieman [2]:

$$1/(k\log k) \ll \omega_f \ll (\log k)/k, \tag{1.7}$$

we easily see that

$$\begin{cases} \widetilde{F}_k(t)/\log k \ll F_k(t) \ll \widetilde{F}_k(t)\log k, \\ \widetilde{G}_k(t)/\log k \ll G_k(t) \ll \widetilde{G}_k(t)\log k. \end{cases} \tag{1.8}$$

This shows that in order to prove (1.5) it is sufficient to establish corresponding estimates of the same quality for $\widetilde{F}_k(t)$ and $\widetilde{G}_k(t)$.

Our main result is the following one.

**Theorem 1.** *For any $A \ge 1$ there are two positive constants $c = c(A)$ and $C = C(A)$ such that the estimate*

$$\widetilde{F}_k(t) = \{1 + \Delta_k(t)\} \exp\Big\{ -\frac{e^{t-\gamma_0}}{t}\Big(1 + O\Big(\frac{1}{t}\Big)\Big)\Big\} \tag{1.9}$$

*holds uniformly for $k \ge 16, 2 \mid k$ and $t \le T(k)$, where $\gamma_0$ is given by (1.24) below, $|\theta| \le 1$ and*

$$\begin{cases} \Delta_k(t) := \theta e^{t-T(k)-C}(t/T(k))^{1/2} + O_A\big(e^{-ce^{t/5}} + (\log k)^{-A}\big), \\ T(k) := \log_2 k - \frac{5}{2}\log_3 k - \log_4 k - 3C. \end{cases} \tag{1.10}$$

*In particular there are two positive constants $c_1$ and $c_2$ such that*

$$e^{-c_1(\log k)/\{(\log_2 k)^{7/2}\log_3 k\}} \ll F_k(T(k)) \ll e^{-c_2(\log k)/\{(\log_2 k)^{7/2}\log_3 k\}}. \tag{1.11}$$

*The similar estimates for $\widetilde{G}_k(t)$ and $G_k(T(k))$ hold also.*

**Remark 1**. The estimates (1.11) of Theorem 1 can be considered as a weaker form of Montgomery-Vaughan's conjecture (1.5) for $m = 1$, since $T(k) \sim \log_2 k$ as $k \to \infty$. Moreover, if we could take $T(k) = \log_2 k$ in (1.11) then (1.9) would lead to the Montgomery-Vaughan's conjecture (1.5). Hence we fail from a shift

$$\frac{5}{2}\log_3 k + \log_4 k + 3C.$$

It seems however to be rather difficult to resolve completely this conjecture. One of the main difficulties is that there are no analogues of the quadratic reciprocity law and Graham-Ringrose's estimates for short characters sums of friable moduli [3], which have been exploited by Granville & Soundararajan [4].

In order to prove Theorem 1, we need to introduce a probabilistic model as in [1]. Consider a probability space $(\Omega, \mu)$, with measure $\mu$. Let $\mathrm{SU}(2)^\natural$ be the set of conjugacy classes of $\mathrm{SU}(2)$. The group $\mathrm{SU}(2)$ is endowed with its Haar measure $\mu_{\mathrm{H}}$ and

$$\mathrm{SU}(2)^\natural = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in [0,\pi] \right\} \Big/ \sim$$

is endowed with the Sato-Tate measure $\mathrm{d}\mu_{\mathrm{st}}(\theta) := (2/\pi)\sin^2\theta\,\mathrm{d}\theta$, *i.e.,* the direct image of $\mu_{\mathrm{H}}$ by the canonical projection $\mathrm{SU}(2) \to \mathrm{SU}(2)^\natural$. On the space $(\Omega, \mu)$, define a sequence indexed by the prime numbers, $g^\natural(\omega) = \{g_p^\natural(\omega)\}_p$ of random matrices taking values in $\mathrm{SU}(2)^\natural$, given by

$$g_p^\natural(\omega) := \begin{pmatrix} e^{i\vartheta_p(\omega)} & 0 \\ 0 & e^{-i\vartheta_p(\omega)} \end{pmatrix}^\natural.$$

We assume that each function $g_p^\natural(\omega)$ is distributed according to the Sato-Tate measure. This means that, for each integrable function $\phi : \mathrm{SU}(2)^\natural \to \mathbb{R}$, the expected value of $\phi \circ g_p^\natural$ is

$$\mathbb{E}(\phi \circ g_p^\natural) := \int_\Omega \phi \circ g_p^\natural(\omega)\,\mathrm{d}\mu(\omega) = \int_0^\pi \phi\left( \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right) \cdot (2/\pi)\sin^2\theta\,\mathrm{d}\theta.$$

Moreover, we assume that the sequence $g^\natural(\omega)$ is made of independent random variables. This means that, for any sequence of integrable functions $\{G_p : \mathrm{SU}(2)^\natural \to \mathbb{R}\}_p$, we have

$$\begin{aligned} \mathbb{E}\Big(\prod_p G_p \circ g_p^\natural\Big) &:= \int_\Omega \prod_p G_p \circ g_p^\natural(\omega)\,\mathrm{d}\mu(\omega) \\ &= \prod_p \int_\Omega G_p \circ g_p^\natural(\omega)\,\mathrm{d}\mu(\omega) \\ &= \prod_p \int_0^\pi G_p\left( \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right) \cdot (2/\pi)\sin^2\theta\,\mathrm{d}\theta. \end{aligned} \tag{1.12}$$

Let $I$ be the identity matrix. Then for $\Re e\, s > \frac{1}{2}$, the random Euler product

$$L(s, g^\natural(\omega)) := \prod_p \det\big(I - p^{-s} g_p^\natural(\omega)\big)^{-1} =: \prod_p L_p(s, g^\natural(\omega))$$

turns out to be absolutely convergent a.s.

Now we define our probabilistic distribution functions

$$\begin{cases} \Phi(t) := \mathrm{Prob}\big(\{L(1, g^\natural(\cdot)) \geq (e^\gamma t)^2\}\big), \\ \Psi(t) := \mathrm{Prob}\big(\{L(1, g^\natural(\cdot)) \leq (6\pi^{-2} e^\gamma t)^{-2}\}\big). \end{cases}$$

We shall prove Theorem 1 in two steps. The first one is to compare $\widetilde{F}_k(t)$ with $\Phi(t)$ (resp. $\widetilde{G}_k(t)$ with $\Psi(t)$).

**Theorem 2.** *For any $A \geq 1$ there are two positive constants $c = c(A)$ and $C = C(A)$ such that the asymptotic formulas*

$$\widetilde{F}_k(t) = \Phi(t)\{1 + \Delta_k(t)\} \qquad \text{and} \qquad \widetilde{G}_k(t) = \Psi(t)\{1 + \Delta_k(t)\} \tag{1.13}$$

*hold uniformly for $k \geq 16, 2 \mid k$ and $t \leq T(k)$, where $\Delta_k(t)$ and $T(k)$ are defined by (1.10).*

The second step of the proof of Theorem 1 is the evaluation of $\Phi(t)$ (resp. $\Psi(t)$). For this, we consider a truncated random Euler product

$$L(s, g^\natural(\omega); y) := \prod_{p \leq y} L_p(s, g^\natural(\omega))$$

and the corresponding distribution functions

$$\begin{cases} \Phi(t, y) := \mathrm{Prob}\big(\{L(1, g^\natural(\omega); y) \geq (e^\gamma t)^2\}\big), \\ \Psi(t, y) := \mathrm{Prob}\big(\{L(1, g^\natural(\omega); y) \leq (6\pi^{-2} e^\gamma t)^{-2}\}\big). \end{cases}$$

We have

$$\Phi(t) = \Phi(t, \infty) \qquad \text{and} \qquad \Psi(t) = \Psi(t, \infty). \tag{1.14}$$

We shall use the saddle-point method (introducted by Hildebrand & Tenenbaum [6]) to evaluate $\Phi(t, y)$ and $\Psi(t, y)$. For this, we need to introduce some notation. For $s \in \mathbb{C}$ and $y \geq 2$, define

$$E(s, y) := \mathbb{E}\big(L(1, g^\natural(\omega); y)^s\big) \qquad \text{and} \qquad E(s) := E(s, \infty), \tag{1.15}$$

where $\mathbb{E}(\cdot)$ denotes the expected value. We define also

$$\phi(s, y) := \log E(s, y), \qquad \phi_n(s, y) := \frac{\partial^n \phi}{\partial s^n}(s, y) \quad (n \geq 0). \tag{1.16}$$

According to Lemmas 2.3 and 8.1 below, there is an absolute constant $c \geq 2$ such that for $t \geq 4 \log c$ and $y \geq ce^t$, the equation

$$\phi_1(\kappa, y) = 2(\log t + \gamma) \tag{1.17}$$

has a unique positive solution $\kappa = \kappa(t, y)$ and for each integer $J \geq 1$, there are computable constants $\gamma_0, \gamma_1, \ldots, \gamma_J$ such that the asymptotic formula

$$\kappa(t, y) = e^{t - \gamma_0} \bigg\{ 1 + \sum_{j=1}^{J} \frac{\gamma_j}{t^j} + O_J\bigg( \frac{1}{t^{J+1}} + \frac{e^t t}{y \log y} \bigg) \bigg\} \tag{1.18}$$

holds uniformly for $t \geq 1$ and $y \geq 2e^t$, the constant $\gamma_0$ beign given by (1.24) below.

Finally write $\sigma_n := \phi_n(\kappa, y)$.

**Theorem 3.** *We have*

$$\Phi(t,y)=\frac{E(\kappa,y)}{\kappa\sqrt{2\pi\sigma_2}(e^\gamma t)^{2\kappa}}\biggl\{1+O\Bigl(\frac{t}{e^t}\Bigr)\biggr\}$$

*uniformly for $t\ge 1$ and $y\ge 2e^t$.*

**Theorem 4.** *For each integer $J\ge 1$, we have*

$$\Phi(t,y)=\exp\biggl\{-\kappa\Bigl[\sum_{j=1}^{J}\frac{a_j}{(\log\kappa)^j}+O_J\bigl(R_J(\kappa,y)\bigr)\Bigr]\biggr\} \tag{1.19}$$

*uniformly for $t\ge 1$ and $y\ge 2e^t$, where the error term $R_J(\kappa,y)$ is given by*

$$R_J(\kappa,y):=\frac{1}{(\log\kappa)^{J+1}}+\frac{\kappa}{y\log y} \tag{1.20}$$

*and*

$$a_j:=\int_0^\infty\Bigl(\frac{h(u)}{u}\Bigr)'(\log u)^{j-1}\,\mathrm{d}u \tag{1.21}$$

*with*

$$h(u):=\begin{cases}\log\Bigl(\dfrac{2}{\pi}\displaystyle\int_0^\pi e^{2u\cos\theta}\sin^2\theta\,\mathrm{d}\theta\Bigr) & \text{if}\quad 0\le u<1,\\[2ex] \log\Bigl(\dfrac{2}{\pi}\displaystyle\int_0^\pi e^{2u\cos\theta}\sin^2\theta\,\mathrm{d}\theta\Bigr)-2u & \text{if}\quad u\ge 1.\end{cases} \tag{1.22}$$

As a corollary of Theorem 4, we can obtain an asymptotic developpment for $\log\Phi(t,y)$ in $t^{-1}$. In particular we see that the probabilistic distribution function $\Phi(t)$ decays double exponentially as $t\to\infty$.

**Corollary 5.** *For each integer $J\ge 1$, there are computable constants $a_1^*,\dots,a_J^*$ such that the asymptotic formula*

$$\Phi(t,y)=\exp\biggl\{-e^{t-\gamma_0}\Bigl[\sum_{j=1}^{J}\frac{a_j^*}{t^j}+O_J\bigl(R_J(e^t,y)\bigr)\Bigr]\biggr\} \tag{1.23}$$

*holds uniformly for $t\ge 1$ and $y\ge 2e^t$. Further we have*

$$\gamma_0:=\frac12\int_0^\infty\frac{h'(u)}{u}\,\mathrm{d}u,\qquad a_1^*:=1,\qquad a_2^*:=\gamma_0-\frac{\gamma_0^2}{2}-\int_0^\infty\frac{h(u)}{u^2}(\log u)\,\mathrm{d}u. \tag{1.24}$$

*In particular for each integer $J\ge 1$, we have*

$$\Phi(t)=\exp\biggl\{-e^{t-\gamma_0}\Bigl[\sum_{j=1}^{J}\frac{a_j^*}{t^j}+O_J\Bigl(\frac{1}{t^{J+1}}\Bigr)\Bigr]\biggr\} \tag{1.25}$$

*uniformly for $t\ge 1$.*

**Remark 2**. (i) The same results hold also for $\Psi(t,y)$.

(ii) Taking $t = \log_2 k$ and $J = 1$ in (1.25) of Corollary 5, we see that the probabilistic distribution function $\Phi(t)$ (resp. $\Psi(t)$) verifies Montgomery-Vaughan's conjecture (1.5). But (1.13) is too weak to derive this conjecture for $F_k(t)$ (resp. $G_k(t)$). This means that we must take $T(k) = \log_2 k$ in Theorem 2, which seems be rather difficult.

(iii) Our method can be generalized to prove that Theorems 1 and 2 hold for $L(1, \mathrm{sym}^m f)$ when $m = 1, 2, 3, 4$ and that Theorems 3, 4 and Corollary 5 are true for $L(1, \mathrm{sym}^m g^\natural(\omega); y)$ when $m \geq 1$.

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## § 2. Expression of $E(s, y)$ and existence of saddle-point

The aim of this section is to prove the existence of the saddle-point $\kappa(t, y)$, defined by equation (1.17). The first step is to give an explicite expression of $E(s, y)$, which is (1.24) of [1]. For the convenience of readers, we state it here as a lemma.

**Lemma 2.1.** *For prime $p$, real $\theta$ and complex number $s$, we define*

$$D_p(\theta) := \prod_{0\leq j\leq 1} \big(1 - e^{i(1-2j)\theta} p^{-1}\big)^{-1} \qquad \text{and} \qquad E_p(s) := \frac{2}{\pi}\int_0^\pi D_p(\theta)^s \sin^2\theta \,\mathrm{d}\theta. \tag{2.1}$$

*Then for all $s \in \mathbb{C}$ and $y \geq 2$, we have*

$$E(s, y) = \prod_{p\leq y} E_p(s). \tag{2.2}$$

*Proof.* Taking

$$G_p(M^\natural) = \begin{cases} \det\big(I - p^{-s'} M^\natural\big)^{-s} & \text{if } p \leq y \\ 1 & \text{otherwise} \end{cases}$$

in (1.12), we get

$$\begin{aligned} \mathbb{E}\big(L(s', g^\natural(\omega); y)^s\big) &= \prod_{p\leq y} \mathbb{E}\big(L_p(s', g_p^\natural(\omega))^s\big) \\ &= \prod_{p\leq y} \int_\Omega \det\big(1 - p^{-s'} g_p^\natural(\omega)\big)^{-s} \,\mathrm{d}\mu(\omega) \\ &= \prod_{p\leq y} \frac{2}{\pi}\int_0^\pi \big(1 - 2p^{-s'}\cos\theta + p^{-2s'}\big)^{-s} \sin^2\theta \,\mathrm{d}\theta. \end{aligned}$$

Taking $s' = 1$ and noticing (1.15) and (2.1), we get the desired result. □

**Lemma 2.2.** *For all $p$ and $\sigma > 0$, we have*

$$E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2 > 0.$$

*In particular for all $\sigma > 0$ and $y \geq 2$, we have $\phi_2(\sigma, y) > 0$.*

*Proof.* By using the definition (2.1) of $E_p(\sigma)$, it is easy to see that

$$\begin{aligned} E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2 &= \frac{4}{\pi^2}\int_0^\pi D_p(\theta)^\sigma \log^2 D_p(\theta)\sin^2\theta\,\mathrm{d}\theta \int_0^\pi D_p(\theta)^\sigma \sin^2\theta\,\mathrm{d}\theta \\ &\quad - \Big(\frac{2}{\pi}\int_0^\pi D_p(\theta)^\sigma \log D_p(\theta)\sin^2\theta\,\mathrm{d}\theta\Big)^2 \\ &= \frac{4}{\pi^2}\int_0^\pi\int_0^\pi D_p(\theta_1)^\sigma D_p(\theta_2)^\sigma \big(\log^2 D_p(\theta_1) - \log D_p(\theta_1)\log D_p(\theta_2)\big)\times \\ &\quad \times \sin^2\theta_1 \sin^2\theta_2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2. \end{aligned}$$

In view of the symmetry in $\theta_1$ and $\theta_2$, the same formula holds if we exchange the roles of $\theta_1$ and $\theta_2$. Thus it follows that

$$E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2 = \frac{2}{\pi^2}\int_0^\pi\int_0^\pi D_p(\theta_1)^\sigma D_p(\theta_2)^\sigma \log^2\Big(\frac{D_p(\theta_1)}{D_p(\theta_2)}\Big)\sin^2\theta_1\sin^2\theta_2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2.$$

This proves the first assertion and the second follows immediately. □

**Lemma 2.3.** *There is an absolute constant $c \geq 2$ such that for $t \geq 4\log c$ and $y \geq ce^t$, the equation $\phi_1(\sigma, y) = 2(\log t + \gamma)$ has a unique positive solution in $\sigma$. Denoting by $\kappa(t, y)$ this solution, we have $\kappa(t, y) \asymp e^t$ uniformly for $t \geq 4\log c$ and $y \geq ce^t$.*

*Proof.* According to Lemma 4.3 below with the choice of $J = 1$, we have

$$\phi_1(\sigma, y) = 2(\log_2\sigma + \gamma) + O(1/\log\sigma)$$

for $y \geq \sigma \geq 2$. Thus

$$\begin{aligned} \phi(ce^t, y) &= 2\log(t + \log c) + 2\gamma + O\Big(\frac{1}{t + \log c}\Big) \\ &> 2\log t + 2\gamma, \\ \phi(c^{-1}e^t, y) &= 2\log(t - \log c) + 2\gamma + O\Big(\frac{1}{t - \log c}\Big) \\ &< 2\log t + 2\gamma, \end{aligned}$$

provided that $c$ is a large constant and $t \geq 4\log c$. On the other hand, in view of Lemma 2.2, we know that for any $y \geq 2$, $\phi_1(\sigma, y)$ is an increasing function of $\sigma$ in $(0, \infty)$. Hence the equation $\phi_1(\sigma, y) = 2(\log t + \gamma)$ has a unique positive solution $\kappa(t, y)$ and $c^{-1}e^t \leq \kappa(t, y) \leq ce^t$ for $t \geq 4\log c$ and $y \geq ce^t$. This completes the proof. □

## § 3. Preliminary lemmas

This section is devoted to establish some preliminary lemmas, which will be useful later.

**Lemma 3.1.** *Let $j \geq 0$ be a fixed real number. Then we have*

$$\int_0^\pi e^{2u\cos\theta}(1 - \cos\theta)^j \sin^2\theta\,\mathrm{d}\theta \asymp_j e^{2u}u^{-(j+3/2)} \qquad (u \geq 1). \tag{3.1}$$

*The implied constant depends on $j$ only.*

*Proof.* First we write

$$\begin{aligned}\int_0^\pi e^{2u\cos\theta}(1-\cos\theta)^j\sin^2\theta\,\mathrm{d}\theta &= \int_0^{\pi/2}\left(e^{2u\cos\theta}(1-\cos\theta)^j + e^{-2u\cos\theta}(1+\cos\theta)^j\right)\sin^2\theta\,\mathrm{d}\theta\\ &= \int_0^1\left(e^{2ut}(1-t)^j + e^{-2ut}(1+t)^j\right)(1-t^2)^{1/2}\,\mathrm{d}t\\ &\asymp \int_0^1 e^{2ut}(1-t)^{j+1/2}\,\mathrm{d}t + \int_0^1 e^{-2ut}(1-t)^{1/2}\,\mathrm{d}t.\end{aligned}$$

By the change of variables $u(1-t)=v$, it follows that

$$\begin{aligned}\int_0^1 e^{2ut}(1-t)^{j+1/2}\,\mathrm{d}t &= e^{2u}u^{-(j+3/2)}\int_0^u e^{-2v}v^{j+1/2}\,\mathrm{d}v\\ &\asymp e^{2u}u^{-(j+3/2)},\\ \int_0^1 e^{-2ut}(1-t)^{1/2}\,\mathrm{d}t &\le \int_0^1 e^{-2ut}\,\mathrm{d}t \ll u^{-1}.\end{aligned}$$

We obtain the desired result by insertion of these estimates into the preceeding relation. □

**Lemma 3.2.** *Let $j\ge 0$ be an integer and*

$$E_{p,j}(\sigma) := \frac{2}{\pi}\int_0^\pi D_p(\theta)^\sigma(1-\cos\theta)^j\sin^2\theta\,\mathrm{d}\theta. \tag{3.2}$$

*(In particular $E_{p,0}(\sigma)=E_p(\sigma)$.) Then we have*

$$E_{p,j}(\sigma) = \frac{2^{j+3}}{\pi}\int_0^1\left[\left(1-\frac{1}{p}\right)^2+\frac{4u}{p}\right]^{-\sigma}u^{j+1/2}(1-u)^{1/2}\,\mathrm{d}u$$

*and the estimate*

$$E_{p,j}(\sigma)/E_p(\sigma) \ll (p/\sigma)^j \tag{3.3}$$

*holds uniformly for all primes $p$ and $\sigma>0$. Further if $p\ge\sigma\ge 0$, we have*

$$E_p(\sigma)\asymp 1. \tag{3.4}$$

*The implied constant in (3.3) depends on $j$ only and the one in (3.4) is absolute.*

*Proof.* By the change of variables $u=\sin^2(\theta/2)$, a simple computation shows that the first assertion is true. Obviously (3.3) holds for $j=0$.

Now assume that it is true for $j$. An integration by parts leads to

$$\begin{aligned}E_p(\sigma) &\gg_j \left(\frac{\sigma}{p}\right)^j\int_0^1\left[\left(1-\frac{1}{p}\right)^2+\frac{4u}{p}\right]^{-\sigma}u^{j+1/2}(1-u)^{1/2}\,\mathrm{d}u\\ &\gg_j \left(\frac{\sigma}{p}\right)^j\int_0^1\left\{\left[\left(1-\frac{1}{p}\right)^2+\frac{4u}{p}\right]^{-1}\frac{4\sigma}{p}+\frac{1}{2(1-u)}\right\}\times\\ &\qquad\times\left[\left(1-\frac{1}{p}\right)^2+\frac{4u}{p}\right]^{-\sigma}u^{j+1+1/2}(1-u)^{1/2}\,\mathrm{d}u.\end{aligned}$$

On the other hand, we have

$$0<u<1 \;\Rightarrow\; \left[\left(1-\frac{1}{p}\right)^2+\frac{4u}{p}\right]^{-1}\frac{4\sigma}{p}+\frac{1}{2(1-u)}\ge\left(1+\frac{1}{p}\right)^{-2}\frac{4\sigma}{p}\ge\frac{16\sigma}{9p}.$$

Inserting it into the preceeding estimate, we see that

$$\begin{aligned}E_p(\sigma)&\gg_j\left(\frac{\sigma}{p}\right)^{j+1}\int_0^1\left[\left(1-\frac{1}{p}\right)^2+\frac{4u}{p}\right]^{-\sigma}u^{j+1+1/2}(1-u)^{1/2}\,\mathrm{d}u\\&\asymp_j\left(\frac{\sigma}{p}\right)^{j+1}E_{p,j+1}(\sigma).\end{aligned}$$

Thus (3.3) holds also for $j+1$.

Since $(1+1/p)^{-2}\le D_p(\theta)\le(1-1/p)^{-2}$ for all primes $p$ and any $\theta\in\mathbb{R}$, we have $D_p(\theta)^\sigma\asymp 1$ uniformly for $p\ge\sigma\ge 0$ and $\theta\in\mathbb{R}$. This implies (3.4). □

Introduce the function

$$g(u):=\log\left(\frac{2}{\pi}\int_0^\pi e^{2u\cos\theta}\sin^2\theta\,\mathrm{d}\theta\right)\qquad(u\ge 0)\tag{3.5}$$

and let $h(u)$ be defined as in (1.22). Clearly we have

$$h(u)=\begin{cases}g(u) & \text{if}\quad 0\le u<1,\\ g(u)-2u & \text{if}\quad u\ge 1,\end{cases}\tag{3.6}$$

$$h'(u)=\begin{cases}g'(u) & \text{if}\quad 0\le u<1,\\ g'(u)-2 & \text{if}\quad u\ge 1,\end{cases}\tag{3.7}$$

$$h''(u)=g''(u)\quad(u\ge 0,\,u\ne 1).\tag{3.8}$$

**Lemma 3.3.** *We have*

$$h(u)\asymp\begin{cases}u^2 & \textit{if}\quad 0\le u<1,\\ \log(2u) & \textit{if}\quad u\ge 1,\end{cases}\tag{3.9}$$

$$h'(u)\asymp\begin{cases}u & \textit{if}\quad 0\le u<1,\\ u^{-1} & \textit{if}\quad u\ge 1,\end{cases}\tag{3.10}$$

$$h''(u)\asymp\begin{cases}1 & \textit{if}\quad 0\le u<1,\\ u^{-2} & \textit{if}\quad u\ge 1,\end{cases}\tag{3.11}$$

$$h'''(u)\asymp\begin{cases}u & \textit{if}\quad 0\le u<1,\\ u^{-3} & \textit{if}\quad u\ge 1.\end{cases}\tag{3.12}$$

*Proof.* The last three relations are trivial. We will prove only the first four estimates.

When $0\le u<1$, we have

$$e^{2u\cos\theta}=\sum_{n=0}^\infty\frac{(u\cos\theta)^n}{n!}.$$

¿From this we deduce that

$$\begin{aligned}h(u)&=\log\left(\frac{2}{\pi}\sum_{n=0}^\infty\frac{u^n}{n!}\int_0^\pi(\cos\theta)^n\sin^2\theta\,\mathrm{d}\theta\right)\\&=\log\left(1+\sum_{\ell=1}^\infty\frac{2\cdot(2\ell-1)!!}{(2\ell)!(2\ell+2)!!}u^{2\ell}\right),\end{aligned}\tag{3.13}$$

where we have used the following facts:

$$\int_0^\pi (\cos\theta)^{2\ell+1}\sin^2\theta\,\mathrm{d}\theta = 0$$

and

$$\frac{2}{\pi}\int_0^\pi (\cos\theta)^{2\ell}\sin^2\theta\,\mathrm{d}\theta = \begin{cases} 1 & \text{if } \ell = 0, \\ 2\dfrac{(2\ell-1)!!}{(2\ell+2)!!} & \text{if } \ell \geq 1. \end{cases}$$

Now we easily deduce, from (3.13), the desired results in the case of $0 \leq u < 1$.

The estimates of (3.9)–(3.12) for $u > 1$ are simple consequences of (3.1), by noticing the following relations

$$h'(u) = -2\,\frac{\displaystyle\int_0^\pi e^{2u\cos\theta}(1-\cos\theta)\sin^2\theta\,\mathrm{d}\theta}{\displaystyle\int_0^\pi e^{2u\cos\theta}\sin^2\theta\,\mathrm{d}\theta},$$

$$h''(u) = 4\,\frac{\displaystyle\int_0^\pi e^{2u\cos\theta}(1-\cos\theta)^2\sin^2\theta\,\mathrm{d}\theta}{\displaystyle\int_0^\pi e^{2u\cos\theta}\sin^2\theta\,\mathrm{d}\theta} - 4\left(\frac{\displaystyle\int_0^\pi e^{2u\cos\theta}(1-\cos\theta)\sin^2\theta\,\mathrm{d}\theta}{\displaystyle\int_0^\pi e^{2u\cos\theta}\sin^2\theta\,\mathrm{d}\theta}\right)^2.$$

This completes the proof. □

## § 4. Estimates of $\phi_n(\sigma, y)$

The aim of this section is to prove some estimates of $\phi_n(\sigma, y)$ for $n = 0, 1, 2, 3, 4$.

**Lemma 4.1.** *For any fixed integer $J \geq 1$, we have*

$$\phi_0(\sigma, y) = \sigma\left\{2\log_2\sigma + 2\gamma + \sum_{j=1}^{J}\frac{b_{j,0}}{(\log\sigma)^j} + O_J\big(R_J(\sigma, y)\big)\right\} \tag{4.1}$$

*uniformly for $y \geq \sigma \geq 3$, where $R_J(\sigma, y)$ is defined as in (1.20) and*

$$b_{j,0} := \int_0^\infty \frac{h(u)}{u^2}(\log u)^{j-1}\,\mathrm{d}u. \tag{4.2}$$

*Proof.* By the definition (2.1) of $D_p(\theta)$ and the one of $E_p(\sigma)$, it is easy to see that for $p \geq \sigma^{1/2}$, we have

$$D_p(\theta)^\sigma = e^{2(\sigma/p)\cos\theta}\left\{1 + O\left(\frac{\sigma}{p^2}\right)\right\}, \tag{4.3}$$

$$E_p(\sigma) = \left\{1 + O\left(\frac{\sigma}{p^2}\right)\right\}\frac{2}{\pi}\int_0^\pi e^{2(\sigma/p)\cos\theta}\sin^2\theta\,\mathrm{d}\theta. \tag{4.4}$$

¿From these, we deduce that

$$\sum_{\sigma^{1/2}<p\leq y}\log E_p(\sigma) = \sum_{\sigma^{1/2}<p\leq y} g(\sigma/p) + O(\sigma^{1/2}/\log\sigma) \tag{4.5}$$

where $g(u)$ is defined as in (3.5).

In order to treat the sum over $p \le \sigma$, we write

$$E_p(\sigma) = (1-1/p)^{-2\sigma} E_p^*(\sigma),$$

where

$$E_p^*(\sigma) := \frac{2}{\pi}\int_0^\pi \left\{1 + \frac{2(1-\cos\theta)}{p}\left(1-\frac{1}{p}\right)^{-2}\right\}^{-\sigma} \sin^2\theta\,\mathrm{d}\theta.$$

By using the change of variables $u = \sin^2(\theta/2)$, we have

$$\begin{aligned} E_p^*(\sigma) &= \frac{8}{\pi}\int_0^\pi \left\{1 + \frac{4}{p}\left(1-\frac{1}{p}\right)^{-2}\sin^2(\theta/2)\right\}^{-\sigma} \sin^2(\theta/2)\cos^2(\theta/2)\,\mathrm{d}\theta \\ &\ge \frac{8}{\pi}\int_0^{p/2\sigma} \left\{1 + \frac{4}{p}\left(1-\frac{1}{p}\right)^{-2}u\right\}^{-\sigma} \sqrt{u(1-u)}\,\mathrm{d}u \\ &\ge \frac{8}{\pi}\left(1+\frac{8}{\sigma}\right)^{-\sigma}\int_0^{p/2\sigma} \sqrt{u(1-u)}\,\mathrm{d}u \\ &\ge C\left(\frac{p}{\sigma}\right)^{3/2}, \end{aligned}$$

where $C > 0$ is a constant. On the other hand, we have trivially $E_p^*(\sigma) \le 1$ for all $p$ and $\sigma > 0$. Thus $|\log E_p^*(\sigma)| \ll \log(\sigma/p)$ for $p \le \sigma^{1/2}$ and

$$\sum_{p\le\sigma^{1/2}} \big|\log E_p^*(\sigma)\big| \ll \sum_{p\le\sigma^{1/2}} \log(\sigma/p) \ll \sigma^{1/2}. \tag{4.6}$$

Combining (4.5) and (4.6), we can write

$$\sum_{p\le y}\log E_p(\sigma) = 2\sigma\sum_{p\le\sigma^{1/2}}\log(1-1/p)^{-1} + \sum_{\sigma^{1/2}<p\le y} g(\sigma/p) + O(\sigma^{1/2}).$$

In view of (3.6) and the following estimate

$$\sum_{\sigma^{1/2}<p\le\sigma}\big(2\sigma\log(1-1/p)^{-1} - 2\sigma/p\big) \ll \sum_{\sigma^{1/2}<p\le\sigma}\sigma/p^2 \ll \sigma^{1/2}/\log\sigma,$$

the preceeding estimate can be written as

$$\sum_{p\le y}\log E_p(\sigma) = 2\sigma\sum_{p\le\sigma}\log(1-1/p)^{-1} + \sum_{\sigma^{1/2}<p\le y} h(\sigma/p) + O(\sigma^{1/2}). \tag{4.7}$$

By using the prime number theorem in the form

$$\pi(t) := \sum_{p\le t} 1 = \int_2^t \frac{\mathrm{d}v}{\log v} + O\Big(te^{-8\sqrt{\log t}}\Big), \tag{4.8}$$

it follows that

$$\sum_{\sigma^{1/2}<p\le y} h\left(\frac{\sigma}{p}\right) = \int_{\sigma^{1/2}}^y \frac{h(\sigma/t)}{\log t}\,\mathrm{d}t + R_0, \tag{4.9}$$

where

$$\begin{aligned} R_0 &:= h\Big(\frac{\sigma}{y}\Big)e^{-8\sqrt{\log y}} + h\big(\sigma^{1/2}\big)e^{-4\sqrt{\log \sigma}} + \int_{\sigma^{1/2}}^{y} \sigma\frac{|h'(\sigma/t)|}{t^2}e^{-8\sqrt{\log t}}\,\mathrm{d}t \\ &\ll \frac{\sigma^2}{y^2}e^{-8\sqrt{\log y}} + e^{-2\sqrt{\log \sigma}} + \int_{\sigma^{1/2}}^{\sigma} \frac{e^{-2\sqrt{\log t}}}{t}\,\mathrm{d}t + \sigma^2\int_{\sigma}^{y}\frac{e^{-8\sqrt{\log t}}}{t^3}\,\mathrm{d}t \\ &\ll e^{-\sqrt{\log \sigma}} \end{aligned}$$

by use of Lemma 3.3.

In order to evaluate the integral of (4.9), we use the change of variables $u = \sigma/t$ to write

$$\begin{aligned} \int_{\sigma^{1/2}}^{y} \frac{h(\sigma/t)}{\log t}\,\mathrm{d}t &= \sigma\int_{\sigma/y}^{\sigma^{1/2}} \frac{h(u)}{u^2\log(\sigma/u)}\,\mathrm{d}u \\ &= \sigma\int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2\log(\sigma/u)}\,\mathrm{d}u + O\big(R_0'\big) \end{aligned}$$

where

$$\begin{aligned} R_0' &:= \sigma\int_{0}^{\sigma/y} \frac{|h(u)|}{u^2\log(\sigma/u)}\,\mathrm{d}u + \sigma\int_{0}^{\sigma^{-1/2}} \frac{|h(u)|}{u^2\log(\sigma/u)}\,\mathrm{d}u \\ &\ll \frac{\sigma^2}{y\log y} + \frac{\sigma^{1/2}}{\log\sigma}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2\log(\sigma/u)}\,\mathrm{d}u &= \frac{1}{\log\sigma}\int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2(1-(\log u)/\log\sigma)}\,\mathrm{d}u \\ &= \sum_{j=1}^{J}\frac{1}{(\log\sigma)^j}\int_{\sigma^{-1/2}}^{\sigma^{1/2}}\frac{h(u)}{u^2}(\log u)^{j-1}\,\mathrm{d}u + O\bigg(\frac{1}{(\log\sigma)^{J+1}}\bigg). \end{aligned}$$

Extending the interval of integration $[\sigma^{-1/2}, \sigma^{1/2}]$ to $(0,\infty)$ and bounding the contributions of $(0,\sigma^{-1/2}]$ and $[\sigma^{1/2},\infty)$ by using (3.9) of Lemma 3.3, we have

$$\int_{\sigma^{-1/2}}^{\sigma^{1/2}}\frac{h(u)}{u^2}(\log u)^{j-1}\,\mathrm{d}u = b_{j,0} + O\bigg(\frac{(\log\sigma)^j}{\sigma^{1/2}}\bigg).$$

Combining these estimates, we find that

$$\sum_{\sigma^{1/2}<p\le y} h\Big(\frac{\sigma}{p}\Big) = \sigma\bigg\{\sum_{j=1}^{J}\frac{b_{j,0}}{(\log\sigma)^j} + O_J\big(R_J(\sigma,y)\big)\bigg\}. \tag{4.10}$$

Now the desired result follows from (4.7), (4.10) and the prime number theorem in the form

$$\sum_{p\le\sigma}\log(1-1/p)^{-1} = \log_2\sigma + \gamma + O\Big(e^{-2\sqrt{\log\sigma}}\Big). \tag{4.11}$$

This completes the proof. □

**Remark 3**. In view of (1.3), we can write (4.1) as

$$\phi_0(\sigma,y) = \sigma\bigg\{\log(B_1^+\log\sigma)^{A_1^+} + \sum_{j=1}^{J}\frac{b_{j,0}}{(\log\sigma)^j} + O_J\big(R_J(\sigma,y)\big)\bigg\}$$

uniformly for $y \ge \sigma \ge 3$. In the case $\sigma < 0$, a similar asymptotic formula (with $A_1^-, B_1^-$ and corresponding $b_{j,0}^-$ in place of $A_1^+, B_1^+$ and $b_{j,0}$) can be established uniformly for $y \ge -\sigma \ge 3$. As indicated in the introduction, Lemma 4.1 can be easily generalised to the general case $m \ge 1$. Thus we give an improvement and generalisation of Corollaries A and C of [15], of Theorem B of [5], and an improvement of Theorem 1.12 of [1]. It is worthy to indicate that our method seems to be simpler and more natural.

**Lemma 4.2.** *We have*

$$\frac{E_p'(\sigma)}{E_p(\sigma)} = \begin{cases} \log D_p(0) + O\Big(\dfrac{1}{\sigma}\Big) & \text{for all } p \text{ and } \sigma > 0, \\ \dfrac{1}{2} g'\Big(\dfrac{\sigma}{p}\Big) \log D_p(0) + O\Big(\dfrac{1}{p^2} + \dfrac{\sigma}{p^3}\Big) & \text{if } \ p \ge \sigma^{1/2}, \end{cases} \tag{4.12}$$

*where $g(u)$ is defined as in (3.5).*

*Proof.* First we write

$$\begin{aligned} E_p'(\sigma) &= \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log D_p(\theta) \sin^2\theta \, \mathrm{d}\theta \\ &= E_p(\sigma) \log D_p(0) + R', \end{aligned} \tag{4.13}$$

where

$$R' := \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log\Big(\frac{D_p(\theta)}{D_p(0)}\Big) \sin^2\theta \, \mathrm{d}\theta. \tag{4.14}$$

Since

$$\Big|\log\Big(\frac{D_p(\theta)}{D_p(0)}\Big)\Big| = \Big|-\log\Big(1 + \frac{2p(1-\cos\theta)}{(p-1)^2}\Big)\Big| \le \frac{2p(1-\cos\theta)}{(p-1)^2} \le \frac{8(1-\cos\theta)}{p},$$

it follows from (3.3) of Lemma 3.2 with $j = 1$ that

$$\frac{R'}{E_p(\sigma)} \ll \frac{E_{p,1}(\sigma)}{pE_p(\sigma)} \ll \frac{1}{\sigma}$$

for all $p$ and $\sigma > 0$. This implies, via (4.13), the first estimate of (4.12).

We have

$$\begin{aligned} \log D_p(\theta) &= (\cos\theta)(2/p) + O\big(1/p^2\big) \\ &= (\cos\theta) \log D_p(0) + O\big(1/p^2\big). \end{aligned}$$

Inserting it and (4.3) into the first relation of (4.13) and in view of (4.4), we can write, for $p \ge \sigma^{1/2}$,

$$\begin{aligned} E_p'(\sigma) &= \Big\{1 + O\Big(\frac{\sigma}{p^2}\Big)\Big\} \frac{2}{\pi} \int_0^\pi e^{2(\sigma/p)\cos\theta} \Big\{(\cos\theta) \log D_p(0) + O\Big(\frac{1}{p^2}\Big)\Big\} \sin^2\theta \, \mathrm{d}\theta \\ &= \Big\{1 + O\Big(\frac{\sigma}{p^2}\Big)\Big\} \frac{2}{\pi} \int_0^\pi e^{2(\sigma/p)\cos\theta} (\cos\theta) \sin^2\theta \, \mathrm{d}\theta \log D_p(0) + O\Big(\frac{E_p(\sigma)}{p^2}\Big). \end{aligned}$$

¿From this and (4.4), we deduce

$$\frac{E_p'(\sigma)}{E_p(\sigma)} = \Big\{1 + O\Big(\frac{\sigma}{p^2}\Big)\Big\} \frac{1}{2} g'\Big(\frac{\sigma}{p}\Big) \log D_p(0) + O\Big(\frac{1}{p^2}\Big),$$

which implies the second estimate of (4.12). This completes the proof. □

**Lemma 4.3.** *Let $J \geq 1$ be a fixed integer. Then we have*

$$\phi_1(\sigma, y) = 2\log_2 \sigma + 2\gamma + \sum_{j=1}^{J} \frac{b_{j,1}}{(\log \sigma)^j} + O_J\big(R_J(\sigma, y)\big)$$

*uniformly for $y \geq \sigma \geq 3$, where the constant $b_{j,1}$ is given by*

$$b_{j,1} := \int_0^\infty \frac{h'(u)}{u} (\log u)^{j-1}\, \mathrm{d}u \tag{4.15}$$

*and $R_J(\sigma, y)$ is defined as in (1.20).*

*Proof.* We have

$$\phi_1(\sigma, y) = \sum_{p \leq y} E_p'(\sigma)/E_p(\sigma).$$

Using the first relation of (4.12) for $p \leq \sigma^{2/3}$ and the second for $\sigma^{2/3} < p \leq y$, we obtain

$$\phi_1(\sigma, y) = \sum_{p \leq \sigma^{2/3}} \log D_p(0) + \frac{1}{2} \sum_{\sigma^{2/3} < p \leq y} g'\Big(\frac{\sigma}{p}\Big) \log D_p(0) + O\Big(\frac{1}{\sigma^{1/3}}\Big).$$

In view of (3.7), the preceeding formula can be written as

$$\phi_1(\sigma, y) = \sum_{p \leq \sigma} \log D_p(0) + \sum_{\sigma^{2/3} < p \leq y} h'\Big(\frac{\sigma}{p}\Big) \log\Big(1 - \frac{1}{p}\Big)^{-1} + O\Big(\frac{1}{\sigma^{1/3}}\Big). \tag{4.16}$$

Similarly to (4.10), we can prove that

$$\sum_{\sigma^{2/3} < p \leq y} h'\Big(\frac{\sigma}{p}\Big) \log\Big(1 - \frac{1}{p}\Big)^{-1} = \sum_{j=1}^{J} \frac{b_{j,1}}{(\log \sigma)^j} + O_J\big(R_J(\sigma, y)\big), \tag{4.17}$$

using (3.10), (3.11) and (4.11) instead of (3.9), (3.10) and (4.8). Now the desired result follows from (4.16), (4.10) and (4.17). □

**Lemma 4.4.** *We have*

$$\frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \begin{cases} O\Big(\dfrac{1}{\sigma^2}\Big) & \text{if } \ p \leq \sigma^{1/2}, \\ \dfrac{1}{p^2} g''\Big(\dfrac{\sigma}{p}\Big) + O\Big(\min\Big\{\dfrac{1}{\sigma^2 p}, \dfrac{1}{\sigma p^2}\Big\}\Big) & \text{if } \ p > \sigma^{1/2}, \end{cases} \tag{4.18}$$

*where $g(u)$ is defined as in (3.5).*

*Proof.* First we write

$$\begin{aligned} E_p''(\sigma) &= \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log^2 D_p(\theta) \sin^2\theta\, \mathrm{d}\theta \\ &= E_p(\sigma) \log^2 D_p(0) + R'', \end{aligned} \tag{4.19}$$

where

$$R'' := \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \Big(\log^2 D_p(\theta) - \log^2 D_p(0)\Big) \sin^2\theta\, \mathrm{d}\theta.$$

Using (4.13) and (4.19), we can deduce

$$\frac{E_p''(\sigma)E_p(\sigma)-E_p'(\sigma)^2}{E_p(\sigma)^2}=\frac{R''-2R'\log D_p(0)}{E_p(\sigma)}-\left(\frac{R'}{E_p(\sigma)}\right)^2, \tag{4.20}$$

where $R'$ is defined as in (4.14).

¿From the definitions of $R'$ and $R''$, a simple calculation shows that

$$R''-2R'\log D_p(0)=\frac{2}{\pi}\int_0^\pi D_p(\theta)^\sigma\log^2\left(\frac{D_p(\theta)}{D_p(0)}\right)\sin^2\theta\,\mathrm{d}\theta.$$

Since

$$\log^2\left(\frac{D_p(\theta)}{D_p(0)}\right)=\log^2\left(1+\frac{2p(1-\cos\theta)}{(p-1)^2}\right)=\frac{4(1-\cos\theta)^2}{p^2}+O\Big(\frac{(1-\cos\theta)^2}{p^3}\Big),$$

we have

$$R''-2R'\log D_p(0)=\frac{4}{p^2}E_{p,2}(\sigma)+O\Big(\frac{E_{p,2}(\sigma)}{p^3}\Big),$$

where $E_{p,j}(\sigma)$ is defined as in (3.2). By using (3.3) with the choice of $j=2$ and the trivial estimate $E_{p,2}(\sigma)\le 4E_p(\sigma)$, we deduce

$$\frac{R''-2R'\log D_p(0)}{E_p(\sigma)}=\frac{4}{p^2}\frac{E_{p,2}(\sigma)}{E_p(\sigma)}+O\Big(\min\Big\{\frac{1}{\sigma^2p},\ \frac{1}{p^3}\Big\}\Big). \tag{4.21}$$

Similarly we have

$$\log\left(\frac{D_p(\theta)}{D_p(0)}\right)=-\log\left(1+\frac{2p(1-\cos\theta)}{(p-1)^2}\right)=-\frac{2(1-\cos\theta)}{p}+O\Big(\frac{(1-\cos\theta)}{p^2}\Big),$$

and therefore

$$R'=-\frac{2}{p}E_{p,1}(\sigma)+O\Big(\frac{E_{p,1}(\sigma)}{p^2}\Big).$$

Now (3.3) with $j=1$ and the trivial estimate $E_{p,1}(\sigma)\le 2E_p(\sigma)$ imply

$$\begin{aligned}\left(\frac{R'}{E_p(\sigma)}\right)^2&=\frac{4}{p^2}\left(\frac{E_{p,1}(\sigma)}{E_p(\sigma)}\right)^2+O\Big(\frac{E_{p,1}(\sigma)^2}{p^3E_p(\sigma)^2}\Big)\\&=\frac{4}{p^2}\left(\frac{E_{p,1}(\sigma)}{E_p(\sigma)}\right)^2+O\Big(\min\Big\{\frac{1}{\sigma^2p},\ \frac{1}{p^3}\Big\}\Big).\end{aligned} \tag{4.22}$$

Inserting (4.21) and (4.22) into (4.20) and in view of (4.14), we deduce

$$\frac{E_p''(\sigma)E_p(\sigma)-E_p'(\sigma)^2}{E_p(\sigma)^2}=\frac{4}{p^2}h_p(\sigma)+O\Big(\min\Big\{\frac{1}{\sigma^2p},\ \frac{1}{p^3}\Big\}\Big) \tag{4.23}$$

for all $p$ and $\sigma>0$, where

$$h_p(\sigma):=\frac{E_{p,2}(\sigma)}{E_p(\sigma)}-\left(\frac{E_{p,1}(\sigma)}{E_p(\sigma)}\right)^2.$$

When $p\le\sigma^{1/2}$, the inequality (3.3) of Lemma 3.2 implies that $h_p(\sigma)\ll(p/\sigma)^2$. ¿From this and (4.23) we deduce the first estimate of (4.18).

If $p\ge\sigma^{1/2}$, we can use (4.3), (3.11) and (3.8) to write

$$\begin{aligned}4h_p(\sigma)&=g''\Big(\frac{\sigma}{p}\Big)\Big\{1+O\Big(\frac{\sigma}{p^2}\Big)\Big\}\\&=g''\Big(\frac{\sigma}{p}\Big)+O\Big(\min\Big\{\frac{\sigma}{p^2},\frac{1}{\sigma}\Big\}\Big).\end{aligned}$$

Inserting it into (4.23) and in view of Lemma 3.1, we get, for $p\ge\sigma^{1/2}$,

$$\frac{E_p''(\sigma)E_p(\sigma)-E_p'(\sigma)^2}{E_p(\sigma)^2}=\frac{1}{p^2}g''\Big(\frac{\sigma}{p}\Big)+O\Big(\min\Big\{\frac{1}{\sigma^2p},\ \frac{1}{\sigma p^2}\Big\}\Big).$$

This completes the proof. □

**Lemma 4.5.** *Let $J \ge 1$ be a fixed integer. Then we have*

$$\phi_2(\sigma, y) = \frac{1}{\sigma}\Big\{ \sum_{j=1}^{J} \frac{b_{j,2}}{(\log \sigma)^j} + O_J\big(R_J(\sigma, y)\big)\Big\}$$

*uniformly for $y \ge \sigma \ge 2$, where*

$$b_{j,2} := \int_0^\infty h''(u)(\log u)^{j-1}\,\mathrm{d}u.$$

*In particular $b_{1,2} = 2$.*

*Proof.* ¿From Lemma 4.4 and (3.8), we deduce easily that

$$\begin{aligned}\phi_2(\sigma, y) &= \sum_{p\le y} \frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2}\\ &= \sum_{\sigma^{1/2}<p\le y} \frac{g''(\sigma/p)}{p^2} + O\Big(\frac{1}{\sigma^{3/2}\log\sigma}\Big)\\ &= \sum_{\sigma^{1/2}<p\le y} \frac{h''(\sigma/p)}{p^2} + O\Big(\frac{1}{\sigma^{3/2}\log\sigma}\Big).\end{aligned}$$

Similarly to (4.10), we can prove that

$$\sum_{\sigma^{1/2}<p\le y} \frac{h''(\sigma/p)}{p^2} = \frac{1}{\sigma}\Big\{ \sum_{j=1}^{J} \frac{b_{j,2}}{(\log \sigma)^j} + O_J\big(R_J(\sigma, y)\big)\Big\},$$

by using (3.11), (3.12) and (4.8). Now the desired result follows from the preceeding two estimates.

Finally

$$\begin{aligned}b_{1,2} &= \int_0^1 h''(u)\,\mathrm{d}u + \int_1^\infty h''(u)\,\mathrm{d}u\\ &= h'(1-) - h'(1+) = h'(1-) - \big(h'(1-) - 2\big) = 2.\end{aligned}$$

This completes the proof. □

Similarly (even more easily, since we only need an upper bound instead of an asymptotic formula), we can prove the following result.

**Lemma 4.6.** *We have*

$$\phi_n(\sigma, y) \ll 1/(\sigma^{n-1}\log\sigma) \qquad (n = 3, 4) \tag{4.24}$$

*uniformly for $y \ge \sigma \ge 3$.*

## § 5. Estimate of $|E(\kappa + i\tau, y)|$

**Lemma 5.1.** *For any $\delta \in (0, \frac{1}{4})$, there are two absolute positive constants $c_1, c_2$ and a positive constant $c_3 = c_3(\delta)$ such that for all $y \ge \sigma \ge 3$ we have*

$$\left|\frac{E(\sigma + i\tau, y)}{E(\sigma, y)}\right| \le \begin{cases} 1 & \text{if } \ |\tau| \le c_1\sigma^{1/2}\log\sigma \ \text{ or } \ |\tau| \ge y^{1/\delta},\\ e^{-c_2\tau^2/[\sigma(\log\sigma)^2]} & \text{if } \ c_1\sigma^{1/2}\log\sigma \le |\tau| \le \sigma,\\ e^{-c_3|\tau|^\delta} & \text{if } \ \sigma \le |\tau| \le y^{1/\delta}.\end{cases} \tag{5.1}$$

*Proof.* First we write

$$\begin{aligned} E_p(s) &= \frac{2}{\pi}\int_0^\pi \big(D_p(\theta)^{-1}\big)^{-s}\sin^2\theta\,\mathrm{d}\theta \\ &= \frac{2}{\pi}\int_0^\pi \frac{\sin^2\theta}{(1-s)(D_p(\theta)^{-1})'}\,\mathrm{d}\big(D_p(\theta)^{-1}\big)^{1-s}. \end{aligned}$$

Since $(D_p(\theta)^{-1})' = 2p^{-1}\sin\theta$, after a simplification and an integration by parts it follows that

$$\begin{aligned} E_p(s) &= \frac{p}{\pi(s-1)}\int_0^\pi D_p(\theta)^{s-1}\cos\theta\,\mathrm{d}\theta \\ &= \frac{p}{\pi(s-1)}\int_0^{\pi/2}\big\{D_p(\theta)^{s-1} - D_p(\pi-\theta)^{s-1}\big\}\cos\theta\,\mathrm{d}\theta. \end{aligned}$$

This implies that

$$\left|\frac{E_p(s)}{E_p(\sigma)}\right| = \left|\frac{\sigma-1}{s-1}\right|\left|\frac{E_p^*(s)}{E_p^*(\sigma)}\right| \tag{5.2}$$

with

$$E_p^*(s) := \int_0^{\pi/2}\big\{D_p(\theta)^{s-1} - D_p(\pi-\theta)^{s-1}\big\}\cos\theta\,\mathrm{d}\theta.$$

1° Case of $\sigma^{1/\delta} < |\tau| \le y^{1/\delta}$

Write

$$E_p^*(s) = \int_0^{\pi/2} D_p(\theta)^{s-1}\big\{1-\Delta_p(\theta)^{s-1}\big\}\cos\theta\,\mathrm{d}\theta$$

with

$$\Delta_p(\theta) := \frac{1-2p^{-1}\cos\theta+p^{-2}}{1+2p^{-1}\cos\theta+p^{-2}}.$$

It is clear that for all $p$, the function $\theta\mapsto\Delta_p(\theta)$ is increasing on $[0,\pi/2]$. It follows that

$$\begin{aligned} E_p^*(\sigma) &\ge \int_0^{\pi/4} D_p(\theta)^{\sigma-1}\big\{1-\Delta_p(\theta)^{\sigma-1}\big\}\cos\theta\,\mathrm{d}\theta \\ &\ge \big\{1-\Delta_p(\pi/4)^{\sigma-1}\big\}\int_0^{\pi/4} D_p(\theta)^{\sigma-1}\cos\theta\,\mathrm{d}\theta \end{aligned}$$

for all $p$ and $\sigma\ge 1$. This implies that

$$\left|\frac{1}{E_p^*(\sigma)}\int_0^{\pi/4} D_p(\theta)^{\sigma-1}\cos\theta\,\mathrm{d}\theta\right| \le \frac{1}{1-\Delta_p(\pi/4)^{\sigma-1}}. \tag{5.3}$$

Similarly since the function $\theta\mapsto D_p(\theta)^{\sigma-1}\cos\theta$ is decreasing on $[0,\pi/2]$ for all $p$ and $\sigma\ge 2$, we can deduce, via (5.3), that

$$\left|\frac{1}{E_p^*(\sigma)}\int_{\pi/4}^{\pi/2} D_p(\theta)^{\sigma-1}\cos\theta\,\mathrm{d}\theta\right| \le \frac{1}{1-\Delta_p(\pi/4)^{\sigma-1}}. \tag{5.4}$$

¿From (5.3) and (5.4), we deduce that

$$\left|\frac{E_p^*(s)}{E_p^*(\sigma)}\right| \le \frac{2}{1-\Delta_p(\pi/4)^{\sigma-1}}.$$

It is easy to verify that for all $p \geq \sigma \geq 2$, we have

$$\Delta_p\Big(\frac{\pi}{4}\Big)^{\sigma-1} \leq \Big(1 - \frac{\sqrt{2}}{p} + \frac{1}{p^2}\Big)^{\sigma-1} \leq 1 - \frac{\sigma-1}{4p}.$$

Combining these estimates with (5.2), we obtain

$$\left|\frac{E_p(s)}{E_p(\sigma)}\right| \leq \frac{8p}{|s-1|} \leq \frac{p^4}{|\tau|} \qquad (p \geq \sigma).$$

By multiplying this inequality for $\sigma < p \leq |\tau|^\delta$ $(\leq y)$ and the trivial inequality $|E_p(s)| \leq |E_p(\sigma)|$ for the others $p$, we deduce, via the prime number theorem, that

$$\begin{aligned}\left|\frac{E(s,y)}{E(\sigma,y)}\right| &\leq \exp\Big\{ - \sum_{\sigma<p\leq|\tau|^\delta} \log|\tau| + 4 \sum_{\sigma<p\leq|\tau|^\delta} \log p\Big\} \\ &\leq e^{-\{1/\delta-4+o(1)\}|\tau|^\delta}.\end{aligned}$$

2° Case of $c_1\sigma^{1/2}\log\sigma \leq |\tau| \leq \sigma^{1/\delta}$

For $p \geq \sigma^{1/2} \geq 2$, we can write

$$\begin{aligned}|E_p^*(s)| &\leq \int_0^{\pi/2} \big\{D_p(\theta)^{\sigma-1} + D_p(\pi-\theta)^{\sigma-1}\big\}\cos\theta\,\mathrm{d}\theta \\ &= \Big\{1 + O\Big(\frac{\sigma}{p^2}\Big)\Big\}\int_0^{\pi/2} \big(e^{2[(\sigma-1)/p]\cos\theta} + e^{-2[(\sigma-1)/p]\cos\theta}\big)\cos\theta\,\mathrm{d}\theta\end{aligned}$$

and

$$\begin{aligned}|E_p^*(\sigma)| &= \int_0^{\pi/2} \big\{D_p(\theta)^{\sigma-1} - D_p(\pi-\theta)^{\sigma-1}\big\}\cos\theta\,\mathrm{d}\theta \\ &= \Big\{1 + O\Big(\frac{\sigma}{p^2}\Big)\Big\}\int_0^{\pi/2} \big(e^{2[(\sigma-1)/p]\cos\theta} - e^{-2[(\sigma-1)/p]\cos\theta}\big)\cos\theta\,\mathrm{d}\theta.\end{aligned}$$

¿From these, we deduce that

$$\left|\frac{E_p^*(s)}{E_p^*(\sigma)}\right| \leq \Big\{1 + O\Big(\frac{\sigma}{p^2} + \frac{1}{e^{\sigma/p}}\Big)\Big\} \qquad (2 \leq \sigma^{1/2} \leq p \leq \sigma) \tag{5.5}$$

where we have used the following facts

$$\int_0^{\pi/2} e^{2[(\sigma-1)/p]\cos\theta}\cos\theta\,\mathrm{d}\theta \gg e^{\sigma/p} \qquad \text{and} \qquad \int_0^{\pi/2} e^{-2[(\sigma-1)/p]\cos\theta}\cos\theta\,\mathrm{d}\theta \ll 1.$$

Inserting (5.5) into (5.2), for $2 \leq \sigma^{1/2} \leq p \leq \sigma$ we obtain

$$\begin{aligned}\left|\frac{E_p(s)}{E_p(\sigma)}\right| &\leq \exp\Big\{ - \log\left|\frac{s-1}{\sigma-1}\right| + C\Big(\frac{\sigma}{p^2} + \frac{1}{e^{\sigma/p}}\Big)\Big\} \\ &\leq \begin{cases} e^{-\tau^2/\sigma^2 + C\sigma/p^2 + Ce^{-\sigma/p}} & \text{if} \quad 3 \leq |\tau| \leq \sigma, \\ e^{-\frac{1}{2}\log(1+\tau^2/\sigma^2) + C\sigma/p^2 + Ce^{-\sigma/p}} & \text{if} \quad \sigma \leq |\tau| \leq \sigma^{1/\delta}, \end{cases}\end{aligned}$$

where $C > 0$ is an absolute constant.

Now by multiplying these inequalities for $\sigma/(4\log\sigma) \le p \le \sigma/(2\log\sigma)$ and the trivial inequality $|E_p(s)| \le E_p(\sigma)$ for the other $p$, we get

$$\begin{aligned}\left|\frac{E(s,y)}{E(\sigma,y)}\right| &\le \exp\Big\{ - \sum_{\sigma/(4\log\sigma)\le p\le\sigma/(2\log\sigma)} \Big(\frac{\tau^2}{\sigma^2} - \frac{C\sigma}{p^2} - \frac{C}{e^{\sigma/p}}\Big)\Big\}\\ &\le \exp\Big\{ - \Big(\frac{\tau^2}{8\sigma(\log\sigma)^2} - 10C - \frac{10C}{\sigma\log\sigma}\Big)\Big\}\\ &\le \exp\Big\{ - \frac{c_2\tau^2}{\sigma(\log\sigma)^2}\Big\}\end{aligned}$$

if $c_1\sigma^{1/2}\log\sigma \le |\tau| \le \sigma$, and

$$\begin{aligned}\left|\frac{E(s,y)}{E(\sigma,y)}\right| &\le \exp\Big\{ - \sum_{\sigma/(4\log\sigma)\le p\le\sigma/(2\log\sigma)} \Big[\frac12\log\Big(1+\frac{\tau^2}{\sigma^2}\Big) - \frac{C\sigma}{p^2} - \frac{C}{e^{\sigma/p}}\Big]\Big\}\\ &\le \exp\Big\{ - \Big[\frac{\sigma}{8\log\sigma}\log\Big(1+\frac{\tau^2}{\sigma^2}\Big) - 10C - \frac{10C}{\sigma\log\sigma}\Big]\Big\}\\ &\le \exp\big\{ - c_3|\tau|^\delta\big\}\end{aligned} \tag{5.6}$$

if $\sigma \le |\tau| \le \sigma^{1/\delta}$. This completes the proof. □

## § 6. Proof of Theorem 3

We follow the argument of Granville & Soundararajan [4] to prove Theorem 3. We shall divide the proof in several steps which are embodied in the following lemmas.

The first one is a classic integration formula (see [4], page 1019).

**Lemma 6.1.** *Let $c > 0$, $\lambda > 0$ and $N \in \mathbb{N}$. Then we have*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^s\Big(\frac{e^{\lambda s}-1}{\lambda s}\Big)^N\frac{\mathrm{d}s}{s} = \begin{cases} 0 & \text{if } 0<y<e^{-\lambda N},\\ \in[0,1] & \text{if } e^{-\lambda N}\le y<1,\\ 1 & \text{if } y\ge 1.\end{cases} \tag{6.1}$$

The second one is an analogue for (3.6) and (3.7) of [4] (see also Lemma 3.1 of [20]).

**Lemma 6.2.** *Let $t \ge 1$, $y \ge 2e^t$ and $0 < \lambda \le e^{-t}$. Then we have*

$$\Phi(t,y) \le \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}\frac{E(s,y)}{(e^\gamma t)^{2s}}\frac{e^{\lambda s}-1}{\lambda s}\frac{\mathrm{d}s}{s} \le \Phi(te^{-\lambda},y), \tag{6.2}$$

$$\Phi(te^{-\lambda},y) - \Phi(t,y) \le \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}\frac{E(s,y)}{(e^\gamma t)^{2s}}\frac{e^{\lambda s}-1}{\lambda s}\big(e^{\lambda s}-e^{-\lambda s}\big)\frac{\mathrm{d}s}{s}. \tag{6.3}$$

*Proof.* Denote by $\mathbf{1}_X(\omega)$ the characteristic function of the set $X \subset \Omega$. Then by Lemma 6.1 with $N = 1$ and $c = \kappa$, we have

$$\mathbf{1}_{\{\omega\in\Omega: L(1,g^\natural(\omega);y)>(e^\gamma t)^2\}}(\omega) \le \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}\Big(\frac{L(1,g^\natural(\omega);y)}{(e^\gamma t)^2}\Big)^s\frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s.$$

Integrating over $\Omega$ and interchanging the order of integrations yield

$$\begin{aligned}\Phi(t,y) &\le \int_\Omega \left\{ \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \left( \frac{L(1, g^\natural(\omega); y)}{(e^\gamma t)^2} \right)^s \frac{e^{\lambda s}-1}{\lambda s^2} \, \mathrm{d}s \right\} \mathrm{d}\mu(\omega) \\ &= \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s,y)}{(e^\gamma t)^{2s}} \frac{e^{\lambda s}-1}{\lambda s^2} \, \mathrm{d}s.\end{aligned}$$

This proves the first inequality of (6.2). The second can be treated by noticing that

$$\begin{aligned}\mathbf{1}_{\{\omega\in\Omega: L(1,g^\natural(\omega);y) > (e^{\gamma-\lambda}t)^2\}}(\omega) &= \mathbf{1}_{\{\omega\in\Omega: L(1,g^\natural(\omega);y) > (e^{\gamma}t)^2\}}(\omega) \\ &\quad + \mathbf{1}_{\{\omega\in\Omega: (e^\gamma t)^2 \ge L(1,g^\natural(\omega);y) > (e^{\gamma-\lambda}t)^2\}}(\omega) \\ &\ge \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \left( \frac{L(1, g^\natural(\omega); y)}{(e^\gamma t)^2} \right)^s \frac{e^{\lambda s}-1}{\lambda s^2} \, \mathrm{d}s.\end{aligned}$$

¿From (6.2), we can deduce

$$\begin{aligned}\Phi(te^{-\lambda}, y) - \Phi(t,y) &\le \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s,y)}{(e^{\gamma-\lambda}t)^{2s}} \frac{e^{\lambda s}-1}{\lambda s^2} \, \mathrm{d}s - \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s,y)}{(e^{\gamma+\lambda}t)^{2s}} \frac{e^{\lambda s}-1}{\lambda s^2} \, \mathrm{d}s \\ &= \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s,y)}{(e^\gamma t)^{2s}} \frac{e^{\lambda s}-1}{\lambda s^2} \big(e^{\lambda s} - e^{-\lambda s}\big) \, \mathrm{d}s.\end{aligned}$$

This completes the proof. □

**Lemma 6.3.** *Let $t \ge 1$, $y \ge 2e^t$ and $0 < \kappa\lambda \le 1$. Then we have*

$$\frac{1}{2\pi i} \int_{\kappa-i\kappa}^{\kappa+i\kappa} \frac{E(s,y)}{(e^\gamma t)^{2s}} \frac{e^{\lambda s}-1}{\lambda s^2} \, \mathrm{d}s = \frac{E(\kappa, y)}{\kappa\sqrt{2\pi\sigma_2}(e^\gamma t)^{2\kappa}} \left\{ 1 + O\left( \kappa\lambda + \frac{\log\kappa}{\kappa} \right) \right\}.$$

*Proof.* First we write, for $s = \kappa + i\tau$ and $|\tau| \le \kappa$,

$$\begin{aligned}E(s,y) &= \exp\left\{ \sigma_0 + i\sigma_1\tau - \frac{\sigma_2}{2}\tau^2 - i\frac{\sigma_3}{6}\tau^3 + O\big(\sigma_4\tau^4\big) \right\}, \\ \frac{e^{\lambda s}-1}{\lambda s^2} &= \frac{1}{\kappa} \left\{ 1 - \frac{i}{\kappa}\tau + O\left( \kappa\lambda + \frac{\tau^2}{\kappa^2} \right) \right\}.\end{aligned}$$

Since $\sigma_1 = \log t + \gamma$, we have

$$\frac{E(s,y)}{(e^\gamma t)^{2s}} \frac{e^{\lambda s}-1}{\lambda s^2} = \frac{E(\kappa,y)}{\kappa(e^\gamma t)^{2\kappa}} e^{-(\sigma_2/2)\tau^2} \left\{ 1 - \frac{i}{\kappa}\tau - i\frac{\sigma_3}{6}\tau^3 + O\big(R(\tau)\big) \right\}$$

with

$$R(\tau) := \kappa\lambda + \kappa^{-2}\tau^2 + \sigma_4\tau^4 + \sigma_3^2\tau^6.$$

Now we integrate the last expression over $|\tau| \le \kappa$ to obtain

$$\frac{1}{2\pi i} \int_{\kappa-i\kappa}^{\kappa+i\kappa} \frac{E(s,y)}{(e^\gamma t)^{2s}} \frac{e^{\lambda s}-1}{\lambda s^2} \, \mathrm{d}s = \frac{E(\kappa,y)}{2\pi\kappa(e^\gamma t)^{2\kappa}} \int_{-\kappa}^{\kappa} e^{-(\sigma_2/2)\tau^2} \big\{ 1 + O\big(R(\tau)\big) \big\} \, \mathrm{d}\tau, \tag{6.4}$$

where we have used the fact that the integrals involving $(i/\kappa)\tau$ and $(i\sigma_3/6)\tau^3$ vanish.

On the other hand, using lemmas 4.5 and 4.6 we have

$$\begin{aligned}\int_{-\kappa}^{\kappa} e^{-(\sigma_2/2)\tau^2} \, \mathrm{d}\tau &= \sqrt{\frac{2\pi}{\sigma_2}} \left\{ 1 + O\left( \exp\left\{ -\frac{1}{2}\kappa^2\sigma_2 \right\} \right) \right\}, \\ \int_{\kappa-i\kappa}^{\kappa+i\kappa} e^{-(\sigma_2/2)\tau^2} R(\tau) \, \mathrm{d}\tau &\ll \frac{1}{\sqrt{\sigma_2}} \left( \kappa\lambda + \frac{1}{\kappa^2\sigma_2} + \frac{\sigma_3^2}{\sigma_2^3} + \frac{\sigma_4}{\sigma_2^2} \right) \\ &\ll \frac{1}{\sqrt{\sigma_2}} \left( \kappa\lambda + \frac{\log\kappa}{\kappa} \right).\end{aligned}$$

Inserting these into (6.4), we obtain the desired result. □

**Lemma 6.4.** *Let $\delta$ and $c_3$ be two constants determined by Lemma 5.1. Then we have*

$$\int_{\kappa\pm i\kappa}^{\kappa\pm i\infty} \frac{E(s,y)}{(e^\gamma t)^{2s}} \frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s \ll \frac{E(\kappa,y)}{\kappa\sqrt{\sigma_2}(e^\gamma t)^{2\kappa}} R_1, \tag{6.5}$$

$$\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s,y)}{(e^\gamma t)^{2s}} \frac{e^{\lambda s}-1}{\lambda s^2}\big(e^{\lambda s}-e^{-\lambda s}\big)\,\mathrm{d}s \ll \frac{E(\kappa,y)}{\kappa\sqrt{\sigma_2}(e^\gamma t)^{2\kappa}} R_2, \tag{6.6}$$

*uniformly for $t \geq 1$, $y \geq 2e^t$, $\kappa \geq 2$ and $0 < \lambda\kappa \leq 1$, where*

$$R_1 := \lambda^{-1} e^{-c_3\kappa^\delta} + \lambda^{-1}(\kappa/\log\kappa)^{1/2} y^{-1/\delta},$$
$$R_2 := \lambda\kappa(\log\kappa)^{1/2} + e^{-(c_3/2)\kappa^\delta} + \lambda^{-1}(\kappa/\log\kappa)^{1/2} y^{-1/\delta}.$$

*Proof.* We split the integral in (6.5) into two parts according to $\kappa \leq |\tau| \leq y^{1/\delta}$ or $|\tau| \geq y^{1/\delta}$. Using Lemma 5.1 with $\sigma = \kappa$ and the inequality $(e^{\lambda s}-1)/s^2 \ll 1/\tau^2$, the integral in (6.5) is

$$\ll \frac{E(\kappa,y)}{(e^\gamma t)^{2\kappa}\lambda}\bigg(\frac{e^{-c_3\kappa^\delta}}{\kappa} + \frac{1}{y^{1/\delta}}\bigg),$$

which implies (6.5), in view of Lemma 4.5 with $J = 1$.

Similarly we split the integral in (6.6) into four parts according to

$$|\tau| \leq c_1\kappa^{1/2}\log\kappa, \qquad c_1\kappa^{1/2}\log\kappa < |\tau| \leq \kappa, \qquad \kappa < |\tau| \leq y^{1/\delta}, \qquad |\tau| \geq y^{1/\delta}.$$

By Lemma 5.1 with $\sigma = \kappa$ and the inequalities

$$(e^{\lambda s}-1)/\lambda s \ll \min\{1,\ 1/(\lambda|\tau|)\}, \qquad (e^{\lambda s}-e^{-\lambda s})/s \ll \min\{\lambda,\ 1/|\tau|\},$$

the integral in (6.6) is, as before,

$$\ll_\varepsilon \frac{E(\kappa,y)}{(e^\gamma t)^{2\kappa}}\Big(\lambda\kappa^{1/2}\log\kappa + e^{-c_3\kappa^\delta} + \lambda^{-1}y^{-1/\delta}\Big),$$

which implies (6.6), as before. □

Now we are ready to complete the proof of Theorem 3. Lemma 6.3 and (6.5) of Lemma 6.4 give

$$\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s,y)}{(e^\gamma t)^{2s}} \frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s = \frac{E(\kappa,y)}{\kappa\sqrt{2\pi\sigma_2}(e^\gamma t)^{2\kappa}}\big\{1 + O\big(R'\big)\big\} \tag{6.7}$$

where

$$R' := \frac{\log\kappa}{\kappa} + \kappa\lambda + \frac{e^{-c_3\kappa^\delta} + (\kappa/\log\kappa)^{1/2}y^{-1/\delta}}{\lambda}.$$

Taking $\lambda = \kappa^{-2}$ and noticing $y \geq 2e^t \asymp \kappa$ and $1/\delta > 4$, we deduce

$$R' \ll t/e^t. \tag{6.8}$$

Combining (6.7) and (6.8) with (6.2), we obtain

$$\Phi(t,y) \leq \frac{E(\kappa,y)}{\kappa\sqrt{2\pi\sigma_2}(e^\gamma t)^{2\kappa}}\bigg\{1 + O\bigg(\frac{t}{e^t}\bigg)\bigg\} \leq \Phi(te^{-\lambda},y) \tag{6.9}$$

uniformly for $t \geq 1$, $y \geq 2e^t$ and $0 < \lambda \leq e^{-t}$.

On the other hand, (6.3) of Lemma 6.2 and (6.6) of Lemma 6.3 imply

$$\begin{aligned}\Phi(te^{-\lambda}, y) - \Phi(t, y) &\ll \frac{E(\kappa, y)}{\kappa\sqrt{\sigma_2}(e^\gamma t)^{2\kappa}}\bigg(\lambda\kappa(\log\kappa)^{1/2} + \frac{(\kappa/\log\kappa)^{1/2}}{e^{c_3\kappa^\delta}} + \frac{(\kappa/\log\kappa)^{1/2}}{\lambda y^{1/\delta}}\bigg)\\ &\ll_\varepsilon \frac{E(\kappa, y)}{\kappa\sqrt{\sigma_2}(e^\gamma t)^{2\kappa}}\bigg(\lambda\kappa(\log\kappa)^{1/2} + \frac{(\kappa/\log\kappa)^{1/2}}{e^{c_3\kappa^\delta}}\bigg)\end{aligned}$$

when $y^{-1/(2\delta)}\kappa^{-1/2}(\log\kappa)^{-1} \leq \lambda \leq \kappa^{-1}$. Since $\Phi(te^{-\lambda}, y) - \Phi(t, y)$ is a non-decreasing function of $\lambda$, we deduce

$$\Phi(te^{-\lambda}, y) - \Phi(t, y) \ll \frac{E(\kappa, y)}{\kappa\sqrt{\sigma_2}(e^\gamma t)^{2\kappa}}\bigg(\lambda\kappa(\log\kappa)^{1/2} + \frac{(\kappa/\log\kappa)^{1/2}}{e^{c_3\kappa^\delta}} + \frac{\kappa(\log\kappa)^{1/2}}{y^{1/(2\delta)}}\bigg) \tag{6.10}$$

uniformly for $t \geq 1$, $y \geq 2e^t$ and $0 < \lambda \leq e^{-t}$. Obviously the estimates (6.9) and (6.10) imply the desired result. This completes the proof of Theorem 3. □

## § 7. Proof of Theorem 4

Using Lemmas 4.1 and 4.5, we can write

$$\begin{aligned}\frac{E(\kappa, y)}{\kappa\sqrt{2\pi\sigma_2}(e^\gamma t)^{2\kappa}} &= \exp\bigg\{\phi(\kappa, y) - 2\kappa(\gamma + \log t) + O(\log\kappa)\bigg\}\\ &= \exp\bigg\{\kappa\bigg(2\log_2\kappa - 2\log t + \sum_{j=1}^{J}\frac{b_{j,0}}{(\log\kappa)^j} + O_J\big(R_J(\kappa, y)\big)\bigg)\bigg\}.\end{aligned}$$

On the other hand, Lemma 4.3 and (1.17) imply that

$$2\log_2\kappa + 2\gamma + \sum_{j=1}^{J}\frac{b_{j,1}}{(\log\kappa)^j} + O_J\big(R_J(\kappa, y)\big) = 2(\log t + \gamma).$$

Combining these estimates, we can obtain

$$\frac{E(\kappa, y)}{\kappa\sqrt{2\pi\sigma_2}(e^\gamma t)^{2\kappa}} = \exp\bigg\{-\kappa\bigg[\sum_{j=1}^{J}\frac{b_{j,1} - b_{j,0}}{(\log\kappa)^j} + O_J\big(R_J(\kappa, y)\big)\bigg]\bigg\}.$$

In view of (1.21), (4.2) and (4.15), we have $b_{j,1} - b_{j,0} = a_j$. This completes the proof. □

## § 8. Proof of Corollary 5

We first prove an asymptotic developpment of $\kappa(t, y)$ in $t$.

**Lemma 8.1.** *For each integer $J \geq 1$, there are computable constants $\gamma_0, \gamma_1, \ldots, \gamma_J$ such that the asymptotic formula*

$$\kappa(t, y) = e^{t-\gamma_0}\bigg\{1 + \sum_{j=1}^{J}\frac{\gamma_j}{t^j} + O_J\big(R_J^*(t, y)\big)\bigg\} \tag{8.1}$$

*holds uniformly for $t \geq 1$ and $y \geq 2e^t$, where*

$$R_N^*(t,y) := \frac{1}{t^{N+1}} + \frac{e^t t}{y \log y}.$$

*Further $\gamma_0$ is given by (1.24) and $\gamma_1 = -\frac{1}{8}b_{1,1}^2 - \frac{1}{4}b_{2,1}$.*

*Proof.* By Lemma 4.3 and (1.17), we have

$$2\log t = 2\log_2 \kappa + \sum_{j=1}^{J+1} \frac{b_{j,1}}{(\log\kappa)^j} + O_J\big(R_{J+1}(\kappa,y)\big), \tag{8.2}$$

where $R_J(\kappa,y)$ is defined as in (1.20). ¿From (8.2), we easily deduce that

$$\begin{aligned} t &= (\log\kappa) \prod_{j=1}^{J+1} \exp\bigg\{ \sum_{j=1}^{J} \frac{b_{j,1}}{2(\log\kappa)^j} \bigg\} \exp\big\{ O_J\big(R_{J+1}(\kappa,y)\big)\big\} \\ &= (\log\kappa) \prod_{j=1}^{J+1} \bigg\{ \sum_{m_j=0}^{J+1} \frac{1}{m_j!} \bigg( \frac{b_{j,1}}{2(\log\kappa)^j} \bigg)^{m_j} + O_J\big(R_{J+1}(\kappa,y)\big) \bigg\}. \end{aligned}$$

Developping the product, we get

$$t = (\log\kappa) \bigg\{ \sum_{j=0}^{J+1} \frac{b'_j}{(\log\kappa)^j} + O_J\big(R_{J+1}(\kappa,y)\big) \bigg\},$$

where

$$\begin{aligned} b'_j &:= \sum_{\substack{m_1\geq 0,\dots,m_{J+1}\geq 0 \\ m_1+2m_2+\cdots+(J+1)m_{J+1}=j}} \frac{b_{1,1}^{m_1}\cdots b_{J+1,1}^{m_{J+1}}}{(2m_1)!!\cdots(2m_{J+1})!!} \\ &= \sum_{\substack{m_1\geq 0,\dots,m_j\geq 0 \\ m_1+2m_2+\cdots+jm_j=j}} \frac{b_{1,1}^{m_1}\cdots b_{j,1}^{m_j}}{(2m_1)!!\cdots(2m_j)!!}. \end{aligned}$$

Since $b'_0 = 1$ and $b'_1 = b_{1,1}/2 = \gamma_0$, the preceeding aymptotic formula can be written as

$$t = \log\kappa + \gamma_0 + \sum_{j=1}^{J} \frac{b'_{j+1}}{(\log\kappa)^j} + O_J\big(R_J^*(t,y)\big), \tag{8.3}$$

where we have used the fact that $\kappa(t,y) \asymp e^t$ (see Lemma 2.3) and $(\log k)R_{J+1}(\kappa,y) \asymp R_J^*(t,y)$.

With the help of (8.3), a simple recurrence argument shows that there are constants $\gamma'_n$ such that

$$t = \log\kappa + \sum_{j=0}^{J} \frac{\gamma'_j}{t^j} + O_J\big(R_J^*(t,y)\big). \tag{8.4}$$

In fact taking $J = 0$ in (8.3), we see that (8.4) holds for $J = 0$. Suppose that it holds for $0,\dots,J-1$, i.e.

$$t = \log\kappa + \sum_{i=0}^{J-j-1} \frac{\gamma'_i}{t^i} + O\big(R_{J-j-1}^*(t,y)\big) \qquad (j = 0,\dots,J-1),$$

which is equivalent to

$$\log\kappa = t\Big\{1 - \sum_{i=1}^{J-j}\frac{\gamma'_{i-1}}{t^i} + O\Big(\frac{R^*_{J-j-1}(t,y)}{t}\Big)\Big\} \qquad (j=0,\dots,J-1). \tag{8.5}$$

This holds also for $j=J$ if we use the convention:

$$\sum_{i=0}^{-1} = 0 \qquad \text{and} \qquad R^*_{-1}(t,y) := 1,$$

since $\log\kappa = t + O(1)$. Inserting it into (8.3), we easily see that (8.4) holds also for $J$. In particular we have $\gamma'_1 = b'_2 = b_{1,1}^2/8 + b_{2,1}/4$.

Now (8.1) is an immediate consequence of (8.4) with

$$\gamma_j := \sum_{\substack{m_1\ge0,\dots,m_J\ge0\\ m_1+2m_2+\cdots+Jm_J=j}} (-1)^{m_1+\cdots+m_J}\frac{{\gamma'_1}^{m_1}\cdots{\gamma'_J}^{m_J}}{m_1!\cdots m_J!}.$$

This completes the proof. □

Now we are ready to prove Corollary 5.

Using (8.5), we have

$$\begin{aligned}\sum_{j=1}^{J}\frac{a_j}{(\log\kappa)^j} &= \sum_{j=1}^{J}\frac{a_j}{t^j}\Big\{1 - \sum_{i=1}^{J-j}\frac{\gamma'_{i-1}}{t^i} + O_N\Big(\frac{R^*_{J-j-1}(t,y)}{t}\Big)\Big\}^{-j}\\ &= \sum_{j=1}^{J}\frac{\rho_j}{t^j} + O_J\Big(\frac{R^*_{J-2}(t,y)}{t^2}\Big),\end{aligned} \tag{8.6}$$

where the $\rho_n$ are constants. In particular we have $\rho_1 = a_1 = 1$ and $\rho_2 = \gamma_0 + a_2$.

Now Theorem 4, (8.1) and (8.6) imply the result of Corollary with

$$a^*_1 = \rho_1 = 1, \qquad a^*_j = \rho_j + \sum_{i=1}^{j-1}\gamma_i\rho_{j-i} \quad (j\ge2).$$

This completes the proof of Corollary 5. □

## § 9. Proof of Theorem 2

For each $\eta\in(0,\frac12)$, define

$$\mathrm{H}^+_k(1;\eta) := \big\{f\in\mathrm{H}^*_k(1) : L(s,f)\ne0,\ s\in\mathcal{S}\big\},$$

where $\mathcal{S} := \{s:=\sigma+i\tau:\ \sigma\ge1-\eta,\ |\tau|\le100k^\eta\}\cup\{s:=\sigma+i\tau:\ \sigma\ge1,\ \tau\in\mathbb{R}\}$, and

$$\mathrm{H}^-_k(1;\eta) := \mathrm{H}^*_k(1)\setminus\mathrm{H}^+_k(1;\eta).$$

Then we have (see [10], (1.11))

$$\big|\mathrm{H}^-_k(1;\eta)\big| \ll_\eta k^{31\eta}. \tag{9.1}$$

Our starting point in the proof of Theorem 2 is the evaluation of the moments of $L(1,f)$. For this, we recall a particular case of Proposition 6.1 of [10].

**Lemma 9.1.** *Let $\eta \in (0, \frac{1}{31})$ be fixed. There are two positive constants $c_i = c_i(\eta)$ $(i = 4, 5)$ such that*

$$\sum_{f\in \mathrm{H}_k^+(1;\eta)} \omega_f L(1,f)^s = E(s) + O_\eta\big(e^{-c_4 \log k/\log_2 k}\big) \tag{9.2}$$

*uniformly for*

$$k \ge 16, \qquad 2 \mid k \qquad \text{and} \qquad |s| \le 2T_k \tag{9.3}$$

*with*

$$T_k := c_5 \log k/(\log_2 k \log_3 k).$$

*Here $E(s)$ is defined by (1.15).*

Let $\kappa(t,y)$ be the saddle-point determined by (1.17) and $\kappa_t := \kappa(t,\infty)$. For $k \ge 16, 2 \mid k$, $\lambda > 0$, $N \in \mathbb{N}$ and $t > 0$, introduce the two integrals

$$I_1(k,t;\lambda,N) := \frac{1}{2\pi i}\int_{\kappa_t - i\infty}^{\kappa_t+i\infty} \sum_{f\in \mathrm{H}_k^+(1;\eta)} \omega_f \Big(\frac{L(1,f)}{(e^\gamma t)^2}\Big)^s \Big(\frac{e^{\lambda s}-1}{\lambda s}\Big)^{2N} \frac{\mathrm{d}s}{s},$$

$$I_2(k,t;\lambda,N) := \frac{1}{2\pi i}\int_{\kappa_t - i\infty}^{\kappa_t+i\infty} \frac{E(s)}{(e^\gamma t)^{2s}} \Big(\frac{e^{\lambda s}-1}{\lambda s}\Big)^{2N} \frac{\mathrm{d}s}{s}.$$

**Lemma 9.2.** *Let $\eta \in (0, \frac{1}{200}]$ be fixed. Then we have*

$$\widetilde{F}_k(t) + O_\eta\big(k^{-5/6}\big) \le I_1(k,t;\lambda,N) \le \widetilde{F}_k(te^{-\lambda N}) + O_\eta\big(k^{-5/6}\big), \tag{9.4}$$

$$\Phi(t) \le I_2(k,t;\lambda,N) \le \Phi(te^{-\lambda N}) \tag{9.5}$$

*uniformly for $k \ge 16$, $2 \mid k$, $\lambda > 0$, $N \in \mathbb{N}$ and $t > 0$. The implied constants depend on $\eta$ only.*

*Proof.* By exchanging the order of sommation and by using Lemma 6.1 with $c = \kappa_t$, we obtain

$$\begin{aligned} I_1(k,t;\lambda,N) &= \sum_{f\in \mathrm{H}_k^+(1;\eta)} \frac{\omega_f}{2\pi i}\int_{\kappa_t - i\infty}^{\kappa_t+i\infty} \Big(\frac{L(1,f)}{(e^\gamma t)^2}\Big)^s \Big(\frac{e^{\lambda s}-1}{\lambda s}\Big)^{2N} \frac{\mathrm{d}s}{s}, \\ &\ge \sum_{f\in \mathrm{H}_k^+(1;\eta),\ L(1,f)\ge (e^\gamma t)^2} \omega_f. \end{aligned}$$

In view of the second estimate of (1.7) and of (9.1), we reintroduce the missing forms

$$\begin{aligned} I_1(k,t;\lambda,N) &\ge \sum_{f\in \mathrm{H}_k^*(1),\ L(1,f)\ge (e^\gamma t)^2} \omega_f + O\Big(\sum_{f\in \mathrm{H}_k^*\smallsetminus \mathrm{H}_k^+(1;\eta)} \omega_f\Big) \\ &\ge \sum_{f\in \mathrm{H}_k^*(1),\ L(1,f)\ge (e^\gamma t)^2} \omega_f + O\big(k^{-1+31\eta}\log k\big). \end{aligned}$$

Clearly this implies the first inequality of (9.4), thanks to (1.6) and (1.7).

Similarly, using Lemma 6.1 with $c = \kappa_t$, we find

$$\begin{aligned} I_1(k,t;\lambda,N) &\le \sum_{\substack{f\in \mathrm{H}_k^+(1;\eta)\\ L(1,f)\ge (e^\gamma t)^2}} \omega_f + \sum_{\substack{f\in \mathrm{H}_k^+(1;\eta)\\ (e^\gamma te^{-\lambda N})^2 \le L(1,f) < (e^\gamma t)^2}} \omega_f \\ &= \sum_{\substack{f\in \mathrm{H}_k^+(1;\eta)\\ L(1,f)\ge (e^\gamma te^{-\lambda N})^2}} \omega_f. \end{aligned}$$

As before, we can easily show that the last sum is $\le \widetilde{F}_k(te^{-\lambda N}) + O\big(k^{-5/6}\big)$.

The estimates (9.5) can be proved in the same way as (6.2). □

**Lemma 9.3.** *Let $\eta \in (0, \frac{1}{200}]$ be fixed and $c_4$ be the positive constant given by Lemma 9.1. Then we have*

$$\begin{aligned}|I_1(k,t;\lambda,N) - I_2(k,t;\lambda,N)| &\ll e^{-c_4(\log k)/\log_2 k}\frac{(1+e^{\lambda\kappa_t})^{2N}\log T_k}{(e^\gamma t)^{2\kappa_t}} \\ &\quad + \frac{E(\kappa_t)+e^{-c_4(\log k)/\log_2 k}}{N(e^\gamma t)^{2\kappa_t}}\Big(\frac{1+e^{\lambda\kappa_t}}{\lambda T_k}\Big)^{2N}\end{aligned} \tag{9.6}$$

*uniformly for $\lambda > 0$, $N \in \mathbb{N}$, $k \geq 16$, $2 \mid k$ and $t \leq T(k)$, where $T(k)$ is given by (1.10). The implied constant depends on $\eta$ only.*

*Proof.* By the definitions of $I_1$ and $I_2$, we can write

$$\begin{aligned}&I_1(k,t;\lambda,N) - I_2(k,t;\lambda,N) \\ &\quad = \frac{1}{2\pi i}\int_{\kappa_t - i\infty}^{\kappa_t + i\infty}\Big(\sum_{f\in \mathrm{H}_k^+(1;\eta)}\omega_f L(1,f)^s - E(s)\Big)\Big(\frac{e^{\lambda s}-1}{\lambda s}\Big)^{2N}\frac{\mathrm{d}s}{s(e^\gamma t)^{2s}}.\end{aligned}$$

In order to estimate the last integral, we split it into two parts according to $|\tau| \leq T_k$ or $|\tau| > T_k$.

In view of (1.18), it is easy to see that $\kappa_t \leq T_k$ for $t \leq T(k)$. Thus we may apply (9.2) of Lemma 9.1 for $s = \kappa_t + i\tau$ with $|\tau| \leq T_k$. Note that $|(e^{\lambda s}-1)/(\lambda s)| \leq 1 + e^{\lambda\kappa_t}$ for $s = \kappa_t + i\tau$, which is easily seen by looking at the cases $|\lambda s| \leq 1$ and $|\lambda s| > 1$. The contribution of $|\tau| \leq T_k$ to $|I_1(k,t;\lambda,N) - I_2(k,t;\lambda,N)|$ is

$$\ll e^{-c_4(\log k)/\log_2 k}\frac{(1+e^{\lambda\kappa_t})^{2N}\log T_k}{(e^\gamma t)^{2\kappa_t}}. \tag{9.7}$$

Since $\kappa_t \leq T_k$ for $t \leq T(k)$, we can apply (9.2) of Lemma 9.1 to write, for $s = \kappa_t + i\tau$ with $\tau \in \mathbb{R}$,

$$\begin{aligned}\Big|\sum_{f\in \mathrm{H}_k^+(1;\eta)}\omega_f L(1,f)^s - E(s)\Big| &\leq \sum_{f\in \mathrm{H}_k^+(1;\eta)}\omega_f L(1,f)^{\kappa_t} + E(\kappa_t) \\ &\leq 2E(\kappa_t) + O\big(e^{-c_4(\log k)/\log_2 k}\big).\end{aligned}$$

Thus the contribution of $|\tau| > T_k$ to $|I_1(k,t;\lambda,N) - I_2(k,t;\lambda,N)|$ is

$$\begin{aligned}&\ll \frac{E(\kappa_t)+e^{-c_4(\log k)/\log_2 k}}{(e^\gamma t)^{2\kappa_t}}\int_{|\tau|\geq T_k}\Big(\frac{1+e^{\lambda\kappa_t}}{\lambda|\tau|}\Big)^{2N}\frac{\mathrm{d}\tau}{|\tau|} \\ &\ll \frac{E(\kappa_t)+e^{-c_4(\log k)/\log_2 k}}{N(e^\gamma t)^{2\kappa_t}}\Big(\frac{1+e^{\lambda\kappa_t}}{\lambda T_k}\Big)^{2N}.\end{aligned} \tag{9.8}$$

Combining (9.7) and (9.8) yields to the required estimate. □

*End of the proof of Theorem 2*

For simplicity of notation, we write

$$I_j := I_j(k,t;\lambda,N) \qquad \text{and} \qquad I_j^+ := I_j(k,te^{\lambda N};\lambda,N) \qquad (j=1,2).$$

By using Lemma 9.2, we have

$$\begin{aligned}\widetilde{F}_k(t) &\leq I_1 + O\big(k^{-5/6}\big) \\ &= I_2 + O\big(|I_1 - I_2| + k^{-5/6}\big) \\ &\leq \Phi(te^{-\lambda N}) + O\big(|I_1 - I_2| + k^{-5/6}\big) \\ &\leq \Phi(t) + |\Phi(te^{-\lambda N}) - \Phi(t)| + O\big(|I_1 - I_2| + k^{-5/6}\big)\end{aligned} \tag{9.9}$$

and

$$\begin{aligned}\widetilde{F}_k(t) &\ge I_1^+ + O\big(k^{-5/6}\big)\\ &= I_2^+ + O\big(|I_1^+ - I_2^+| + k^{-5/6}\big)\\ &\ge \Phi(te^{\lambda N}) + O\big(|I_1^+ - I_2^+| + k^{-5/6}\big)\\ &\ge \Phi(t) - |\Phi(t) - \Phi(te^{\lambda N})| + O\big(|I_1^+ - I_2^+| + k^{-5/6}\big).\end{aligned} \tag{9.10}$$

In view of (6.10) and Theorem 3, we have

$$\big|\Phi(t) - \Phi(te^{-\lambda N})\big| \ll \Phi(t)\big\{\lambda N\kappa_t(\log\kappa_t)^{1/2} + e^{-(c_3/2)\kappa_t^\delta}\big\}$$

for $\lambda N \le e^{-t}$. Take

$$\lambda = e^{5A}/T_k \qquad \text{and} \qquad N = [\log_2 k]. \tag{9.11}$$

Since $T_k = e^{T(k)+\frac{3}{2}\log_3 k + 2C + \log c_5}$, it is easy to see that

$$\lambda N \le e^{-T(k)-2C}T(k)^{-1/2} \qquad \text{and} \qquad \kappa_t \asymp e^t.$$

Inserting these estimates into the preceeding inequality, a simple calculation shows that

$$\big|\Phi(t) - \Phi(te^{-\lambda N})\big| \le \Phi(t)\big\{e^{t-T(k)-C}(t/T(k))^{1/2} + O\big(e^{-c_6e^{\delta t}}\big)\big\}, \tag{9.12}$$

provided the constant $C$ is suitably large, where $c_6 = c_6(\eta,\delta)$ is a positive constant.

Similarly by using (6.10) with $te^{\lambda N}$ in place of $t$, we have

$$\big|\Phi(t) - \Phi(te^{\lambda N})\big| \ll \Phi(te^{\lambda N})\big\{\lambda N\kappa_{te^{\lambda N}}(\log\kappa_{te^{\lambda N}})^{1/2} + e^{-(c_3/2)\kappa_{te^{\lambda N}}^\delta}\big\}.$$

Since for $t \le T(k)$ we have

$$te^{\lambda N} = t + O\big((\log_2 k)^3(\log_3 k)/\log k\big) \qquad \text{and} \qquad \kappa_{te^{\lambda N}} \asymp e^{te^{\lambda N}} \asymp e^t,$$

the preceeding estimate can be writen as

$$\begin{aligned}\big|\Phi(t) - \Phi(te^{\lambda N})\big| &\le \tfrac14\Phi(te^{\lambda N})\big\{e^{t-T(k)-C}(t/T(k))^{1/2} + O\big(e^{-c_6e^{\delta t}}\big)\big\}\\ &\le \tfrac14\Phi(t)\big\{e^{t-T(k)-C}(t/T(k))^{1/2} + O\big(e^{-c_6e^{\delta t}}\big)\big\}\\ &\quad + \tfrac14\big|\Phi(t) - \Phi(te^{\lambda N})\big|\big\{e^{t-T(k)-C}(t/T(k))^{1/2} + O\big(e^{-c_6e^{\delta t}}\big)\big\},\end{aligned}$$

from which we deduce that

$$\big|\Phi(t) - \Phi(te^{\lambda N})\big| \le \Phi(t)\big\{e^{t-T(k)-C}(t/T(k))^{1/2} + O\big(e^{-c_6e^{\delta t}}\big)\big\}. \tag{9.13}$$

By using Lemma 9.3 with $te^{\lambda N}$ in place of $t$, we have

$$\begin{aligned}|I_1^+ - I_2^+| &\ll e^{-c_4(\log k)/\log_2 k}\frac{(1+e^{\lambda\kappa_{te^{\lambda N}}})^{2N}\log T_k}{(e^\gamma te^{\lambda N})^{2\kappa_{te^{\lambda N}}}}\\ &\quad + \frac{E(\kappa_{te^{\lambda N}}) + e^{-c_4(\log k)/\log_2 k}}{N(e^\gamma te^{\lambda N})^{2\kappa_{te^{\lambda N}}}}\left(\frac{1+e^{\lambda\kappa_{te^{\lambda N}}}}{\lambda T_k}\right)^{2N}.\end{aligned}$$

On the other hand, by using Theorem 3 and (1.25), it is easy to see that there is a positive constant $c$ such that

$$\Phi(te^{\lambda N}) \asymp \Phi(t) \sim \frac{E(\kappa_t)}{\kappa_t\sqrt{2\pi\sigma_2}(e^\gamma t)^{2\kappa_t}} \gg e^{-c_8 e^t/t} \gg e^{-c_9(\log k)/[(\log_2 k)^{7/2}\log_3 k]}$$

for $t \le T(k)$. Thanks to Lemma 4.5, the previous estimate can be written as

$$\begin{aligned} |I_1^+ - I_2^+| &\ll \Phi(t)\frac{1}{N}\left(\frac{\kappa_{te^{\lambda N}}}{\log\kappa_{te^{\lambda N}}}\right)^{1/2}\left(\frac{1+e^{\lambda\kappa_{te^{\lambda N}}}}{\lambda T_k}\right)^{2N} \\ &\ll \frac{\Phi(t)}{(\log k)^A}. \end{aligned} \tag{9.14}$$

Similarly we can prove (even more easily)

$$|I_1 - I_2| \ll \Phi(t)/(\log k)^A. \tag{9.15}$$

Inserting (9.12) and (9.16) into (9.9) and (9.13) and (9.15) into (9.10), we obtain

$$\begin{aligned} \widetilde{F}_k(t) &\le \Phi(t)\big\{1 + e^{t-T(k)-C}(t/T(k))^{1/2} + O\big(e^{-c_6 e^{\delta t}} + (\log k)^{-A}\big)\big\}, \\ \widetilde{F}_k(t) &\ge \Phi(t)\big\{1 - e^{t-T(k)-C}(t/T(k))^{1/2} + O\big(e^{-c_6 e^{\delta t}} + (\log k)^{-A}\big)\big\}. \end{aligned}$$

This implies the first asymptotic formula of (1.13) by taking $\eta = \frac{1}{200}$ and $\delta = \frac{1}{5}$.

The second can be established similarly. This completes the proof of Theorem 2. □

## § 10. Proof of Theorem 1

The formula (1.9) is an immediate consequence of Theorem 2 and (1.25).

Taking $t = T(k)$ in (1.9), we find that

$$e^{-c_1'(\log k)/\{(\log_2 k)^{7/2}\log_3 k\}} \ll \widetilde{F}_k(T(k)) \ll e^{-c_2'(\log k)/\{(\log_2 k)^{7/2}\log_3 k\}}, \tag{10.1}$$

where $c_1'$ and $c_2'$ are two positive constants. Clearly (10.1) and (1.8) imply (1.11).

The related results on $\widetilde{G}_k(t)$ and $G_k(T(k))$ can be proved similarly. This completes the proof of Theorem 1. □

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