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On the additive theory of prime numbers II

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Abstract

The undecidability of the additive theory of primes (with identity) as well as the theory $\text{Th}(\mathbb{N}, +, n \mapsto p_n)$, where p_n denotes the $(n + 1)$ -th prime, are open questions. As a possible approach, we extend the latter theory by adding some extra function. In this direction we show the undecidability of the existential part of the theory $\text{Th}(\mathbb{N}, +, n \mapsto p_n, n \mapsto r_n)$, where r_n is the remainder of p_n divided by n in the euclidian division.

Résumé

L'indécidabilité de la théorie additive des nombres premiers ainsi que de la théorie $\text{Th}(\mathbb{N}, +, n \mapsto p_n)$, où p_n désigne le $(n + 1)$ -ième premier, sont deux questions ouvertes. Nous étendons cette dernière théorie en lui ajoutant une fonction supplémentaire et nous montrons l'indécidabilité de la théorie $\text{Th}(\mathbb{N}, +, n \mapsto p_n, n \mapsto r_n)$, où r_n désigne le reste de p_n de la division euclidienne de p_n par n , et même de sa seule partie existentielle.

Introduction - The additive theory of primes contains longtime open classical conjectures of Number Theory, as famous GOLDBACH's binary conjecture or TWIN PRIMES conjecture, and so on. Some authors provided [BJW,BM,LM] conditional proofs (through SCHINZEL's Hypothesis [SS]) of the undecidability of the additive theory of primes $\text{Th}(\mathbb{N}, +, \mathbb{P})$, where \mathbb{P} is the set of all primes. Weakening the problem by strengthening this theory, we introduced [CRV] the theory $\text{Th}(\mathbb{N}, +, n \mapsto p_n)$, where p_n is the $(n + 1)$ -th prime, and posed the problem of its (un)decidability. As usual for a language containing a function symbol, we suppose it contains identity. Note that \mathbb{P} is existentially definable within $\langle \mathbb{N}, n \mapsto p_n \rangle$, hence $\text{Th}(\mathbb{N}, +, \mathbb{P})$ is a subtheory of $\text{Th}(\mathbb{N}, +, n \mapsto p_n)$. At the moment, the undecidability of the latter theory is still an open question, and our approach in [CRV] was to consider several approximations of the function $n \mapsto p_n$ as, for instance, $n \lfloor \log n \rfloor$ and on this way we showed the undecidability of theories $\text{Th}(\mathbb{N}, +, nf(n))$ for a family of functions f including $\lfloor \log \rfloor$ mentioned above. Another approach consists of extending the language $\{+, n \mapsto p_n\}$ to $\{+, n \mapsto p_n, n \mapsto r_n\}$, where r_n is the remainder of p_n divided by n . The main result of this paper is the following:

Theorem 1 *Multiplication is existentially $\langle \mathbb{N}, +, n \mapsto p_n, n \mapsto r_n \rangle$ -definable at first-order.*

This leads to the following (without use of conjectures) result:

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Corollary 1 $\text{Th}_{\exists}(\mathbb{N}, +, n \mapsto p_n, n \mapsto r_n)$ is undecidable.

Remark Actually all positive integer constants are existentially $\{+, \mathbb{P}\}$ -definable in the following manner:

$$\begin{aligned} x = 0 & \Leftrightarrow x + x = x; \\ x = 1 & \Leftrightarrow \exists y(y = x + x \wedge y \in \mathbb{P}); \\ x = 2 & \Leftrightarrow \exists y(y = 1 \wedge x = y + y); \\ & \vdots \\ x = n + 1 & \Leftrightarrow \exists y \exists z(y = n \wedge z = 1 \wedge x = y + z). \end{aligned}$$

As we mentioned above, \mathbb{P} is existentially definable within the language $\{+, n \mapsto p_n\}$, hence all positive integer constants are also existentially $\{+, n \mapsto p_n\}$ -definable. Note, that $n \lfloor \frac{p_n}{n} \rfloor = p_n - r_n$. We intend to define (section 2, see Lemma 3) $\lfloor \frac{p_n}{n} \rfloor$ from $+$ and $n \lfloor \frac{p_n}{n} \rfloor$. Then the strategy will be to define multiplication through the function $n \mapsto cn^2$ (where c is a fixed constant), which is to be proved $\{+, \lfloor \frac{p_n}{n} \rfloor, n \lfloor \frac{p_n}{n} \rfloor\}$ -definable. Consequently, multiplication will be existentially $\{+, n \mapsto p_n, n \mapsto r_n\}$ -definable at first-order.

Remark. In the previous paper [CRV] we consider continuous real strictly increasing functions and their inverses. Since we work with integer parts we have to introduce pseudo-inverses of discrete functions. For such a discrete unbounded function f from \mathbb{N} into \mathbb{N} , we define its pseudo-inverse f^{-1} from \mathbb{N} into \mathbb{N} by $f^{-1}(n) = \mu m[f(m+1) > n]$, where μ means “the smallest ... such that”. Due to the unboundness of f such an f^{-1} is correctly defined.

1) Some preliminary results in Number Theory

Contrarily to what happens with \log , the behavior of $\lfloor \frac{p_n}{n} \rfloor$ is *a priori* irregular but we shall prove it is not too much chaotic. Namely, we prove:

Proposition 1 Let us denote the mapping $n \mapsto \lfloor \frac{p_n}{n} \rfloor$ by f .

- 1) For $m > n$, we have $f(m) - f(n) \geq -1$;
- 2) For $n \geq 11$, we have $f^{-1}(n+1) - f^{-1}(n) > n$.

Proof 1) We use the following estimates for p_n ([RP], p. 249):

$$\begin{aligned} p_m &\geq m \log m + m \log \log m - 1.0072629m \quad \text{for } m \geq 2; \\ p_m &\leq m \log m + m \log \log m - 0.9385m \quad \text{for } m \geq 7022. \end{aligned} \tag{1}$$

For $m > n \geq 7022$, we have $f(m) - f(n) = \lfloor \frac{p_m}{m} \rfloor - \lfloor \frac{p_n}{n} \rfloor$
 $\geq \frac{p_m}{m} - \frac{p_n}{n} - 1 \geq \log(\frac{m}{n}) - \log(\frac{\log m}{\log n}) - 0.9385 + 1.0072629 - 1$.

Hence $f(m) - f(n) \geq -1$ because the sum of the two first terms is positive as is the sum of terms three and four.

If $n < 7022$, one may check the desired inequality by a direct computation.

2) Let m be $f^{-1}(n)$. By the very definition of f^{-1} , the equality $m = f^{-1}(n)$ is equivalent to the conjunction of the two following conditions:

$$\left\{ \begin{array}{l} \lfloor \frac{p_{m+1}}{m+1} \rfloor \geq n + 1; \\ \forall k \leq m \quad \lfloor \frac{p_k}{k} \rfloor \leq n. \end{array} \right. \tag{2}$$

For $k \leq 7022$, the maximum of $\frac{p_k}{k}$ is attained for $k = 7012$ and equal to $\frac{p_{7012}}{7012} < 10.102824 < 11$. Consequently, we see that $m = f^{-1}(n) \geq f^{-1}(11) \geq 7022$ and this is the reason why in the hypothesis of Proposition 1, item 2) we assume $n \geq 11$.

To prove the inequality, it is sufficient to prove that for $k = m + n$ we have $\lfloor \frac{p_k}{k} \rfloor \leq n + 1$, or in other words,

$$\frac{p_k}{k} < n + 2. \quad (3)$$

Note that for $m \geq 7022$, we have by (2):

$$n + 1 \leq \left\lfloor \frac{p_{m+1}}{m+1} \right\rfloor + 1 \leq \frac{p_{m+1}}{m+1} + 1 \leq \log(m+1) + \log \log(m+1) - 0.07 < m.$$

Consequently it is sufficient – and actually more convenient – to prove a somehow stronger result, namely the same inequality (3) but for $m \geq 7022$ and $m + 1 \leq k \leq 2m$.

From the second estimate of (1) we have, since $k \geq m \geq 7022$, the following inequalities:

$$\begin{aligned} \frac{p_k}{k} &< \log k + \log \log k - 0.9385 \\ &\leq \log 2m + \log \log 2m - 0.9385 \\ &= \log m + \log \log m + \log 2 + \log\left(1 + \frac{\log 2}{\log m}\right) - 0.9385; \end{aligned}$$

using the first estimate of (1) and $\frac{\log 2}{\log m} \leq \frac{\log 2}{\log 7022}$, we have:

$$\log m + \log \log m - 1.0072629 \leq \frac{p_m}{m};$$

consequently:

$$\frac{p_k}{k} \leq \frac{p_m}{m} + 0.07 + \log 2 + \log\left(1 + \frac{\log 2}{\log 7022}\right) \leq \frac{p_m}{m} + 1$$

by an easy computation and finally, due to (2), we obtain $\frac{p_k}{k} < n + 2$. \square

Item 1) of previous proposition emphasizes on the fact that $f : n \mapsto \lfloor \frac{p_n}{n} \rfloor$ is “almost” increasing and item 2) shows that the difference $f^{-1}(n+1) - f^{-1}(n)$ is big enough with respect to n . This suggests to introduce a new class of functions, containing f , for which we prove that the existential part of the theory $\text{Th}(\mathbb{N}, +, n \mapsto nf(n))$ is undecidable.

2) The class $C(k, d, n_0)$ and some its properties

2.1) The class $C(k, d, n_0)$

Let $k \geq 0$ be a fixed nonnegative integer. We shall say f is k -almost increasing if and only if

$$\forall y \geq x [f(y) - f(x) \geq -k]. \quad (4)$$

In this sense 0-almost increasing means increasing (not necessarily strictly) and $n \mapsto \lfloor \frac{p_n}{n} \rfloor$ is 1-almost increasing (due to Proposition 1).

Still looking at $n \mapsto \lfloor \frac{p_n}{n} \rfloor$, we intend to consider functions whose pseudo-inverse is defined and asymptotically increases quickly enough with respect to its argument. Let us say that f^{-1} has at least $(1/d)$ -linear difference, if

$$\exists n_0 \in \mathbb{N} \forall n \geq n_0 [f^{-1}(n+1) - f^{-1}(n) > \frac{n}{d}]. \quad (5)$$

In fact, for $\lfloor \frac{p_n}{n} \rfloor$, the constant d is 1 and $n_0 = 11$, but results and proofs hold for an arbitrary (fixed) d .

Now let us define the class $C(k, d, n_0)$ as the set of functions from \mathbb{N} into \mathbb{N} satisfying conditions (4) of being k -almost increasing and (5) of having its pseudo-inverse with an at least $(1/d)$ -linear difference.

In order to prove FUNDAMENTAL LEMMA of section 3, whose Theorem 1 is a corollary, we show some properties of the class $C(k, d, n_0)$. Firstly, in section 2.2 we present in three lemmas these properties and comment them. Afterwards, in section 2.3, we prove them.

2.2) Properties of $C(k, d, n_0)$

Lemma 1 For any function $f \in C(k, d, n_0)$ the following items hold:

- (i) For any $n \geq n_0$, we have $f^{-1}(n+d) - f^{-1}(n) > n$;
- (ii) For any $n \geq n_0 + 1$, the set $\{x \in \mathbb{N} \mid f(x) = n\}$ is nonempty;
- (iii) For any $n \geq n_0 + 1$, the equality $f(x) = n$ implies

$$x > \frac{1}{2d}[(n-1)(n-2) - n_0(n_0-1)].$$

Lemma 2 If $f \in C(k, d, n_0)$ and $f(x) = n \geq n_0$, then for any c such that $1 \leq c \leq n$, we have:

$$-k \leq f(x+c) - f(x) \leq k+d. \quad (6)$$

Lemma 3 For any $f \in C(k, d, n_0)$, let $x_0 = f^{-1}(2 + 4d + n_0^2 + k)$.

Consider $\tilde{f} : [x_0 + 1, +\infty[\cap \mathbb{N} \rightarrow \mathbb{N}$ with $\tilde{f}(x) = f(x)$. Then \tilde{f} is existentially definable at first-order within $\langle \mathbb{N}, +, 1, x \mapsto xf(x) \rangle$.

Remarks 1) Item (i) of Lemma 1 provides a linear lower bound of values of f^{-1} when difference of arguments is the parameter d of the considered class.

Item (ii) of the same lemma insure that f is asymptotically onto, and item (iii) gives a quadratic lower bound for solutions of the equation $f(x) = n$, that we need in section 3.

2) Actually, as the reader will see within the proof, Lemma 1 does not use condition (4) of being k -almost increasing.

3) Lemma 2 provides asymptotical bounds for the difference of two values of f with arguments taken in a short interval with respect to the values of these arguments. Referring to the previous Lemma 1 we see that n is at most $\sqrt{2dx + n_0^2} + 2$.

4) Lemma 3 generalizes the situation of the main result of the previous paper [CRV] of the same authors when $\lfloor \log n \rfloor$ was “extracted”, *i.e.* defined, from $+$ and $n \lfloor \log n \rfloor$.

2.3) Proofs of the three Lemmas

Proof of Lemma 1 (i) By condition (5):

$$\begin{aligned} f^{-1}(n+d) - f^{-1}(n) &= [f^{-1}(n+d) - f^{-1}(n+d-1)] \\ &\quad + [f^{-1}(n+d-1) - f^{-1}(n+d-2)] \\ &\quad + \dots \\ &\quad + [f^{-1}(n+1) - f^{-1}(n)] \\ &> \frac{n+d-1}{d} + \frac{n+d-2}{d} + \dots + \frac{n}{d} \\ &> n. \end{aligned} \quad 4$$

(ii) If there was no x such that $f(x) = n$, we would have $f^{-1}(n) = f^{-1}(n - 1)$. But $f^{-1}(n) > f^{-1}(n - 1)$ according to condition (5).

(iii) By definition of f^{-1} , we have: $x > f^{-1}(n - 1)$.

As in (i), we have:

$$\begin{aligned} f^{-1}(n - 1) - f^{-1}(n_0) &= [f^{-1}(n - 1) - f^{-1}(n - 2)] \\ &\quad + \dots \\ &\quad + [f^{-1}(n_0 + 1) - f^{-1}(n_0)] \\ &> \frac{n-2}{d} + \frac{n_0}{d} + \dots + \frac{n}{d} \\ &= \frac{(n-2)(n-1) - n_0(n_0+1)}{2d}. \end{aligned}$$

and the result. \square

Proof of Lemma 2 The left-hand side of the inequality is an immediate consequence of the very definition of a k -almost increasing function. For proving the right-hand side, note that, using k -almost increasing property of f together with $f(x) = n$, we obtain:

$$\max_{y \leq x} f(y) \leq f(x) + k = n + k,$$

so that $f^{-1}(n + k) \geq x$, by the definition of f^{-1} . By previous Lemma 1, item (i) and the latter inequality, we have:

$$f^{-1}(n + k + d) > f^{-1}(n + k) + n + k \geq x + n + k \geq x + n \geq x + c$$

since $1 \leq c \leq n$. Using again the definition of f^{-1} , we see that $f(x + c) \leq n + k + d = f(x) + k + d$ and we are done. \square

Proof of Lemma 3 To define \tilde{f} within the structure $\langle \mathbb{N}, +, x \mapsto xf(x) \rangle$ we shall make use of the inequality:

$$0 \leq f(x) < x$$

together with the remainder of $f(x)$ modulo $x + 1$, which we must define in the considered structure.

Fact 1.- $f(x) < x$.

By the definition of f^{-1} , we have $f(x_0 + 1) > k + 2 + 4d + n_0^2$ and by the k -almost increasing property we deduce, for $x \geq x_0 + 1$,

$$n = f(x) \geq f(x_0 + 1) - k > 2 + 4d + n_0^2. \quad (7)$$

Hence $\frac{n-2}{2d} > 2$.

From (7), we obtain $n > n_0 + 1$ so that by Lemma 1, item (iii), we have:

$$x > \frac{1}{2d} [(n - 1)(n - 2) - n_0(n_0 - 1)],$$

hence:

$$x > 2(n - 1) - \frac{n_0(n_0 - 1)}{2d} > 2(n - 1) - n_0^2 = n + (n - 2 - n_0^2) > n = f(x). \quad \square \square$$

Fact2.- We have:

$$f(x) \equiv (x + 1)f(x + 1) - xf(x) \pmod{x + 1}. \quad (8)$$

It is sufficient to note that $(x+1)f(x+1) - xf(x) = f(x) + (x+1)[f(x+1) - f(x)]$. \square

We are still unable to define general congruences, fortunately here the difference $|f(x+1) - f(x)|$ is bounded, namely,

$$|f(x+1) - f(x)| \leq k + d, \quad (9)$$

due to Lemme 2, with $c = 1$. This suggests to introduce the notion of a restricted congruence, namely, for a, b, m in \mathbb{N} and some fixed integer c , we define $a \equiv_c b \pmod{m}$ by:

$$\bigvee_{h=0}^c \{ [a = b + \underbrace{m + \dots + m}_{h \text{ times}}] \vee [b = a + \underbrace{m + \dots + m}_{h \text{ times}}] \}.$$

Obviously, the first-order latter formula is expressible within the structure $\langle \mathbb{N}, + \rangle$, since c is fixed. The congruence (8) and inequality (9) provide together the following restricted congruence:

$$f(x) \equiv_{k+d} (x+1)f(x+1) - xf(x) \pmod{x+1},$$

which is a definition of $f(x)$ within $\langle \mathbb{N}, +, 1, x \mapsto xf(x) \rangle$ since $1 \leq f(x) < x$. Finally, we provide explicitly an existential first-order definition of f , namely:

$$[x > x_0 \wedge y = f(x)] \leftrightarrow$$

$$\{ x > x_0 \wedge y \leq x \wedge \bigvee_{h=0}^{k+d} [(y + xf(x) = (x+1)f(x+1) + \underbrace{(x+1) + \dots + (x+1)}_{h \text{ times}}) \vee ((x+1)f(x+1) = y + xf(x) + \underbrace{(x+1) + \dots + (x+1)}_{h \text{ times}})] \}.$$

3) Fundamental Lemma and the proof of the Main Theorem

In order to prove the undecidability of $\text{Th}(\mathbb{N}, n \mapsto p_n, n \mapsto r_n)$, we prove a more general result, namely:

Lemma 4 (Fundamental Lemma) *For any $f \in C(k, d, n_0)$ [see §2], multiplication is existentially $\{+, 1, x \mapsto xf(x)\}$ -definable at first-order.*

As shown by Y. MATIYASEVICH, the existential true theory of numbers is exactly the set of arithmetical relations, which are definable by diophantine equations. Therefore the negative solution of the 10-th Hilbert's problem [MY] implies the following corollary.

Corollary 2 *The existential theory $\text{Th}_{\exists}(\mathbb{N}, +, 1, x \mapsto xf(x))$ is undecidable.*

Proof of Lemma 4 It suffices to show that for some constants c and n_1 the function $n \mapsto cn^2$ from $[n_1, +\infty[\cap \mathbb{N}$ into \mathbb{N} is $\{+, 1, x \mapsto xf(x)\}$ -definable. More precisely, we shall take $c = 5d$ and $n_1 = 2 + 5d + n_0^2$. Consider $n \geq n_1$. Since $n_1 > n_0 + 1$, we can apply Lemma 1, item (ii), proving there exists x such that $f(x) = 5dn$. Let x_0 be the same as in Lemma 3, namely $x_0 = f^{-1}(2 + 4d + n_0^2 + k)$. Let us show $x > x_0$. Otherwise $x \leq x_0$, so that by the k -almost increasing property $f(x) \leq f(x_0) - k$, implying, by the definitions of f^{-1} and x_0 ,

$$f(x) \leq 2 + 4d + n_0^2 + k - k \underset{6}{<} n_1 < 5dn_1 \leq 5dn = f(x),$$

which is impossible.

Note that $5dn$ is $\{+\}$ -definable as the sum of $5d$ terms equal to n (d is a fixed constant). Now thanks to Lemma 3, an x such that $f(x) = 5dn$ is $\{+, 1, x \mapsto xf(x)\}$ -definable.

On the other hand:

$$(x+n)f(x+n) - xf(x) = (x+n)[f(x+n) - f(x)] + nf(x) = (x+n)[f(x+n) - f(x)] + 5dn^2.$$

By Lemma 2 applied to $c = n$, we have $|f(x+n) - f(x)| \leq k + d$, so that:

$$5dn^2 \equiv_{k+d} (x+n)f(x+n) - xf(x) \pmod{x+n}. \quad (10)$$

According to Lemma 1 and item (iii) since $f(x) = 5dn$ and $5dn > n_1 > n_0 + 1$ the inequalities $n \geq n_1 > n_0^2$ and:

$$\begin{aligned} x+n &> \frac{(5dn-1)(5dn-2)}{2d} - \frac{n_0(n_0-1)}{2d} + n \\ &> \frac{25d^2n^2 - 15nd}{2d} > 5dn^2 \end{aligned} \quad (11)$$

hold.

Using (10) and (11), a similar argument as in Lemma 3 shows that the function $n \mapsto 5dn^2 = cn^2$ with domain $[n_1, +\infty[\cap \mathbb{N}$ is existentially $\{+, 1, x \mapsto xf(x)\}$ -definable. By a routine argument, multiplication is clearly existentially $\{+, 1, x \mapsto xf(x)\}$ -definable. \square

Proof of the Main-Theorem We remind the reader that 1 was existentially $\{+, \mathbb{P}\}$ and $\{+, n \mapsto p_n\}$ -defined in the introduction.

We also noted that $n \lfloor \frac{p_n}{n} \rfloor = p_n - r_n$ and $n \mapsto n \lfloor \frac{p_n}{n} \rfloor$ belongs to $C(1, 1, 11)$, the latter due to Proposition 1, §1. Then Fundamental Lemma can be applied and multiplication is existentially $\{+, n \mapsto p_n, n \mapsto r_n\}$ -definable. \square

Conclusion: Our main result is absolute in the sense that does not depend on any conjecture. In order to shed more light on the considered theories $\text{Th}_{\exists}(\mathbb{N}, +, \mathbb{P})$ and $\text{Th}_{\exists}(\mathbb{N}, n \mapsto p_n, n \mapsto r_n)$, we recall a conditional result of A. WOODS. Let us recall that DICKSON'S CONJECTURE [DL] claims that if $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ are integers with all $a_i > 0$ and

$$\forall y \neq 1 \exists x [y \nmid \prod_{1 \leq i \leq n} (a_i x + b_i)]$$

then there exist infinitely many x such that $a_i x + b_i$ are primes for all i . Let us call *DC* this conjecture, then A. WOODS proved [WA]:

If DC is true then the existential theory $\text{Th}_{\exists}(\mathbb{N}, +, \mathbb{P})$ is decidable.

Now, the question is to know whether there is a gap between $\text{Th}_{\exists}(\mathbb{N}, +, n \mapsto p_n, n \mapsto r_n)$ or whether they are exactly the same. In the case of equality between these two theories, *DC* is false (and hence SCHINZEL'S HYPOTHESIS on primes, whose *DC* is the linear case, is also false).

Open problem: *Is $\text{Th}_{\exists}(\mathbb{N}, +, \mathbb{P})$ equal to $\text{Th}_{\exists}(\mathbb{N}, +, n \mapsto p_n, n \mapsto r_n)$?*

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