

# On the additive theory of prime numbers II

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## Abstract

The undecidability of the additive theory of primes (with identity) as well as the theory  $\text{Th}(\mathbb{N}, +, n \mapsto p_n)$ , where  $p_n$  denotes the  $(n + 1)$ -th prime, are open questions. As a possible approach, we extend the latter theory by adding some extra function. In this direction we show the undecidability of the existential part of the theory  $\text{Th}(\mathbb{N}, +, n \mapsto p_n, n \mapsto r_n)$ , where  $r_n$  is the remainder of  $p_n$  divided by  $n$  in the euclidian division.

## Résumé

L'indécidabilité de la théorie additive des nombres premiers ainsi que de la théorie  $\text{Th}(\mathbb{N}, +, n \mapsto p_n)$ , où  $p_n$  désigne le  $(n + 1)$ -ième premier, sont deux questions ouvertes. Nous étendons cette dernière théorie en lui ajoutant une fonction supplémentaire et nous montrons l'indécidabilité de la théorie  $\text{Th}(\mathbb{N}, +, n \mapsto p_n, n \mapsto r_n)$ , où  $r_n$  désigne le reste de  $p_n$  de la division euclidienne de  $p_n$  par  $n$ , et même de sa seule partie existentielle.

**Introduction** - The additive theory of primes contains longtime open classical conjectures of Number Theory, as famous GOLDBACH's binary conjecture or TWIN PRIMES conjecture, and so on. Some authors provided [BJW,BM,LM] conditional proofs (through SCHINZEL's Hypothesis [SS]) of the undecidability of the additive theory of primes  $\text{Th}(\mathbb{N}, +, \mathbb{P})$ , where  $\mathbb{P}$  is the set of all primes. Weakening the problem by strengthening this theory, we introduced [CRV] the theory  $\text{Th}(\mathbb{N}, +, n \mapsto p_n)$ , where  $p_n$  is the  $(n + 1)$ -th prime, and posed the problem of its (un)decidability. As usual for a language containing a function symbol, we suppose it contains identity. Note that  $\mathbb{P}$  is existentially definable within  $\langle \mathbb{N}, n \mapsto p_n \rangle$ , hence  $\text{Th}(\mathbb{N}, +, \mathbb{P})$  is a subtheory of  $\text{Th}(\mathbb{N}, +, n \mapsto p_n)$ . At the moment, the undecidability of the latter theory is still an open question, and our approach in [CRV] was to consider several approximations of the function  $n \mapsto p_n$  as, for instance,  $n \lfloor \log n \rfloor$  and on this way we showed the undecidability of theories  $\text{Th}(\mathbb{N}, +, nf(n))$  for a family of functions  $f$  including  $\lfloor \log \rfloor$  mentioned above. Another approach consists of extending the language  $\{+, n \mapsto p_n\}$  to  $\{+, n \mapsto p_n, n \mapsto r_n\}$ , where  $r_n$  is the remainder of  $p_n$  divided by  $n$ . The main result of this paper is the following:

**Theorem 1** *Multiplication is existentially  $\langle \mathbb{N}, +, n \mapsto p_n, n \mapsto r_n \rangle$ -definable at first-order.*

This leads to the following (without use of conjectures) result:

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**Corollary 1**  $\text{Th}_{\exists}(\mathbb{N}, +, n \mapsto p_n, n \mapsto r_n)$  is undecidable.

Remark Actually all positive integer constants are existentially  $\{+, \mathbb{P}\}$ -definable in the following manner:

$$\begin{aligned} x = 0 & \Leftrightarrow x + x = x; \\ x = 1 & \Leftrightarrow \exists y(y = x + x \wedge y \in \mathbb{P}); \\ x = 2 & \Leftrightarrow \exists y(y = 1 \wedge x = y + y); \\ & \vdots \\ x = n + 1 & \Leftrightarrow \exists y \exists z(y = n \wedge z = 1 \wedge x = y + z). \end{aligned}$$

As we mentioned above,  $\mathbb{P}$  is existentially definable within the language  $\{+, n \mapsto p_n\}$ , hence all positive integer constants are also existentially  $\{+, n \mapsto p_n\}$ -definable. Note, that  $n \lfloor \frac{p_n}{n} \rfloor = p_n - r_n$ . We intend to define (section 2, see Lemma 3)  $\lfloor \frac{p_n}{n} \rfloor$  from  $+$  and  $n \lfloor \frac{p_n}{n} \rfloor$ . Then the strategy will be to define multiplication through the function  $n \mapsto cn^2$  (where  $c$  is a fixed constant), which is to be proved  $\{+, \lfloor \frac{p_n}{n} \rfloor, n \lfloor \frac{p_n}{n} \rfloor\}$ -definable. Consequently, multiplication will be existentially  $\{+, n \mapsto p_n, n \mapsto r_n\}$ -definable at first-order.

Remark. In the previous paper [CRV] we consider continuous real strictly increasing functions and their inverses. Since we work with integer parts we have to introduce pseudo-inverses of discrete functions. For such a discrete unbounded function  $f$  from  $\mathbb{N}$  into  $\mathbb{N}$ , we define its pseudo-inverse  $f^{-1}$  from  $\mathbb{N}$  into  $\mathbb{N}$  by  $f^{-1}(n) = \mu m[f(m+1) > n]$ , where  $\mu$  means “the smallest ... such that”. Due to the unboundness of  $f$  such an  $f^{-1}$  is correctly defined.

## 1) Some preliminary results in Number Theory

Contrarily to what happens with  $\log$ , the behavior of  $\lfloor \frac{p_n}{n} \rfloor$  is *a priori* irregular but we shall prove it is not too much chaotic. Namely, we prove:

**Proposition 1** Let us denote the mapping  $n \mapsto \lfloor \frac{p_n}{n} \rfloor$  by  $f$ .

- 1) For  $m > n$ , we have  $f(m) - f(n) \geq -1$ ;
- 2) For  $n \geq 11$ , we have  $f^{-1}(n+1) - f^{-1}(n) > n$ .

**Proof** 1) We use the following estimates for  $p_n$  ([RP], p. 249):

$$\begin{aligned} p_m & \geq m \log m + m \log \log m - 1.0072629m \quad \text{for } m \geq 2; \\ p_m & \leq m \log m + m \log \log m - 0.9385m \quad \text{for } m \geq 7022. \end{aligned} \tag{1}$$

For  $m > n \geq 7022$ , we have  $f(m) - f(n) = \lfloor \frac{p_m}{m} \rfloor - \lfloor \frac{p_n}{n} \rfloor$   
 $\geq \frac{p_m}{m} - \frac{p_n}{n} - 1 \geq \log(\frac{m}{n}) - \log(\frac{\log m}{\log n}) - 0.9385 + 1.0072629 - 1$ .

Hence  $f(m) - f(n) \geq -1$  because the sum of the two first terms is positive as is the sum of terms three and four.

If  $n < 7022$ , one may check the desired inequality by a direct computation.

2) Let  $m$  be  $f^{-1}(n)$ . By the very definition of  $f^{-1}$ , the equality  $m = f^{-1}(n)$  is equivalent to the conjunction of the two following conditions:

$$\left\{ \begin{array}{l} \lfloor \frac{p_{m+1}}{m+1} \rfloor \geq n + 1; \\ \forall k \leq m \quad \lfloor \frac{p_k}{k} \rfloor \leq n. \end{array} \right. \tag{2}$$

For  $k \leq 7022$ , the maximum of  $\frac{p_k}{k}$  is attained for  $k = 7012$  and equal to  $\frac{p_{7012}}{7012} < 10.102824 < 11$ . Consequently, we see that  $m = f^{-1}(n) \geq f^{-1}(11) \geq 7022$  and this is the reason why in the hypothesis of Proposition 1, item 2) we assume  $n \geq 11$ .

To prove the inequality, it is sufficient to prove that for  $k = m + n$  we have  $\lfloor \frac{p_k}{k} \rfloor \leq n + 1$ , or in other words,

$$\frac{p_k}{k} < n + 2. \quad (3)$$

Note that for  $m \geq 7022$ , we have by (2):

$$n + 1 \leq \left\lfloor \frac{p_{m+1}}{m+1} \right\rfloor + 1 \leq \frac{p_{m+1}}{m+1} + 1 \leq \log(m+1) + \log \log(m+1) - 0.07 < m.$$

Consequently it is sufficient – and actually more convenient – to prove a somehow stronger result, namely the same inequality (3) but for  $m \geq 7022$  and  $m + 1 \leq k \leq 2m$ .

From the second estimate of (1) we have, since  $k \geq m \geq 7022$ , the following inequalities:

$$\begin{aligned} \frac{p_k}{k} &< \log k + \log \log k - 0.9385 \\ &\leq \log 2m + \log \log 2m - 0.9385 \\ &= \log m + \log \log m + \log 2 + \log\left(1 + \frac{\log 2}{\log m}\right) - 0.9385; \end{aligned}$$

using the first estimate of (1) and  $\frac{\log 2}{\log m} \leq \frac{\log 2}{\log 7022}$ , we have:

$$\log m + \log \log m - 1.0072629 \leq \frac{p_m}{m};$$

consequently:

$$\frac{p_k}{k} \leq \frac{p_m}{m} + 0.07 + \log 2 + \log\left(1 + \frac{\log 2}{\log 7022}\right) \leq \frac{p_m}{m} + 1$$

by an easy computation and finally, due to (2), we obtain  $\frac{p_k}{k} < n + 2$ .  $\square$

Item 1) of previous proposition emphasizes on the fact that  $f : n \mapsto \lfloor \frac{p_n}{n} \rfloor$  is “almost” increasing and item 2) shows that the difference  $f^{-1}(n+1) - f^{-1}(n)$  is big enough with respect to  $n$ . This suggests to introduce a new class of functions, containing  $f$ , for which we prove that the existential part of the theory  $\text{Th}(\mathbb{N}, +, n \mapsto nf(n))$  is undecidable.

## 2) The class $C(k, d, n_0)$ and some its properties

### 2.1) The class $C(k, d, n_0)$

Let  $k \geq 0$  be a fixed nonnegative integer. We shall say  $f$  is  $k$ -almost increasing if and only if

$$\forall y \geq x [f(y) - f(x) \geq -k]. \quad (4)$$

In this sense 0-almost increasing means increasing (not necessarily strictly) and  $n \mapsto \lfloor \frac{p_n}{n} \rfloor$  is 1-almost increasing (due to Proposition 1).

Still looking at  $n \mapsto \lfloor \frac{p_n}{n} \rfloor$ , we intend to consider functions whose pseudo-inverse is defined and asymptotically increases quickly enough with respect to its argument. Let us say that  $f^{-1}$  has at least  $(1/d)$ -linear difference, if

$$\exists n_0 \in \mathbb{N} \forall n \geq n_0 [f^{-1}(n+1) - f^{-1}(n) > \frac{n}{d}]. \quad (5)$$

In fact, for  $\lfloor \frac{p_n}{n} \rfloor$ , the constant  $d$  is 1 and  $n_0 = 11$ , but results and proofs hold for an arbitrary (fixed)  $d$ .

Now let us define the class  $C(k, d, n_0)$  as the set of functions from  $\mathbb{N}$  into  $\mathbb{N}$  satisfying conditions (4) of being  $k$ -almost increasing and (5) of having its pseudo-inverse with an at least  $(1/d)$ -linear difference.

In order to prove FUNDAMENTAL LEMMA of section 3, whose Theorem 1 is a corollary, we show some properties of the class  $C(k, d, n_0)$ . Firstly, in section 2.2 we present in three lemmas these properties and comment them. Afterwards, in section 2.3, we prove them.

## 2.2) Properties of $C(k, d, n_0)$

**Lemma 1** For any function  $f \in C(k, d, n_0)$  the following items hold:

- (i) For any  $n \geq n_0$ , we have  $f^{-1}(n+d) - f^{-1}(n) > n$ ;
- (ii) For any  $n \geq n_0 + 1$ , the set  $\{x \in \mathbb{N} \mid f(x) = n\}$  is nonempty;
- (iii) For any  $n \geq n_0 + 1$ , the equality  $f(x) = n$  implies

$$x > \frac{1}{2d}[(n-1)(n-2) - n_0(n_0-1)].$$

**Lemma 2** If  $f \in C(k, d, n_0)$  and  $f(x) = n \geq n_0$ , then for any  $c$  such that  $1 \leq c \leq n$ , we have:

$$-k \leq f(x+c) - f(x) \leq k+d. \quad (6)$$

**Lemma 3** For any  $f \in C(k, d, n_0)$ , let  $x_0 = f^{-1}(2 + 4d + n_0^2 + k)$ .

Consider  $\tilde{f} : [x_0 + 1, +\infty[ \cap \mathbb{N} \rightarrow \mathbb{N}$  with  $\tilde{f}(x) = f(x)$ . Then  $\tilde{f}$  is existentially definable at first-order within  $\langle \mathbb{N}, +, 1, x \mapsto xf(x) \rangle$ .

Remarks 1) Item (i) of Lemma 1 provides a linear lower bound of values of  $f^{-1}$  when difference of arguments is the parameter  $d$  of the considered class.

Item (ii) of the same lemma insure that  $f$  is asymptotically onto, and item (iii) gives a quadratic lower bound for solutions of the equation  $f(x) = n$ , that we need in section 3.

2) Actually, as the reader will see within the proof, Lemma 1 does not use condition (4) of being  $k$ -almost increasing.

3) Lemma 2 provides asymptotical bounds for the difference of two values of  $f$  with arguments taken in a short interval with respect to the values of these arguments. Referring to the previous Lemma 1 we see that  $n$  is at most  $\sqrt{2dx + n_0^2} + 2$ .

4) Lemma 3 generalizes the situation of the main result of the previous paper [CRV] of the same authors when  $\lfloor \log n \rfloor$  was ‘‘extracted’’, *i.e.* defined, from  $+$  and  $n \lfloor \log n \rfloor$ .

## 2.3) Proofs of the three Lemmas

**Proof of Lemma 1** (i) By condition (5):

$$\begin{aligned} f^{-1}(n+d) - f^{-1}(n) &= [f^{-1}(n+d) - f^{-1}(n+d-1)] \\ &\quad + [f^{-1}(n+d-1) - f^{-1}(n+d-2)] \\ &\quad + \dots \\ &\quad + [f^{-1}(n+1) - f^{-1}(n)] \\ &> \frac{n+d-1}{d} + \frac{n+d-2}{d} + \dots + \frac{n}{d} \\ &> n. \end{aligned} \quad 4$$

(ii) If there was no  $x$  such that  $f(x) = n$ , we would have  $f^{-1}(n) = f^{-1}(n - 1)$ . But  $f^{-1}(n) > f^{-1}(n - 1)$  according to condition (5).

(iii) By definition of  $f^{-1}$ , we have:  $x > f^{-1}(n - 1)$ .

As in (i), we have:

$$\begin{aligned} f^{-1}(n - 1) - f^{-1}(n_0) &= [f^{-1}(n - 1) - f^{-1}(n - 2)] \\ &\quad + \dots \\ &\quad + [f^{-1}(n_0 + 1) - f^{-1}(n_0)] \\ &> \frac{n-2}{d} + \frac{n_0}{d} + \dots + \frac{n}{d} \\ &= \frac{(n-2)(n-1) - n_0(n_0+1)}{2d}. \end{aligned}$$

and the result.  $\square$

**Proof of Lemma 2** The left-hand side of the inequality is an immediate consequence of the very definition of a  $k$ -almost increasing function. For proving the right-hand side, note that, using  $k$ -almost increasing property of  $f$  together with  $f(x) = n$ , we obtain:

$$\max_{y \leq x} f(y) \leq f(x) + k = n + k,$$

so that  $f^{-1}(n + k) \geq x$ , by the definition of  $f^{-1}$ . By previous Lemma 1, item (i) and the latter inequality, we have:

$$f^{-1}(n + k + d) > f^{-1}(n + k) + n + k \geq x + n + k \geq x + n \geq x + c$$

since  $1 \leq c \leq n$ . Using again the definition of  $f^{-1}$ , we see that  $f(x + c) \leq n + k + d = f(x) + k + d$  and we are done.  $\square$

**Proof of Lemma 3** To define  $\tilde{f}$  within the structure  $\langle \mathbb{N}, +, x \mapsto xf(x) \rangle$  we shall make use of the inequality:

$$0 \leq f(x) < x$$

together with the remainder of  $f(x)$  modulo  $x + 1$ , which we must define in the considered structure.

**Fact 1.-**  $f(x) < x$ .

By the definition of  $f^{-1}$ , we have  $f(x_0 + 1) > k + 2 + 4d + n_0^2$  and by the  $k$ -almost increasing property we deduce, for  $x \geq x_0 + 1$ ,

$$n = f(x) \geq f(x_0 + 1) - k > 2 + 4d + n_0^2. \quad (7)$$

Hence  $\frac{n-2}{2d} > 2$ .

From (7), we obtain  $n > n_0 + 1$  so that by Lemma 1, item (iii), we have:

$$x > \frac{1}{2d} [(n - 1)(n - 2) - n_0(n_0 - 1)],$$

hence:

$$x > 2(n - 1) - \frac{n_0(n_0 - 1)}{2d} > 2(n - 1) - n_0^2 = n + (n - 2 - n_0^2) > n = f(x). \quad \square \square$$

**Fact2.-** We have:

$$f(x) \equiv (x + 1)f(x + 1) - xf(x) \pmod{x + 1}. \quad (8)$$

It is sufficient to note that  $(x+1)f(x+1) - xf(x) = f(x) + (x+1)[f(x+1) - f(x)]$ .  $\square$

We are still unable to define general congruences, fortunately here the difference  $|f(x+1) - f(x)|$  is bounded, namely,

$$|f(x+1) - f(x)| \leq k + d, \quad (9)$$

due to Lemme 2, with  $c = 1$ . This suggests to introduce the notion of a restricted congruence, namely, for  $a, b, m$  in  $\mathbb{N}$  and some fixed integer  $c$ , we define  $a \equiv_c b \pmod{m}$  by:

$$\bigvee_{h=0}^c \{ [a = b + \underbrace{m + \dots + m}_{h \text{ times}}] \vee [b = a + \underbrace{m + \dots + m}_{h \text{ times}}] \}.$$

Obviously, the first-order latter formula is expressible within the structure  $\langle \mathbb{N}, + \rangle$ , since  $c$  is fixed. The congruence (8) and inequality (9) provide together the following restricted congruence:

$$f(x) \equiv_{k+d} (x+1)f(x+1) - xf(x) \pmod{x+1},$$

which is a definition of  $f(x)$  within  $\langle \mathbb{N}, +, 1, x \mapsto xf(x) \rangle$  since  $1 \leq f(x) < x$ . Finally, we provide explicitly an existential first-order definition of  $f$ , namely:

$$[x > x_0 \wedge y = f(x)] \leftrightarrow$$

$$\{ x > x_0 \wedge y \leq x \wedge \bigvee_{h=0}^{k+d} [(y + xf(x) = (x+1)f(x+1) + \underbrace{(x+1) + \dots + (x+1)}_{h \text{ times}}) \vee ((x+1)f(x+1) = y + xf(x) + \underbrace{(x+1) + \dots + (x+1)}_{h \text{ times}})] \}.$$

### 3) Fundamental Lemma and the proof of the Main Theorem

In order to prove the undecidability of  $\text{Th}(\mathbb{N}, n \mapsto p_n, n \mapsto r_n)$ , we prove a more general result, namely:

**Lemma 4 (Fundamental Lemma)** *For any  $f \in C(k, d, n_0)$  [see §2], multiplication is existentially  $\{+, 1, x \mapsto xf(x)\}$ -definable at first-order.*

As shown by Y. MATIYASEVICH, the existential true theory of numbers is exactly the set of arithmetical relations, which are definable by diophantine equations. Therefore the negative solution of the 10-th Hilbert's problem [MY] implies the following corollary.

**Corollary 2** *The existential theory  $\text{Th}_{\exists}(\mathbb{N}, +, 1, x \mapsto xf(x))$  is undecidable.*

**Proof of Lemma 4** It suffices to show that for some constants  $c$  and  $n_1$  the function  $n \mapsto cn^2$  from  $[n_1, +\infty[ \cap \mathbb{N}$  into  $\mathbb{N}$  is  $\{+, 1, x \mapsto xf(x)\}$ -definable. More precisely, we shall take  $c = 5d$  and  $n_1 = 2 + 5d + n_0^2$ . Consider  $n \geq n_1$ . Since  $n_1 > n_0 + 1$ , we can apply Lemma 1, item (ii), proving there exists  $x$  such that  $f(x) = 5dn$ . Let  $x_0$  be the same as in Lemma 3, namely  $x_0 = f^{-1}(2 + 4d + n_0^2 + k)$ . Let us show  $x > x_0$ . Otherwise  $x \leq x_0$ , so that by the  $k$ -almost increasing property  $f(x) \leq f(x_0) - k$ , implying, by the definitions of  $f^{-1}$  and  $x_0$ ,

$$f(x) \leq 2 + 4d + n_0^2 + k - k \underset{6}{<} n_1 < 5dn_1 \leq 5dn = f(x),$$

which is impossible.

Note that  $5dn$  is  $\{+\}$ -definable as the sum of  $5d$  terms equal to  $n$  ( $d$  is a fixed constant).

Now thanks to Lemma 3, an  $x$  such that  $f(x) = 5dn$  is  $\{+, 1, x \mapsto xf(x)\}$ -definable.

On the other hand:

$$(x+n)f(x+n) - xf(x) = (x+n)[f(x+n) - f(x)] + nf(x) = (x+n)[f(x+n) - f(x)] + 5dn^2.$$

By Lemma 2 applied to  $c = n$ , we have  $|f(x+n) - f(x)| \leq k + d$ , so that:

$$5dn^2 \equiv_{k+d} (x+n)f(x+n) - xf(x) \pmod{x+n}. \quad (10)$$

According to Lemma 1 and item (iii) since  $f(x) = 5dn$  and  $5dn > n_1 > n_0 + 1$  the inequalities  $n \geq n_1 > n_0^2$  and:

$$\begin{aligned} x+n &> \frac{(5dn-1)(5dn-2)}{2d} - \frac{n_0(n_0-1)}{2d} + n \\ &> \frac{25d^2n^2 - 15nd}{2d} > 5dn^2 \end{aligned} \quad (11)$$

hold.

Using (10) and (11), a similar argument as in Lemma 3 shows that the function  $n \mapsto 5dn^2 = cn^2$  with domain  $[n_1, +\infty[ \cap \mathbb{N}$  is existentially  $\{+, 1, x \mapsto xf(x)\}$ -definable. By a routine argument, multiplication is clearly existentially  $\{+, 1, x \mapsto xf(x)\}$ -definable.  $\square$

**Proof of the Main-Theorem** We remind the reader that 1 was existentially  $\{+, \mathbb{P}\}$  and  $\{+, n \mapsto p_n\}$ -defined in the introduction.

We also noted that  $n \lfloor \frac{p_n}{n} \rfloor = p_n - r_n$  and  $n \mapsto n \lfloor \frac{p_n}{n} \rfloor$  belongs to  $C(1, 1, 11)$ , the latter due to Proposition 1, §1. Then Fundamental Lemma can be applied and multiplication is existentially  $\{+, n \mapsto p_n, n \mapsto r_n\}$ -definable.  $\square$

**Conclusion:** Our main result is absolute in the sense that does not depend on any conjecture. In order to shed more light on the considered theories  $\text{Th}_{\exists}(\mathbb{N}, +, \mathbb{P})$  and  $\text{Th}_{\exists}(\mathbb{N}, n \mapsto p_n, n \mapsto r_n)$ , we recall a conditional result of A. WOODS. Let us recall that DICKSON'S CONJECTURE [DL] claims that if  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$  are integers with all  $a_i > 0$  and

$$\forall y \neq 1 \exists x [y \nmid \prod_{1 \leq i \leq n} (a_i x + b_i)]$$

then there exist infinitely many  $x$  such that  $a_i x + b_i$  are primes for all  $i$ . Let us call *DC* this conjecture, then A. WOODS proved [WA]:

*If DC is true then the existential theory  $\text{Th}_{\exists}(\mathbb{N}, +, \mathbb{P})$  is decidable.*

Now, the question is to know whether there is a gap between  $\text{Th}_{\exists}(\mathbb{N}, +, n \mapsto p_n, n \mapsto r_n)$  or whether they are exactly the same. In the case of equality between these two theories, *DC* is false (and hence SCHINZEL'S HYPOTHESIS on primes, whose *DC* is the linear case, is also false).

**Open problem:** *Is  $\text{Th}_{\exists}(\mathbb{N}, +, \mathbb{P})$  equal to  $\text{Th}_{\exists}(\mathbb{N}, +, n \mapsto p_n, n \mapsto r_n)$ ?*

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