



HAL
open science

Non-Symmetric Hall-Littlewood Polynomials

Francois Descouens, Alain Lascoux

► **To cite this version:**

Francois Descouens, Alain Lascoux. Non-Symmetric Hall-Littlewood Polynomials. 2006. hal-00096385

HAL Id: hal-00096385

<https://hal.science/hal-00096385>

Preprint submitted on 19 Sep 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

NON-SYMMETRIC HALL-LITTLEWOOD POLYNOMIALS

Francois Descouens

Alain Lascoux

À Adriano Garsia, en toute amitié

Abstract

Using the action of the Yang-Baxter elements of the Hecke algebra on polynomials, we define two bases of polynomials in n variables. The Hall-Littlewood polynomials are a subfamily of one of them. For $q = 0$, these bases specialize into the two families of classical Key polynomials (i.e. Demazure characters for type A). We give a scalar product for which the two bases are adjoint of each other.

1 Introduction

We define two linear bases of the ring of polynomials in x_1, \dots, x_n , with coefficients in q .

These polynomials, that we call *q-Key polynomials*, and denote U_v, \widehat{U}_v , $v \in \mathbb{N}^n$, specialize at $q = 0$ into key polynomials K_v, \widehat{K}_v . The polynomials U_v which are symmetrical in x_1, \dots, x_n are precisely the Hall-Littlewood polynomials P_λ , indexed by partitions $\lambda \in \mathfrak{Part}$, the relation between the two indices being $\lambda = [\lambda_1, \dots, \lambda_n] = [v_n, \dots, v_1]$.

Our main tool is the Hecke algebra $\mathcal{H}_n(q)$ of the symmetric group, acting on polynomials by deformation of divided differences. This algebra contains two adjoint bases of Yang-Baxter elements (Th. 2.1). The q -Key polynomials are the images of dominant monomials under these Yang-Baxter elements (Def. 3.1). These polynomials are clearly two linear bases of polynomials, since the transition matrix to monomials is uni-triangular.

We show in the last section that $\{U_v\}$ and $\{\widehat{U}_v\}$ are two adjoint bases with respect to a certain scalar product reminiscent of Weyl's scalar product on symmetric functions.

We have intensively used MuPAD (package `MuPAD-Combinat` [11]) and Maple (package `ACE` [10]).

2 The Hecke algebra $\mathcal{H}_n(q)$

Let $\mathcal{H}_n(q)$ be the Hecke algebra of the symmetric group \mathfrak{S}_n , with coefficients the rational functions in a parameter q . It has generators T_1, \dots, T_{n-1} satisfying the braid relations

$$\begin{cases} T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \\ T_i T_j = T_j T_i \quad (|j - i| > 1), \end{cases} \quad (1)$$

and the Hecke relations

$$(T_i + 1)(T_i - q) = 0, \quad 1 \leq i \leq n - 1 \quad (2)$$

For a permutation σ in \mathfrak{S}_n , we denote by T_σ the element $T_\sigma = T_{i_1} \dots T_{i_p}$ where (i_1, \dots, i_p) is any reduced decomposition of σ . The set $\{T_\sigma : \sigma \in \mathfrak{S}_n\}$ is a linear basis of $\mathcal{H}_n(q)$.

2.1 Yang-Baxter bases

Let s_1, \dots, s_{n-1} denote the simple transpositions, $\ell(\sigma)$ denote the length of $\sigma \in \mathfrak{S}_n$, and let ω be the permutation of maximal length.

Given any set of indeterminates $\mathbf{u} = (u_1, \dots, u_n)$, let $\mathcal{H}_n(q)[u_1, \dots, u_n] = \mathcal{H}_n(q) \otimes \mathbb{C}[u_1, \dots, u_n]$.

One defines recursively a *Yang-Baxter basis* $(Y_\sigma^{\mathbf{u}})_{\sigma \in \mathfrak{S}_n}$, depending on \mathbf{u} , by

$$Y_{\sigma s_i}^{\mathbf{u}} = Y_\sigma^{\mathbf{u}} \left(T_i + \frac{1 - q}{1 - u_{\sigma_{i+1}}/u_{\sigma_i}} \right), \quad \text{when } \ell(\sigma s_i) > \ell(\sigma), \quad (3)$$

starting with $Y_{id}^{\mathbf{u}} = 1$.

Let φ be the anti-automorphism of $\mathcal{H}_n(q)[u_1, \dots, u_n]$ such that

$$\begin{cases} \varphi(T_\sigma) = T_{\sigma^{-1}}, \\ \varphi(u_i) = u_{n-i+1}. \end{cases}$$

We define a bilinear form \langle, \rangle on $\mathcal{H}_n(q)[u_1, \dots, u_n]$ by

$$\langle h_1, h_2 \rangle := \text{coefficient of } T_\omega \text{ in } h_1 \cdot \varphi(h_2). \quad (4)$$

The main result of [6, Th. 5.1] is the following duality property of Yang-Baxter bases.

Theorem 2.1 *For any set of parameters $\mathbf{u} = (u_1, \dots, u_n)$, the basis adjoint to $(Y_\sigma^{\mathbf{u}})_{\sigma \in \mathfrak{S}_n}$ with respect to \langle, \rangle is the basis $(\widehat{Y}_\sigma^{\mathbf{u}})_{\sigma \in \mathfrak{S}_n} = (Y_\sigma^{\varphi(\mathbf{u})})_{\sigma \in \mathfrak{S}_n}$. More precisely, one has*

$$\forall \sigma, \nu \in \mathfrak{S}_n, \quad \langle Y_\sigma^{\mathbf{u}}, \widehat{Y}_\nu^{\mathbf{u}} \rangle = \delta_{\lambda, \nu\omega} \quad .$$

Let us fix from now on the parameters u to be $\mathbf{u} = (1, q, q^2, \dots, q^{n-1})$. Write \mathcal{H}_n for $\mathcal{H}_n(q)[1, q, \dots, q^{n-1}]$.

In that case, the Yang-Baxter basis $(Y_\sigma)_{\sigma \in \mathfrak{S}_n}$ and its adjoint basis $(\widehat{Y}_\sigma)_{\sigma \in \mathfrak{S}_n}$ are defined recursively, starting with $Y_{id} = 1 = \widehat{Y}_{id}$, by

$$Y_{\sigma s_i} = Y_\sigma (T_i + 1/[k]_q) \quad \text{and} \quad \widehat{Y}_{\sigma s_i} = \widehat{Y}_\sigma (T_i + q^{k-1}/[k]_q), \quad \ell(\sigma s_i) > \ell(\sigma), \quad (5)$$

with $k = \sigma_{i+1} - \sigma_i$ and $[k]_q = (1 - q^k)/(1 - q)$.

Notice that the maximal Yang-Baxter elements have another expression [2] :

$$Y_\omega = \sum_{\sigma \in \mathfrak{S}_n} T_\sigma \quad \text{and} \quad \widehat{Y}_\omega = \sum_{\sigma \in \mathfrak{S}_n} (-q)^{\ell(\sigma\omega)} T_\sigma \quad .$$

Example 2.2 For \mathcal{H}_3 , the transition matrix between $\{Y_\sigma\}_{\sigma \in \mathfrak{S}_3}$ and $\{T_\sigma\}_{\sigma \in \mathfrak{S}_3}$ is

$$\begin{array}{l} 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{array} \left| \begin{array}{cccccc} 1 & 1 & 1 & \frac{1}{q+1} & \frac{1}{q+1} & 1 \\ \cdot & 1 & \cdot & 1 & \frac{1}{q+1} & 1 \\ \cdot & \cdot & 1 & \frac{1}{q+1} & 1 & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{array} \right| ,$$

writing ‘ \cdot ’ for 0. Each column represents the expansion of some element Y_σ .

2.2 Action of \mathcal{H}_n on polynomials

Let \mathfrak{Pol} be the ring of polynomials in the variables x_1, \dots, x_n with coefficients the rational functions in q . We write monomials exponentially: $x^v = x_1^{v_1} \dots x_n^{v_n}$, $v = (v_1, \dots, v_n) \in \mathbb{Z}^n$. A monomial x^v is *dominant* if $v_1 \geq \dots \geq v_n$.

We extend the natural order on partitions to elements of \mathbb{Z}^n by

$$u \leq v \quad \text{iff} \quad \forall k > 0, \quad \sum_{i=k}^n (v_i - u_i) \geq 0.$$

For any polynomial P in \mathfrak{Pol} , we call *leading term* of P all the monomials (multiplied by their coefficients) which are maximal with respect to this partial order. This order is compatible with the right-to-left lexicographic order, that we shall also use. We also use the classical notation $\mathbf{n}(v) = 0v_1 + 1v_2 + 2v_3 + \dots + (n-1)v_n$.

Let i be an integer such that $1 \leq i \leq n-1$. As an operator on \mathfrak{Pol} , the simple transposition s_i acts by switching x_i and x_{i+1} , and we denote this action by $f \rightarrow f^{s_i}$. The i -th *divided difference* ∂_i and the i -th *isobaric divided difference* π_i , written on the right of the operand, are the following operators :

$$\partial_i : f \mapsto f \partial_i := \frac{f - f^{s_i}}{x_i - x_{i+1}} \quad , \quad \pi_i : f \mapsto f \pi_i := \frac{x_i f - x_{i+1} f^{s_i}}{x_i - x_{i+1}} .$$

The Hecke algebra \mathcal{H}_n has a faithful representation as an algebra of operators on \mathfrak{Pol} given by the following equivalent formulas [2, 8]

$$\left\{ \begin{array}{llll} T_i & = \square_i - 1 & = (x_i - qx_{i+1}) \partial_i - 1 & = (1 - qx_{i+1}/x_i) \pi_i - 1 , \\ Y_{s_i} & = \square_i & = (x_i - qx_{i+1}) \partial_i & = (1 - qx_{i+1}/x_i) \pi_i , \\ \hat{Y}_{s_i} & = \nabla_i & = \square_i - (1 + q) & = \partial_i (x_{i+1} - qx_i) . \end{array} \right.$$

The Hecke relations imply

$$\square_i^2 = (1 + q) \square_i \quad , \quad \nabla_i^2 = -(1 + q) \nabla_i \quad \text{and} \quad \square_i \nabla_i = \nabla_i \square_i = 0 .$$

One easily checks that the operators $R_i(a, b)$ and $S_i(a, b)$ defined by

$$R_i(a, b) = \square_i - q \frac{[b - a - 1]_q}{[b - a]_q} \quad \text{and} \quad S_i(a, b) = \nabla_i + q \frac{[b - a - 1]_q}{[b - a]_q}$$

satisfy the Yang-Baxter equation

$$R_i(a, b) R_{i+1}(a, c) R_i(b, c) = R_{i+1}(c, b) R_i(a, c) R_{i+1}(a, b). \quad (6)$$

We have implicitly used these equations in the recursive definition of Yang-Baxter elements (5).

This realization comes from geometry [3], where the maximal Yang-Baxter elements are interpreted as Euler-Poincaré characteristic for the flag variety of $GL_n(\mathbb{C})$. This gives still another expression of the maximal Yang-Baxter elements :

$$Y_\omega = \prod_{1 \leq i < j \leq n} (x_i - qx_j) \partial_\omega \quad , \quad \widehat{Y}_\omega = \partial_\omega \prod_{1 \leq i < j \leq n} (x_j - qx_i). \quad (7)$$

Example 2.3 Let $\sigma = (3412) = s_2 s_3 s_1 s_2$. The elements Y_{3412} and \widehat{Y}_{3412} can be written

$$\begin{aligned} Y_{3412} &= \square_2 \left(\square_3 - \frac{q}{1+q} \right) \left(\square_1 - \frac{q}{1+q} \right) \left(\square_2 - \frac{q+q^2}{1+q+q^2} \right), \\ \widehat{Y}_{3412} &= \nabla_2 \left(\nabla_3 + \frac{q}{1+q} \right) \left(\nabla_1 + \frac{q}{1+q} \right) \left(\nabla_2 + \frac{q+q^2}{1+q+q^2} \right). \end{aligned}$$

We shall now identify the images of dominant monomials under the maximal Yang-Baxter operators with Hall-Littlewood polynomials. Recall that there are two proportional families $\{P_\lambda\}$ and $\{Q_\lambda\}$ of Hall-Littlewood polynomials. Given a partition $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_r] = (0^{m_0}, 1^{m_1}, \dots, n^{m_n})$, with $m_0 = n - r = n - m_1 - \dots - m_n$, then

$$Q_\lambda = \prod_{1 \leq i \leq n} \prod_{j=1}^{m_i} (1 - q^j) P_\lambda.$$

Let moreover $d_\lambda(q) = \prod_{0 \leq i \leq n} \prod_{j=1}^{m_i} [j]_q$. The definition of Hall-Littlewood polynomials with raising operators [7],[9, III.2] can be rewritten, thanks to (7), as follows.

Proposition 2.4 Let λ be a partition of n . Then one has

$$x^\lambda Y_\omega d_\lambda(q)^{-1} = P_\lambda(x_1, \dots, x_n; q) \quad (8)$$

The family of the Hall-Littlewood functions $\{Q_\lambda\}$ indexed by partitions can be extended into a family $\{Q_v : v \in \mathbb{Z}^n\}$, using the following relations due to Littlewood ([7], [9, II.2.Ex. 2])

$$Q_{(\dots, u_i, u_{i+1}, \dots)} = -Q_{(\dots, u_{i+1}-1, u_{i+1}, \dots)} + q Q_{(\dots, u_{i+1}, u_i, \dots)} + q Q_{(\dots, u_i+1, u_{i+1}-1, \dots)} \quad \text{if } u_i < u_{i+1}, \quad (9)$$

$$Q_{(u_1, \dots, u_n)} = 0 \quad \text{if } u_n < 0 \quad . \quad (10)$$

By iteration of the first relation, one can write any Q_u in terms of Hall-Littlewood functions indexed by decreasing vectors v such that $|v| = |u|$. Consequently, if u such that $|u| = 0$, Q_u must be proportionnal to $Q_{0\dots 0} = 1$, i.e. is a constant that one can note as the specialisation $Q_u(0)$ in $x_1 = 0 = \dots = x_n$.

The final expansion of Q_u , after iterating (9) many times, is not easy to predict. In particular, one needs to know whether $Q_u \neq 0$. For that purpose, we shall isolate a distinguished term in the expansion of Q_u . Given a sum $\sum_{\lambda \in \mathfrak{Part}} c_\lambda(t) Q_\lambda$, call *top term* the image of the leading term $\sum c_\mu(t) Q_\mu$ after restricting each coefficient $c_\mu(t)$ to its term in highest degree in t .

Given $u \in \mathbb{Z}^n$, define recursively $\mathfrak{p}(u) \in \mathfrak{Part} \cup \{-\infty\}$ by

- if $u \not\geq [0, \dots, 0]$ then $\mathfrak{p}(u) = -\infty$
- if $u_2 \geq u_3 \geq \dots \geq u_n > 0$ then $\mathfrak{p}(u)$ is the maximal partition of length $\leq n$, of weight $|u|$ (eventual zero terminal parts are suppressed).
- $\mathfrak{p}(u) = \mathfrak{p}(u \mathfrak{p}([u_2, \dots, u_n]))$

Lemma 2.5 *Let $u \in \mathbb{Z}^n$. Then*

- if $u \not\geq [0, \dots, 0]$ then $Q_u = 0$,
- if $u \geq [0, \dots, 0]$, let $v = \mathfrak{p}(u)$. Then $Q_u \neq 0$ and its leading term is $q^{n(u)-n(v)} Q_v$.

Proof. Given any decomposition $u = u'.u''$, then one can apply (9) to u'' and write Q_u as a linear combination of terms $Q_{u'v}$ with v decreasing, with $|v| = |u''|$. Therefore, if $|u''| = 0$, then the last components of such v are negative, all $Q_{u'v}$ are 0, and $Q_u = 0$.

If $u \geq [0, \dots, 0]$ and u is not a partition, write $u = [\dots, a, b, \dots]$, with a, b the rightmost increase in u . We apply relation (9), assuming the validity of lemma for the three terms in the RHS :

$$Q_{\dots, a, b, \dots} = -Q_{\dots, b-1, a+1, \dots} + qQ_{\dots, b, a, \dots} + qQ_{\dots, a+1, b-1, \dots}$$

Notice that the first two terms have not necessarily an index $\geq [0, \dots, 0]$, but that $[\dots, a+1, b-1, \dots] \geq [0, \dots, 0]$.

In any case, it is clear that $\mathfrak{p}([\dots, b-1, a+1, \dots]) = p_1 \leq v$, $\mathfrak{p}([\dots, b, a, \dots]) = p_2 \leq v$, and $\mathfrak{p}([\dots, a+1, b-1, \dots]) = v$.

Restricted to top terms, the expansion of the RHS in the basis Q_λ becomes

$$-\left((q^{n(u)+a+1-b-n(v)} + \dots) Q_v\right) + q\left((q^{n(u)+a-b-n(v)} + \dots) Q_v\right) + q\left((q^{n(u)-1-n(v)} + \dots) Q_v\right) ,$$

where one or two of the first two terms may be replaced by 0, depending on the value of p_1 , or p_2 . In final, the top term of the RHS is $q^{n(u)-n(v)} Q_v$, as wanted. Q.E.D

Example 2.6 For $v = [-2, 3, 2]$,

$$Q_{2,3,2} = (q^3 - q^2)Q_3 + (q^5 + q^4 - q^3 - 2q^2 + q)Q_{21} + (q^4 - q^3 - q^2 + q)Q_{111},$$

and the top term is q^4Q_{111} , since $4 = (0(-2) + 1(3) + 2(2)) - (0(1) + 1(1) + 2(1))$ and $[1, 1, 1] > [2, 1]$, $[1, 1, 1] > [3]$. Notice that the coefficient of Q_{21} is of higher degree.

3 q -Key Polynomials

In this section, we show that the images of dominant monomials under the Yang-Baxter elements Y_σ (resp. \widehat{Y}_σ), $\sigma \in \mathfrak{S}_n$ constitute two bases of \mathfrak{Pol} , which specialize into the two families of Demazure characters.

We have already identified in the preceding section the images of dominant monomials under Y_ω to Hall-Littlewood polynomial, using the relation between Y_ω and ∂_ω . The other polynomials are new.

3.1 Two bases

The dimension of the linear span of the image of a monomial x^v under all permutations depends upon the stabilizer of v . We meet the same phenomenon when taking the images of a monomial under Yang-Baxter elements.

Let $\lambda = [\lambda_1, \dots, \lambda_n]$ be a decreasing partition (adding eventual parts equal to 0). Denote its orbit under permutations of components by $\mathcal{O}(\lambda)$. Given any v in $\mathcal{O}(\lambda)$, let $\zeta(v)$ be the permutation of maximal length such that $\lambda \zeta(v) = v$ and $\eta(v)$ be the permutation of minimal length such that $\lambda \eta(v) = v$. These two permutations are representative of the same coset of \mathfrak{S}_n modulo the stabilizer of λ .

Definition 3.1 For all v in \mathbb{N}^n , the q -Key polynomials U_v and \widehat{U}_v are the following polynomials :

$$U_v(x; q) = \left(\frac{1}{d_\lambda(q)} x^\lambda \right) Y_{\zeta(v)} \quad , \quad \widehat{U}_v(x; q) = x^\lambda \widehat{Y}_{\eta(v)},$$

where λ is the dominant reordering of v .

In particular, if v is (weakly) increasing, then $\zeta(v) = \omega$ and U_v is a Hall-Littlewood polynomial.

Lemma 3.2 The leading term of U_v and \widehat{U}_v is x^v . Consequently, the transition matrix between the U_v (resp. the \widehat{U}_v) and the monomials is upper unitriangular with respect to the right-to-left lexicographic order.

Proof. Let k be an integer and u be a weight such that $u_k > u_{k+1}$. Suppose by induction that x^u is the leading term of U_u . Recall the the explicit action of \square_k is (noting only the

two variables x_k, x_{k+1})

$$\begin{aligned} x^{\beta\alpha} \square_k &= x^{\beta\alpha} + (1-t)(x^{\beta-1, \alpha+1} + \dots + x^{\alpha+1, \beta-1}) + x^{\alpha\beta}, \quad \beta > \alpha \\ x^{\beta\beta} \square_k &= (1+t)x^{\beta\beta} \\ x^{\alpha\beta} \square_k &= tx^{\beta\alpha} + (t-1)(x^{\beta-1, \alpha+1} + \dots + x^{\alpha+1, \beta-1}) + tx^{\alpha\beta}, \quad \alpha < \beta. \end{aligned}$$

From these formulas, it is clear that for any constant c , the leading term of $x^u (\square_k + c)$ is $(x^u)^{s_k}$, and, for any v such that $v < u$, all the monomials in $x^v (\square_k + c)$ are strictly less (with respect to the partial order) than $(x^u)^{s_k}$. \square

Example 3.3 For $n = 3$, Figures 1 and 2 show the case of a regular dominant weight x^{210} and Figures 3 and 4 correspond to a case, x^{200} , where the stabilizer is not trivial. In this last case, the polynomials belonging to the family are framed, the extra polynomials denoted A, B do not belong to the basis.

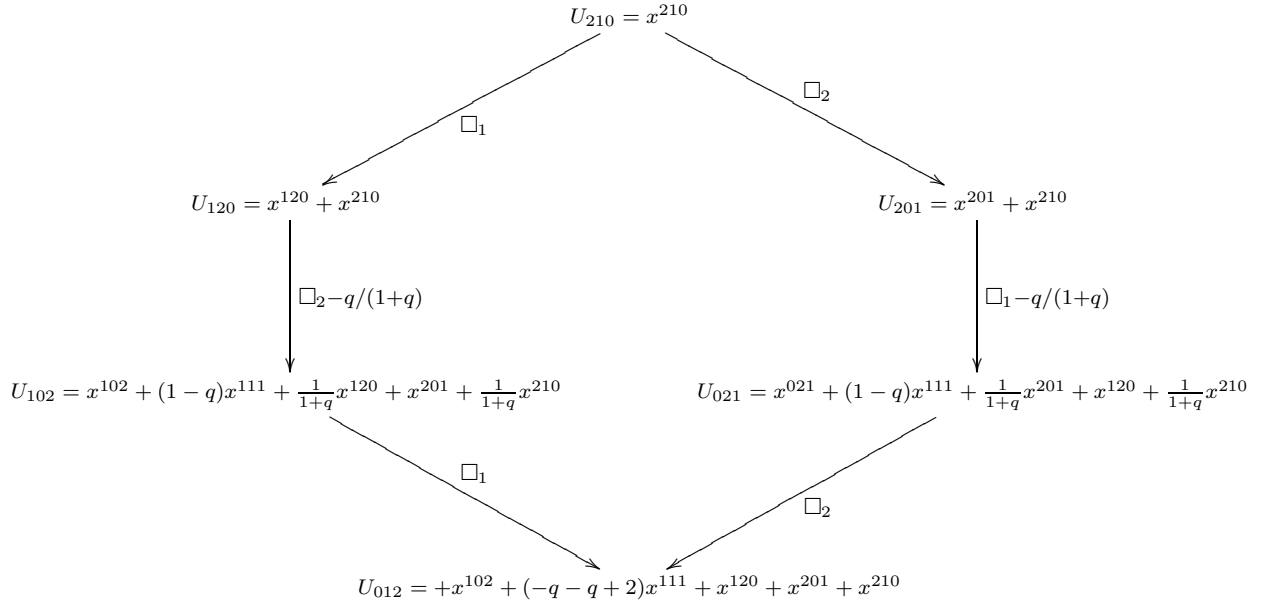


Figure 1: q -Key polynomials generated from x^{210} .

3.2 Specialization at $q = 0$

The specialization at $q = 0$ of the Hecke algebra is called the *0-Hecke algebra*. The elementary Yang-Baxter elements specialize in that case into

$$Y_{s_i} = T_i + 1 = \square_i \rightarrow x_i \partial_i = \pi_i, \quad (11)$$

$$\hat{Y}_{s_i} = T_i = \nabla_i \rightarrow \partial_i x_{i+1} = \hat{\pi}_i. \quad (12)$$

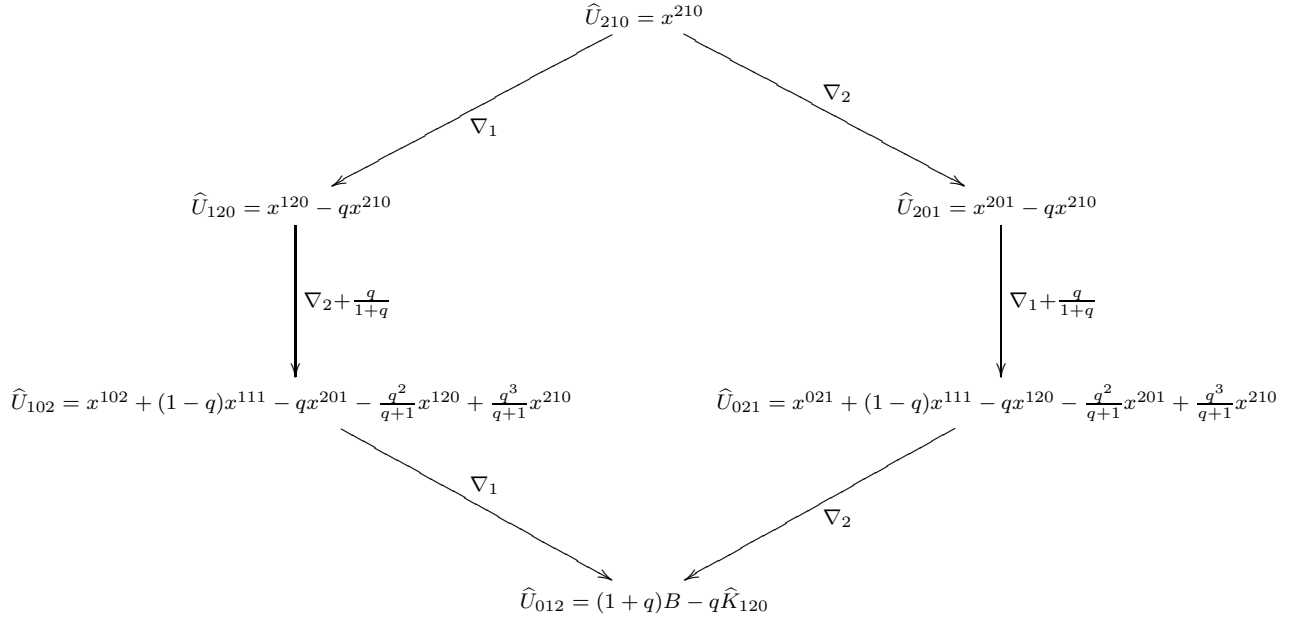


Figure 2: Dual q -Key polynomials generated from x^{210} .

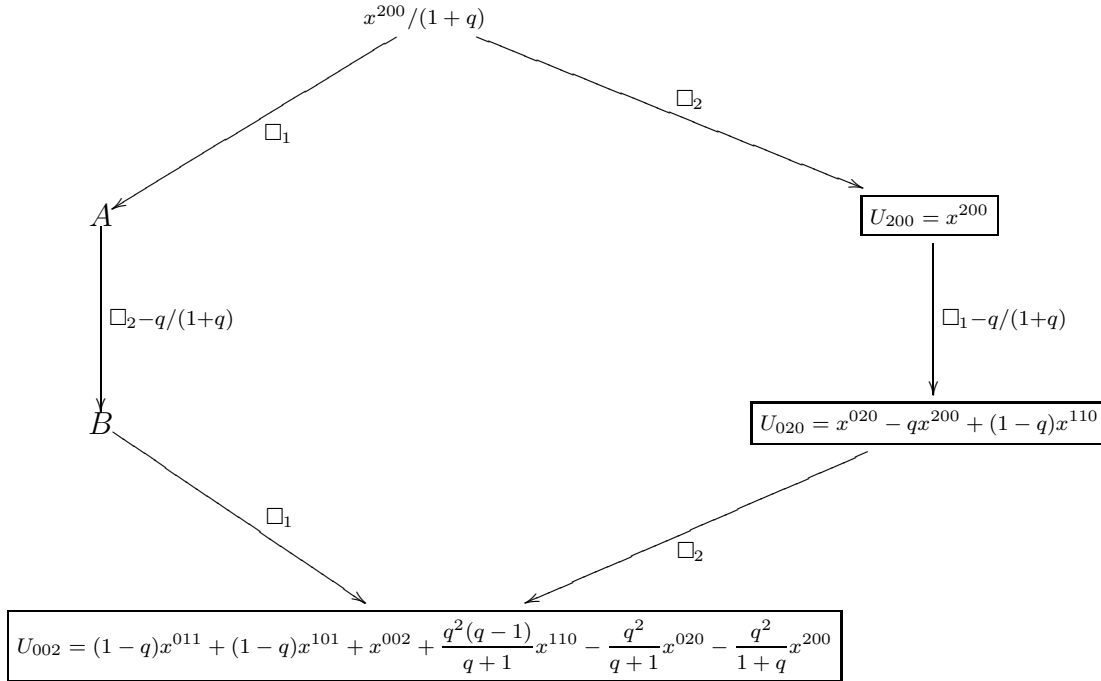


Figure 3: q -Key polynomials generated from $x^{200}/(1+q)$.

Definition 3.4 (Key polynomials) Let $v \in \mathbb{N}^n$. The Key polynomials K_v and \hat{K}_v are

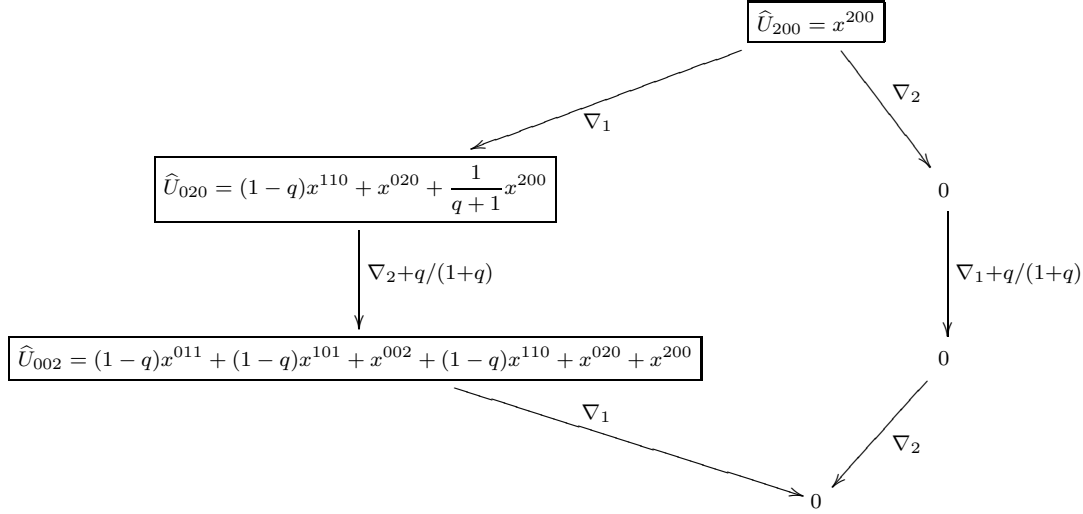


Figure 4: Dual q -Key polynomials generated from x^{200} .

defined recursively, starting with $K_v = x^v = \widehat{K}_v$ if x^v dominant, by

$$K_{vs_i} = K_v \pi_i \quad , \quad \widehat{K}_{vs_i} = \widehat{K}_v \widehat{\pi}_i \quad , \quad \text{for } i \text{ such that } v_i > v_{i+1} .$$

In particular, the subfamily (K_v) for v increasing, is the family of Schur functions in x_1, \dots, x_n . Demazure [1] defined Key polynomials (using another terminology) for all the classical groups, and not only the type A_{n-1} which is our case.

Lemma 3.2 specializes into :

Lemma 3.5 *The transition matrix between the U_v and the K_v (resp. from \widehat{U}_v to \widehat{K}_v) is upper unitriangular with respect to the lexicographic order.*

Example 3.6 *For $n = 3$, the transition matrix between $\{U_v\}$ and $\{K_v\}$ in weight 3 is (reading a column as the expansion of some U_v)*

$$\begin{array}{l}
 300 \\
 210 \\
 201 \\
 120 \\
 111 \\
 102 \\
 030 \\
 021 \\
 012 \\
 003
 \end{array}
 \left|
 \begin{array}{cccccccc}
 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{-q}{(q+1)} & \cdot & \cdot & \cdot \\
 \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \frac{-q}{(q+1)} & \cdot & \cdot \\
 \cdot & \cdot & \cdot & 1 & \cdot & \frac{-q}{(q+1)} & -q & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & 1 & -q & \cdot & -q & -q(q+1) & q^2 \\
 \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -q \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
 \end{array}
 \right. ,$$

and the transition matrix between $\{\widehat{U}_v\}$ and $\{\widehat{K}_v\}$ is

$$\begin{array}{l}
300 \\
210 \\
201 \\
120 \\
111 \\
102 \\
030 \\
021 \\
012 \\
003
\end{array}
\left| \begin{array}{cccccccccc}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & -q & \cdot & \cdot & \frac{-q^2}{(q+1)} \\
\cdot & 1 & -q & -q & \cdot & \frac{q^3}{(q+1)} & -q & \frac{q^3}{(q+1)} & -q^3 & \frac{q^3}{(q+1)} \\
\cdot & \cdot & 1 & \cdot & \cdot & -q & \cdot & \frac{-q^2}{(q+1)} & q^2 & -q \\
\cdot & \cdot & \cdot & 1 & \cdot & \frac{-q^2}{(q+1)} & -q & -q & q^2 & \frac{q^3}{(q+1)} \\
\cdot & \cdot & \cdot & \cdot & 1 & -q & \cdot & -q & q(q+1) & q^2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & -q & -q \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \frac{-q^2}{(q+1)} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -q & -q \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -q \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array} \right|.$$

4 Orthogonality properties for the q -Key polynomials

We show in this section that the q -Key polynomials U_v and \widehat{U}_v are two adjoint bases with respect to a certain scalar product.

4.1 A scalar product on \mathfrak{Pol}

For any Laurent series $f = \sum_{i=k}^{\infty} f_i x^i$, we denote by $CT_x(f)$ the coefficient f_0 .

Let

$$\Theta := \prod_{1 \leq i < j \leq n} \frac{1 - x_i/x_j}{1 - qx_i/x_j}.$$

Therefore, for any Laurent polynomial $f(x_1, \dots, x_n)$, the expression

$$CT(f\Theta) := CT_{x_n} (CT_{x_{n-1}} (\dots (CT_{x_1} (f\Theta)) \dots))$$

is well defined. Let us use it to define a bilinear form $(,)_q$ on \mathfrak{Pol} by

$$(f, g)_q = CT \left(f g^{\clubsuit} \prod_{1 \leq i < j \leq n} \frac{1 - x_i/x_j}{1 - qx_i/x_j} \right) \quad (13)$$

where \clubsuit is the automorphism defined by $x_i \mapsto 1/x_{n+1-i}$ for $1 \leq i \leq n$.

Since Θ is invariant under \clubsuit , the form $(,)_q$ is symmetrical. Under the specialization $q = 0$, the previous scalar product becomes

$$(f, g) := (f, g)|_{q=0} = CT \left(f g^{\clubsuit} \prod_{1 \leq i < j \leq n} (1 - x_i/x_j) \right). \quad (14)$$

We can also write $(f, g)_q = (f, g\Omega)$ with $\Omega = \prod_{1 \leq i < j \leq n} (1 - qx_i/x_j)^{-1}$.

Notice that, interpreting Schur functions as characters of unitary groups, Weyl defined the scalar product of two symmetric functions f, g in n variables as the constant term of

$$\frac{1}{n!} f g^{\clubsuit} \prod_{i,j:i \neq j} (1 - x_i/x_j).$$

Essentially, Weyl takes the square of the Vandermonde, while we are taking the quotient of the Vandermonde by the q -Vandermonde.

We now examine the compatibility of \square_i and ∇_i with the scalar product.

Lemma 4.1 *For i such that $1 \leq i \leq n-1$, \square_i (resp. ∇_i) is adjoint to \square_{n-i} (resp. ∇_{n-i}) with respect to $(,)_q$.*

Proof. Since π_i (resp. $\hat{\pi}_i$) is adjoint to π_{n-i} (resp. $\hat{\pi}_{n-i}$) with respect to $(,)$ (see [5] for more details), we have

$$\begin{aligned} (f \square_i, g)_q &= (f, g \Omega \pi_{n-i}(1 - qx_{n-i+1}/x_{n-i})) \\ &= (f, g \frac{(1 - qx_{n-i+1}/x_{n-i})}{(1 - qx_{n-i+1}/x_{n-i})} \Omega \pi_{n-i}(1 - qx_{n-i+1}/x_{n-i})) \end{aligned}$$

Since the polynomial $\Omega/(1 - qx_{n-i+1}/x_{n-i})$ is symmetrical in the indeterminates x_{n-i} and x_{n-i+1} , it commutes with the action of π_{n-i} . Therefore

$$(f \square_i, g)_q = (f, g (1 - qx_{n-i+1}/x_{n-i}) \pi_{n-i} \Omega) = (f, g \square_{n-i})_q.$$

This proves that \square_i is adjoint to \square_{n-i} , and, equivalently, that ∇_i is adjoint to ∇_{n-i} . Q.E.D

We shall need to characterize whether the scalar product of two monomials vanishes or not. Notice that, by definition,

$$(x^u, x^v) = (x^{u-v}, 1),$$

so that one of the two monomials can be taken equal to 1.

Lemma 4.2 *For any $u \in \mathbb{Z}^n$, then $(x^u, 1)_q \neq 0$ iff $|u| = 0$ and $u \geq [0, \dots, 0]$. In that case, $(x^u, 1)_q = Q_u(0)$.*

Proof. Let us first show that the scalar products $(x^u, 1)_q$ satisfy the same relations (9) as the Hall-Littlewood functions Q_u .

Let k be a positive integer less than n . Write $x_k = y$, $x_{k+1} = z$. Any monomial x^v can be written $x^t y^a z^b$, with x^t of degree 0 in x_k, x_{k+1} . The product

$$x^t (y^a z^b + y^b z^a) (z - qy) \prod_{1 \leq i < j \leq n} \frac{1 - x_i/x_j}{1 - qx_i/x_j}$$

is equal to

$$(y^a z^b + y^b z^a)(z - qy) \frac{1 - y/z}{1 - qy/z} F_1 = (y^a z^b + y^b z^a)(z - y) F_1$$

with F_1 symmetrical in y, z . The constant term $CT_{x_{k-1}} \dots CT_{x_1}(x^t(y^a z^b + y^b z^a)F_1) = F_2$ is still symmetric in x_k, x_{k+1} . Therefore

$$CT_y \left(CT_z \left((z - y) F_2 \right) \right)$$

is null, and in final

$$CT \left(x^t (y^a z^b + y^b z^a) (z - qy) \prod_{1 \leq i < j \leq n} \frac{1 - x_i/x_j}{1 - qx_i/x_j} \right) = 0.$$

This relation can be rewritten

$$(y^a z^{b+1} x^t, 1)_q + (y^{b+1} z^{a+1} x^t, 1)_q - q(y^{b+1} z^a x^t, 1)_q - q(y^{a+1} z^b x^t, 1)_q = 0,$$

which is, indeed, relation (9).

On the other hand, if $u_n < 0$, then there is no term of degree 0 in x_n in $x^u \prod_{1 \leq i < j \leq n} (1 - x_i/x_j)(1 - qx_i/x_j)^{-1}$, and $(x^u, 1) = 0$, so that rule (10) is also satisfied.

In consequence, the function $u \in \mathbb{Z}^n \rightarrow (x^u, 1)$ is determined by the values $(x^\lambda, 1)$, λ partition, as the function $u \in \mathbb{Z}^n \rightarrow Q_u$ is determined by its restriction to partitions. However, for degree reasons, $(x^\lambda, 1) = 0$ if $\lambda \neq 0$. Since $(x^0, 1) = 1$, one has in final that $(x^u, 1) = Q_u(0)$. Q.E.D

Example 4.3 For $u = [1, 0, 3]$ and $v = [0, 1, 3]$,

$$(x^{103}, x^{013})_q = (x^{-2, -1, 3}, 1)_q = Q_{-2, -1, 3}(0) = q^2(1 - q)(1 - q^2).$$

4.2 Duality between $(U_v)_{v \in \mathbb{N}^n}$ and $(\widehat{U}_v)_{v \in \mathbb{N}^n}$

Using that \square_i is adjoint to \square_{n-i} , we are going to prove in this section that U_v and \widehat{U}_v are two adjoint bases of \mathfrak{Pol} with respect to the scalar product $(,)_q$.

We first need some technical lemmas, to allow an induction on the q -Key polynomials, starting with dominant weights.

Lemma 4.4 Let i be an integer such that $1 \leq i \leq n-1$, let f_1, f_2, g_1 be three polynomials and b be a constant such that

$$f_2 = f_1(\square_i + b), \quad (f_1, g_1)_q = 0 \quad \text{and} \quad (f_2, g_1)_q = 1.$$

Then the polynomial $g_2 = g_1(\nabla_{n-i} - b)$ is such that

$$(f_1, g_2)_q = 1, \quad (f_2, g_2)_q = 0.$$

Proof. Using that ∇_{n-i} is adjoint to \square_i and that $\square_i \nabla_i = 0$, one has

$$(f_2, g_2)_q = (f_1(\square_i + b), g_1(\nabla_{n-i} - b))_q = (f_1(\square_i + b)(\nabla_i - b), g_1)_q \\ = (f_1(-b(1+q) - b^2), g_1)_q = 0.$$

Similarly, we have

$$(f_1, g_2)_q = (f_1, g_1(\nabla_{n-i} - b))_q \\ = (f_1, g_1(\square_{n-i} - 1 - q - b))_q \\ = (f_1(\square_i + b - 1 - q - 2b), g_1)_q = (f_2, g_1)_q = 1.$$

Q.E.D

Corollary 4.5 *Let i be an integer such that $1 \leq i \leq n-1$, let V be a vector space such that $V = V' \oplus \langle f_1, f_2 \rangle$ with $f_2 = f_1(\square_i + b)$ and V' stable under \square_i , and let g_1 such that*

$$(f_1, g_1)_q = 0 \quad \text{and} \quad (f_2, g_1)_q = 1 \quad \text{and} \quad (v, g_1)_q = 0, \quad \forall v \in V'.$$

Then the element $g_2 = g_1(\nabla_{n-i} - b)$ is such that

$$(f_2, g_2)_q = 0 \quad \text{and} \quad (f_1, g_2)_q = 1 \quad \text{and} \quad (v, g_2)_q = 0, \quad \forall v \in V'.$$

Lemma 4.6 *Let u and λ be two dominant weights and v and μ two permutations of u and λ respectively. If $(x^v, x^\lambda) \neq 0$ and $(x^u, x^\mu) \neq 0$ then*

$$u = \lambda \quad , \quad v = \lambda\omega \quad \text{and} \quad \mu = u\omega.$$

Proof. Using lemma 4.2, the condition $(x^v, x^\lambda)_q \neq 0$ and $(x^u, x^\mu)_q \neq 0$ implies two systems of inequalities

$$\left\{ \begin{array}{l} v_n \geq \lambda_1, \\ v_n + v_{n-1} \geq \lambda_1 + \lambda_2, \\ \vdots \\ v_n + \dots + v_1 \geq \lambda_1 + \dots + \lambda_n. \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \mu_n \geq u_1, \\ \mu_n + \mu_{n-1} \geq u_1 + u_2, \\ \vdots \\ \mu_n + \dots + \mu_1 \geq u_1 + \dots + u_n. \end{array} \right.$$

The first inequalities of the systems give $v_n \geq \lambda_1 \geq \mu_n \geq u_1 \geq v_n$. Consequently $u_1 = \lambda_1 = v_n = u_n$. By recursion, using the other inequalities, one gets the lemma. Q.E.D

Corollary 4.7 *Let v be a weight and λ a dominant weight. Then,*

$$(U_v, x^\lambda)_q = \delta_{v, \lambda\omega}$$

Proof. Let u be the decreasing reordering of v and σ the permutation such that $U_v = x^u Y_\sigma$. As the leading term of U_v is x^v and using lemma 4.6, we have that $(x^u Y_\sigma, x^\lambda)_q \neq 0$ implies $(x^v, x^\lambda)_q \neq 0$. By denoting Δ_σ the adjoint of Y_σ with respect to $(,)_q$, we have $(x^u Y_\sigma, x^\lambda)_q = (x^u, x^\lambda \Delta_\sigma)_q \neq 0$. As the leading term of $x^\lambda \Delta_\sigma$ is $x^{\lambda\sigma'}$, where $\lambda\sigma'$ is a permutation of λ , we obtain that $(x^u, x^{\lambda\sigma'})_q \neq 0$. Using lemma 4.6 we conclude that $v = \lambda\omega$.

Q.E.D

Our main result is the following duality property between U_v and \widehat{U}_v .

Theorem 4.8 *The two sets of polynomials $(U_v)_{v \in \mathbb{N}^n}$ and $(\widehat{U}_v)_{v \in \mathbb{N}^n}$ are two adjoint bases of \mathfrak{Pol} with respect to the scalar product $(\cdot, \cdot)_q$. More precisely, they satisfy*

$$(U_v, \widehat{U}_{u\omega})_q = \delta_{v,u} .$$

Proof. Let λ be a dominant weight and V the vector space spanned by the U_v for v in $\mathcal{O}(\lambda)$. The idea of the proof is to build by iteration the elements $(\widehat{U}_v)_{v \in \mathcal{O}(\lambda)}$ starting with $x^\lambda = \widehat{U}_\lambda$. By definition of the q -Key polynomials, it exists a constant b such that $U_{\lambda\omega} = U_{\lambda\omega s_1} (\square_1 + b)$. One can write the decomposition $V = V' \oplus \langle U_{\lambda\omega}, U_{\lambda\omega s_1} \rangle$, with V' stable under the action of \square_1 . Using the previous lemma, we have that $(U_{\lambda\omega}, x^\lambda)_q = (U_{\lambda\omega}, \widehat{U}_\lambda)_q = 1$ and $(U_{\lambda\omega s_1}, x^\lambda)_q = (U_{\lambda\omega s_1}, \widehat{U}_\lambda)_q = 0$. Consequently, by lemma 4.5, the function $x^\lambda (\nabla_{n-1} - b) = \widehat{U}_{\lambda s_1}$ satisfy the duality conditions

$$(U_{\lambda\omega}, \widehat{U}_{\lambda s_1})_q = 0 \quad , \quad (U_{\lambda\omega s_1}, \widehat{U}_{\lambda s_1})_q = 1 \quad \text{and} \quad (v, \widehat{U}_{\lambda s_1})_q = 0 \quad \forall v \in V' .$$

By iteration, this proves that for all u, v , one has $(U_v, \widehat{U}_{u\omega})_q = \delta_{v,u}$. Q.E.D

This theorem implies that the space of symmetric functions and the linear span of dominant monomials are dual of each other, the Hall-Littlewood functions being the basis dual to dominant monomials.

We finally mention that in the case $q = 0$, one has a reproducing kernel, as stated by the following theorem of [4], which gives another implicit definition of the scalar product (\cdot, \cdot) .

Theorem 4.9 *The two families of polynomials $(K_v)_{v \in \mathbb{N}^n}$ and $(\widehat{K}_v)_{v \in \mathbb{N}^n}$ satisfy the following Cauchy formula*

$$\sum_{u \in \mathbb{N}^n} K_u(x) \widehat{K}_{u\omega}(y) = \prod_{i+j \leq n+1} \frac{1}{1 - x_i y_j} . \quad (15)$$

References

- [1] M. DEMAZURE, *Une nouvelle formule des caractères*, Bull. Sci. Math. **98** (1974), 163-172.
- [2] G. DUCHAMP, D. KROB, A. LASCoux, B. LECLERC, T. SCHARF and J.-Y. THIBON, *Euler-Poincaré characteristic and polynomial representations of Iwahori-Hecke algebras*, Publ. RIMS Kyoto **31** (1995), 179-201.
- [3] A. LASCoux *About the y in the χ_y -characteristic of Hirzebruch*, Contemporary Mathematics **241**, (1999) 285–296.
- [4] A. LASCoux. *Double Crystal graphs*, Studies in Memory of Issai Schur, Progress In Math. **210**, Birkhäuser(2003) 95–114.

- [5] A. LASCoux *Symmetric functions & Combinatorial operators on polynomials*, CBMS/AMS Lectures Notes **99**, (2003).
- [6] A. LASCoux, B. LECLERC AND J.Y. THIBON. *Flag Varieties and the Yang-Baxter Equation*, Letters in Math. Phys., **40** (1997) 75–90.
- [7] D.E. Littlewood. *On certain symmetric functions*, Proc. London. M.S. **43** (1961) 485–498.
- [8] G. LUSZTIG. *Equivariant K-theory and representations of Hecke Algebras*, Proc. Amer. Math. Soc. **94** (1985), 337-342.
- [9] I. G. MACDONALD. *Symmetric functions and Hall polynomials*. Oxford University Press, Oxford, 2nd edition (1995).
- [10] Package ACE <http://phalanstere.univ-mlv.fr/ace>
- [11] Package MUPAD-COMBINAT <http://mupad-combinat.sourceforge.net/>