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# Tetromino tilings and the Tutte polynomial

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## Abstract

We consider tiling rectangles of size  $4m \times 4n$  by T-shaped tetrominoes. Each tile is assigned a weight that depends on its orientation and position on the lattice. For a particular choice of the weights, the generating function of tilings is shown to be the evaluation of the multivariate Tutte polynomial  $Z_G(Q, \mathbf{v})$  (known also to physicists as the partition function of the  $Q$ -state Potts model) on an  $(m-1) \times (n-1)$  rectangle  $G$ , where the parameter  $Q$  and the edge weights  $\mathbf{v}$  can take arbitrary values depending on the tile weights.

The problem of evaluating the number of ways a given planar domain can be covered with a prescribed set of tiles (without leaving any holes or overlaps) has a long history in recreational mathematics, enumerative combinatorics, and theoretical physics.

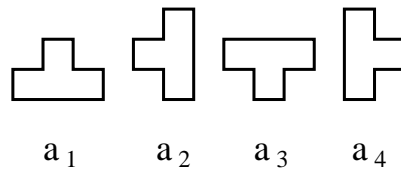


Figure 1: Weights of the four types of T-shaped tetrominoes.

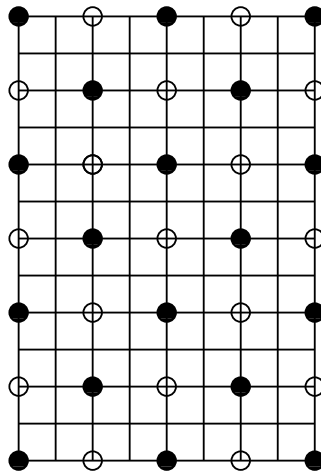


Figure 2: The domain  $\mathcal{D}$  to be tiled (here with  $m = 2, n = 3$ ). Some of the vertices are distinguished by black or white circles.

In this note we consider tiling an  $M \times N$  rectangle of the square lattice (henceforth denoted  $\mathcal{D}$ ) with T-shaped tetrominoes, i.e., tiles of size four lattice faces in the shape of the letter T. The four different orientations of the tiles are shown in Fig. 1. Note that the boundary of each tile comprises ten vertices, of which eight are corners. Fig. 2 shows the domain  $\mathcal{D}$  to be tiled. The circles distinguish two subsets of the vertices which we shall henceforth refer to as either black (filled circles) or white (empty circles).

We have the following remarkable

**Theorem 1 (Walkup [1]).**  *$\mathcal{D}$  is tileable iff  $M = 4m$  and  $N = 4n$ . In a valid tiling, no tile corner covers a white vertex, whereas black vertices are covered by tile corners only.*

Theorem 1 implies that tiles can be distinguished not only by their orientation (cf. Fig. 1) but also by their position relative the the white vertices. There are two possible situations, as shown in Fig. 3, depending on which of the two cornerless vertices of the tile covers the white vertex. (This applies to any of the four orientations of the tiles, although Fig. 3 only illustrates the first orientation.)

Note that the lattice  $B$  of black vertices is a tilted square lattice of edge length  $2\sqrt{2}$ . This can be divided into two sublattices,  $B_{\text{even}}$  and  $B_{\text{odd}}$ , which are both straight square lattices of edge length 4. Our convention is that all black vertices on the boundary of  $\mathcal{D}$  belong to  $B_{\text{odd}}$ .

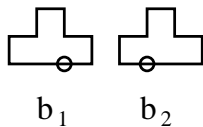


Figure 3: Weights depending on the relative position of a white vertex within a tile.

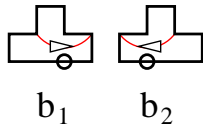


Figure 4: Two ways of decorating a tile by an oriented half-arch, depending on its position relative to a white vertex.

We similarly divide the lattice  $W$  of white vertices into two sublattices  $W_{\text{even}}$  and  $W_{\text{odd}}$ . Referring again to Fig. 2, we adopt the convention that white vertices on the lower and upper boundaries of  $\mathcal{D}$  belong to  $W_{\text{odd}}$  and that those on the left and right boundaries belong to  $W_{\text{even}}$ .

Equivalently, any vertex in  $B$  or  $W$  belongs to the odd (resp. even) sublattice whenever its distance from the bottom of  $\mathcal{D}$  is twice an even (resp. odd) integer.

Finally, we define the generating function

$$F_{\mathcal{D}}(\{a\}, \{b\}) = \sum_{T \in \mathcal{T}(\mathcal{D})} \prod_{t \in T} a_{o(t)}^{w(t)} \prod_{j=1}^2 b_j^{B_j} \quad (1)$$

where  $\mathcal{T}(\mathcal{D})$  is the set of all T-tetromino tilings of  $\mathcal{D}$ , and  $t$  is a single tile within the tiling  $T$ . The weights  $a_{o(t)}^{w(t)}$  depend on the orientation  $o(t) = 1, 2, 3, 4$  of the tile  $t$  (cf. Fig. 1) and on the (unique) white vertex  $w(t)$  that it covers. Moreover,  $B_j$  is the number of tiles of position  $j = 1, 2$  relative to the white vertices (cf. Fig. 3). Note that the weights  $b_j$  are not vertex dependent.

Evidently, the tiles covering the white vertices of the boundary  $\mathcal{B}$  of the domain  $\mathcal{D}$  have their orientation fixed (their long side is parallel to  $\mathcal{B}$ ). It follows that those boundary tiles contribute the same  $a$ -type weight to any tiling. Without loss of generality, we can therefore henceforth set  $a_{o(t)}^{w(t)} = 1$  for any  $t$  such that  $w(t) \in \mathcal{B}$ .

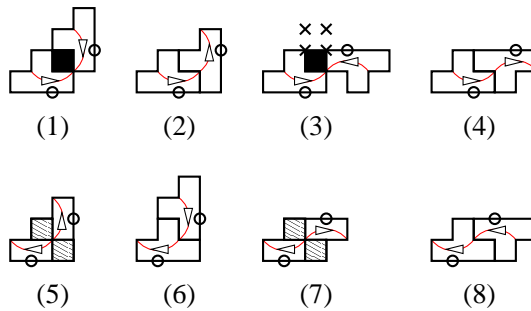


Figure 5: Arrow conservation at tile junctions.

It is convenient to illustrate the distinction of Fig. 3 in another way, by decorating the interior of each tile by an oriented half-arch (see Fig. 4). We then have the following

**Lemma 1.** *In any valid tiling, the orientation of the half-arches is conserved across the junctions of the tiles.*

*Proof.* By rotational and reflectional symmetry, it suffices to consider the junction between a tile in the  $a_1$ -type position (cf. Fig. 1) and the neighbouring tile immediately North-East of it.

Consider then the situation where the first tile is of the  $b_1$  type (cf. Fig. 4). The position of the neighbouring tile is constrained by Theorem 1 to have the correct location of its white vertex. When the neighbouring tile has its long side vertical, there are just two possibilities, shown in Fig. 5, panels (1) and (2). The possibility (1) is disallowed since it leaves an untiled hole (shown in black), whereas (2) is allowed. When the neighbour tile's long side is horizontal, we have the possibilities (3) and (4). But (3) is actually disallowed, since, when adding a further tile to cover the lattice face shown in black, a white point would be placed in one of the four vertices marked by a cross, and neither of these vertices is a valid position for a white point according to Theorem 1. In summary, only the possibilities (2) and (4) are allowed, and both of those conserve the arrow orientation across the tile junction.

Consider next the situation where the first tile is of the  $b_2$  type. A neighbour tile with a vertical long side leads to the possibilities (5) and (6), but (5) is disallowed since it leads to two overlaps (shown hatched). Similarly,

when the neighbour's long side is horizontal, one can rule out (7) due to two overlaps, whereas (8) is allowed. In summary, only (6) and (8) are allowed and compatible with arrow conservation.  $\square$

**Definition 1.** Let  $G = (V, E)$  be a finite undirected graph with vertex set  $V$  and edge set  $E$ . The multivariate Tutte polynomial of  $G$  with parameter  $Q$  and edge weights  $\mathbf{v} = \{v_e\}_{e \in E}$  is

$$Z_G(Q, \mathbf{v}) = \sum_{A \subseteq E} Q^{k(A)} \prod_{e \in A} v_e, \quad (2)$$

where  $k(A)$  denotes the number of connected components in the subgraph  $(V, A)$ .

For the relation of the multivariate Tutte polynomial to the more standard two-parameter Tutte polynomial  $T_G(x, y)$  we refer to Section 2.5 of [2]. In the physics literature,  $Z_G(Q, \mathbf{v})$  is better known as the partition function of the  $Q$ -state Potts model in the Fortuin-Kasteleyn formulation [3].

In the remainder of the paper we let  $V = B_{\text{even}}$  and  $E$  the associated natural set of horizontal and vertical edges of length 4. In other words,  $G$  is an  $(m-1) \times (n-1)$  rectangle  $G$ , as referred to in the abstract. Furthermore, let  $w(e)$  be the natural bijection between edges  $e \in E$  and white vertices  $w \in W \setminus \mathcal{B}$  that are *not* on the boundary  $\mathcal{B}$  of  $\mathcal{D}$ .

We can now state our main result which is

**Theorem 2.** *The generating function  $F_{\mathcal{D}}(\{a\}, \{b\})$  of  $T$ -tetromino tilings of the  $4m \times 4n$  rectangle  $\mathcal{D}$  depends only on the parameters  $\{a\}$  through the combinations  $a_1 a_3$  and  $a_2 a_4$ . For the specific choice of tile weights*

$$a_1 a_3 = \begin{cases} x_e & \text{for } w(e) \in W_{\text{even}} \setminus \mathcal{B} \\ 1 & \text{otherwise} \end{cases} \quad a_2 a_4 = \begin{cases} x_e & \text{for } w(e) \in W_{\text{odd}} \setminus \mathcal{B} \\ 1 & \text{otherwise} \end{cases} \quad (3)$$

and

$$b_1 = (b_2)^{-1} = q^{1/4} \quad (4)$$

we have the identity

$$Q^{mn/2} F_{\mathcal{D}}(\{a\}, \{b\}) = Z_G(Q; \mathbf{v}) \quad (5)$$

with the correspondence between parameters

$$Q = (q + q^{-1})^2, \quad v_e = (q + q^{-1})x_e \text{ for } e \in E \quad (6)$$

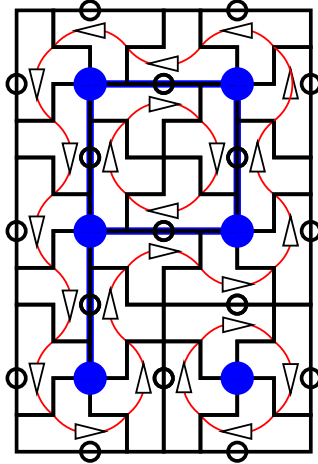


Figure 6: A valid tetromino tiling of the domain  $\mathcal{D}$  shown in Fig. 1. The vertices  $V$  entering the definition of the Tutte polynomial (2) are shown as fat solid circles, and the edges of the subset  $A \subseteq E$  are shown as fat lines.

*Proof.* By Theorem 1, any white vertex  $w \in W \setminus \mathcal{B}$  is at the junction between the long sides of exactly two tiles. If the long side is horizontal, the contribution of the  $a$ -type weight is  $a_1^w a_3^w$ , and if it is vertical the contribution is  $a_2^w a_4^w$ . This proves the first part of the theorem. (Above we have already explained that one may set all  $a_i^w = 1$  when  $w \in W \cap \mathcal{B}$ .)

By Lemma 1 the oriented half-arches form continuous curves with a consistent orientation (see Fig. 6 for an example). Moreover, by the consistency of the position of the white vertices (cf. Figs. 2–3), these curves cannot end on  $\mathcal{B}$ . Therefore, the half-arches form a set of oriented cycles.

Let  $C$  be the set of all possible configurations of cycles arising from valid tilings of  $\mathcal{D}$ , but disregarding the orientations of the cycles. The summation in the generating function (1) can then be written  $\sum_{\mathcal{T}(\mathcal{D})} = \sum_C \sum_{\mathcal{T}(\mathcal{D})|C}$ , where the last sum is over cycle orientations only. A close inspection of panels (2), (4), (6) and (8) in Fig. 5 reveals that the orientation of a given oriented cycle  $c_0$  can be changed independently of all other oriented cycles in the tiling  $T \in \mathcal{T}(\mathcal{D})$  by shifting the tiles traversed by  $c_0$  by one lattice unit to the left, right, up or down, but without moving any tiles not traversed by  $c_0$ . In particular, this transformation does not change the orientation of any tile

(cf. Fig. 1), and so does not change the  $a$ -type weights. We have therefore

$$F_{\mathcal{D}}(\{a\}, \{b\}) = \sum_C \prod a_{o(t)}^{w(t)} \sum_{\mathcal{T}(\mathcal{D})|C} \prod_{j=1}^2 b_j^{B_j}. \quad (7)$$

[The remainder of the proof parallels a construction used in [4] to show the equivalence between the Potts and six-vertex model partition functions.]

The partial summation  $\sum_{\mathcal{T}(\mathcal{D})|C} \prod_{j=1}^2 b_j^{B_j}$  appearing in (7) can be carried out by noting that the  $b$ -type weights (4) simply amount to weighting each turn of  $c_0$  through an angle  $\pm\pi/2$  by the factor  $q^{\pm 1/4}$ . Since the complete cycle turns a total angle of  $\pm 2\pi$  depending on its orientation, the product of the  $b$ -type weights associated with the cycle  $c_0$ , summed over its two possible orientations, yields the weight  $q + q^{-1} = Q^{1/2}$ . Denoting by  $\ell(C)$  the number of cycles in  $C$ , we have therefore found that

$$\sum_{\mathcal{T}(\mathcal{D})|C} \prod_{j=1}^2 b_j^{B_j} = Q^{\ell/2}. \quad (8)$$

It now remains to sum over the un-oriented cycles  $C$ . There is an easy bijection between  $C$  and the edge subsets  $A \subseteq E$  appearing in (2). Namely, an edge  $e \in E$  belongs to  $A$  iff it is not cut by any cycle in  $C$  (see Fig. 6). Conversely, given  $A$ , the cycles are constructed so that they cut no edge in  $A$ , cut all edges in  $E \setminus A$ , and are reflected off the boundary  $\mathcal{B}$  in the vertices  $W \cap \mathcal{B}$ . Under this construction, the  $a$ -type weights (3) simply give  $\prod_{e \in A} x_e$ . Collecting the pieces this amounts to

$$F_{\mathcal{D}}(\{a\}, \{b\}) = \sum_{A \subseteq E} Q^{\ell(A)/2} \prod_{e \in A} \frac{v_e}{Q^{1/2}}. \quad (9)$$

Using finally the Euler relation, and remarking that  $|V| = mn$ , one obtains (5).  $\square$

Let us discuss a couple of special cases of Theorem 2. First, when there are no  $a$ -type weights (i.e., setting  $a_{o(t)}^{w(t)} = 1$  for any  $o(t)$  and  $w(t)$ ), the Tutte polynomial has  $v_e = q + q^{-1}$  for any  $e \in E$ . The tiling entropy defined by

$$S_G(q) \equiv \lim_{m, n \rightarrow \infty} (F_{\mathcal{D}})^{\frac{1}{mn}} \quad (10)$$



is then related to the one found for the Tutte polynomial (*alias* Potts model) by Baxter [5] using the Bethe Ansatz method. For  $0 < Q < 4$  one sets  $q \equiv e^{i\mu}$  with  $\mu \in (0, \pi/2)$  and finds

$$\log S_G(q) = \int_{-\infty}^{\infty} dt \frac{\sinh [(\pi - \mu)t] \tanh(\mu t)}{t \sinh(\pi t)}, \quad (11)$$

whereas for  $Q > 4$  one sets  $q \equiv e^\lambda$  and finds

$$\log S_G(q) = \lambda + 2 \sum_{n=1}^{\infty} \frac{e^{-n\lambda} \tanh(n\lambda)}{n}. \quad (12)$$

Finally, for  $Q = 4$  one has

$$S_G(1) = \left( \frac{\Gamma(5/4)}{\Gamma(3/4)} \right)^4. \quad (13)$$

An even more special case of the identity (5) arises when  $b_1 = b_2 = 1$  as well.  $F_{\mathcal{D}}$  is then the unweighted sum over all T-tetromino tilings of  $\mathcal{D}$ . In terms of the standard Tutte polynomial  $T_G(x, y)$  one then has  $F_{\mathcal{D}} = 2T_G(3, 3)$ , as first proved by Korn and Pak [6].

Finally, let us remark that the construction used in the proof of Theorem 2 makes it evident how to go beyond Theorem 1. Namely, a (not necessarily simply) connected region  $\mathcal{D}$  is tileable by T-tetrominoes iff its boundary  $\mathcal{B}$  is a set of non-intersecting cycles on the lattice  $B_{\text{odd}}$ .

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