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# INTEGRABILITY OF INVARIANT METRICS ON THE DIFFEOMORPHISM GROUP OF THE CIRCLE

ADRIAN CONSTANTIN AND BORIS KOLEV

ABSTRACT. Each  $H^k$  Sobolev inner product ( $k \geq 0$ ) defines a Hamiltonian vector field  $X_k$  on the regular dual of the Lie algebra of the diffeomorphism group of the circle. We show that only  $X_0$  and  $X_1$  are bi-Hamiltonian relatively to a modified Lie-Poisson structure.

## 1. INTRODUCTION

Often motions of inertial mechanical systems are described in Lagrangian variables by paths on a configuration space  $G$  that is a Lie group. The velocity phase space is the tangent bundle  $TG$  and the kinetic energy

$$\mathcal{K} = \frac{1}{2} \langle v, v \rangle$$

for  $v \in TG$ . For example, in continuum mechanics the state of a system at time  $t \geq 0$  can be specified by a diffeomorphism  $x \mapsto \varphi(t, x)$  of the ambient space, giving the configuration of the particles with respect to their initial positions at time  $t = 0$ . Here  $x$  is a label identifying a particle, taken to be the position of the particle at time  $t = 0$  so that  $\varphi(0, x) = x$ . In this setting  $G$  would be the group of diffeomorphisms. The material (Lagrangian) velocity field is given by  $(t, x) \mapsto \varphi_t(t, x)$  while the spatial (Eulerian) velocity field is  $u(t, y) = \varphi_t(t, x)$ , where  $y = \varphi(t, x)$ , i.e.  $u = \varphi_t \circ \varphi^{-1}$ . Observe that for any fixed time-independent diffeomorphism  $\eta$ , the spatial velocity field  $u = \varphi_t \circ \varphi^{-1}$  along the path  $t \mapsto \varphi(t)$  remains unchanged if we replace this path by  $t \mapsto \varphi(t) \circ \eta$ . This right-invariance property suggests to extend the kinetic energy  $\mathcal{K}$  by right translation to a right-invariant Lagrangian  $\mathcal{K} : TG \rightarrow \mathbb{R}$ , obtaining a Lagrangian system on  $G$ . The length of a path  $\{\varphi(t)\}_{t \in [a, b]}$  in  $G$  is defined as

$$l(\varphi) = \int_a^b \langle \varphi_t, \varphi_t \rangle^{1/2} dt.$$

The Least Action Principle holds if the equation of motion is the geodesic equation. The set  $\text{Diff}(\mathbb{S}^1)$  of all smooth orientation-preserving diffeomorphisms of the circle represents the configuration space for the spatially periodic motion of inertial one-dimensional mechanical systems.  $\text{Diff}(\mathbb{S}^1)$  is an infinite dimensional Lie group, the group operation being composition [19] and its Lie algebra  $\text{Vect}(\mathbb{S}^1)$  being the space of all smooth vector fields on  $\mathbb{S}^1$

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cf. [29]. On the regular (or  $L^2$ ) dual  $\text{Vect}^*(\mathbb{S}^1)$  of the Lie algebra  $\text{Vect}(\mathbb{S}^1)$  there are some affine canonical Lie-Poisson structures, called *modified Lie-Poisson structures*, which are all compatible. On the other hand, we can consider on the regular dual  $\text{Vect}^*(\mathbb{S}^1)$  a countable family  $\{X_k\}_{k \geq 0}$  of Hamiltonian vector fields defined by Sobolev inner products. The importance of these inner products lies in that each gives rise via right translation to a geodesic flow on  $\text{Diff}(\mathbb{S}^1)$ , the Riemannian exponential map of which defines a local chart for every  $k \geq 1$  cf. [10] - a property which fails for the Lie group exponential map [19, 27] as well as for the Riemannian exponential map if  $k = 0$  [9]. In this paper we show that the Hamiltonian vector field  $X_k$  is bi-Hamiltonian relatively to a modified Lie-Poisson structure if and only if  $k \in \{0, 1\}$ .

## 2. PRELIMINARIES

In this section, we review some fundamental aspects of finite dimensional smooth Poisson manifolds.

**Definition 2.1.** A *symplectic manifold* is a pair  $(M, \omega)$ , where  $M$  is a manifold and  $\omega$  is a closed nondegenerate 2-form on  $M$ , that is  $d\omega = 0$  and for each  $m \in M$ ,  $\omega_m : T_m M \times T_m M \rightarrow \mathbb{R}$  is a continuous bilinear skew-symmetric map such that the induced linear map  $\tilde{\omega}_v : T_m M \rightarrow T_m^* M$  defined by  $\tilde{\omega}_v(w) = \omega(v, w)$  is an isomorphism for all  $v \in T_m M$ .

*Example 2.2.* In the general study of variational problems, extensive use is made of the canonical symplectic structure on the cotangent bundle  $T^*M$  (representing the phase space) of the manifold  $M$  (representing the configuration space). This symplectic form is given in any local trivialization  $(q, p) \in U \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n$  of  $T^*M$  by

$$\omega_{(q,p)}\left((Q, P), (\tilde{Q}, \tilde{P})\right) = \tilde{P} \cdot Q - P \cdot \tilde{Q}, \quad (Q, P), (\tilde{Q}, \tilde{P}) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Since a symplectic form  $\omega$  is nondegenerate, it induces an isomorphism

$$(2.1) \quad \flat : TM \rightarrow T^*M, \quad X \mapsto X^\flat,$$

defined via  $X^\flat(Y) = \omega(X, Y)$ . The *symplectic gradient*  $X_f$  of a function  $f$  is defined by the relation  $X_f^\flat = -df$ . The inverse of the isomorphism  $\flat$  defines a skew-symmetric bilinear form  $W$  on the cotangent space of  $M$ . This bilinear form  $W$  induces itself a bilinear mapping on  $C^\infty(M)$ , the space of smooth functions  $f : M \rightarrow \mathbb{R}$ , given by

$$(2.2) \quad \{f, g\} = W(df, dg) = \omega(X_f, X_g), \quad f, g \in C^\infty(M),$$

and called the *Poisson bracket* of the functions  $f$  and  $g$ .

*Example 2.3.* In Example 2.2, the Poisson bracket is given by

$$(2.3) \quad \{f, g\} = \sum_{i=1}^n \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).$$

The observation that a bracket like (2.3) could be introduced on  $C^\infty(M)$  for a smooth manifold  $M$ , without the use of a symplectic form, leads to the general notion of a *Poisson structure* [26].

**Definition 2.4.** A *Poisson structure* on a  $C^\infty$  manifold  $M$  is a skew-symmetric bilinear mapping  $(f, g) \mapsto \{f, g\}$  on the space  $C^\infty(M)$ , which satisfies the *Jacobi identity*

$$(2.4) \quad \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0,$$

as well as the *Leibnitz identity*

$$(2.5) \quad \{f, gh\} = \{f, g\}h + g\{f, h\}.$$

When the Poisson structure is induced by a symplectic structure  $\omega$ , the *Leibnitz identity* is a direct consequence of (2.2), whereas the *Jacobi identity* (2.4) corresponds to the condition  $d\omega = 0$  satisfied by the symplectic form  $\omega$ . In the general case, the fact that the mapping  $g \mapsto \{f, g\}$  satisfies (2.5) means that it is a derivation of  $C^\infty(M)$ . Each derivation on  $C^\infty(M)$  for a  $C^\infty$  manifold (even in the infinite dimensional case cf. [1]) corresponds to a smooth vector field, that is, to each  $f \in C^\infty(M)$  is associated a vector field  $X_f : M \rightarrow TM$ , called the *Hamiltonian vector field* of  $f$ , such that

$$(2.6) \quad \{f, g\} = X_f \cdot g = dg \cdot X_f,$$

where  $dg \cdot X_f = L_{X_f}g$  is the *Lie derivative* of  $g$  along  $X_f$ . Conversely, a vector field  $X : M \rightarrow TM$  on a Poisson manifold  $M$  is said to be *Hamiltonian* if there exists a function  $f$  such that  $X = X_f$ .

Recall [29] that for a smooth vector field  $X : M \rightarrow TM$ , the Lie derivative operator  $L_X : C^\infty(M) \rightarrow C^\infty(M)$  acts on smooth functions  $g : M \rightarrow \mathbb{R}$  with differentials  $dg : M \rightarrow T^*M$  by  $(L_Xg)(m) = dg(m) \cdot X(m)$  for  $m \in M$ . The space  $\text{Vect}(M)$  of smooth vector fields on  $M$  and the space of operators  $\{L_X : X \in \text{Vect}(M)\}$  are isomorphic as real vector spaces, the linear isomorphism between them being  $X \mapsto L_X$  [1]. Therefore the elements of  $\text{Vect}(M)$  can be regarded as operators on  $C^\infty(M)$  via  $X \cdot f = L_Xf$ , forming a Lie algebra if endowed with the bracket  $[X, Y] = L_X \circ L_Y - L_Y \circ L_X$ . Notice that (2.4) yields

$$(2.7) \quad [X_f, X_g] = X_{\{f, g\}}.$$

From (2.7) it follows (see [29]) that  $g \in C^\infty(M)$  is a constant of motion for  $X_f$  if and only if  $\{f, g\} = 0$ .

Just [21] pointed out that, just like a derivation on  $C^\infty(M)$  corresponds to a vector field, a bilinear bracket  $\{f, g\}$  satisfying the Leibnitz rule (2.5) corresponds to a skew-symmetric bilinear form on  $TM$ . That is, there exists a  $C^\infty$  tensor field  $W \in \Gamma(\wedge^2 TM)$ , called the *Poisson bivector* of  $(M, \{\cdot, \cdot\})$ , such that

$$\{f, g\} = W(df, dg).$$

Using the unique local extension of the Lie bracket of vector fields to skew-symmetric multivector fields, called the *Schouten-Nijenhuis bracket* [30], the condition (2.4) becomes

$$(2.8) \quad [W, W] = 0.$$

Conversely, any  $W \in \Gamma(\wedge^2 TM)$  that satisfies (2.8) induces a Poisson structure on  $M$  via (2.2). The only condition that must be satisfied by  $W$  is (2.8) since (2.5) holds automatically. A Poisson structure on  $M$  is therefore

equivalent to a bivector  $W$  that satisfies (2.8). This induces a homomorphism

$$(2.9) \quad \# : T^*M \rightarrow TM, \quad \alpha \mapsto \alpha^\#,$$

such that  $\beta(\alpha^\#) = W(\beta, \alpha)$  for every  $\beta \in T^*M$ . Notice that for  $f \in C^\infty(M)$  we have  $(df)^\# = X_f$ . If the homomorphism (2.9) is an isomorphism we call the Poisson structure *nondegenerate*. A nondegenerate Poisson structure on  $M$  is equivalent to a symplectic structure where the symplectic form  $\omega$  is just  $\#W$ , the closedness condition corresponding to the Jacobi identity [30].

*Remark 2.5.* The notion of a Poisson manifold is more general than that of a symplectic manifold. For example, in the symplectic case the Poisson bracket satisfies the additional property that  $\{f, g\} = 0$  for all  $g \in C^\infty(M)$  only if  $f \in C^\infty(M)$  is constant, whereas for Poisson manifolds such non-constant functions  $f$  might exist, in which case they are called *Casimir functions*. To highlight this, notice that by Darboux' theorem [29] a finite dimensional symplectic manifold  $M$  has to be even dimensional and locally there are coordinates  $\{q_1, \dots, q_n, p_1, \dots, p_n\}$  such that  $\{f, g\}$  is given by (2.3). On the other hand, on  $M = \mathbb{R}^{2n+1}$  with coordinates  $\{q_1, \dots, q_n, p_1, \dots, p_n, \zeta\}$  we determine a Poisson structure defining the Poisson bracket of  $f, g \in C^\infty(\mathbb{R}^{2n+1})$  by the same formula (2.3). Notice that any  $f \in C^\infty(\mathbb{R}^{2n+1})$  which depends only on  $\zeta$  is a Casimir function.

Two Poisson bivectors  $W_1$  and  $W_2$  define *compatible* Poisson structures if

$$(2.10) \quad [W_1, W_2] = 0.$$

This is equivalent to say that for any  $\lambda, \mu \in \mathbb{R}$ ,

$$\{f, g\}_{\lambda, \mu} = \lambda \{f, g\}_1 + \mu \{f, g\}_2$$

is also a Poisson bracket. On a manifold  $M$  equipped with two compatible Poisson structures, a vector field  $X$  is said to be (formally) *integrable* or *bi-Hamiltonian* if it is Hamiltonian for both structures.

On a symplectic manifold  $(M, \omega)$ , a necessary condition for a vector field  $X$  to be Hamiltonian is that  $L_X\omega = 0$  [29]. A similar criterion exists for a Poisson manifold  $(M, W)$ . It is instructive for later considerations to present a short proof of this known result.

**Proposition 2.6.** *On a Poisson manifold  $(M, W)$  a necessary condition for a vector field  $X$  to be Hamiltonian is*

$$(2.11) \quad L_X W = 0.$$

*Proof.* If  $X$  is Hamiltonian, there is a function  $h \in C^\infty(M)$  such that  $X = X_h$ . Let  $f$  and  $g$  be arbitrary smooth functions on  $M$ . We have

$$L_X W(df, dg) = L_X(W(df, dg)) - W(L_X df, dg) - W(df, L_X dg).$$

But  $L_{X_h} f = \{h, f\}$  and  $L_{X_h} df = dL_{X_h} f = d\{h, f\}$ . Therefore

$$\begin{aligned} L_X W(df, dg) &= L_X \{f, g\} - W(d\{h, f\}, dg) - W(df, d\{h, g\}) \\ &= \{h, \{f, g\}\} - \{\{h, f\}, g\} - \{f, \{h, g\}\}. \end{aligned}$$

This last expression equals zero because of the Jacobi identity.  $\square$

The fundamental example of a non-symplectic Poisson structure is the *Lie-Poisson structure* on the dual  $\mathfrak{g}^*$  of a Lie algebra  $\mathfrak{g}$ .

**Definition 2.7.** On the dual space  $\mathfrak{g}^*$  of a Lie algebra  $\mathfrak{g}$  of a Lie group  $G$ , there is a Poisson structure defined by

$$(2.12) \quad \{f, g\}(m) = m([d_m f, d_m g])$$

for  $m \in \mathfrak{g}^*$  and  $f, g \in C^\infty(\mathfrak{g}^*)$ , called the *canonical Lie-Poisson structure*<sup>1</sup>.

*Remark 2.8.* The canonical Lie-Poisson structure has the remarkable property to be *linear*. A Poisson bracket on a vector space is said to be *linear* if the bracket of two linear functionals is itself a linear functional.

Each element  $\gamma \in \wedge^2 \mathfrak{g}^*$  can be viewed as a Poisson bivector. Indeed,  $[\gamma, \gamma] = 0$  since  $\gamma$  is a constant tensor field. As such,  $\gamma$  defines a Poisson structure on  $\mathfrak{g}^*$ . The condition of compatibility with the canonical Lie-Poisson structure,  $[W_0, \gamma] = 0$ , can be written as (see [30], Chapter 3)

$$(2.13) \quad \gamma([u, v], w) + \gamma([v, w], u) + \gamma([w, u], v) = 0, \quad u, v, w \in \mathfrak{g}.$$

On a Lie group  $G$ , a right-invariant  $k$ -form  $\omega$  is completely defined by its value at the unit element  $e$ , and hence by an element of  $\wedge^k \mathfrak{g}^*$ . In other words, there is a natural isomorphism between the space of right-invariant  $k$ -forms on  $G$  and  $\wedge^k \mathfrak{g}^*$ . Moreover, since the exterior differential  $d$  commutes with right translations, it induces a linear operator  $\partial : \wedge^k \mathfrak{g}^* \rightarrow \wedge^{k+1} \mathfrak{g}^*$  that satisfies  $\partial \circ \partial = 0$  and

- (1)  $\partial\gamma = 0$  for  $\gamma \in \wedge^0 \mathfrak{g}^* = \mathbb{R}$ ;
- (2)  $\partial\gamma(u, v) = -\gamma([u, v])$  for  $\gamma \in \wedge^1 \mathfrak{g}^* = \mathfrak{g}^*$ ;
- (3)  $\partial\gamma(u, v, w) = \gamma([u, v], w) + \gamma([v, w], u) + \gamma([w, u], v)$  for  $\gamma \in \wedge^2 \mathfrak{g}^*$ ,

where  $u, v, w \in \mathfrak{g}$ , as one can check by direct computation (see [18], Chapter 24). The kernel  $Z^n(\mathfrak{g})$  of  $\partial : \wedge^n(\mathfrak{g}^*) \rightarrow \wedge^{n+1}(\mathfrak{g}^*)$  is the space of *n-cocycles* and the range  $B^n(\mathfrak{g})$  of  $\partial : \wedge^{n-1}(\mathfrak{g}^*) \rightarrow \wedge^n(\mathfrak{g}^*)$  is the spaces of *n-coboundaries*. Notice that  $B^n(\mathfrak{g}) \subset Z^n(\mathfrak{g})$ . The quotient space  $H_{CE}^n(\mathfrak{g}) = Z^n(\mathfrak{g})/B^n(\mathfrak{g})$  is the *n-th Lie algebra cohomology* or *Chevalley-Eilenberg cohomology group* of  $\mathfrak{g}$ . Notice that in general the Lie algebra cohomology is different from the de Rham cohomology  $H_{DR}^n$ . For example,  $H_{DR}^1(\mathbb{R}) = \mathbb{R}$  but  $H_{CE}^1(\mathbb{R}) = 0$ .

Each 2-cocycle  $\gamma$  defines a Poisson structure on  $\mathfrak{g}^*$  compatible with the canonical one. Indeed (2.13) can be recast as  $\partial\gamma = 0$ . Notice that the Hamiltonian vector field  $X_f$  of a function  $f \in C^\infty(\mathfrak{g}^*)$  computed with respect to the Poisson structure defined by the 2-cocycle  $\gamma$  is

$$(2.14) \quad X_f(m) = \gamma(d_m f, \cdot).$$

**Definition 2.9.** A *modified Lie-Poisson structure* is a Poisson structure on  $\mathfrak{g}^*$  whose Poisson bivector is given by  $W_\gamma = W_0 + \gamma$ , where  $W_0$  is the canonical Poisson bivector and  $\gamma$  is a 2-cocycle.

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<sup>1</sup>Here,  $d_m f$ , the differential of a function  $f \in C^\infty(\mathfrak{g}^*)$  at  $m \in \mathfrak{g}^*$  is to be understood as an element of the Lie algebra  $\mathfrak{g}$

*Example 2.10.* A special case of modified Lie-Poisson structure is given by a 2-cocycle  $\gamma$  which is a coboundary. If  $\gamma = \partial m_0$  for some  $m_0 \in \mathfrak{g}^*$ , the expression

$$\{f, g\}_\gamma(m) = m_0([d_m f, d_m g])$$

looks like if the Lie-Poisson bracket had been "frozen" at a point  $m_0 \in \mathfrak{g}^*$  and for this reason some authors call it a "freezing" structure.

### 3. MODIFIED LIE-POISSON STRUCTURES ON $\text{Vect}(\mathbb{S}^1)$

The group  $\text{Diff}(\mathbb{S}^1)$  of smooth orientation-preserving diffeomorphisms of the circle  $\mathbb{S}^1$  is endowed with a smooth manifold structure based on the Fréchet space  $C^\infty(\mathbb{S}^1)$ . The composition and the inverse are both smooth maps  $\text{Diff}(\mathbb{S}^1) \times \text{Diff}(\mathbb{S}^1) \rightarrow \text{Diff}(\mathbb{S}^1)$ , respectively  $\text{Diff}(\mathbb{S}^1) \rightarrow \text{Diff}(\mathbb{S}^1)$ , so that  $\text{Diff}(\mathbb{S}^1)$  is a Lie group [19]. Its Lie algebra  $\text{Vect}(\mathbb{S}^1)$  is the space of smooth vector fields on  $\mathbb{S}^1$ , which is isomorphic to the space  $C^\infty(\mathbb{S}^1)$  of periodic functions. The Lie bracket on  $\text{Vect}(\mathbb{S}^1)$  is given by

$$[u, v] = uv_x - u_x v.$$

Since the topological dual of the Fréchet space  $\text{Vect}(\mathbb{S}^1)$  is too big, being isomorphic to the space of distributions on the circle, we restrict our attention in the following to the *regular dual*  $\text{Vect}^*(\mathbb{S}^1)$ , the subspace of distributions defined by linear functionals of the form

$$u \mapsto \int_{\mathbb{S}^1} m u dx$$

for some function  $m \in C^\infty(\mathbb{S}^1)$ . The regular dual  $\text{Vect}^*(\mathbb{S}^1)$  is therefore isomorphic to  $C^\infty(\mathbb{S}^1)$  by means of the  $L^2$  inner product <sup>2</sup>

$$\langle u, v \rangle = \int_{\mathbb{S}^1} uv dx.$$

Let  $f$  be a smooth real valued function on  $C^\infty(\mathbb{S}^1)$ . Its *Fréchet* derivative at  $m$ ,  $df(m)$  is a linear functional on  $C^\infty(\mathbb{S}^1)$ . We say that  $f$  is a *regular function* if there exists a smooth map  $\delta f : C^\infty(\mathbb{S}^1) \rightarrow C^\infty(\mathbb{S}^1)$  such that

$$df(m) M = \int_{\mathbb{S}^1} M \cdot \delta f(m) dx, \quad m, M \in C^\infty(\mathbb{S}^1).$$

That is, the Fréchet derivative  $df(m)$  belongs to the regular dual  $\text{Vect}^*(\mathbb{S}^1)$  and the mapping  $m \mapsto \delta f(m)$  is smooth. The map  $\delta f$  is a vector field on  $C^\infty(\mathbb{S}^1)$ , called the *gradient* of  $f$  for the  $L^2$ -metric. In other words, a regular function is a smooth function on  $C^\infty(\mathbb{S}^1)$  which has a smooth gradient.

*Example 3.1.* Typical examples of *regular functions* are nonlinear *functionals* over the space  $C^\infty(\mathbb{S}^1)$ , like

$$f(m) = \int_{\mathbb{S}^1} (m^2 + mm_x^2) dx \quad \text{with} \quad \delta f(m) = 2m - m_x^2 - 2mm_{xx},$$

as well as linear functionals

$$f(m) = \int_{\mathbb{S}^1} um dx \quad \text{with} \quad \delta f(m) = u \in C^\infty(\mathbb{S}^1).$$

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<sup>2</sup>In the sequel, we use the notation  $u, v, \dots$  for elements of  $\text{Vect}(\mathbb{S}^1)$  and  $m, n, \dots$  for elements of  $\text{Vect}^*(\mathbb{S}^1)$  to distinguish them, although they all belong to  $C^\infty(\mathbb{S}^1)$ .



Notice that the smooth function  $f_\theta : C^\infty(\mathbb{S}^1) \rightarrow \mathbb{R}$  defined by  $f_\theta(m) = m(\theta)$  for some fixed  $\theta \in \mathbb{S}^1$  is not regular as  $\delta f_\theta$  is the Dirac measure at  $\theta$ .

Conversely, a smooth vector field  $X$  on  $\text{Vect}^*(\mathbb{S}^1)$  is called a *gradient* if there exists a *regular function*  $f$  on  $\text{Vect}^*(\mathbb{S}^1)$  such that  $X(m) = \delta f(m)$  for all  $m \in \text{Vect}^*(\mathbb{S}^1)$ . Observe that if  $f$  is a smooth real valued function on  $C^\infty(\mathbb{S}^1)$  then its second Fréchet derivative is symmetric [19], that is,

$$d^2 f(m)(M, N) = d^2 f(m)(N, M), \quad m, M, N \in C^\infty(\mathbb{S}^1).$$

For a regular function, this property can be written as

$$(3.1) \quad \int_{\mathbb{S}^1} (d\delta f(m)M)N \, dx = \int_{\mathbb{S}^1} (d\delta f(m)N)M \, dx,$$

for all  $m, M, N \in C^\infty(\mathbb{S}^1)$ . Hence the linear operator  $d\delta f(m)$  is symmetric for the  $L^2$ -inner product on  $C^\infty(\mathbb{S}^1)$  for each  $m \in C^\infty(\mathbb{S}^1)$ . We will resume this fact in the following lemma.

**Lemma 3.2.** *A necessary condition for a vector field  $X$  on  $C^\infty(\mathbb{S}^1)$  to be a gradient is that its Fréchet derivative  $dX(m)$  is a symmetric linear operator.*

To define a *Poisson bracket* on the space of *regular functions* on  $\text{Vect}^*(\mathbb{S}^1)$ , we consider a one-parameter family of linear operators  $J(m)$  and set

$$(3.2) \quad \{f, g\}(m) = \int_{\mathbb{S}^1} \delta f(m) J(m) \delta g(m) \, dx.$$

The operators  $J(m)$  must satisfy certain conditions in order for (3.2) to be a valid Poisson structure on  $\text{Vect}^*(\mathbb{S}^1)$ .

**Definition 3.3.** A family of linear operators  $J(m)$  on  $\text{Vect}^*(\mathbb{S}^1)$  defines a Poisson structure on  $\text{Vect}^*(\mathbb{S}^1)$  if (3.2) satisfies

- (1)  $\{f, g\}$  is regular if  $f$  and  $g$  are regular,
- (2)  $\{g, f\} = -\{f, g\}$ ,
- (3)  $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$ .

Notice that the second condition above simply means that  $J(m)$  is a skew-symmetric operator for each  $m$ .

*Example 3.4.* The canonical Lie-Poisson structure on  $\text{Vect}^*(\mathbb{S}^1)$  given by

$$\{f, g\}(m) = m([\delta f, \delta g]) = \int_{\mathbb{S}^1} \delta f(m) (mD + Dm) \delta g(m) \, dx$$

is represented by the one-parameter family of skew-symmetric operators

$$(3.3) \quad J(m) = mD + Dm$$

where  $D = \partial_x$ . It can be checked that all the three required properties are satisfied. In particular, we have

$$\delta \{f, g\} = d\delta f(J\delta g) - d\delta g(J\delta f) + \delta f \delta g_x - \delta g \delta f_x.$$

**Definition 3.5.** The *Hamiltonian* of a *regular function*  $f$ , for a Poisson structure defined by  $J$  is defined as the vector field

$$X_f(m) = J(m) \delta f(m).$$



**Proposition 3.6.** *A necessary condition for a smooth vector field  $X$  on  $\text{Vect}^*(\mathbb{S}^1)$  to be Hamiltonian with respect to the Poisson structure defined by a constant linear operator  $K$  is the symmetry of the operator  $dX(m) \circ K$  for each  $m \in \text{Vect}^*(\mathbb{S}^1)$ .*

*Proof.* If  $X$  is Hamiltonian, we can find a regular function  $f$  such that

$$X(m) = K \delta f(m).$$

Moreover, since  $K$  is a constant linear operator, we have

$$d(K \delta f)(m) M = K \circ (d \delta f(m)) M.$$

Therefore, we get

$$\begin{aligned} \langle dX(m) \circ K M, N \rangle &= \langle K \circ d \delta f(m) \circ K M, N \rangle \\ &= \langle M, K \circ d \delta f(m) \circ K N \rangle \\ &= \langle M, dX(m) \circ K N \rangle, \end{aligned}$$

since  $K$  is skew-symmetric and  $d \delta f(m)$  is symmetric.  $\square$

A 2-cocycle on  $\text{Vect}(\mathbb{S}^1)$  is a bilinear functional  $\gamma$  represented by a skew-symmetric operator  $K : C^\infty(\mathbb{S}^1) \rightarrow C^\infty(\mathbb{S}^1)$  such that

$$\gamma(u, v) = \langle u, K v \rangle = \int_{\mathbb{S}^1} u K v \, dx,$$

and satisfying the Jacobi identity

$$\langle [u, v], K w \rangle + \langle [v, w], K u \rangle + \langle [w, u], K v \rangle = 0.$$

If  $K$  is a differential operator we call  $\gamma$  a *differential cocycle*. Gelfand and Fuks [16] observed that all differential 2-cocycles of  $\text{Vect}(\mathbb{S}^1)$  belong to the one-dimensional cohomology class generated by  $[D^3]$ . Moreover, each regular 2-coboundary is represented by the skew-symmetric operator

$$m_0 D + D m_0,$$

for some  $m_0 \in C^\infty(\mathbb{S}^1)$ . Therefore, each differential 2-cocycle of  $\text{Vect}(\mathbb{S}^1)$  is represented by an operator of the form

$$(3.4) \quad K = m_0 D + D m_0 + \beta D^3$$

where  $m_0 \in C^\infty(\mathbb{S}^1)$  and  $\beta \in \mathbb{R}$  (see also [17]).

For  $k \geq 0$  and  $u, v \in \text{Vect}(\mathbb{S}^1) \equiv C^\infty(\mathbb{S}^1)$ , let us now define the  $H^k$  (Sobolev) inner product by

$$\langle u, v \rangle_k = \int_{\mathbb{S}^1} \sum_{i=0}^k (\partial_x^i u) (\partial_x^i v) \, dx = \int_{\mathbb{S}^1} A_k(u) v \, dx,$$

where

$$(3.5) \quad A_k = 1 - \frac{d^2}{dx^2} + \dots + (-1)^k \frac{d^{2k}}{dx^{2k}}$$

is a continuous linear isomorphism of  $C^\infty(\mathbb{S}^1)$ . Note that  $A_k$  is a symmetric operator for the  $L^2$  inner product since

$$\int_{\mathbb{S}^1} A_k(u) v \, dx = \int_{\mathbb{S}^1} u A_k(v) \, dx.$$

The operator  $A_k$  gives rise to a Hamiltonian function on  $\text{Vect}^*(\mathbb{S}^1)$  given by

$$h_k(m) = \int_{\mathbb{S}^1} \frac{1}{2} m(A_k^{-1}m) dx.$$

The corresponding Hamiltonian vector field  $X_k$  is given by

$$X_k(m) = (mD + Dm)(A_k^{-1}m) = 2mu_x + um_x,$$

if we let  $m = A_k u$ .

**Theorem 3.7.** *The Hamiltonian vector field  $X_k$  is bi-Hamiltonian relatively to a modified Lie-Poisson structure if and only if  $k \in \{0, 1\}$ .*

*Proof.* It is well known (see [28]) that  $X_0$  is bi-Hamiltonian with respect to the operator  $D$  which represents a coboundary. It is also known that  $X_1$  is a bi-Hamiltonian vector field with respect to the cocycle represented by the operator  $D(1 - D^2)$  cf. [2, 11, 14]. Notice that this cocycle is not a coboundary.

We will now show that there is no differential cocycle

$$K = m_0 D + Dm_0 + \beta D^3$$

for which  $X_k$  could be Hamiltonian unless  $k \in \{0, 1\}$ . We have

$$dX_k(m) = 2u_x I + uD + 2mDA_k^{-1} + m_x A_k^{-1},$$

and in particular, for  $m = 1$ ,

$$dX_k(1) = D + 2DA_k^{-1}.$$

Letting

$$P(m) = dX_k(m) \circ K,$$

we obtain that

$$P(1) = (D + 2DA_k^{-1}) \circ (m_0 D + Dm_0) + \beta D^4(1 + 2A_k^{-1}),$$

whereas

$$P(1)^* = (m_0 D + Dm_0) \circ (D + 2DA_k^{-1}) + \beta D^4(1 + 2A_k^{-1}).$$

Therefore, denoting  $m'_0 = \partial_x m_0$ , we have

$$\begin{aligned} P(1) - P(1)^* &= (m'_0 D + Dm'_0) + 2(A_k^{-1} Dm_0 D - Dm_0 D A_k^{-1}) + \\ &\quad + 2(A_k^{-1} D^2 m_0 - m_0 D^2 A_k^{-1}). \end{aligned}$$

If this operator is zero, we must have in particular the relation

$$A_k(P(1) - P(1)^*)A_k(e^{irx}) = 0,$$

for all  $r \in \mathbb{Z}$ . But, for  $r \neq \pm 1$ ,

$$A_k(e^{irx}) = f_k(r) e^{irx} \quad \text{with} \quad f_k(r) = \frac{r^{2k+2} - 1}{r^2 - 1},$$

and

$$A_k(P(1) - P(1)^*)A_k(e^{irx})$$

is of the form  $e^{irx}$  times a polynomial expression in  $r$  with highest order term  $2i m'_0(x) r^{4k+1}$ . Therefore, a necessary condition for  $X_k$  to be Hamiltonian relatively to the Poisson operator  $K$  defined by (3.4) is that  $m_0$  is a constant.

Let  $\alpha = 2m_0 \in \mathbb{R}$ . Then

$$P(m) = dX_k(m) \circ K = \alpha \{2u_x D + uD^2 + 2mD^2 A_k^{-1} + m_x D A_k^{-1}\} + \\ + \beta \{2u_x D^3 + uD^4 + 2mD^4 A_k^{-1} + m_x D^3 A_k^{-1}\}$$

because  $D$  and  $A_k$  commute. By virtue of Proposition 3.6, a necessary condition for  $X_k$  to be Hamiltonian with respect to the cocycle represented by  $K$  is that  $P(m)$  is symmetric. That is

$$(3.6) \quad \langle P(m)M, N \rangle = \langle M, P(m)N \rangle,$$

for all  $m, M, N \in C^\infty(\mathbb{S}^1)$ . Since this last expression is tri-linear in the variables  $m, M, N$ , the equality can be checked for complex periodic functions  $m, M, N$  where the  $L^2$  inner product is extended naturally into a complex bilinear functional. That is, the extension is not a hermitian product, we just allow homogeneity with respect to the complex scalar field in both components. Let  $m = A_k u$ ,  $u = \exp(iax)$ ,  $M = \exp(ibx)$  and  $N = \exp(icx)$  with  $a, b, c \in \mathbb{Z}$ . We have

$$\langle P(m)M, N \rangle = \left[ (2ab^3 + b^4)\beta - (2ab + b^2)\alpha + \right. \\ \left. + \left( (ab^3 + 2b^4)\beta - (ab + 2b^2)\alpha \right) \frac{f_k(a)}{f_k(b)} \right] \int_{\mathbb{S}^1} e^{i(a+b+c)x} dx,$$

whereas

$$\langle M, P(m)N \rangle = \left[ (2ac^3 + c^4)\beta - (2ac + c^2)\alpha + \right. \\ \left. + \left( (ac^3 + 2c^4)\beta - (ac + 2c^2)\alpha \right) \frac{f_k(a)}{f_k(c)} \right] \int_{\mathbb{S}^1} e^{i(a+b+c)x} dx.$$

For  $a = n$ ,  $b = -2n$  and  $c = n$ , we obtain

(3.7)

$$\langle P(m)M, N \rangle = (24n^4\beta - 6n^2\alpha) \frac{f_k(n)}{f_k(2n)}, \quad \langle M, P(m)N \rangle = 6n^4\beta - 6n^2\alpha.$$

The equality of the two expressions in (3.7) for all  $n \in \mathbb{N}$  is ensured by means of (3.6). For  $k = 1$  this leads to the condition  $\alpha + \beta = 0$  and we recover the second Poisson structure given by  $K = D - D^3$  for which  $X_1$  is known to be Hamiltonian with Hamiltonian function

$$\tilde{h}_1(m) = \frac{1}{2} \int_{\mathbb{S}^1} \left( (A_1^{-1}m)^3 + (A_1^{-1}m) [(A_1^{-1}m)_x]^2 \right) dx.$$

In the general case, if  $\beta \neq 0$ , the leading term with respect to  $n$  in the first expression in (3.7) is  $(-48\beta 2^{-2k})$ , whereas in the second it is  $(-12\beta)$ . Thus unless  $\beta = 0$  we must have  $k = 1$ . On the other hand, if  $\beta = 0$ , from (3.6)-(3.7) we infer that  $\alpha f_k(n) = \alpha f_k(2n)$  for all  $n \in \mathbb{N}$ . Thus  $\alpha = 0$  unless  $k = 0$ . For  $k = 0$  we recover the Poisson structure given by  $K = D$  for which  $X_0$  is Hamiltonian with Hamiltonian function

$$\tilde{h}_0(m) = \frac{1}{2} \int_{\mathbb{S}^1} m^3 dx.$$

This completes the proof.  $\square$

## 4. CONCLUSION

We showed that among all  $H^k$  Sobolev inner products on  $C^\infty(\mathbb{S}^1)$ , only for  $k \in \{0, 1\}$  is the associated vector field bi-Hamiltonian relatively to a modified Lie-Poisson structure. Endowing  $\text{Diff}(\mathbb{S}^1)$  with the  $H^1$  right-invariant metric, the associated geodesic equation turns out to be the Camassa-Holm equation [23] (see also [22])

$$u_t + uu_x + \partial_x(1 - \partial_x^2)^{-1}(u^2 + \frac{1}{2}u_x^2) = 0,$$

a model for shallow water waves (see [2] and the alternative derivations in [5, 13, 15, 20]) that accommodates waves that exist indefinitely in time [3, 7] as well as breaking waves [6, 8]. The bi-Hamiltonian structure is reflected in the existence of infinitely many conserved integrals for the equation [2, 11, 14, 24] which are very useful in the qualitative analysis of its solutions. Both global existence results and blow-up results can be obtained using certain conservation laws [3, 7, 31], while the proof of stability of traveling wave solutions relies on the specific form of some conserved quantities [4, 11, 12, 25]. On the other hand, the geodesic equation on  $\text{Diff}(\mathbb{S}^1)$  for the  $L^2$  right-invariant metric is the inviscid Burgers equation

$$u_t + 3uu_x = 0.$$

This model of gas dynamics has been thoroughly studied (see [9] and references therein). In contrast to the case of the  $H^1$  right-invariant metric [10], the Riemannian exponential map is not a  $C^1$  local diffeomorphism in the case of the  $L^2$  right-invariant metric [9]. This means that of the two bi-Hamiltonian vector fields  $X_0$  and  $X_1$ , the second generates a flow on  $\text{Diff}(\mathbb{S}^1)$  with properties that parallel those of geodesic flows on finite-dimensional Lie groups.

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