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# ON THE CODIMENSION-ONE FOLIATION THEOREM OF W. THURSTON

FRANÇOIS LAUDENBACH

ABSTRACT. We are dealing with an  $n$ -manifold equipped with a  $\Gamma_n$ -structure in the sense of Haefliger with a transverse geometric structure. Here we mainly consider  $\Gamma_n$ -structures transversally codimension-one foliated. We prove a regularization theorem which implies the celebrated theorem of W. Thurston.

## 1. INTRODUCTION

The mentioned theorem [9] states that every  $(n - 1)$ -plane field on an  $n$ -dimensional compact manifold  $M^n$  is homotopic to an integrable plane field, that is to a plane field which is tangent to a foliation. Both proofs presented by Thurston consist of creating holes in  $M$ , foliating the complement of the holes and finally filling the holes. In dimension 3 the holes are solid tori and they are filled by Reeb components. One of the proofs brings triangulations to play an important part. But, *a posteriori*, a fine enough triangulation cannot detect the existence of Reeb components, even when they are forced to exist (according for instance to Novikov's theorem when  $M^n = S^3$ ). The aim of this paper is to produce a proof where no holes are created, working more in the spirit of the  $h$ -principle of Gromov [4] (see 4.3).

There is an equivalent statement involving the concept of  $\Gamma$ -structure in the sense of A. Haefliger [5]. A  $\Gamma$ -structure is something like a singular foliation; a precise definition will be recalled later. At the beginning I was interested in  $\Gamma$ -structure equipped with some extra transverse structure in the same way as a foliation may have a transverse geometric structure which is invariant by the holonomy. In [8] W. Thurston proved a general theorem of deformation of  $\Gamma$ -structures to *regular*  $\Gamma$ -structures. A natural question is to discuss such a theorem for  $\Gamma$ -structures equipped with a transverse geometry. This is not true in general, for instance for transverse symplectic structures. It should depend on the flexibility of the considered geometry. Here we limit ourselves to prove the following theorems for which precise definitions will be introduced in the next section.

**Theorem A.** *Every  $\Gamma_n$ -structure  $\xi$  on  $M^n$  which is transversally codimension-one foliated and whose normal bundle  $\nu\xi$  is isomorphic to the tangent bundle  $\tau M$  is homotopic to a regular one.*

**Theorem B.** (Thurston [9]) *Every  $\Gamma_1$ -structure  $\xi$  on  $M^n$  whose normal bundle  $\nu\xi$  injects into the tangent bundle  $\tau M$  is homotopic to a regular  $\Gamma_1$ -structure, that is to a codimension-one foliation  $\mathcal{H}_1$  of  $M$ . The normal bundle of  $\mathcal{H}_1$  is isomorphic to  $\nu\xi$  as a sub-bundle of  $\tau M$ .*

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As it is explained in section 5, Theorem B is an easy consequence of Theorem A.

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## 2. $\Gamma_k$ -STRUCTURES

**Definition 2.1.** A  $\Gamma_k$ -structure  $\xi$  on a manifold  $M^n$  is given by the following data:

- (1) a (non-linear)  $\mathbb{R}^k$ -bundle  $\pi : E \rightarrow M$  provided with a zero section  $Z$ ;
- (2) a germ of codimension  $k$  foliation  $\mathcal{L}$  of  $E$  along  $Z$ , transversal to the fibers of  $\pi$ .

The normal bundle of  $\mathcal{L}$  restricted to  $Z$  will be denoted by  $\nu\xi$ . It is isomorphic to  $(E, \pi, M)$ . As a matter of fact such a data is equivalent to a Chech cocycle on  $M$  with values in the groupoid of germs of diffeomorphisms of  $\mathbb{R}^k$  and for this reason the definition extends to all CW-complexes.

**Definition 2.2.** A  $\Gamma_k$ -structure is said *regular* when the foliation  $\mathcal{L}$  is transversal to  $Z$ .

In that case,  $Z$  (or  $M$ ) is foliated by  $\mathcal{L} \cap Z$ . Conversely, if  $\mathcal{F}$  is a foliation of codimension  $k$  on  $M$ , it is induced by a regular  $\Gamma_k$ -structure whose underlying fiber bundle is  $\nu\mathcal{F}$ , the usual normal bundle of  $\mathcal{F}$ . Indeed the exponential map  $\nu\mathcal{F} \rightarrow M$  is transversal to  $\mathcal{F}$  along the zero section and  $\mathcal{L} := \exp^{-1}(\mathcal{F})$  meets the required conditions.

**Definition 2.3.** A homotopy between two  $\Gamma_k$ -structures  $\xi_0$  and  $\xi_1$  is a  $\Gamma_k$ -structure on  $M \times [0, 1]$  which induces  $\xi_i$  on  $M \times \{i\}$  for  $i = 0, 1$ . It is not required that the zero section be  $Z \times [0, 1]$ .

Though it is not said in Thurston's work (but it is used), the transversality to the fibers can be relaxed from the definition of a homotopy, according to the following proposition.

**Proposition 2.4.** *Let  $\xi_i = (E, Z, \mathcal{L}_i)$ ,  $i = 0, 1$ , be two  $\Gamma_k$ -structures on  $M$  with the same normal bundle. If there exists a germ of codimension- $k$  foliation  $\mathcal{L}$  along  $Z \times [0, 1]$  in  $E \times [0, 1]$  which is transversal to  $E \times \{0, 1\}$  and induces  $\mathcal{L}_i$  on  $E \times \{i\}$ ,  $i = 0, 1$ , then  $\xi_0$  and  $\xi_1$  are homotopic up to isomorphism.*

**Proof.** The normal bundle  $\nu\mathcal{L}$  over  $Z \times [0, 1] \cong M \times [0, 1]$  is isomorphic to  $E \times [0, 1]$ . An exponential map associated with some Riemannian metric  $\mu$  on the manifold  $E \times [0, 1]$  yields a codimension- $k$  foliation  $\mathcal{L}'$  on the total space of  $\nu\mathcal{L}$  near  $Z \times [0, 1]$ , transversal to the fibers. If  $\mu$  makes  $\mathcal{L}_i$  orthogonal to the fibers of  $E \times \{i\}$ ,  $i = 0, 1$ , then  $\mathcal{L}'$  induces  $\mathcal{L}_i$  there.  $\square$

**Definition 2.5.** A  $\Gamma_k$ -structure  $\xi = (E, Z, \mathcal{L})$  has a transverse geometric structure  $\mathcal{G}$  if each transversal  $T$  to  $\mathcal{L}$  is endowed with a geometric structure  $\mathcal{G}_T$  invariant by holonomy: if  $T$  and  $T'$  are two transversal discs and if  $\varphi : T \rightarrow T'$  is some holonomy diffeomorphism then  $\varphi$  has to be an isomorphism from  $\mathcal{G}_T$  to  $\mathcal{G}_{T'}$ .

The corresponding definition of homotopy is obvious: the foliation contributing to the homotopy has to carry a transverse geometric structure  $\mathcal{G}$  inducing the ones which are given at  $t = 0$  and  $t = 1$ . When  $\xi$  is homotopic to a regular structure, then  $Z$  (and hence  $M$ ) carries a regular foliation with a transverse  $\mathcal{G}$ -structure; in particular when  $k = n = \dim M$ ,  $M$  itself carries the geometric structure  $\mathcal{G}$ .

**Examples 2.6.** 1) A  $\Gamma_n$ -structure transversally codimension- $k$  foliated ( $k < n$ ) is given by the following data:

- (1) A  $\mathbb{R}^n$ -bundle  $\pi : E \rightarrow M$  ( $\dim E = 2n$ ), with a zero section  $Z$ ;
- (2) A germ of codimension- $n$  foliation  $\mathcal{L}$  on  $E$  along  $Z$ ;
- (3) A germ of codimension- $k$  foliation  $\mathcal{G}$  on  $E$  along  $Z$  containing  $\mathcal{L}$  in the sense that each leaf of  $\mathcal{L}$  is contained in a leaf of  $\mathcal{G}$ .

2) If  $n = 2k + 1$ , a  $\Gamma_n$ -structure transversally contact is given by a  $\Gamma_n$ -structure and a germ of hyperplane field along  $Z$  locally defined as the kernel of a 1-form  $\alpha$  which is  $\mathcal{L}$ -basic and satisfies  $\alpha \wedge (d\alpha)^k \neq 0$ .

3) If  $n = 2k$ , a  $\Gamma_n$ -structure transversally symplectic is given by a germ of 2-form  $\omega$  which is  $\mathcal{L}$ -basic, closed and satisfies  $\omega^k \neq 0$ .

The definitions from examples 2 and 3 above have been introduced by D. McDuff [6]. Of course when the normal bundle is trivial these three types of structure exist. Are they homotopic to a regular one? Certainly not in the symplectic case. For instance  $S^3 \times S^3$  is parallelizable and hence it has a  $\Gamma_6$ -structure  $\xi$  transversally symplectic. If  $\xi$  would be homotopic to a regular one, then  $S^3 \times S^3$  would carry a symplectic structure. It is not the case since  $H^2(S^3 \times S^3, \mathbb{R}) = 0$ . The case of  $\Gamma_n$ -structures transversally contact will be considered elsewhere. Here we only consider the case of  $\Gamma_n$ -structures transversally codimension-one foliated.

### 3. PREPARATION FOR PROVING THEOREM A

**3.1.** We consider a  $\Gamma_n$ -structure  $\xi$  on  $M^n$  which is transversally codimension-one foliated. The structure  $\xi = (E, Z, \mathcal{L}, \mathcal{H})$  is made of a codimension- $n$  foliation  $\mathcal{L}$  on  $E$  near  $Z$  and a codimension-one foliation  $\mathcal{H}$  containing  $\mathcal{L}$ . The goal is to construct a pair of foliations  $\overline{\mathcal{L}} \subset \overline{\mathcal{H}}$ , on  $E \times [0, 1]$  near  $Z \times [0, 1]$ , respectively of codimension  $n$  and 1, extending  $(\mathcal{L}, \mathcal{H})$  and such that  $\overline{\mathcal{L}}$  be transversal to  $Z \times \{1\}$ . Hence  $\overline{\mathcal{H}}$  will be also transversal to  $Z \times \{1\}$ .

As in Thurston's paper [8], we have a codimension- $n$  plane field  $\overline{\mathcal{L}}$ , defined near  $Z \times [0, 1]$  in  $E \times [0, 1]$  which is tangent to  $\mathcal{L}$  along  $E \times \{0\}$  and transversal to  $Z \times \{1\}$ . This is precisely the place where the assumption on  $\nu\xi$  is used. Such a field is for instance yielded by the following formula

$$\overline{\mathcal{L}}(x, t) = \mathbb{R} \frac{\partial}{\partial t} \oplus [t\Delta(x) + (1-t)L(x)].$$

In this formula  $L(x)$  stands for the tangent plane to  $\mathcal{L}$  at point  $x$  and  $\Delta(x)$  is the "diagonal" of  $T_x E \cong T_x M \oplus T_x M$ ; the barycentric combination refers to the affine structure on the set of  $n$ -planes in  $T_x E$  which are transversal to the vertical tangent plane. Without loss of generality, we may assume that  $\overline{\mathcal{L}}$  and  $\overline{\mathcal{H}}$  are already defined near  $Z \times \{0\}$  in  $E \times [0, 1]$ ,  $\overline{\mathcal{L}}$

being tangent to  $\overline{\mathcal{L}}$  there; of course it requires a slight change in the above formula. Clearly  $\overline{L}$  is contained in a codimension-one distribution  $\overline{H}$  which is tangent to  $\overline{H}$  where it makes sense.

**3.2. Product triangulation, crystalline subdivision.** One starts with a *product triangulation*  $\theta$  of a neighborhood of  $Z \times [0, 1]$  in  $E \times [0, 1]$  having  $Z \times [0, 1]$  as a subcomplex. A product triangulation means that, for each  $k$ -simplex  $\sigma$  of  $E \times \{0\}$ ,  $\sigma \times [0, 1]$  is triangulated in a *crystalline* way in the sense of Thurston - Whitney ([11], p. 365, and [8], p. 227). Before any subdivision such a triangulation only depends on a triangulation of  $E$  with a numbering of its vertices; it is figured below. Each  $(k+1)$ -simplex in  $\sigma \times [0, 1]$  is obtained by “raising” one vertex of  $\sigma$ , then another one, ...; the slogan is: at each step the “lowest” vertex (for the lexicographic order on  $\{0, 1\} \times \{\text{numbering}\}$ ) is raised first.

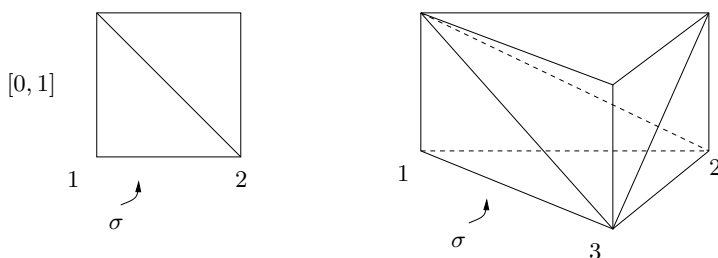


Figure 1 – Here  $\sigma$  is 1- or 2-dimensional.

The first *crystalline subdivision* consists of adding vertices at the middle of each edge; each new face lying in  $\sigma \times [0, 1]$  is parallel to some face from the initial product triangulation. This process can be iterated as many times as we want in order to produce an arbitrarily fine subdivision. All these triangulations will be called *product triangulations*.

**3.3. Different kind of simplices.** A  $k$ -simplex of  $(Z \times [0, 1])$  is of *second kind* if it belongs to  $(Z^{[k-1]} \times [0, 1])$  where  $Z^{[k-1]}$  denotes the  $(k-1)$ -skeleton of  $Z$ . It is of *first kind* if it projects onto a  $k$ -simplex of  $Z \times \{0\}$ .

By definition of a crystalline subdivision, if  $\sigma_0$  is a  $k$ -simplex in  $Z \times \{0\}$ , each  $(k+1)$ -simplex  $\sigma$  in  $\sigma_0 \times [0, 1]$  has exactly two  $k$ -faces of first kind; the one which lies the nearest to  $\sigma_0$  will be called the *base face* and the other one will be called the *free face*. Moreover all the second kind  $k$ -faces of  $\sigma$  have a common edge which projects onto a point; it is the unique second kind edge of  $\sigma$ . Finally each 1st kind  $k$ -simplex is the face of exactly two 2nd kind  $(k+1)$ -simplices.

**3.4. Good position.** The triangulation  $\theta$  of  $Z \times [0, 1]$  is said in *good position* with respect to  $(\overline{H}, \overline{L})$  when the following conditions hold for each  $(n+1)$ -simplex  $\sigma$  and each point  $x \in \sigma$ :

- (1)  $\overline{H}(x)$  is transversal to  $\sigma$  and to any  $k$ -face  $\sigma' \subset \sigma$  ( $k \geq 1$ ) when  $x \in \sigma'$ .
- (2)  $\overline{L}(x)$  is transversal to  $\sigma$  and to any  $n$ -face  $\sigma' \subset \sigma$  up to parallel transport associated with the affine structure of  $\sigma$ .

Good position is preserved when  $(\overline{H}, \overline{L})$  is changed by a  $C^0$ -small deformation.

If  $\theta$  is fine enough it can be *jiggled* to  $\theta^{jig}$  (it is the word used by Thurston meaning a slight move of the vertices) so that  $\theta^{jig}$  is in good position with respect to  $(\overline{H}, \overline{L})$ . After jiggling,

$Z \times [0, 1]$  is no longer a subcomplex. The jiggling can be performed keeping  $Z \times \{1\}$  invariant since  $\bar{L}$  is already transversal to it. In contrary  $Z \times \{0\}$  is not preserved. The jiggling lemma is proved in [8] for one plane distribution. The proof can be easily adapted to a pair made of a distribution and a sub-distribution.

Let  $\bar{Z} := (Z \times [0, 1])^{jig}$  denote the product subcomplex from the jiggled triangulation, which is based on  $Z \times \{1\}$ . Its own triangulation is a product triangulation but its product structure is no longer compatible with the product  $E \times [0, 1]$ , due to the jiggling. The subcomplex  $\bar{Z}_0 := (Z \times \{0\})^{jig}$  is required to coincide with  $\bar{Z} \cap E \times \{0\}$ . We can think of  $\bar{Z}$  as a  $PL$ -section of  $E \times [0, 1]$  over  $M \times [0, 1]$ . But as a section it can be smoothed yielding a zero section for a homotopy of  $\Gamma_n$ -structure.

A simplex in  $\bar{Z}$  is said of first kind (resp. second kind) when it comes from a simplex of the same kind in  $Z \times [0, 1]$  through the jiggling.

**3.5. Security.** Let  $(\bar{H}, \bar{L})$  be defined on some open neighborhood  $U$  of  $\bar{Z}$ . If  $\bar{Z}$  is in good position with respect to  $(\bar{H}, \bar{L})$ , there exists  $\varepsilon > 0$  such that each move of the vertices of less than  $\varepsilon$ , extended by linearity to each simplex, keeps  $\bar{Z}$  inside of  $U$  and in good position with respect to  $(\bar{H}, \bar{L})$ . In particular the edges of an  $\varepsilon$ -displaced  $\bar{Z}$  remain transverse to  $\bar{H}$ . For use in the sequel it is convenient to make  $(\bar{H}, \bar{L})$  constant in each  $\varepsilon$ -ball about the vertices of  $\bar{Z}$ .

### 3.6. Integrability of $\bar{L}$ .

The trace of  $\bar{L}$  on each  $(n + 1)$ -simplex is 1-dimensional, hence integrable. Then, using a *thickening* of the triangulation (cf. [8] section 6), it is shown that after a  $C^0$ -small perturbation  $\bar{L}$  becomes integrable near  $\bar{Z}$ . The corresponding foliation is denoted by  $\bar{\mathcal{L}}$  and is defined on some neighborhood  $U$  of  $\bar{Z}$ . We observe that any change of the trace of  $\bar{L}$  on  $\bar{Z}$  whose support avoids the thickening of the  $n$ -skeleton extends as a change of  $\bar{\mathcal{L}}$  over the same  $U$ ; one just crosses the new trace by the fibers of the thickening. Nevertheless there is no reason that the modified  $\bar{\mathcal{L}}$  be transversal to the fibers of  $E \times [0, 1] \rightarrow M \times [0, 1]$ . It is the reason why proposition 2.4 is useful.

**3.7. Profile.** Let  $\sigma$  be a  $(n + 1)$ -simplex in  $\bar{Z}$ . The *profile* of  $\sigma$  is the quotient space of  $\sigma$  by the 1-foliation  $\bar{\mathcal{L}} \cap \sigma$  endowed with the image of the  $n$ -faces of  $\sigma$ . Let  $\Pi : \sigma \rightarrow \Pi(\sigma)$  denote the projection onto the quotient. Due to the definition of good position (3.4), the figure of the profile is very simple:  $\Pi(\sigma)$  is an  $n$ -ball with corners; its boundary is the image of the *apparent contour*  $C$  of  $\Pi$ , which is made of some  $(n - 1)$ -faces of  $\sigma$ . Indeed each leaf of  $\bar{\mathcal{L}} \cap \sigma$  being nowhere parallel to any  $n$ -face of  $\sigma$  makes at each point an angle with it which is positively bounded from below. Hence each leaf which enters  $\sigma$  at a point of some  $n$ -face must get out of  $\sigma$  through another face. The apparent contour has the following characterization: a leaf of  $\bar{\mathcal{L}} \cap \sigma$  which meets  $C$  consists of one point only.

The profile has  $n + 2$  vertices which all lie on  $\Pi(C)$  but at most one. Moreover  $C$  separates the boundary of  $\sigma$  in two  $n$ -discs  $\Sigma_+$  and  $\Sigma_-$ , and each 1-leaf from the foliation connects a point of  $\Sigma_+$  to a point of  $\Sigma_-$ . We have the following possibilities:

- a) In the case when  $C$  contains  $n + 1$  vertices, one hemisphere, let us say  $\Sigma_-$ , consists of exactly one  $n$ -face; then  $\Sigma_+$  contains the last vertex  $v$  in its interior and all  $n$ -faces from  $\Sigma_+$  have  $v$  as a common vertex.
- b) In the case when  $C$  contains  $n + 2$  vertices,  $\Sigma^\pm$  is made of all  $n$ -faces containing some  $k_\pm$ -face  $\tau_\pm$ , where  $k_+ + k_- = n$ ;  $\tau_+$  and  $\tau_-$  lie opposite each other in  $\sigma$ . Moreover the open face  $\text{int } \tau_\pm$  lies in the open hemisphere  $\text{int } \Sigma_\pm$ .

In case b), here is the combinatoric structure of the apparent contour:  $C$  is the join  $\partial\tau_+ * \partial\tau_-$ . In particular, except when  $\tau_+$  or  $\tau_-$  is 1-dimensional, each edge of  $\sigma$  is contained in the contour.

**3.8. Weak good position.** The triangulation  $\theta$  of  $Z \times [0, 1]$  is said in a *weak good position* with respect to  $(\overline{H}, \overline{L})$  if condition (2) in 3.4 is replaced with the following:

- (2') For every  $(n + 1)$ -simplex  $\sigma$ ,  $\overline{L}$  is transversal to  $\sigma$  and to its faces and, moreover, each 1-dimensional leaf of  $\overline{L} \cap \sigma$  traverses  $\sigma$  from boundary to boundary.

With that definition,  $\sigma$  has still a profile, with the same structure as above, though a leaf of  $\overline{L} \cap \sigma$  might be somewhere parallel to one face.

**3.9. An ordering.** Another consequence of good position is that, for any simplex  $\sigma$ , there is a total ordering on  $\sigma^{[0]}$  (the set of its vertices) up to reversing. Indeed if a co-orientation of  $\overline{H}$  is chosen along  $\sigma$ , its edges are oriented allowing us to write  $a < b$  when an edge starts from  $a$  and ends in  $b$ . This relation is transitive for the following reason: if  $a < b$ ,  $b < c$  and  $c < a$ , then the trace of  $\overline{H}$  on the triangle  $[a, b, c]$  must have a singularity contradicting the transversality of  $\overline{H}$  to the 2-simplex  $[a, b, c]$ . Thus the vertices are ordered as the number of edges ending there.

A vertex  $a \in \sigma$  is called a *min* (resp. *max*) vertex of  $\sigma$  if all edges of  $\sigma$  passing through  $a$  start from it (resp. point to it). Clearly each simplex contains one *min* and one *max* vertex; they are respectively denoted  $v_{min}(\sigma)$  and  $v_{max}(\sigma)$ . Moreover if  $\sigma$  is  $(n + 1)$ -dimensional both distinguished vertices lie on the apparent contour. When  $\overline{H}$  is integrable, as it is along  $\overline{Z}_0$ , good position implies that  $\overline{H}$  traces a foliation by  $(k - 1)$ -discs on each  $k$ -simplex (Reeb Stability Theorem [7], [3]).

**3.10. Embedding into an affine space.** In the sequel it will be useful to embed  $E$  into  $\mathbb{R}^N$ , the fibers of  $E$  being mapped onto affine subspaces, and think of  $\overline{H}$  (resp.  $\overline{L}$ ) as a codimension 1 (resp. codimension  $n$ ) distribution in  $\mathbb{R}^N \times [0, 1]$  which are both obtained by summing with the normal bundle to  $E$ . Let  $\pi : V \rightarrow Z \times [0, 1]$  be a tubular neighborhood of  $Z \times [0, 1]$  in  $\mathbb{R}^N \times [0, 1]$ , whose fibers  $V_z$  are open sets in affine subspaces; we may arrange  $V \cap E_z \subset V_z$ . We think of  $\overline{Z}$  as a Lipschitz graph, that is the image of a Lipschitz section  $s$  of the fibration  $\pi$ ;  $s$  is smooth on each simplex. If the product triangulation is fine enough, one can replace  $\overline{Z}$  by its *secant approximation* (same vertices and recilinear simplices); it is still a graph noted  $\overline{Z}$ . The new  $\overline{Z}_1 := \overline{Z} \cap \mathbb{R}^N \times \{1\}$  is no longer smooth. But, according to Whitehead [10],  $\overline{Z}$  has a  $C^0$ -approximation which is a smooth graph. For the arguments of genericity which will be used in the next section we take  $N$  very large, for instance larger than the total number of simplices in  $\overline{Z}$ .



## 4. PROOF OF THEOREM A

We start in the setting of 3.10. The codimension-one foliation  $\overline{\mathcal{H}}$  is given near  $\overline{Z}_0 := \overline{Z} \cap \mathbb{R}^N \times \{0\}$ . We will extend it along  $\overline{Z}$  so that it contains  $\overline{\mathcal{L}}$ . Hence, roughly speaking, the transversally foliated  $\Gamma_n$ -structure induced by  $(\overline{\mathcal{H}}, \overline{\mathcal{L}})$  on a smoothing of  $\overline{Z}_1$  will be regular and theorem A will be proved. We refer to 4.12 for a more precise explanation.

We emphasize the role played by the codimension- $n$  foliation: if we only have a codimension-one foliation transversal to each simplex of  $\overline{Z}$ , then it has no reason to be transversal to a smoothing of  $\overline{Z}_1$ . But transversality of  $\overline{\mathcal{L}}$  to a smoothing of  $\overline{Z}_1$  implies the same for  $\overline{\mathcal{H}}$ .

The construction of  $\overline{\mathcal{H}}$  will need in general to change  $\overline{\mathcal{L}}$  into a new codimension- $n$  foliation  $\overline{\mathcal{L}'}$  inside the  $(n+1)$ -simplexes of  $\overline{Z}$ . But we insist to keep  $\overline{\mathcal{L}} = \overline{\mathcal{L}'}$  near the  $n$ -th skeleton, in particular near  $\overline{Z}_1$ , so that the above conclusion remains true. As  $\overline{\mathcal{L}'}$  could be not transversal to the fibers of  $E \times [0, 1]$  we apply proposition 2.4 as it is said in 3.6.

We are going to construct  $\overline{\mathcal{H}}$  step by step along  $\overline{Z}$  by taking the simplices in a convenient order which is defined below. Actually we shall explain later that the simplices of  $\overline{Z}$  will be not themselves foliated but there is a *pleated* version of them that will be foliated.

**4.1. Second kind skeleta.** For  $1 \leq k \leq n+1$ ,  $\overline{Z}^{[k,2]}$  will denote the union of the 2nd kind  $k$ -simplices in  $\overline{Z}$ . We will successively foliate  $\overline{Z}^{[k,2]}$ ,  $k = 1, 2, \dots, n+1$ . When we speak of foliating a domain  $K$  we think of foliating a neighborhood of  $K$  with a codimension-one foliation, containing  $\overline{\mathcal{L}}$  when  $k \leq n$ . When we foliate  $\overline{Z}^{[k+1,2]}$  we do not change the foliation near  $\overline{Z}^{[k,2]} \cup \overline{Z}_0$ . Therefore if  $\sigma_1$  and  $\sigma_2$  are two  $k$ -simplices in  $Z$ , foliating  $(\sigma_1 \times [0, 1])^{jig}$  and  $(\sigma_2 \times [0, 1])^{jig}$  can be performed independently. On the other hand foliating  $(\sigma_1 \times [0, 1])^{jig}$  is performed step by step for each of its  $(k+1)$ -simplices in the obvious order: if we are going to foliate a new  $(k+1)$ -simplex from  $(\sigma_1 \times [0, 1])^{jig}$  its base is assumed to be already foliated.

At the beginning  $\overline{\mathcal{H}}$  is given along  $\overline{Z}_0 \cup \overline{Z}^{[0]}$ . We extend it to  $\overline{Z}^{[1,2]}$  and then to  $\overline{Z}^{[2,2]}$ . The distribution  $\overline{H}$  has an integrable trace on  $\overline{Z}^{[2,2]}$ . So, after a  $C^0$ -small deformation of  $\overline{H}$  keeping  $\overline{\mathcal{L}}$  fixed, it becomes integrable near  $\overline{Z}_0 \cup \overline{Z}^{[2,2]}$ .

**4.2. About foliating a 3-simplex, the main difficulty.** Let us now consider a 2nd kind 3-simplex  $\sigma = [a, a^*, b, c]$  whose base is  $[a, b, c]$  and free face is  $\tau := [a^*, b, c]$ . The base is assumed to be foliated. Let  $\partial'\sigma$  denote the union of its 2-faces except  $\tau$ ; it is foliated. We have  $a^* > a$  for some local co-orientation of  $\overline{H}$ . According to 3.9 there are two distinguished vertices  $v_{min}$  and  $v_{max}$  in  $\sigma$ ; certainly  $a \neq v_{max}$ . We have different cases according to  $v_{min} = a$  or not.

Case 1:  $v_{min} \neq a$ . The free face contains  $v_{min}$  and  $v_{max}$ . In that case there is an isotopy  $\rho_t : \sigma \rightarrow \sigma$  fixing a neighborhood of  $\partial'\sigma$  and pushing  $\tau$  into a foliated neighborhood of  $\partial'\sigma$  transversally to the foliation. Pulling  $\overline{\mathcal{H}}$  back by  $\rho_1$  yields the trace on  $\sigma$  of an extended  $\overline{\mathcal{H}}$  which is non-singular and transversal to the faces of  $\sigma$ . In order to foliate a neighborhood of  $\sigma$  we just have to insist that the enlarged foliation should contain  $\overline{\mathcal{L}}$ .

Case 2:  $v_{min} = a$ . In that case  $\overline{\mathcal{H}}$  traces on  $\partial'\sigma$  a foliation which is made of circles just above  $a$  because this region lies in a foliation chart, but which, higher, may look spiraling. In case



2 we assume that there are no spiraling leaves traced on  $\partial'\sigma$ . Let us say  $b$  is the min vertex of the free face  $\tau$  (the other cases work alike). The foliation traced on  $\partial'\sigma$  is made of circles crossing  $[a, b]$  and intervals whose end points lie in the frontier of  $\tau$ . Near  $a$ ,  $\sigma$  is foliated by 2-discs because we are inside a foliation chart. As the circle leaves on  $\partial\sigma$  are all isotopic, by an isotopy extension we get a foliation by 2-discs of some domain  $A \subset \sigma$  which contains all circle leaves of  $\partial'\sigma$ . Let  $\hat{A} \subset \sigma$  be the union  $A \cup \partial'\sigma$ . There is an isotopy  $(\rho_t)$  of  $\sigma$  into itself, fixing  $\hat{A}$  and pushing  $\tau$  into a foliated neighborhood of  $\hat{A}$  transversally to the foliation. Pulling back the foliation by  $\rho_1$  yields the wanted foliation of  $\sigma$ . The germ of  $\overline{\mathcal{H}}$  along  $\sigma$  is chosen so that it contains  $\overline{\mathcal{L}}$ .

Case 3: We now consider the case when spiraling leaves do exist on  $\partial'\sigma$ . In that case, the Reeb stability theorem prevents us from any extension of  $\overline{\mathcal{H}}$  over  $\sigma$  transversally to its faces. We then appeal the *pleating process* which will be defined below (4.3) and, instead of foliating  $\sigma$  we do it for the pleated simplex  $\sigma^{pl}$  associated to  $\sigma$ .

**4.3. Pleating and foliating a 2nd kind 3-simplex.** The following construction is very much in the spirit of the *zig-zag* construction which appears in the  $h$ -principle of Gromov [4], as it is revisited by Eliashberg and Mishachev [2].

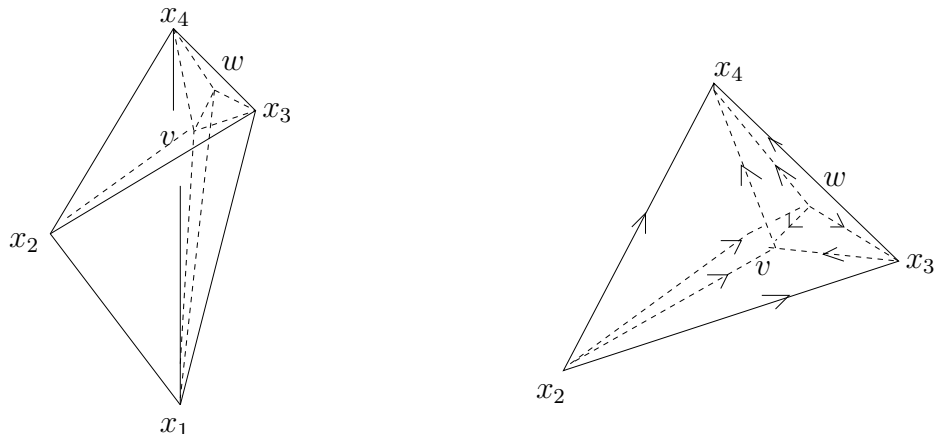
Let  $\sigma := [x_1, x_2, x_3, x_4]$  be a 3-simplex of  $\overline{\mathcal{Z}}$  and  $\tau = [x_2, x_3, x_4]$ . The numbering and the co-orientation of  $\overline{H}$  are so that the sequence of vertices is increasing:  $x_1 < x_2 < x_3 < x_4$ . Assume that all 2-faces through  $x_1$  are foliated; spiraling leaves may exist around  $x_1$  making an obstruction for foliating  $\sigma$ . We are going to cancel the obstruction by the *pleating process*. We first define the pleated simplex  $\tilde{\sigma}^{pl}$  which is isomorphic to a double stellar subdivision of  $\sigma$ . We add a pair of new vertices  $\{v, w\}$ ;  $v \in \tau$  defines the first stellar subdivision and  $w \in [v, x_3, x_4]$  defines the second one. Actually  $v$  and  $w$  are chosen in the  $\varepsilon$ -ball about  $x_3$  (see 3.5) and a small jiggling is performed on them meeting some generic conditions which are precised below. We define  $\tilde{\tau}^{pl}$  as  $\tau$  subdivided by  $\{v, w\}$  and jiggled, and we set  $\tilde{\sigma}^{pl} = x_1 * \tilde{\tau}^{pl}$  (Figure 2), where  $*$  stands for the affine join operation. Our pleated 3-simplex associated to  $\sigma$  will be the union

$$\sigma^{pl} := \tilde{\sigma}^{pl} \cup x_2 * ([x_3, x_4, w] \cup [v, x_3, w] \cup [v, x_4, w]).$$

This polyhedron is also isomorphic to another double stellar subdivision of  $\sigma$ : take  $v$  in the *interior* of  $[x_1, x_2, x_3, x_4]$  for the first stellar subdivision and  $w \in [v, x_3, x_4]$  for the second one. Observe that  $\sigma^{pl}$  is not starshaped at  $x_3$ . The boundary of  $\sigma^{pl}$  is unpleated as it coincides with  $\partial\sigma$ . The jiggling conditions are the following:

- 1)  $\sigma^{pl}$  is embedded in  $\mathbb{R}^N \times [0, 1]$ ;
- 2)  $\sigma^{pl}$  takes place in the open set  $U$  of  $\mathbb{R}^N \times [0, 1]$  where  $\overline{\mathcal{L}}$  and  $\overline{H}$  are defined; in particular the edges of  $\sigma^{pl}$  are oriented (up to reversing);
- 3)  $\sigma^{pl}$  is in good position with respect to  $(\overline{H}, \overline{\mathcal{L}})$ .

These conditions are generic for  $\{v, w\}$ . Notice that, when  $n = 3$ ,  $\sigma^{pl}$  might not embed into the quotient modulo  $\overline{\mathcal{L}}$ ; more precisely, if  $U(\sigma)$  denotes the  $\varepsilon$ -neighborhood of  $\sigma$  in  $\mathbb{R}^N \times [0, 1]$ , only each of its 3-simplices embeds into  $U(\sigma)/\overline{\mathcal{L}} \cap U(\sigma)$ . But, in any dimension, this quotient is convenient for proving the genericity of property 3).



( $\tilde{\sigma}^{pl}$  on the left and the closure of  $\sigma^{pl} \setminus \tilde{\sigma}^{pl}$  on the right)

Figure 2

For the co-orientation of  $\overline{H}$  making  $x_1 < x_2 < x_3 < x_4$  we require the following condition:

- 4)  $w < x_3 < v < x_4$ .

Here it is understood that the order of two points is induced by the co-orientation of  $\overline{H}$  on the edge joining them. Observe that by taking  $\{v_0, w_0\} \subset \sigma$  it is easy to fulfill  $x_3 < w_0 < v_0 < x_4$  and impossible to fulfill 4). Recall the tubular neighborhood  $\pi : V \rightarrow Z \times [0, 1]$ ;  $\overline{Z}$  is a Lipschitz graph, that is the image of a Lipschitz section of  $\pi$ . If we allow  $w$  to move in the  $(N - n)$ -dimensional fiber  $V_{\pi(w_0)}$  one can cross the affine hyperplane  $x_3 + \overline{H}(x_3)$  and get  $x_3 > w$ ; indeed both affine subspaces are mutually transverse. As we shall prove below, condition 4) can be fulfilled at the same time as:

- (P<sub>3</sub>)  $\sigma^{pl}$  is a Lipschitz graph over  $\pi(\sigma^{pl})$ .

*Proof.* Let  $\alpha(w_0) = w_0 * \beta$  be a simplex in  $\sigma$  subdivided by  $\{v_0, w_0\} \in \sigma$ ; we set  $\alpha(w) := w * \beta$ , where  $w$  runs inside of  $V_{\pi(w_0)}$ . Assume first that  $x_3$  is a vertex of  $\beta$ . Hence the angle  $A(\alpha(w), V_{\pi(w_0)})$  is bounded below by a constant depending on the angle  $A([x_3, w_0], \beta)$  only. Therefore, if the crystalline subdivision  $\theta$  is fine enough, the fibers  $V_z$  are almost parallel when  $z \in \pi(\sigma)$  and the angle  $A(\alpha(w), V_z)$  is positive for every  $z \in \pi(\sigma)$ . The same is true (and easier) when  $x_3 \notin \beta$ ; in that case  $\alpha(w)$  rotates very few as long as  $w_0$  is very close to  $x_3$ . In the same way one can check that the distribution of the  $V_z$ 's is transverse to  $\sigma^{pl}$  in the sense of Whitehead, making sure that  $\pi|_{\sigma^{pl}}$  is *bilipschitz* (or *regular Lipschitz*, as Whitehead says in [10] Section 2). In particular that projection is injective and  $\sigma^{pl}$  is a Lipschitz graph.  $\square$

Doing the same for all the 2nd kind 3-simplices we get a pleated copy of the skeleton  $\overline{Z}^{[3,2]}$  which is a Lipschitz graph in the tube  $V$ .

*About foliating.* We are now going to extend  $\overline{H}$ , first over  $\tilde{\sigma}^{pl}$  and then over  $\sigma^{pl}$ . We observe that we are free to foliate the 2-simplices through  $[x_1, v]$  and  $[x_1, w]$  as we want, as long as the orientation of the edges is preserved, because these 2-simplices are not contained in  $\overline{Z}^{[2,2]}$ . For instance we keep the trace of  $\overline{H}$  as a foliation of  $[x_1, v, x_3]$ ; we then choose the foliation of  $[x_1, x_2, v]$  so that there are no spiraling leaves on  $\partial[x_1, x_2, x_3, v]$ . Hence we can apply 4.2 case 2 and foliate  $[x_1, x_2, x_3, v]$ . Similarly we choose the foliation of  $[x_1, v, x_4]$  so that there are no spiraling leaves on  $\partial[x_1, x_2, v, x_4]$  and then we are able to foliate  $[x_1, x_2, v, x_4]$ .

Concerning foliating of  $[x_1, x_3, v, x_4]$ , stellarly subdivided by  $w$  and jiggled, we argue as follows. The possibility of foliating this polyhedron transversally to its faces does not depend on the position of  $w$  as long as the order of its vertices is preserved. For instance we can look at the case where  $w$  is very close to  $x_1$  and  $x_1 < w$ . In that case the considered polyhedron is foliated by the germ of  $\overline{\mathcal{H}}$  previously constructed. Observe that the tetrahedra  $[v, x_2, x_3, x_4]$  and  $[v, w, x_3, x_4]$  do not belong to  $\tilde{\sigma}^{pl}$  (nor  $\sigma^{pl}$ ) and remain unfoliated.

In order to extend  $\overline{\mathcal{H}}$  over  $\sigma^{pl}$ , we observe the following. The edge  $[x_2, w]$  is oriented by  $\overline{\mathcal{H}}$  like its neighbor  $[x_2, x_3]$ ; we foliate  $[x_2, v, w]$  in a way compatible with this orientation. We choose this foliation so that no spiraling leaves appear on  $\partial[x_2, x_3, v, w]$  whose triangles  $[v, w, x_3]$  and  $[v, x_2, x_3]$  are already foliated. That condition uniquely determines the holonomy map  $h : [v, w] \rightarrow [v, x_2[$  on  $[v, z]$ , where  $z$  belongs to the leaf going through  $x_3$  in  $[v, w, x_3]$ . So we can foliate  $[x_2, x_3, v, w]$  according to case 2 of 4.2 and we are still free to choose  $h$  on the interval  $[z, w]$ .

The mentioned freedom allows us to fulfill condition (\*) below which will be used later on:

(\*) The star of  $x_3$  in  $\partial[x_3, x_1, x_2, w]$  is foliated without spiraling leaves.

Indeed,  $h|[z, w]$  allows us to choose the holonomy  $k : [x_3, w] \rightarrow [x_3, x_2[$  in the foliated triangle  $[x_2, x_3, w]$  exactly as we want ( $h|[z, w]$  and  $k$  are in 1-1 correspondence).

The three faces of  $[x_2, v, w, x_4]$  through  $v$  are now foliated and the edge orientation allows us to foliate this 3-simplex according to case 1 of 4.2. Finally, in the 3-simplex  $[x_2, x_3, w, x_4]$  the faces through  $w$  are now foliated and, once more, the edge orientation allows us to foliate it together with its free face  $[x_2, x_3, x_4]$ . Once  $\sigma^{pl}$  is foliated, the germ of  $\overline{\mathcal{H}}$  along  $\sigma^{pl}$  is chosen so that it contains  $\overline{\mathcal{L}}$ ; such a germification can be easily performed since  $\overline{\mathcal{L}}$  is transversal to each simplex of  $\sigma^{pl}$ .

*Conclusion.* After this construction we keep in mind the following important facts.

- 1)  $\partial\sigma^{pl} = \partial\sigma$ ; in particular the free face is unpleated.
- 2) The foliation  $\overline{\mathcal{H}}$  is transversal to all faces in  $\sigma^{pl}$  and orientates the edges as  $\overline{\mathcal{H}}$  does.
- 3) Iterating the construction, we can pleat and foliate the skeleton  $\overline{\mathcal{Z}}^{[3,2]}$ . In particular the 2-skeleton of  $\overline{\mathcal{Z}}_1$  is foliated without any pleating, as it is made of free faces from 3-simplices in  $\overline{\mathcal{Z}}^{[3,2]}$ .

**4.4. Higher dimension.** For codimension-one foliating the  $k$ -simplices of  $\overline{\mathcal{Z}}$ ,  $k \geq 4$ , we need the following proposition whose proof mainly relies on the Reeb stability theorem.

**Proposition.**

1) Let  $\sigma = a * \tau$  be a  $k$ -simplex,  $k \geq 4$ . Assume that a germ of codimension-one foliation  $\mathcal{F}$  is given along the partial boundary  $\partial'\sigma = a * \partial\tau$  transversally to each  $\ell$ -face,  $1 \leq \ell \leq k-1$ . Then  $\mathcal{F}$  extends over  $\sigma$  transversally to  $\tau$ .

2) When  $k = 4$  and  $a = v_{\min}(\sigma)$  for some local co-orientation of  $\mathcal{F}$ , we have the following reinforced statement. Let  $\delta_1, \delta_2, \delta_3$  denote the three 2-faces of  $\tau$  through  $b := v_{\min}(\tau)$  and let  $\delta_0$  be its fourth face. Assume that  $\mathcal{F}$  is only given along  $a * (\delta_0 \cup \delta_1 \cup \delta_2)$  transversally to its faces. Then  $\mathcal{F}$  extends over  $a * \delta_3$  and thus over  $\sigma$  transversally to its faces.

**Proof.** Let us only consider the case  $k = 4$ . Each foliated 3-face  $\gamma$  is foliated by 2-discs. Indeed it is true near  $v_{\min}(\gamma)$  and Reeb stability theorem forces it globally.

Case 1: vertex  $a$  is not a distinguished vertex in  $\sigma$ . In that case  $\partial'\sigma$  has a foliated smoothing without critical points in its interior. Hence we can argue exactly as in case 1 from 4.2.

Case 2:  $a = v_{\min}(\sigma)$ . Each leaf of  $\mathcal{F} \cap (a * \delta_i)$ ,  $i = 0, 1, 2$ , is a 2-disc. Then each leaf of  $\mathcal{F}$  crossing  $]a, b[$  is a 2-disc whose boundary is a circle in  $a * \partial\delta_3$ . Therefore  $\mathcal{F} \cap \partial(a * \delta_3)$  has no spiraling leaves and  $\mathcal{F}$  extends over  $a * \delta_3$ . After this extension, all leaves of  $\mathcal{F}$  crossing  $]a, b[$  are 2-spheres. When they are close to  $a$  they bound 3-disc leaves in the germ of  $\mathcal{F} \cap \sigma$  near  $a$ . By an extension of isotopy, one extends  $\mathcal{F}$  to some domain  $A \subset \sigma$  foliated by 3-discs, such that each 2-sphere leaf in  $\mathcal{F} \cap \partial\sigma$  is the boundary of a 3-disc leaf in  $A$ . One easily checks that the leaves of  $\mathcal{F} \cap (\partial'\sigma \setminus A)$ , except the singular ones through  $b$  and  $v_{\max}(\sigma)$ , are 2-discs. Then one extends  $\mathcal{F}$  to  $\sigma$  by the already used pulling back process. Namely there is an isotopy  $\rho_t : \sigma \rightarrow \sigma$ ,  $t \in [0, 1]$ , which is stationary near  $\hat{A} := A \cup \partial'\sigma$  and pushes  $\tau$  into a foliated neighborhood of  $\hat{A}$  transversally to the codimension-one foliation. Point 2) is now proved.  $\square$

In the next corollary we are going to use the above-stated proposition for foliating a pleated 4-simplex obtained by slightly moving one vertex from a pleated 3-simplex:  $x_i \mapsto x_i^d$ ;  $x_i^d$  is called a *displaced vertex* from  $x_i$ . We use the notations of 4.3 and look at  $\sigma^{pl} = [x_1, x_2, x_3, x_4]^{pl}$ , edge oriented as on figure 2, and pleated by  $\{v, w\}$  near  $x_3$ . In next corollary we choose a generic  $x_3^d$ . Let  $\Sigma := x_3^d * \sigma^{pl}$  be the (embedded) affine join; it is a pleated 4-simplex whose 3-faces through  $x_3^d$  are not pleated (since the 2-faces of  $\partial\sigma^{pl}$  are not) and it is a small simplex as the edge  $[x_3, x_3^d]$  is short.

**Corollary 4.5.** *By taking  $x_3^d < w$  with respect to the (locally) co-oriented hyperplane field  $\overline{H}$ , then we can foliate the polyhedron  $\Sigma$  transversally to all its simplices.*

**Proof.** We choose foliations of the triangles through  $[x_3, x_3^d]$  in a way compatible with the edge orientation. One checks that 3-simplices of  $\Sigma$  through  $[x_3, x_3^d]$  are relevant to case 1 of 4.2, except  $\sigma_1 := [x_1, x_2, x_3, x_3^d]$ ,  $\sigma_2 := [x_1, w, x_3, x_3^d]$  and  $\sigma_3 := [x_2, w, x_3, x_3^d]$ . We choose the foliation of  $[x_1, x_3, x_3^d]$  arbitrarily and those of  $[x_2, x_3, x_3^d]$  and  $[w, x_3, x_3^d]$  so that no spiraling leaves appear on the boundaries of  $\sigma_1$  and  $\sigma_2$ . In  $\sigma_3$ , all faces through  $x_3$  are now foliated. But we recall condition (\*) from 4.3 and see that 2) of proposition 4.4 applies to  $\Delta := [x_1, x_2, w, x_3, x_3^d]$ . Indeed, condition (\*) allows one to foliate  $[x_1, x_2, w, x_3]$  (though it is not a simplex from  $\sigma^{pl}$ ),  $w$  is the max vertex of the face  $[x_1, x_2, x_3^d, w]$  which is opposite to  $x_3$  in  $\Delta$  and  $w$  belongs to  $\sigma_3$ . As a consequence there are no spiraling leaves on  $\partial\sigma_3$ . Hence  $\sigma_i$ ,  $i = 1, 2, 3$ , can be foliated.

If we look at the 4-simplices of  $\Sigma$  through  $[x_3, x_3^d]$ , all their 3-faces through  $x_3$  are now foliated. Hence, by applying proposition 4.4 1), they can be foliated.

One checks that the edge orientation allows us to extend the foliation to the 3-simplices of  $\Sigma$  through  $[w, x_3^d]$  (case 1 of 4.2)<sup>1</sup> and then to the 4-simplices through  $[w, x_3^d]$ . Finally, concerning the 3-simplices through  $[v, x_3^d]$  not previously considered, they are already foliated as free faces of foliated 4-simplices. Hence all 4-simplices through  $[v, x_3^d]$  can be foliated.  $\square$

**4.6. Pleating a 2nd kind 4-simplex.** We now explain how to pleat and foliate a 2nd kind 4-simplex  $\sigma = a * \tau$ , whose free face is  $\tau$ . The base face is assumed to be unpleated and foliated; the three 2nd kind 3-faces are pleated (when necessary) and foliated. The main property of this pleating and foliating process of  $\sigma$  is *to keep the free face  $\tau = [a^*, b, c, d]$  unpleated*. As a consequence the process can be repeated for foliating the skeleton  $\overline{\mathcal{Z}}^{[4,2]}$ , up to pleating. The pleating of  $\sigma$  is chosen generic with respect to  $\overline{\mathcal{L}}$  so that each simplex in the pleated simplex  $\sigma^{pl}$  will be transversal to  $\overline{\mathcal{L}}$ . Here are details for the construction of  $\sigma^{pl}$ ; we refer to the next subsection for foliating.

The boundary of  $\sigma$  has a pleated version, noted  $(\partial\sigma)^{pl}$ , which is a piecewise linear 3-sphere; it is made of the base and free face (unpleated) and the possibly pleated 2nd kind 3-faces. We will associate a pleated simplex  $\sigma^{pl}$  with  $(\partial\sigma)^{pl}$  as boundary. As a polyhedron  $\sigma^{pl}$  will consist of the union of a *displaced* 4-simplex  $\sigma^d$  and some small 4-simplices (small meaning that there is at least one edge in a ball of radius  $\varepsilon$ ). The vertices of  $\sigma^d$  which are not vertices of  $\sigma$  (they are really displaced) will be the interior pleating points of  $\sigma^{pl}$ .

Let  $x$  be a vertex of  $\tau$  and  $\tau_1, \tau_2, \tau_3$  be the other 3-faces of  $\sigma$  through  $x$ ; we suppose that at least one of them has been pleated near  $x$ . Let  $\tau_i^{pl}$  denote the pleated 3-simplex associated to  $\tau_i$  with the convention  $\tau_i^{pl} = \tau_i$  when no pleating is necessary for foliating it; say that  $\tau_i^{pl}$  is pleated near  $x$  for  $1 \leq i \leq \ell$  (check  $\ell \leq 2$ ). We choose some generic point  $x^d$  in the  $\varepsilon$ -ball about  $x$ . We define  $X_i := x^d * \tau_i^{pl}$  when  $1 \leq i \leq \ell$ , and  $X_i := x^d * \text{star}(x, \tau_i^{pl})$  when  $i \geq \ell + 1$ . Set also  $X := x^d * \tau$ . Generically the union of  $X$  and the  $X_i$ 's is an embedded 4-ball  $\Delta(x)$ . Its boundary consists of two hemispheres  $\Sigma_{\pm}$ , with  $\Sigma_- \subset S := (\partial\sigma)^{pl}$ . By replacing  $\Sigma_-$  with  $\Sigma_+$  we change  $S$  into a new pleated 3-sphere  $S'$  which passes through  $x^d$  instead of  $x$ . Moreover  $S'$  has no pleating points near  $x^d$ .

Suppose that  $y$  is another vertex of  $\partial\sigma$  having pleating points nearby; at most one such new vertex exists. Starting from  $S'$  we repeat the previous construction about  $y$  instead of  $x$ . It yields a third sphere  $S''$  with only five vertices and no pleating points. The convex hull of  $S''$  is the wanted *displaced* simplex  $\sigma^d$ . Vertex  $a$  belongs to  $\sigma^d$  (no pleating points near  $a$  in  $(\partial\sigma)^{pl}$ ); the opposite face  $\tau^d$ , called the free face, differs from  $\tau$  by the displacements  $x \mapsto x^d$  and  $y \mapsto y^d$ , which generate small 4-simplices  $X$  and  $Y$ . We recover  $\tau$  from  $\tau^d$  by applying the inverse displacements which are taken in the inverse order. Finally  $\sigma^{pl}$  is the union  $\sigma^d \cup \Delta(x) \cup \Delta(y)$ . The genericity of  $x^d, y^d$ , viewed in the quotient  $U(\sigma)/\overline{\mathcal{L}}$  (notations of 4.3), guaranties that  $\sigma^{pl}$  is embedded in  $U(\sigma)$  transversally to  $\overline{\mathcal{L}}$ ; only each of its simplices embeds in the quotient.

For foliating we want that  $x^d$  and  $y^d$  meet some inequalities; here the edge orientation is chosen so that  $a < a^*$ , where  $[a, a^*]$  is the 2nd kind edge of  $\sigma$ . Let  $\{v_i, w_i\}$  be the pair of pleating

<sup>1</sup>For instance  $[x_1, x_2, w, x_3^d]$  does not meet this property, but it is not a simplex of  $\Sigma$ .

points of  $\tau_i$  near  $x$ , (same notations as in 4.3). We ask for  $x^d < w_i$  for every  $1 \leq i \leq \ell$ , and similar inequalities for  $y$ . As for the pleated 3-simplices, we can fulfill these conditions together with the following:

(P<sub>4</sub>)  $\sigma^{pl}$  is a Lipschitz graph in the tube  $V$  over its  $\pi$ -projection into  $Z \times [0, 1]$ .

The proof of this statement is similar to the one for property (P<sub>3</sub>). We also get a global pleating of the skeleton  $\overline{Z}^{[4,2]}$  which is a Lipschitz graph in the tube  $V$ .

**4.7. Foliating a 2nd kind pleated 4-simplex.** We use notations of the preceding subsection. We also denote  $\Delta'(x)$  (resp.  $\Delta'(y)$ ) the closure of  $\Delta(x) \setminus X$  (resp.  $\Delta(y) \setminus Y$ ). We successively foliate  $\Delta'(x)$ ,  $\Delta'(y)$ ,  $\sigma^d$ ,  $Y$  and  $X$ . Suppose both first foliations are done. Then the closure of  $\partial\sigma^d \setminus \tau^d$  is foliated; proposition 4.4 1) applies for foliating  $\sigma^d$ . Now all 3-faces of  $Y$  but one are foliated and the same proposition applies; finally the same works for  $X$ . So we are reduced to foliate  $\Delta'(x)$  which is a cone whose top vertex is  $x^d$  and base, noted  $B(x)$ , is foliated.

All edges in  $\Delta'(x)$  ending at  $x^d$  are oriented; the long ones are oriented as their neighbors ending at  $x$  and the short ones are oriented by the inequalities  $x^d < w_i < x < v_i$  when  $1 \leq i \leq \ell$ . Foliating  $X_1$  directly follows from corollary 4.5. If  $\ell = 2$ , what we assume in the sequel, it is also done for  $X_2$ . But we have to check that the algorithm given in 4.5 allows one to foliate both simultaneously. It is true as it starts in foliating some triangles  $[z, x, x^d]$  submitted to some no-holonomy condition. If such a triangle lies in  $X_1 \cap X_2$ , the 3-simplex  $[a, z, x, x^d]$  lies also in that intersection and the no-holonomy condition is the same for foliating both parts.

We have now to extend  $\overline{\mathcal{H}}$  to  $X_3 = x^d * \text{star}(x, \tau_3^{pl})$  where  $\tau_3^{pl}$  is unpleated near  $x$ . Its common faces with  $X_1$  and  $X_2$  are already foliated; they are unpleated and contain  $a$ ; we denote them  $[x, x^d, a, x_1]$  and  $[x, x^d, a, x_2]$ . The *link*  $L$  of  $x$  in  $\tau_3^{pl}$  is  $[a, x_1, x_2]$  possibly pleated at  $\{v_3, w_3\}$ . Anyway  $L$  collapses onto  $L_0 := [x_1, a] \cup [a, x_2]$ . We have first a sequence of collapsings of  $L$  through 2-simplices onto a maximal tree  $L_1$  followed by  $L_1 \searrow L_0$ . One can use the inverse sequence for foliating. If  $[z_1, z_2]$  is an edge in  $L_1$  and  $[x, x^d, z_1]$  is foliated, foliating  $[x, x^d] * [z_1, z_2]$  requires to foliate  $[x, x^d, z_1]$  with some no-holonomy condition which can easily fulfilled. If  $\delta$  is a 2-simplex in  $L$  with two edges in  $L_1$  then the 4-simplex  $[x, x^d] * \delta$  meets the condition of proposition 4.4 which allows us to foliate it. Repeating this argument we can finish to foliate  $\Delta'(x)$ . When  $\ell = 1$  the argument is very similar.

Recalling that  $\overline{\mathcal{L}}$  is transversal to  $\sigma^{pl}$ , when  $n \geq 4$ ,  $\overline{\mathcal{L}} \cap \sigma^{pl}$  is a foliation by points. Hence, once  $\overline{\mathcal{H}} \cap \sigma^{pl}$  is constructed, it is easy to deduce a germ of  $\overline{\mathcal{H}}$  along  $\sigma^{pl}$  containing  $\overline{\mathcal{L}}$ . When  $n = 3$ , we refer to 4.10.

**4.8. Pleating and foliating the 2nd kind  $k$ -simplices,  $5 \leq k \leq n + 1$ .** One can essentially repeat the same process for pleating and codimension-one foliating the  $k$ -skeleton  $\overline{Z}^{[k,2]}$ , successively for  $k = 5, \dots, n + 1$ . We do it explicitly for  $k = 5$ . Actually there is very little change with respect to  $k = 4$ . Let  $\sigma = a * \tau$  be a 2nd kind 5-simplex with free face  $\tau$ . From 4.6 we have a pleated boundary  $S := (\partial\sigma)^{pl}$  made of the unpleated base and free face, and the possibly pleated 2nd kind 4-faces. Let  $x$  be a vertex of  $\tau$  and  $\tau_1, \dots, \tau_4$  be the four other 4-faces of  $\sigma$  through  $x$ . We suppose that one of them is pleated near  $x$ ; let  $\tau_1, \dots, \tau_\ell$  be the faces which are pleated near  $x$  and  $\tau_{\ell+1}, \dots, \tau_4$  the other ones; we have  $\ell \leq 3$ . For  $1 \leq i \leq \ell$ ,  $x_i$  denotes the



interior pleating point of  $\tau_i$  near  $x$ . We choose a generic point  $x^d$  in the  $\varepsilon$ -ball about  $x$ . We define  $X_i := x^d * star(x_i, \tau_i^{pl})$  when  $1 \leq i \leq \ell$ , and  $X_i := x^d * star(x, \tau_i^{pl})$  when  $\ell + 1 \leq i \leq 4$  (notice the difference with  $k = 4$ ); we also set  $X = x^d * \tau$ . Generically, the union of  $X$  and the  $X_i$ 's is an embedded 5-ball  $\Delta(x)$ . Its boundary has two hemispheres  $\Sigma_{\pm}$  with  $\Sigma_- \subset S$ . By replacing  $\Sigma_-$  with  $\Sigma_+$  we change  $S$  into  $S'$  which is a pleated 4-sphere passing through  $x^d$  instead of  $x$ ;  $S'$  has no pleating points near  $x^d$ . Starting from this new sphere we repeat the above construction with another vertex  $y$  of  $\sigma$  having pleating points nearby. By repeating this process we get a 4-sphere with six vertices and no pleating points. Its convex hull is the wanted *displaced* 5-simplex  $\sigma^d$ . Vertex  $a$  belongs to  $\sigma^d$  and the opposite face  $\tau^d$  is called its free face. It differs from  $\tau$  by the displacements  $x \mapsto x^d$ ,  $y \mapsto y^d, \dots$ . So we recover  $\tau$  from  $\tau^d$  by applying the inverse displacements which are taken in the inverse order. They generate small 5-simplices;  $X$  is the last of them. Finally  $\sigma^{pl}$  is the union of  $\sigma^d$  and all small 5-simplices in  $\Delta(x)$ ,  $\Delta(y), \dots$ . Generically it is a piecewise linear 5-ball which is embedded transversally to  $\overline{\mathcal{L}}$ . The interior pleating points are  $x^d, y^d, \dots$  (at most 3).

For foliating it is useful that the interior pleating points of  $\sigma^{pl}$  meet some inequalities. Those related to  $x^d$  are:  $x^d < x_i$  for  $1 \leq i \leq \ell$ ; in particular  $x^d$  meets the ordering condition of corollary 4.5 with respect to every 3-face of  $\sigma$  which is pleated near  $x$ . At the same time it is possible to fulfill condition (P<sub>5</sub>) similar to (P<sub>4</sub>), and recursively until (P<sub>(n+1)</sub>). The last one yields a global pleating of  $\overline{Z}^{[n+1,2]} = \overline{Z}$ , noted  $\overline{Z}^{pl}$ , which is a Lipschitz graph in the tube  $V$ .

**4.9. Codimension-one foliating the pleated 2nd kind 5-simplices .** As for  $k = 4$ , the only difficulty arises for foliating  $\Delta'(x)$ , the closure of  $\Delta(x) \setminus X$ . It is a cone with top vertex  $x^d$ ; its base is  $B(x) := \Delta'(x) \cap S$ . There is a slight difficulty with the fact that  $B(x)$  is not starshaped, due to the pleating of the 3-simplices with a pair of points. So we foliate in two steps: firstly  $x^d * star(x, B(x))$  and secondly the rest of the  $X_i$ 's,  $1 \leq i \leq \ell$ .

*First step.* Let  $L$  be the link of  $x$  in  $star(x, B(x))$ ; we have  $x^d * star(x, B(x)) = [x, x^d] * L$ . That link is a piecewise linear 3-disc since it is the link of  $x$  in  $S$  with one open simplex removed (opposite to  $x$  in  $\tau$ ). More precisely, if  $\tau = x * \gamma$ , we have  $L = (a * \partial\gamma)^{pl}$ . The effect of pleating can be deduced from the rule of pleating. For instance, with the notation of 4.3,  $link(x_3, [x_1, x_2, x_3, x_4]^{pl})$  is isomorphic as a polyhedron to  $[x_1, x_2, x_4]$  doubly stellar subdivided, firstly by  $w$  and secondly by  $v$  in the interior of  $[x_1, x_2, w]$ . The link of  $x$  in a pleated 4-simplex like  $\tau_i^{pl}$  is obtained from the preceding rule and a cone operation with  $x_i$  when  $1 \leq i \leq \ell$ ; for  $i > \ell$ , we have  $link(x, \tau_i^{pl}) = link(x, \tau_i^d)$  where  $\tau_i^d$  is the displaced associated 4-simplex (see 4.6). One can check the following facts:

- 1) There is a maximal tree  $L_1$  in  $L$  rooted at  $a$  with branches with one edge only (as in the unpleated case).
- 2) Each edge not in  $L_1$  belongs to a triangle whose other sides are in  $L_1$ . Let  $L_2$  denote the union of these triangles.

We are now ready for foliating  $[x, x^d] * L$ . We foliate the triangle  $[a, x, x^d]$  in a way compatible with the edge orientation. We now look at  $[a, z_1, z_2] * [x, x^d]$  where  $z_1$  and  $z_2$  belong to  $L_1$ . Are there spiraling leaves on  $\partial[z_1, z_2, x, x^d]$ ? Certainly not, except when  $z_1 < x$  and  $z_2 < x$ . But in that case proposition 4.4 applies and answers no. As a consequence we can foliate  $[x, x^d] * L_2$ ,



in particular  $[x, x^d] * L_1^{[1]}$  (where exponent [1] stands for the 1-skeleton). Foliating is continued by taking the simplices of  $L$ , firstly the 2-dimensional and secondly the 3-dimensional. If  $\alpha$  is one of them, proposition 4.4 applies for foliating  $[x, x^d] * \alpha$ .

*Second step.* We are concerned with extending to  $X_i$ ,  $1 \leq i \leq \ell$ , the foliation previously constructed on  $X_i \cap L * [x, x^d]$ . Let  $L_i$  be the link of  $x_i$  in  $B(x)$ ; it is a 3-sphere since  $x_i$  lies in the interior of the piecewise linear 4-ball  $B(x)$  and we have  $X_i = L_i * [x_i, x^d]$ . The already foliated part reads  $\lambda_i * [x_i, x^d]$  where  $\lambda_i = \text{star}(x, L_i)$ . We have to extend the foliation to  $\lambda'_i * [x_i, X^d]$  where  $\lambda'_i$  is the closure of  $L_i \setminus \lambda_i$ . One checks a property similar to the one in the previous step. Every edge whose  $a$  is not a vertex is a side in a triangle whose  $a$  is a vertex. It allows us to achieve foliating as before.

When  $k \geq 6$ , it goes recursively in the same way. As usual the germ of the extended  $\overline{\mathcal{H}}$  is required to contain the one of  $\overline{\mathcal{L}}$  along  $\sigma^{pl}$ ; it is easy to do when  $k \neq n + 1$ , since  $\sigma^{pl}$  is transversal to  $\overline{\mathcal{L}}$ ; for  $k = n + 1$ , we refer to proposition 4.10.

**4.10.** *A new codimension- $n$  foliation inside the  $(n + 1)$ -simplices.* For the  $(n + 1)$ -simplices there is something special to do because we want  $\overline{\mathcal{H}} \supset \overline{\mathcal{L}}$ . In general this condition is not fulfilled since  $\overline{\mathcal{L}}$  traces a 1-foliation on each of them. Let  $S$  be a 2nd kind  $(n + 1)$ -simplex of  $\overline{\mathcal{Z}}$  with free face  $T$ . We have associated a pleated  $(n + 1)$ -simplex  $S^{pl}$  with free face  $T$  which remains unpleated; it is made of a displaced  $(n + 1)$ -simplex  $S^d$  with free face  $T^d$  and small simplices. We succeeded in codimension-one foliating  $S^{pl}$  transversally to all its simplices (4.9). We will prove the following proposition.

**Proposition.** *There exists a codimension- $n$  foliation  $\overline{\mathcal{L}}'$  near  $S^{pl}$  meeting the following properties:*

- 1)  $\overline{\mathcal{L}}' \subset \overline{\mathcal{H}}$ .
- 2)  $\overline{\mathcal{L}}' = \overline{\mathcal{L}}$  near the  $n$ -skeleton of  $S^{pl}$ .

*Lemma.* 1) *Each  $(n + 1)$ -simplex  $\sigma$  in  $S^{pl}$  is equipped with a “free”  $n$ -face. When  $\sigma = S^d$ , this free face is  $T^d$ .*

2) *Simplex  $\sigma$  is in a weak good position with respect to  $(\overline{\mathcal{H}}, \overline{\mathcal{L}})$  in the sense of 3.8.*

*Proof of the lemma.* 1) There is a collapsing  $S^{pl} \searrow \partial' S^{pl}$  ( $=$  the closure of  $\partial S^{pl} \setminus T$ ). Each  $(n + 1)$ -simplex  $\sigma$  supports an elementary step of that collapsing. The free face of  $\sigma$  is defined as the free face for the elementary collapsing; it depends of course on the chosen sequence of elementary collapsings.

In the construction of  $S^{pl}$  we have an intermediate  $(n + 1)$ -polyhedron  $S^d \cup S_1$ , where  $S_1$  is the union of all small  $(n + 1)$ -simplices encountered during the construction of  $S^d$  (see 4.9). We may factor the total collapsing through

$$S^{pl} \searrow S^d \cup S_1 \cup \partial' S^{pl} \searrow S_1 \cup \partial' S^{pl} \searrow \partial' S^{pl}.$$

Taking such a sequence of collapsings makes  $S^d$  to have  $T^d$  as a free face of collapsing; therefore for that simplex both definitions of free face coincide.

2) We recall that  $U(S)$  is a neighborhood of  $S$ . Looking at the image in the quotient of  $U(S)$  modulo  $\overline{\mathcal{L}}$ , one sees that, generically on the interior pleating points of  $S^{pl}$ ,  $\overline{\mathcal{L}}$  is transversal to the faces of  $\sigma$ . We have now to check that the leaves of  $\overline{\mathcal{L}} \cap \sigma$  traverse  $\sigma$ . Consider  $A(S)$ , the affine space generated by  $S$  and, on the open set  $U(S)$ , consider a codimension- $(n+1)$  foliation  $\mathcal{K}$ , with  $\mathcal{K} \subset \overline{\mathcal{L}}$  and whose leaves cross  $A(S)$  in one point. So we have a projection  $q : U(S) \rightarrow A(S)$  mapping each leaf of  $\mathcal{K}$  to a crossing point. For  $\sigma$  in  $S^{pl}$ ,  $q$  embeds  $\sigma$  into  $U(S) \cap A(S)$  which is an  $\varepsilon$ -enlargement of  $S$  that each 1-leaf of  $\overline{\mathcal{L}} \cap A(S)$  traverses; indeed  $\sigma$  embeds into  $U(S)/\overline{\mathcal{L}}$  and the projection onto this quotient factors through  $q$ . Since  $\overline{\mathcal{L}}$  is transversal to the faces of  $\sigma$ , it is also transversal to those of  $q(\sigma)$ . Hence, due to these facts, each leaf of  $\overline{\mathcal{L}} \cap A(S)$  entering  $q(\sigma)$  must get out through another face. Therefore the same holds for  $\sigma$  itself.  $\square$

**4.11. Codimension- $n$  foliating continued.** Let  $\sigma$  be a  $(n+1)$ -simplex in a weak good position with respect to  $(\overline{H}, \overline{\mathcal{L}})$  and  $\tau$  be its free face in the above sense. Let  $a$  be the min vertex of  $\sigma$ , the edge orientation being chosen so that the max vertex belongs to  $\tau$ . Let us consider the most difficult case when  $a \notin \tau$ . We recall from 3.7 that  $a$  belongs to the apparent contour  $C$  of  $\sigma$ . The leaves of  $\overline{\mathcal{H}} \cap \sigma$  are  $n$ -discs. Each leaf of  $\overline{\mathcal{H}} \cap \sigma$  is 1-foliated by  $\overline{\mathcal{L}} \cap \sigma$  near  $C$  and it is entirely 1-foliated near  $a$ . Let  $A$  be the union of the  $n$ -disc leaves not touching the interior of  $\tau$ . Because they are all isotopic by an isotopy keeping  $C$  and the germ of  $\overline{\mathcal{L}}$  invariant (transversality of  $\overline{\mathcal{H}}$  to the subsimplices), each  $n$ -disc leaf in  $A$  can be continuously foliated by intervals; we arrange these 1-leaves to be contained in  $\overline{\mathcal{L}}$  near their end points and near  $C$ . That yields  $\overline{\mathcal{L}}' \cap A$ .

For going further and get rid a *pseudo-isotopy* problem<sup>2</sup> which could arise if we do it carelessly, we will make use of the free face and a convenient retraction as in case 2 of 4.2. But here it is a little more complicated because we have to take the profile into account. Let us recall its structure (3.7):  $\partial\sigma$  is made of the union of two hemispheres denoted by  $\Sigma_+ \cup \Sigma_-$ , with  $\tau \subset \Sigma_+$ ,  $\Sigma_+ \cap \Sigma_- = C$  and each hemisphere  $\Sigma_{\pm}$  is made of the  $n$ -faces sharing some simplex  $\tau_{\pm}$ . The vertex  $a$  belongs to  $\tau_-$ , as it lies in the contour but not in  $\tau_+$  ( $a \notin \tau$ ). Let  $\tau'$  be the closure of  $\Sigma_+ \setminus \tau$ . The  $n$ -faces in  $\tau'$  have the form  $\tau_+ * \gamma$ , where  $\gamma$  is a hyperface of  $\tau_-$  containing  $a$  (if  $a \notin \gamma$ , thus  $\gamma \subset \tau$  and  $\gamma * \tau^+ = \tau$  which is absurd). The  $(n-1)$ -faces of  $\tau' \cap \tau$  are exactly the  $(n-1)$ -faces which contain  $\tau_+$  and not  $a$ . The other  $(n-1)$ -faces in the boundary of  $\tau'$  lie in the contour. So, the boundary of  $\tau'$  is

$$\partial\tau' = (\tau' \cap \tau) \cup (\tau' \cap C).$$

Let  $\hat{A}$  be the union  $A \cup \Sigma_-$ . The codimension- $n$  foliation  $\overline{\mathcal{L}}' \subset \overline{\mathcal{H}}$  can be extended near  $\hat{A}$  in requiring  $\overline{\mathcal{L}}' = \overline{\mathcal{L}}$  along  $\hat{A} \setminus A$ .

*Claim.* There exists an isotopy  $\rho_t : \sigma \rightarrow \sigma$ ,  $t \in [0, 1]$ , meeting the following conditions:

- i) for each  $t \in [0, 1]$ ,  $\rho_t$  fixes pointwise a small neighborhood of  $\hat{A}$ ;
- ii)  $\rho_1$  maps  $\tau$  into a neighborhood of  $\hat{A}$  where the pair  $(\overline{\mathcal{H}}, \overline{\mathcal{L}}')$  is already defined and  $\rho_1(\tau)$  is transversal  $\overline{\mathcal{L}}'$ ;
- iii)  $\rho_t$  maps  $\tau'$  into itself preserving the germ of  $\overline{\mathcal{L}} = \overline{\mathcal{L}}'$  along it; so  $\rho_1(\tau')$  lies in a neighborhood of the contour  $C$ ;

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<sup>2</sup>This problem is more or less the problem of classification of the non-singular 1-dimensional foliations of the  $(n+1)$ -ball which have no minimal sets and which are “standard” near the boundary [1].

iv)  $\rho_1^*$  keeps  $\overline{\mathcal{H}} \cap \tau'$  invariant.

*Proof of the claim.* The existence of such an isotopy is easy, knowing that  $\partial'\sigma \setminus A$  is foliated by  $(n-1)$ -discs, which are transversal to  $C$  and whose boundaries lie in  $\partial\tau$ . Indeed, if  $R$  denotes such a leaf, one easily checks that  $R \cap \tau'$  collapses onto  $R \cap C$ .  $\square$

We finish to foliate  $\sigma$  using the isotopy  $\rho_t$ . Namely, pulling back  $(\overline{\mathcal{H}}, \overline{\mathcal{L}}')$  by  $\rho_1$  almost yields the wanted  $(\overline{\mathcal{H}}, \overline{\mathcal{L}}')$  in  $\sigma \setminus A$ ; we only have to make  $\overline{\mathcal{L}}'$  coinciding with  $\overline{\mathcal{L}}$  near  $\tau$ , which is easy since they are both transversal to  $\tau$ .

**4.12.** *End of the proof of Theorem A.* We use both following facts:

- 1) The free faces of the pleated 2nd kind simplices are never pleated. As a consequence the pleating process keeps  $\overline{Z}_0$  and  $\overline{Z}_1$  unchanged.
- 2) Both codimension- $n$  foliations  $\overline{\mathcal{L}}'$  and  $\overline{\mathcal{L}}$  coincide near the  $n$ -skeleton of the pleated polyhedron  $\overline{Z}^{pl}$  associated to  $\overline{Z}$ , in particular near  $\overline{Z}_0 \cup \overline{Z}_1$ .

We look at the tubular neighborhood  $\pi : V \rightarrow Z \times [0, 1]$  (see 3.10). As a consequence of conditions  $(P_3), \dots, (P_{n+1})$ ,  $\overline{Z}^{pl}$  is a graph in  $V$  (it crosses each fiber in one point). Along  $V$  we have two integrable distributions  $\zeta$  and  $\nu$ : for  $x \in V$ ,  $\zeta_x$  is parallel to  $E_{\pi(x)}$  and  $\nu_x$  is parallel to  $E_{\pi(x)}^\perp$ , the normal space to  $E_{\pi(x)}$  in  $\text{span}(V_{\pi(x)})$ . We may require  $\overline{\mathcal{L}}'$  to contain  $\nu$ . On the other hand, for  $i = 0, 1$ , we define  $(\overline{\mathcal{H}}_i, \overline{\mathcal{L}}_i)$  as the trace of  $(\overline{\mathcal{H}}, \overline{\mathcal{L}}')$  (or  $(\overline{\mathcal{H}}, \overline{\mathcal{L}})$  since it yields the same trace) on  $V_i := \pi^{-1}(Z \times \{i\})$ . Since  $\overline{\mathcal{L}}_1$  is transversal to  $Z \times \{1\}$ , if tube  $V$  is fine enough the leaves of  $\overline{\mathcal{L}}_1$  are discs, each one crossing  $\overline{Z}_1$  and  $Z \times \{1\}$  in exactly one point. As a consequence,  $\overline{\mathcal{H}}_1$ , which *a priori* is only defined near  $\overline{Z}_1$  and contains  $\overline{\mathcal{L}}_1$ , canonically extends to  $V_1$  by prolongating a leaf  $K$  by the union of the leaves of  $\overline{\mathcal{L}}_1 \cap V_1$  passing through  $K$ . The initial data makes that the same is true on  $V_0$ . Now it is possible to change  $\overline{Z}^{pl}$  into a new pleated manifold  $\widehat{Z}$ , transversal to the fibers of  $\pi$  in the sense of Whitehead ([10] section 1), bounded by  $Z \times \{0\}$  and  $Z \times \{1\}$  (take for instance a homotopy, for  $t \in [0, \eta]$  (resp.  $t \in [1 - \eta, 1]$ ) from the 0-section to  $\overline{Z}_0$  (resp.  $\overline{Z}_1$ )).

Let  $U$  be an open set in  $V$ , containing  $V_0$ ,  $V_1$  and  $\widehat{Z}$ , on which  $(\overline{\mathcal{H}}, \overline{\mathcal{L}}')$  is defined. According to theorem (1.9) of Whitehead in [10],  $\widehat{Z}$  can be smoothed and changed into the image  $\widehat{Z}^{sm}$  of a new section of  $\pi$  with values in  $U$  and vanishing on  $Z \times \{0, 1\}$ . There is a smooth little thickening  $\widehat{E}$  of  $\widehat{Z}^{sm}$  in the direction of  $\zeta$ . By construction  $\widehat{E}$  is transversal to distribution  $\nu$ . Since  $\nu \subset \overline{\mathcal{L}}' \subset \overline{\mathcal{H}}$ ,  $\overline{\mathcal{L}}'$  and  $\overline{\mathcal{H}}$  are transversal to  $\widehat{E}$ . Finally,  $(\widehat{E}, \widehat{Z}^{sm}, \overline{\mathcal{L}}' \cap \widehat{E}, \overline{\mathcal{H}} \cap \widehat{E})$  yields a homotopy from the initial data  $\xi = (E, Z, \mathcal{L}, \mathcal{H})$  to a regular  $\Gamma_n$ -structure transversally foliated. Theorem A is now proved.  $\square$

## 5. PROOF OF THEOREM B

**5.1.** Let us denote  $\Gamma_n^{fol}(M)$  the space of  $\Gamma_n$ -structures transversally codimension-one foliated and  $\Gamma_1(M)$  the space of  $\Gamma_1$ -structures. There is a *forgetful map*

$$\begin{aligned} U : \Gamma_n^{fol}(M) &\longrightarrow \Gamma_1(M) \\ (E, Z, \mathcal{L}, \mathcal{H}) &\longmapsto (E_1, Z, \mathcal{H}_1) \end{aligned}$$

defined as follows:  $E_1$  is the total space of the bundle  $\nu\mathcal{H}|Z$  which can be thought of as a sub-bundle of  $E$ ;  $\mathcal{H}_1$  is the transversal trace of  $\mathcal{H}$  on  $E_1$ .

**Proposition 5.2.** *Let  $\xi \in \Gamma_1(M)$ . If  $\nu\xi$  meets Thurston's condition to embed into  $\tau M$ , then  $\xi = U(\eta)$  for some  $\eta \in \Gamma_n^{fol}(M)$  with  $\nu\eta \cong \tau M$ .*

**5.3.** We can achieve the proof of Theorem B. According to Theorem A,  $\eta$  is homotopic to a regular structure  $\eta' = (TM, Z, \mathcal{L}', \mathcal{H}')$ . As  $\mathcal{L}'$  is transversal to  $Z$ ,  $\mathcal{H}'$  is alike. Applying  $U$  to the homotopy from  $\eta$  to  $\eta'$  we get a homotopy from  $\xi$  to  $\xi' := (E_1, Z, \mathcal{H}'_1) = U(\eta')$ . One checks easily that  $\mathcal{H}'_1$  is transversal to  $Z$ .  $\square$

**5.4. Proof of Proposition 5.2.** Let  $\xi = (E, Z, \mathcal{H}) \in \Gamma_1(M)$ . Here  $E$  is the total space of the 1-bundle  $\nu := \nu\xi$  which is assumed to be a sub-bundle of  $\tau M$ . Let  $\nu^\perp$  be a complementary sub-bundle. If  $p : E \rightarrow M$  denotes the projection,  $p^*(\nu^\perp)$  is a bundle  $q : \tilde{E} \rightarrow E$  whose total space is also the one of  $\nu \oplus \nu^\perp = \tau M$ , that is  $\tilde{E} \cong TM$ . The pulled-back foliation  $q^*\mathcal{H}$  is the wanted codimension-one foliation on  $TM$ . We have to discover the codimension- $n$  foliation. Let  $x \in E$  be a point close to  $Z$ . Let  $\mathcal{H}(x)$  denote the leaf of  $\mathcal{H}$  through  $x$ . The projection  $p$  is a local diffeomorphism from  $\mathcal{H}(x)$  onto an open set in  $M$ . It allows us to lift the line distribution  $\nu \subset \tau M$  as a line distribution tangent to  $\mathcal{H}(x)$  and which integrates as a 1-foliation  $\mathcal{I}_x$  of  $\mathcal{H}(x)$ . The projection also allows us to lift the usual exponential map  $\nu^\perp(p(x)) \rightarrow M$ , yielding

$$\exp_{\mathcal{H}} : p^*\nu^\perp(x) \rightarrow \mathcal{H}(x)$$

which is transversal to  $\mathcal{I}_x$  near the 0-section  $Z$ .

More globally we have  $\exp_{\mathcal{H}} : p^*\nu^\perp|\mathcal{H}(x) \rightarrow \mathcal{H}(x)$  which is transversal to  $\mathcal{I}_x$  near the 0-section  $Z$ . Hence  $\exp_{\mathcal{H}}^{-1}(\mathcal{I}_x)$  is a codimension- $(n-1)$  foliation of  $q^{-1}(\mathcal{H}(x))$  near the 0-section. Moreover it is transversal to the fibers of  $p^*\nu^\perp|\mathcal{H}(x)$ , due to the transversality property of  $\exp_{\mathcal{H}}$ . Doing the same simultaneously for every point  $x$ , we get a codimension- $n$  foliation with the wanted properties.  $\square$

**5.5. Final comment.** There is no relative version of Theorem A stating that, if we start with a structure  $\xi \in \Gamma_n^{fol}(M)$  which is regular along  $K \subset M$ ,  $\xi$  can be regularized relatively to  $K$ . In contrary of what Thurston states in [8], it is already wrong for usual  $\Gamma_n$ -structures. The homotopy formula from 3.1 has no relative version; more precisely there is an obstruction at the level of codimension- $n$  plane fields lying in  $\pi_k(\text{Gl}(n))$ ,  $k \leq n-1$ ; here we think of  $\text{Gl}(n)$  as the space of the  $n$ -planes in  $\mathbb{R}^n \times \mathbb{R}^n$  which are transversal to both factors.

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