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# THE FIRST EIGENVALUE OF DIRAC AND LAPLACE OPERATORS ON SURFACES

J.F. GROSJEAN ET E. HUMBERT

ABSTRACT. Let  $(M, g, \sigma)$  be a compact Riemannian surface equipped with a spin structure  $\sigma$ . For any metric  $\tilde{g}$  on  $M$ , we denote by  $\mu_1(\tilde{g})$  (resp.  $\lambda_1(\tilde{g})$ ) the first positive eigenvalue of the Laplacian (resp. the Dirac operator) with respect to the metric  $\tilde{g}$ . In this paper, we show that

$$\inf \frac{\lambda_1(\tilde{g})^2}{\mu_1(\tilde{g})} \leq \frac{1}{2}.$$

where the infimum is taken over the metrics  $\tilde{g}$  conformal to  $g$ . This answers a question asked by Agricola, Ammann and Friedrich in [AAF99].

<sup>1</sup> **MSC 2000:** 34L15, 53C27, 58J05.

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## 1. INTRODUCTION

Let  $(M, g, \sigma)$  be a compact Riemannian surface equipped with a spin structure  $\sigma$ . For any metric  $\tilde{g}$  on  $M$ , we denote by  $\Sigma_{\tilde{g}}M$  the spinor bundle associated to  $\tilde{g}$ . We let  $\Delta_{\tilde{g}}$  be the Laplace-Beltrami operator acting on smooth functions of  $M$  and  $D_{\tilde{g}}$  be the Dirac operator acting on smooth spinor fields with respect to the metric  $\tilde{g}$ . We also denote by  $\mu_1(\tilde{g})$  (resp.  $\lambda_1(\tilde{g})$ ) the smallest positive eigenvalue of  $\Delta_{\tilde{g}}$  (resp.  $D_{\tilde{g}}$ ). Agricola, Ammann and Friedrich asked the following question in [AAF99]:

*When  $M$  is a two dimensional torus, can we find on  $M$  a Riemannian metric  $\tilde{g}$  for which  $\lambda_1(\tilde{g})^2 < \mu_1(\tilde{g})$  ?*

The main goal of this article is to answer this question. We prove the

**Theorem 1.1.** *There exists a family of metrics  $(g_{\varepsilon})_{\varepsilon}$  conformal to  $g$  for which*

$$\limsup_{\varepsilon \rightarrow 0} \lambda_1(g_{\varepsilon})^2 \text{Vol}_{g_{\varepsilon}}(M) \leq 4\pi$$

$$\liminf_{\varepsilon \rightarrow 0} \mu_1(g_{\varepsilon}) \text{Vol}_{g_{\varepsilon}}(M) \geq 8\pi.$$

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Theorem 1.1 clearly answers the question of [AAF99] but says much more: first, the result is true on any compact Riemannian surface equipped with a spin structure and not only when  $M$  is a two-dimensional torus. In addition, the metric  $\tilde{g}$  can be chosen in a given conformal class. Finally, this metric  $\tilde{g}$  can be chosen such that  $(2 - \delta)\lambda_1(g)^2 < \mu_1(g)$  where  $\delta > 0$  is arbitrary small. More precisely Theorem 1.1 shows

**Corollary 1.2.** *On any compact Riemannian surface  $(M, g)$ , we have*

$$\inf \frac{\lambda_1(\bar{g})^2}{\mu_1(\bar{g})} \leq \frac{1}{2}$$

where the infimum is taken over the metric  $\bar{g}$  conformal to  $g$ .

Theorem 1.1 has other interesting consequences. Indeed, it proves

**Corollary 1.3.** *For any compact surface  $(M, g)$  equipped with a spin structure  $\sigma$ , we let*

$$\lambda_{\min}^+(M, g, \sigma) = \inf \lambda_1(\bar{g}) \text{Vol}_{\bar{g}}^{\frac{1}{2}}(M)$$

where the infimum is taken over the metrics  $\bar{g}$  conformal to  $g$ . Then, we have  $\lambda_{\min}^+(M, g, \sigma) \leq \lambda_{\min}^+(\mathbb{S}^2)$  where  $\lambda_{\min}^+(\mathbb{S}^2)$  is the same invariant computed on the standard sphere  $\mathbb{S}^2$ .

This corollary is an immediate consequence of the fact that  $\lambda_{\min}^+(\mathbb{S}^2) = 2\sqrt{\pi}$  (see [AHM03]). This result was announced in [AHM03]. The conformal invariant  $\lambda_{\min}^+$  has been studied in many papers (see for example [Hij86, Lot86, Bär92, Amm03, AHM03, AH06]). Indeed, it has many relations with Yamabe problem (see [LP87]). Corollary 1.3 has been proved in all dimensions by Ammann in [Amm03] if either  $n \geq 3$  or  $D$  is invertible. Corollary 1.3 extends the result to the remaining case:  $n = 2$  and  $\text{Ker}(D) \neq \{0\}$ . In [AHM03], an alternative proof of the case  $n \geq 3$  is given and the proof of the case  $n = 2$  is sketched.

In the same spirit, a consequence of Theorem 1.1 is

**Corollary 1.4.** *For any compact surface  $(M, g)$ , we let*

$$\mu_{\sup}(M, g) = \sup \mu_1(\bar{g}) \text{Vol}_{\bar{g}}^{\frac{1}{2}}(M)$$

where the infimum is taken over the metrics  $\bar{g}$  conformal to  $g$ . Then, we have  $\mu_{\sup}(M, g) \geq \mu_{\sup}(\mathbb{S}^2)$  where  $\mu_{\sup}(\mathbb{S}^2)$  is the same invariant computed on the standard sphere  $\mathbb{S}^2$ .

The invariant  $\mu_{\sup}$  has been studied in [CoES03] and Corollary 1.4 is a particular case of Theorem A in this paper. We obtain here another proof.

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## 2. GENERALIZED METRICS

Let  $f$  be a smooth positive function and set  $\bar{g} = f^2g$ . Let also for  $u \in C^\infty(M)$

$$I_{\bar{g}}(u) = \frac{\int_M |\nabla u|_{\bar{g}}^2 dv_{\bar{g}}}{\int_M u^2 dv_{\bar{g}}}.$$

It is well known that  $\mu_1(\bar{g}) = \inf I_{\bar{g}}(u)$  where the infimum is taken over the smooth non-zero functions  $u$  for which  $\int_M u dv_{\bar{g}} = 0$ . We now can write all these expressions in the metric  $g$ . We then see that for  $u \in C^\infty(M)$ , we have

$$I_{\bar{g}}(u) = \frac{\int_M |\nabla u|_g^2 dv_g}{\int_M u^2 f^2 dv_g}$$

and  $\mu_1(\bar{g}) = \inf I_{\bar{g}}(u)$  where the infimum is taken over the smooth non-zero functions  $u$  for which  $\int_M u f^2 dv_g = 0$ . Now if  $f$  is only of class  $C^{0,a}(M)$  for some  $a > 0$ , we can define  $\bar{g} = f^2g$ . The 2-form  $\bar{g}$  is not really a metric since  $f$  is not smooth. We then say that  $g$  is a *generalized metric*. We can

define the first eigenvalue  $\mu_1(\bar{g})$  of  $\Delta_{\bar{g}}$  using the definition above. Now, by standard methods, one sees that there exists a function  $u \in C^2(M)$  with  $\int_M u f^2 dv_g = 0$  and such that  $I_{\bar{g}}(u) = \mu_1(\bar{g})$ . Writing the Euler equation for  $u$ , we see that

$$\Delta_g u = \mu_1(\bar{g}) f^2 u. \quad (1)$$

We prove the following result

**Lemma 2.1.** *If  $(f_n)$  is a sequence of smooth positive functions which converges uniformly to  $f$ , then  $\mu_1(f_n^2 g)$  tends to  $\mu_1(\bar{g})$ .*

*Proof.* Let  $u_n$  be a eigenfunction function associated to  $\mu_1(f_n^2 g)$ . Without loss of generality, we can

assume that  $\int_M u_n^2 f_n^2 = 1$ . We set  $v_n = u_n - \frac{\int_M u_n f^2 dv_g}{\int_M f^2 dv_g}$ . We then have  $\int_M v_n f^2 dv_g = 0$  and hence

$$\mu_1(\bar{g}) \leq I_{\bar{g}}(v_n). \quad (2)$$

We have

$$\int_M |\nabla v_n|^2 dv_g = \int_M |\nabla u_n|^2 dv_g = \int_M u_n \Delta_g u_n dv_g.$$

By equation (1), we get that

$$\int_M |\nabla v_n|^2 dv_g = \mu_1(f_n^2 g) \int_M f_n^2 u_n^2 dv_g = \mu_1(f_n^2 g). \quad (3)$$

We also have

$$\int_M f^2 v_n^2 dv_g = \int_M f^2 u_n^2 - \frac{\left( \int_M u_n f^2 dv_g \right)^2}{\int_M f^2 dv_g}.$$

Now,

$$\left| \int_M u_n f^2 dv_g \right| = \left| \int_M u_n (f^2 - f_n^2) dv_g \right| \leq C \int_M |u_n| (f + f_n)^2 \|f - f_n\|_{\infty}.$$

Since the sequence  $(f_n)_n$  tends uniformly to  $f$  and since  $\int_M f_n^2 u_n^2 dv_g = 1$ , we get that  $\lim_n \int_M u_n f^2 dv_g = 0$ . In the same way,

$$\int_M f^2 u_n^2 dv_g = \int_M f_n^2 u_n^2 dv_g + o(1) = 1 + o(1).$$

Finally, we obtain

$$\int_M f^2 v_n^2 dv_g = 1 + o(1).$$

Together with (2) and (3), we obtain that  $\mu_1(\bar{g}) \leq \liminf_n \mu_1(f_n^2 g)$ . Now, let  $u$  be associated to  $\mu_1(\bar{g})$

and set  $v = u - \frac{\int_M u f_n^2 dv_g}{\int_M f_n^2 dv_g}$ . We have  $\int_M v^2 f_n^2 dv_g = 0$  and hence

$$\mu_1(f_n^2 g) \leq I_{f_n^2 g}(v).$$

It is easy to see that  $\lim_n I_{f_n^2 g}(v) = I_{\bar{g}}(u) = \mu_1(\bar{g})$ . We then obtain that  $\mu_1(\bar{g}) \geq \limsup_n \mu_1(f_n^2 g)$ . This proves Lemma 2.1.  $\square$

In the same way, if  $\bar{g} = f^2 g$  is a metric conformal to  $g$  where  $f$  is positive and smooth, we define

$$J'_{\bar{g}}(\psi) = \frac{\int_M |D_{\bar{g}}\psi|_{\bar{g}}^2 f^{-1} dv_{\bar{g}}}{\int_M \langle D_{\bar{g}}\psi, \psi \rangle_{\bar{g}} dv_{\bar{g}}}.$$

The first eigenvalue of the Dirac operator  $D_{\bar{g}}$  is then given by  $\lambda_1^+(\bar{g}) = \inf J'_{\bar{g}}(\psi)$  where the infimum is taken over the smooth spinor fields  $\psi$  for which  $\int_M \langle D_{\bar{g}}\psi, \psi \rangle_{\bar{g}} dv_{\bar{g}} > 0$ . Now, it is well known (see [Hit74, Hij01]) that we can identify isometrically on each fiber spinor fields for the metric  $g$  and spinor fields for the metric  $\bar{g}$ . Moreover, we have for all smooth spinor field:

$$D_{\bar{g}}(f^{-\frac{1}{2}}\psi) = f^{-\frac{3}{2}}D_g\psi.$$

This implies that if we set  $\varphi = f^{-\frac{1}{2}}\psi$ , we have

$$J_{\bar{g}}(\varphi) := \frac{\int_M |D_g\varphi|^2 f^{-1} dv_g}{\int_M \langle D_g\varphi, \varphi \rangle dv_g} = J'_g(\psi)$$

and the first eigenvalue of the Dirac operator  $D_{\bar{g}}$  is given by  $\lambda_1^+(\bar{g}) = \inf J_{\bar{g}}(\varphi)$  where the infimum is taken over the smooth spinor fields  $\varphi$  for which  $\int_M \langle D_g\varphi, \varphi \rangle dv_g > 0$ . With the definition above, we can extend the definition of  $\lambda_1(\bar{g})$  when  $\bar{g}$  is a generalised metric. By standard methods, there exists a spinor fields  $\varphi \in C^1(M)$  such that  $\lambda_1^+(\bar{g}) = J_{\bar{g}}(\varphi)$  and such that

$$D_g\varphi = \lambda_1^+(\bar{g})f\varphi. \quad (4)$$

We then have a result similar to Lemma 2.1:

**Lemma 2.2.** *If  $(f_n)$  is a sequence of smooth positive functions which converges uniformly to  $f$ , then  $\lambda_1(f_n^2 g)$  tends to  $\lambda_1(\bar{g})$ .*

The proof is similar to the one of Lemma 2.1 and we omit it here.

### 3. THE METRICS $(g_{\alpha,\varepsilon})_{\alpha,\varepsilon}$

In this paragraph, we construct the metrics  $(g_{\alpha,\varepsilon})_{\alpha,\varepsilon}$  which will satisfy:

$$\limsup_{\varepsilon \rightarrow 0} \lambda_1(g_{\alpha,\varepsilon})^2 \text{Vol}_{g_{\alpha,\varepsilon}}(M) \leq 4\pi + C(\alpha) \quad (5)$$

where  $C(\alpha)$  is a positive constant which goes to 0 with  $\alpha$  and

$$\liminf_{\varepsilon \rightarrow 0} \mu_1(g_{\alpha,\varepsilon}) \text{Vol}_{g_{\alpha,\varepsilon}}(M) \geq 8\pi. \quad (6)$$

Clearly this implies Theorem 1.1. By Lemmas 2.1 and 2.2, one can assume that the metrics  $(g_{\alpha,\varepsilon})_{\alpha,\varepsilon}$  are generalized metrics. We just have to define the volume of  $M$  for generalized metric by  $\text{Vol}_{f^2 g}(M) = \int_M f^2 dv_g$ . At first, without loss of generality, we can assume that  $g$  is flat near a point  $p \in M$ . Let  $\alpha > 0$  be a small number to be fixed later such that  $g$  is flat on  $B_p(\alpha)$ . We set for all  $x \in M$  and  $\varepsilon > 0$ ,

$$f_{\alpha,\varepsilon}(x) = \begin{cases} \frac{\varepsilon^2}{\varepsilon^2 + r^2} & \text{if } r \leq \alpha \\ \frac{\varepsilon^2}{\varepsilon^2 + \alpha^2} & \text{if } r > \alpha \end{cases}$$

where  $r = d_g(\cdot, p)$ . The function  $f_{\alpha,\varepsilon}$  is of class  $C^{0,a}$  for all  $a \in ]0, 1[$  and is positive on  $M$ . We then define for all  $\varepsilon > 0$ ,  $g_{\alpha,\varepsilon} = f_{\alpha,\varepsilon}^2 g$ . The 2-forms  $(g_{\alpha,\varepsilon})_{\alpha,\varepsilon p}$  will be the desired generalized metrics. For these metrics, we have

$$\text{Vol}_{g_{\alpha,\varepsilon}}(M) = \int_M f_{\alpha,\varepsilon}^2 dv_g = \int_{B_p(\alpha)} f_{\alpha,\varepsilon}^2 dv_g + \int_{M \setminus B_p(\alpha)} f_{\alpha,\varepsilon}^2 dv_g.$$

Since  $g$  is flat on  $B_p(\alpha)$ , we have

$$\int_{B_p(\alpha)} f_{\alpha,\varepsilon}^2 dv_g = \int_0^{2\pi} \int_0^\alpha \frac{\varepsilon^4 r}{(\varepsilon^2 + r^2)^2} dr d\Theta.$$

Setting  $y = \frac{x}{\varepsilon}$  we obtain:

$$\int_{B_p(\alpha)} f_{\alpha,\varepsilon}^2 dv_g = 2\pi\varepsilon^2 \int_0^{\frac{\alpha}{\varepsilon}} \frac{r}{(1+r^2)^2} dr = 2\pi\varepsilon^2 \left( \int_0^{+\infty} \frac{r}{(1+r^2)^2} dr + o(1) \right) = \pi\varepsilon^2 + o(\varepsilon^2).$$

Since  $f_{\alpha,\varepsilon}^2 \leq \frac{\varepsilon^4}{\alpha^4}$  on  $M \setminus B_p(\alpha)$ , we have  $\int_{M \setminus B_p(\alpha)} f_{\alpha,\varepsilon}^2 dv_g = o(\varepsilon^2)$ . We obtain

$$\text{Vol}_{g_\varepsilon}(M) = \pi\varepsilon^2 + o(\varepsilon^2). \quad (7)$$

In the whole paper, the notation “ $o(\cdot)$ ” must be understood as  $\varepsilon$  tends to 0.

#### 4. PROOF OF RELATION (5)

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined by  $f(x) = \frac{2}{1+|x|^2}$ . Let  $\psi_0$  be a non-zero parallel spinor field on  $\mathbb{R}^2$  such that  $|\psi_0|^2 = 1$ . As in [AHM03], we set on  $\mathbb{R}^2$

$$\psi(x) = f^{\frac{n}{2}}(x)(1-x) \cdot \psi_0.$$

As easily computed, we have on  $\mathbb{R}^2$

$$D\psi = f\psi \quad \text{and} \quad |\psi| = f^{\frac{1}{2}}. \quad (8)$$

Now, we fixe a small number  $\delta > 0$  such that  $g$  is flat on  $B_p(\delta)$ . Then, we take  $\varepsilon \leq \alpha \leq \delta$ . We will let  $\varepsilon$  goes to 0. We let also  $\eta$  be a smooth cut-off function defined on  $M$  such that  $0 \leq \eta \leq 1$ ,  $\eta(B_p(\delta)) = \{1\}$ ,  $\eta(M \setminus B_p(2\delta)) = \{0\}$ . Identifying  $B_p(\delta)$  in  $M$  with  $B_0(\delta)$  in  $\mathbb{R}^2$ , we can define a smooth spinor field on  $M$  by  $\psi_\varepsilon = \eta(x)\psi\left(\frac{x}{\varepsilon}\right)$ . Using (8), we have

$$D_g(\psi_\varepsilon) = \nabla\eta \cdot \psi\left(\frac{x}{\varepsilon}\right) + \frac{\eta}{\varepsilon} f\left(\frac{x}{\varepsilon}\right) \psi\left(\frac{x}{\varepsilon}\right). \quad (9)$$

Since  $\langle \nabla\eta \cdot \psi\left(\frac{x}{\varepsilon}\right), \psi\left(\frac{x}{\varepsilon}\right) \rangle \in i\mathbb{R}$  and since  $|D_g\psi_\varepsilon|^2 \in \mathbb{R}$ , we have

$$\int_M |D_g\psi_\varepsilon|^2 f_{\alpha,\varepsilon}^{-1} dv_g = I_1 + I_2 \quad (10)$$

where

$$I_1 = \int_M |\nabla\eta|^2 \left| \psi\left(\frac{x}{\varepsilon}\right) \right|^2 dx \quad \text{and} \quad I_2 = \int_M \frac{\eta^2}{\varepsilon^2} f^2\left(\frac{x}{\varepsilon}\right) \left| \psi\left(\frac{x}{\varepsilon}\right) \right|^2 f_{\alpha,\varepsilon}^{-1} dx.$$

At first, let us deal with  $I_1$ . By (8),

$$I_1 \leq C \int_M f\left(\frac{x}{\varepsilon}\right) f_{\alpha,\varepsilon}^{-1} dx = C \int_{B_p(\alpha)} f\left(\frac{x}{\varepsilon}\right) f_{\alpha,\varepsilon}^{-1} dx + C \int_{B_p(2\delta) \setminus B_p(\alpha)} f\left(\frac{x}{\varepsilon}\right) f_{\alpha,\varepsilon}^{-1} dx$$

where, as in the following,  $C$  denotes a constant independant of  $\alpha$  and  $\varepsilon$ . On  $B_p(\alpha)$ ,  $f\left(\frac{x}{\varepsilon}\right) f_{\alpha,\varepsilon}^{-1} = 2$ . Hence,

$$\int_{B_p(\alpha)} f\left(\frac{x}{\varepsilon}\right) f_{\alpha,\varepsilon}^{-1} dx \leq C\alpha^2.$$

On  $B_p(2\delta) \setminus B_p(\alpha)$ , since  $\varepsilon \leq \alpha$ ,

$$f\left(\frac{x}{\varepsilon}\right) f_{\alpha,\varepsilon}^{-1} \leq \frac{4\alpha^2}{\varepsilon^2 + r^2} = \frac{4\alpha^2}{\varepsilon^2(1 + \left(\frac{r}{\varepsilon}\right)^2)}$$

Hence,

$$\begin{aligned}
\int_{B_p(2\delta) \setminus B_p(\alpha)} f\left(\frac{x}{\varepsilon}\right) f_{\alpha,\varepsilon}^{-1} dx &\leq \frac{4\alpha^2}{\varepsilon^2} \int_0^{2\pi} \int_\alpha^\delta \frac{r}{\left(1 + \left(\frac{r}{\varepsilon}\right)^2\right)} dr d\Theta \\
&\leq 8\pi\alpha^2 \int_{\frac{\alpha}{\varepsilon}}^{\frac{\delta}{\varepsilon}} \frac{r}{(1+r^2)} dr \\
&\leq 8\pi\alpha^2 \ln\left(\frac{\varepsilon^2 + \delta^2}{\varepsilon^2 + \alpha^2}\right).
\end{aligned}$$

We get

$$\int_{B_p(2\delta) \setminus B_p(\alpha)} f\left(\frac{x}{\varepsilon}\right) f_{\alpha,\varepsilon}^{-1} dx \leq C\alpha^2 \ln\left(\frac{2\delta^2}{\alpha^2}\right).$$

Finally, we obtain

$$I_1 \leq C\alpha^2 + C \ln\left(\frac{2\delta^2}{\alpha^2}\right) = a(\alpha) \quad (11)$$

where  $a(\alpha)$  goes to 0 with  $\alpha$ . Now, by (8),

$$I_2 \leq C \int_{B_p(2\delta)} f^3\left(\frac{x}{\varepsilon}\right) f_{\alpha,\varepsilon}^{-1} dx.$$

Since  $f_{\alpha,\varepsilon} \geq \frac{1}{2}f\left(\frac{x}{\varepsilon}\right)$ , we have

$$I_2 \leq \frac{2}{\varepsilon^2} \int_{B_p(2\delta)} f^2\left(\frac{x}{\varepsilon}\right) dx.$$

Mimicking what we did to get (7), we obtain that

$$I_2 \leq 8\pi + o(1)$$

when  $\varepsilon$  tends to 0. Together with (10) and (11), we obtain

$$\int_M |D_g \psi_\varepsilon|^2 f_{\alpha,\varepsilon}^{-1} dv_g \leq 8\pi + a(\alpha) + o(1). \quad (12)$$

In the same way, by (9), since  $\int_M \langle D_g(\psi_\varepsilon), \psi_\varepsilon \rangle dv_g \in \mathbb{R}$  and since  $\langle \nabla \eta \cdot \psi\left(\frac{x}{\varepsilon}\right), \psi\left(\frac{x}{\varepsilon}\right) \rangle \in i\mathbb{R}$ , we have

$$\int_M \langle D_g(\psi_\varepsilon), \psi_\varepsilon \rangle dv_g = \int_M \frac{\eta^2}{\varepsilon} f\left(\frac{x}{\varepsilon}\right) \left| \psi\left(\frac{x}{\varepsilon}\right) \right|^2 dv_g.$$

By (8), this gives

$$\int_M \langle D_g(\psi_\varepsilon), \psi_\varepsilon \rangle dv_g = \int_M \frac{\eta^2}{\varepsilon} f^2\left(\frac{x}{\varepsilon}\right) dv_g.$$

With the computations made above, it follows that

$$\int_M \langle D_g(\psi_\varepsilon), \psi_\varepsilon \rangle dv_g = 4\pi\varepsilon + o(\varepsilon).$$

Together with (12) and (7), we obtain

$$\lambda_1(g_{\alpha,\psi})^2 \text{Vol}_{g_{\alpha,\psi}}(M) \leq (J_{g_{\alpha,\psi}}(\psi_\varepsilon))^2 \text{Vol}_{g_{\alpha,\psi}}(M) \leq \left( \frac{8\pi + a(\alpha) + o(1)}{4\pi\varepsilon + o(\varepsilon)} \right)^2 (\pi\varepsilon^2 + o(\varepsilon^2)) = \frac{1}{\varepsilon} (4\pi + a(\alpha) + o(1)).$$

Relation (5) immediatly follows.

## 5. PROOF OF RELATION (6)

First we need the following estimate

**Lemma 5.1.** *For any  $\varepsilon > 0$  and  $u \in C_c^\infty(B_p(\alpha))$ , then*

$$\int_M u^2 f_{\alpha,\varepsilon}^2 dv_g \leq \frac{\varepsilon^2}{8} \int_M |\nabla u|^2 dv_g + \frac{1}{\pi \varepsilon^2} \left( \int_M u f_{\alpha,\varepsilon}^2 dv_g \right)^2.$$

*Proof.* Let  $g_\varepsilon = f_{\alpha,\varepsilon}^2 g$ . Then  $(B_p(\alpha), g_\varepsilon)$  is embedded in a canonical sphere of volume  $\int_{\mathbb{R}^2} \left( \frac{\varepsilon^2}{\varepsilon^2 + r^2} \right)^2 dx = \pi \varepsilon^2$ . Then from the Poincaré-Sobolev inequality, we have

$$\int_M u^2 dv_{g_\varepsilon} \leq \frac{1}{\mu_{1,\varepsilon}} \int_M |\nabla^\varepsilon u|_{g_\varepsilon}^2 dv_{g_\varepsilon} + \frac{1}{V_\varepsilon} \left( \int_M u dv_{g_\varepsilon} \right)^2$$

where  $\mu_{1,\varepsilon} = \frac{8}{\varepsilon^2}$  is the first nonzero eigenvalue of the Laplacian on the sphere of volume  $V_\varepsilon = \pi \varepsilon^2$  and  $\nabla^\varepsilon u$  denotes the gradient of  $u$  with respect to the metric  $g_\varepsilon$ . Now since  $|\nabla^\varepsilon u|_{g_\varepsilon}^2 = f_{\alpha,\varepsilon}^{-2} |\nabla u|_g^2$  and  $dv_{g_\varepsilon} = f_{\alpha,\varepsilon}^2 dv_g$ , we get the desired result.  $\square$

**Lemma 5.2.** *For any  $u, v \in C^\infty(M)$ , we have*

$$\int_M (\Delta u) u v^2 dv_g = \int_M |\nabla(uv)|_g^2 dv_g - \int_M u^2 |\nabla v|_g^2 dv_g.$$

*Proof.* The proof is an elementary calculation.  $\square$

Because of the relation (7), the inequality (6) is equivalent to the following

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \mu_1(g_\varepsilon) \geq .8 \quad (13)$$

In order to prove this inequality, we assume that for any  $\varepsilon$  small enough, there exists  $k$ ,  $0 < k < 1$  so that

$$\mu_1(g_\varepsilon) < \frac{8}{\varepsilon^2} k. \quad (14)$$

Let  $u_\varepsilon$  be an eigenfunction associated to  $\mu_1(g_\varepsilon)$ . Then  $u_\varepsilon \in C^2(M)$  and  $\Delta_{g_\varepsilon} u_\varepsilon = \mu_1(g_\varepsilon) u_\varepsilon$  where  $\Delta_{g_\varepsilon}$  denotes the Laplacian associated to the metric  $g_\varepsilon$ . Since the dimension is 2,  $\Delta_{g_\varepsilon} = \frac{1}{f_{\alpha,\varepsilon}^2} \Delta$  and

$$\Delta u_\varepsilon = \mu_1(g_\varepsilon) f_{\alpha,\varepsilon}^2 u_\varepsilon. \quad (15)$$

We normalize  $u_\varepsilon$  so that  $\|u_\varepsilon\|_{H_1^2} = 1$ . Up to a subsequence we can assume that  $\int_M |\nabla u_\varepsilon|^2 dv_g \rightarrow l$  and  $\int_M u_\varepsilon^2 dv_g \rightarrow l'$  with  $l + l' = 1$ . Since  $(u_\varepsilon)$  is bounded in  $H_1^2$ , there exists a subsequence so that  $u_\varepsilon \rightarrow u$  weakly in  $H_1^2$ . In the following, all the convergences are up to subsequence. We sometimes omit to recall this fact.

**Lemma 5.3.** *There exists a constant  $c_0$  such that  $u = c_0$ .*



*Proof.* Let  $\varphi \in C^\infty(M)$  and

$$\eta_\rho := \begin{cases} 1 & \text{on } B_p(\rho) \\ 0 & \text{on } M \setminus B_p(2\rho) \end{cases}$$

satisfying  $0 \leq \eta_\rho \leq 1$  and  $|\nabla \eta_\rho| \leq \frac{1}{\rho}$ . We have

$$\int_M \langle \nabla u, \nabla \varphi \rangle = \int_M \langle \nabla u, \nabla (\eta_\rho \varphi) \rangle dv_g + \int_M \langle \nabla u, \nabla ((1 - \eta_\rho) \varphi) \rangle dv_g. \quad (16)$$

Now we have

$$\begin{aligned} \int_M \langle \nabla u, \nabla (\eta_\rho \varphi) \rangle dv_g &= \int_M \langle \nabla u, \nabla \eta_\rho \rangle \varphi dv_g + \int_M \langle \nabla u, \nabla \varphi \rangle \eta_\rho dv_g \\ &\leq C \left( \int_{B_p(2\rho)} |\nabla u|^2 dv_g \right)^{1/2} \left( \int_{B_p(2\rho)} |\nabla \eta_\rho|^2 dv_g \right)^{1/2} \\ &\quad + \left( \int_{B_p(2\rho)} |\nabla u|^2 dv_g \right)^{1/2} \left( \int_{B_p(2\rho)} |\nabla \varphi|^2 dv_g \right)^{1/2}. \end{aligned}$$

The limit of the last term is 0 when  $\rho \rightarrow 0$ . Moreover from the definition of  $\eta_\rho$  and from the fact that  $M$  is a 2-dimensional locally flat domain, the limit of  $\left( \int_{B_p(2\rho)} |\nabla \eta_\rho|^2 dv_g \right)^{1/2}$  is bounded in a neighborhood of 0. Then we deduce that

$$\int_M \langle \nabla u, \nabla (\eta_\rho \varphi) \rangle dv_g \rightarrow 0 \quad (17)$$

when  $\rho \rightarrow 0$ . On the other hand

$$\begin{aligned} \left| \int_M \langle \nabla u, \nabla ((1 - \eta_\rho) \varphi) \rangle dv_g \right| &= \lim_{\varepsilon \rightarrow 0} \left| \int_M \langle \nabla u_\varepsilon, \nabla ((1 - \eta_\rho) \varphi) \rangle dv_g \right| \\ &= \lim_{\varepsilon \rightarrow 0} \left| \int_M (\Delta u_\varepsilon) (1 - \eta_\rho) \varphi dv_g \right| \\ &= \lim_{\varepsilon \rightarrow 0} \left| \mu_1(g_\varepsilon) \int_M f_{\alpha, \varepsilon}^2 u_\varepsilon (1 - \eta_\rho) \varphi dv_g \right|. \end{aligned}$$

Now from the definition of  $f_{\alpha, \varepsilon}$  and from (14) we get

$$\left| \mu_1(g_\varepsilon) \int_M f_{\alpha, \varepsilon}^2 u_\varepsilon (1 - \eta_\rho) \varphi dv_g \right| \leq \frac{8}{\varepsilon^2} k C \varepsilon^4 \left( \int_M u_\varepsilon^2 dv_g \right)^{1/2} \left( \int_M (1 - \eta_\rho) \varphi^2 dv_g \right)^{1/2}$$

where  $C$  is a constant depending on the compact support of  $(1 - \eta_\rho) \varphi$ . Then making  $\varepsilon \rightarrow 0$ , we deduce that

$$\int_M \langle \nabla u, \nabla ((1 - \eta_\rho) \varphi) \rangle dv_g = 0.$$

Now, reporting this and (17) in (16) we obtain that  $\int_M \langle \nabla u, \nabla \varphi \rangle dv_g = 0$  and  $\Delta u = 0$  on  $M$  in the sense of distributions. This implies that  $u \equiv c_0$  on  $M$  for a constant  $c_0$ .

□

**Lemma 5.4.** *Let  $(c_\varepsilon)_\varepsilon$  be a bounded sequence of real numbers. Then*

$$\int_M f_{\alpha,\varepsilon}^2 u_\varepsilon^2 dv_g \leq O(\varepsilon^2 \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \varepsilon^4).$$

*Proof.* Let  $\eta$  be a  $C^\infty$  function defined on  $M$  so that

$$\eta := \begin{cases} 1 & \text{on } B_p(\alpha/2) \\ 0 & \text{on } M \setminus B_p(\alpha) \end{cases}$$

satisfying  $0 \leq \eta \leq 1$  and  $|\nabla \eta| \leq 1$ .

From the lemma 5.1, we have

$$\int_M (u_\varepsilon - c_\varepsilon)^2 f_{\alpha,\varepsilon}^2 \eta^2 dv_g \leq \frac{\varepsilon^2}{8} \int_M |\nabla((u_\varepsilon - c_\varepsilon)\eta)|^2 dv_g + \frac{1}{\pi \varepsilon^2} \left( \int_M (u_\varepsilon - c_\varepsilon) \eta f_{\alpha,\varepsilon}^2 dv_g \right)^2$$

and applying the lemma 5.2 to the first term of the right hand side, we get

$$\begin{aligned} \int_M (u_\varepsilon - c_\varepsilon)^2 f_{\alpha,\varepsilon}^2 \eta^2 dv_g &\leq \\ &\frac{\varepsilon^2}{8} \int_M (\Delta(u_\varepsilon - c_\varepsilon))(u_\varepsilon - c_\varepsilon) \eta^2 dv_g + \frac{\varepsilon^2}{8} \int_M (u_\varepsilon - c_\varepsilon)^2 |\nabla \eta|^2 dv_g + \frac{1}{\pi \varepsilon^2} \left( \int_M (u_\varepsilon - c_\varepsilon) \eta f_{\alpha,\varepsilon}^2 dv_g \right)^2. \end{aligned}$$

From (15) we deduce that

$$\begin{aligned} \int_M (u_\varepsilon - c_\varepsilon)^2 f_{\alpha,\varepsilon}^2 \eta^2 dv_g &\leq \\ &\frac{\varepsilon^2}{8} \mu_1(g_\varepsilon) \int_M u_\varepsilon (u_\varepsilon - c_\varepsilon) \eta^2 f_{\alpha,\varepsilon}^2 dv_g + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{1}{\pi \varepsilon^2} \left( \int_M (u_\varepsilon - c_\varepsilon) \eta f_{\alpha,\varepsilon}^2 dv_g \right)^2. \end{aligned}$$

*First case :* assume that  $\int_M u_\varepsilon (u_\varepsilon - c_\varepsilon) \eta^2 f_{\alpha,\varepsilon}^2 dv_g \geq 0$ .

The relation (14) implies

$$\int_M (u_\varepsilon - c_\varepsilon)^2 f_{\alpha,\varepsilon}^2 \eta^2 dv_g \leq k \int_M u_\varepsilon (u_\varepsilon - c_\varepsilon) \eta^2 f_{\alpha,\varepsilon}^2 dv_g + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{1}{\pi \varepsilon^2} \left( \int_M (u_\varepsilon - c_\varepsilon) \eta f_{\alpha,\varepsilon}^2 dv_g \right)^2.$$

A straightforward computation shows that

$$\begin{aligned} (1-k) \int_M u_\varepsilon^2 f_{\alpha,\varepsilon}^2 \eta^2 dv_g + c_\varepsilon^2 \int_M f_{\alpha,\varepsilon}^2 \eta^2 dv_g &\leq \\ (2-k)c_\varepsilon \int_M u_\varepsilon f_{\alpha,\varepsilon}^2 \eta^2 dv_g + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{1}{\pi \varepsilon^2} \left( \int_M (u_\varepsilon - c_\varepsilon) \eta f_{\alpha,\varepsilon}^2 dv_g \right)^2. \end{aligned} \quad (18)$$

Now note that

$$\begin{aligned}
\int_M u_\varepsilon f_{\alpha,\varepsilon}^2 \eta^2 dv_g &= \int_M u_\varepsilon f_{\alpha,\varepsilon}^2 (\eta^2 - 1) dv_g + \int_M u_\varepsilon f_{\alpha,\varepsilon}^2 dv_g \\
&= \int_M u_\varepsilon f_{\alpha,\varepsilon}^2 (\eta^2 - 1) dv_g + \frac{1}{\mu_1(g_\varepsilon)} \int_M \Delta u_\varepsilon dv_g \\
&= \int_M u_\varepsilon f_{\alpha,\varepsilon}^2 (\eta^2 - 1) dv_g \\
&\leq \int_{M \setminus B_p(\alpha/2)} u_\varepsilon f_{\alpha,\varepsilon}^2 (\eta^2 - 1) dv_g
\end{aligned}$$

and from the definition of  $f_{\alpha,\varepsilon}$  and  $\eta$  and from the fact that  $u_\varepsilon$  is bounded in  $L^2$ , we deduce that

$$\int_M u_\varepsilon f_{\alpha,\varepsilon}^2 \eta^2 dv_g = O(\varepsilon^4).$$

Since  $c_\varepsilon$  is bounded (18) becomes

$$\begin{aligned}
(1-k) \int_M u_\varepsilon^2 f_{\alpha,\varepsilon}^2 \eta^2 dv_g + c_\varepsilon^2 \int_M f_{\alpha,\varepsilon}^2 \eta^2 dv_g &\leq O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{1}{\pi \varepsilon^2} \left( \int_M (u_\varepsilon - c_\varepsilon) \eta f_{\alpha,\varepsilon}^2 dv_g \right)^2 \\
&= O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{1}{\pi \varepsilon^2} \left( \int_M f_{\alpha,\varepsilon}^2 u_\varepsilon (\eta - 1) dv_g + \int_M f_{\alpha,\varepsilon}^2 u_\varepsilon dv_g - c_\varepsilon \int_M f_{\alpha,\varepsilon}^2 \eta \right)^2 \\
&= O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{1}{\pi \varepsilon^2} \left( \int_M f_{\alpha,\varepsilon}^2 u_\varepsilon (\eta - 1) dv_g - c_\varepsilon \int_M f_{\alpha,\varepsilon}^2 \eta \right)^2 \tag{19}
\end{aligned}$$

where in the last equality we have used the fact that  $\int_M f_{\alpha,\varepsilon}^2 u_\varepsilon dv_g = \frac{1}{\mu_1(g_\varepsilon)} \int_M \Delta u_\varepsilon dv_g = 0$ .

Using the same arguments as above we see that  $\int_M f_{\alpha,\varepsilon}^2 u_\varepsilon (\eta - 1) dv_g = O(\varepsilon^4)$ . Reporting this in (19) we get

$$(1-k) \int_M u_\varepsilon^2 f_{\alpha,\varepsilon}^2 \eta^2 dv_g + c_\varepsilon^2 \int_M f_{\alpha,\varepsilon}^2 \eta^2 dv_g \leq O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{O(\varepsilon^4)}{\varepsilon^2} \int_M f_{\alpha,\varepsilon}^2 \eta dv_g + \frac{c_\varepsilon^2}{\pi \varepsilon^2} \left( \int_M f_{\alpha,\varepsilon}^2 \eta dv_g \right)^2.$$

Now

$$\begin{aligned}
\int_M f_{\alpha,\varepsilon}^2 \eta dv_g &= \int_{B_p(\alpha)} f_{\alpha,\varepsilon}^2 dv_g = \int_0^{2\pi} \int_0^\alpha \frac{\varepsilon^4 r}{(\varepsilon^2 + r^2)^2} dr d\Theta \\
&= 2\pi \varepsilon^2 \int_0^{\alpha/\varepsilon} \frac{t}{(1+t^2)^2} dt \\
&\leq 2\pi \varepsilon^2 \int_0^{+\infty} \frac{t}{(1+t^2)^2} dt \\
&= \pi \varepsilon^2.
\end{aligned}$$

This gives

$$\begin{aligned}
(1-k) \int_M u_\varepsilon^2 f_{\alpha,\varepsilon}^2 \eta^2 dv_g + c_\varepsilon^2 \int_M f_{\alpha,\varepsilon}^2 \eta^2 dv_g &\leq O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{c_\varepsilon^2}{\pi \varepsilon^2} \left( \int_M f_{\alpha,\varepsilon}^2 \eta dv_g \right)^2 \\
&= O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + c_\varepsilon^2 \int_M f_{\alpha,\varepsilon}^2 \eta dv_g.
\end{aligned}$$

Finally we have

$$\begin{aligned}
(1-k) \int_M u_\varepsilon^2 f_{\alpha,\varepsilon}^2 \eta^2 dv_g &\leq O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + c_\varepsilon^2 \int_M f_{\alpha,\varepsilon}^2 (\eta - \eta^2) dv_g \\
&\leq O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + c_\varepsilon^2 \int_{B_p(\alpha) \setminus B_p(\alpha/2)} f_{\alpha,\varepsilon}^2 dv_g \\
&= O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2.
\end{aligned} \tag{20}$$

*Second case :* Assume that  $\int_M u_\varepsilon (u_\varepsilon - c_\varepsilon) \eta^2 f_{\alpha,\varepsilon}^2 dv_g \leq 0$ .

In this case, we have

$$\begin{aligned}
\int_M u_\varepsilon^2 f_{\alpha,\varepsilon}^2 \eta^2 dv_g - 2c_\varepsilon \int_M u_\varepsilon f_{\alpha,\varepsilon}^2 \eta^2 dv_g + c_\varepsilon^2 \int_M f_{\alpha,\varepsilon}^2 \eta^2 dv_g &\leq \\
O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{1}{\pi \varepsilon^2} \left( \int_M (u_\varepsilon - c_\varepsilon) \eta f_{\alpha,\varepsilon}^2 dv_g \right)^2 &\tag{21}
\end{aligned}$$

and we conclude as in the previous case.

Then we have proved that

$$\int_M u_\varepsilon^2 f_{\alpha,\varepsilon}^2 \eta^2 dv_g = O(\varepsilon^4 + \varepsilon^2 \|u_\varepsilon - c_\varepsilon\|_{L^2}^2).$$

To finish the proof, we write

$$\int_M u_\varepsilon^2 f_{\alpha,\varepsilon}^2 dv_g = \int_M u_\varepsilon^2 f_{\alpha,\varepsilon}^2 \eta^2 dv_g + \int_M u_\varepsilon^2 f_{\alpha,\varepsilon}^2 (1 - \eta^2) dv_g$$

and the last term is  $O(\varepsilon^4)$  which completes the proof.  $\square$

*Proof. of Relation (13).* First we apply the lemma 5.4 to  $c_\varepsilon = c_0$  and we see that  $c_0 \neq 0$ . Indeed, let us compute the  $L^2$ -norm of the gradient of  $u_\varepsilon$ .

$$\begin{aligned}
\int_M |\nabla u_\varepsilon|^2 dv_g &= \int_M (\Delta u_\varepsilon) u_\varepsilon dv_g \leq \frac{8k}{\varepsilon^2} \int_M f_{\alpha,\varepsilon}^2 u_\varepsilon^2 dv_g \\
&= \frac{8k}{\varepsilon^2} O(\varepsilon^2 \|u_\varepsilon - c_0\|_{L^2}^2 + \varepsilon^4) \\
&= o(1).
\end{aligned}$$

Then we deduce that up to a subsequence

$$\int_M |\nabla u_\varepsilon|^2 dv_g \longrightarrow 0.$$

But we have chosen  $u_\varepsilon$  so that  $\|u_\varepsilon\|_{H_1^2} = 1$ . Then  $\|u_\varepsilon\|_{L^2} \longrightarrow 1$  and  $c_0 \neq 0$ .

Now let us consider  $\bar{u}_\varepsilon = \frac{1}{\text{Vol}(M)} \int_M u_\varepsilon dv_g$  and  $a_\varepsilon = \|u_\varepsilon - \bar{u}_\varepsilon\|_{H_1^2}$ . Then  $u_\varepsilon \longrightarrow c_0$  and  $a_\varepsilon \longrightarrow 0$ . It follows that the function  $v_\varepsilon = \frac{u_\varepsilon - \bar{u}_\varepsilon}{a_\varepsilon}$  satisfies  $\|v_\varepsilon\|_{H_1^2} = 1$  and there exists  $v \in H_1^2$  so that  $v_\varepsilon \longrightarrow v$  weakly in  $H_1^2$  and strongly in  $L^2$ .

To prove (13) we will consider two cases.

*First case :* Assume that up to a subsequence  $a_\varepsilon = O(\varepsilon)$ .

We have

$$\begin{aligned} \int_M (\Delta u_\varepsilon)^2 dv_g &= \mu_1(g_\varepsilon)^2 \int_M f_{\alpha,\varepsilon}^4 u_\varepsilon^2 dv_g \\ &\leq \mu_1(g_\varepsilon)^2 \int_M f_{\alpha,\varepsilon}^2 u_\varepsilon^2 dv_g \\ &\leq \frac{64k}{\varepsilon^4} O(\varepsilon^2 \|u_\varepsilon - \bar{u}_\varepsilon\|_{L^2}^2 + \varepsilon^4) \\ &\leq \frac{64k}{\varepsilon^4} O(\varepsilon^2 a_\varepsilon^2 + \varepsilon^4) \\ &\leq M. \end{aligned}$$

Then  $\|\Delta u_\varepsilon\|_{L^2}$ ,  $\|\nabla u_\varepsilon\|_{L^2}$  and  $\|u_\varepsilon\|_{L^2}$  are bounded. It well known that the norms

$$\|v\| := \|\Delta v\|_{L^2} + \|\nabla v\|_{L^2} + \|v\|_{L^2}$$

and  $\|v\|_{H_2^2}$  are equivalent (it is a direct consequence of Bochner formula). Hence, this implies that  $(u_\varepsilon)_\varepsilon$  is bounded in  $H_2^2$  which is embedded in  $C^0$ . Then  $u_\varepsilon \longrightarrow c_0$  uniformly up to a subsequence. Since  $c_0 \neq 0$  it follows that for  $\varepsilon$  small enough  $u_\varepsilon$  has a constant sign, which is not possible because  $u_\varepsilon$  is an eigenfunction in the metric  $g_\varepsilon$ .

*Second case :* Assume that  $\varepsilon = a_\varepsilon o(1)$ . In this case we have the

**Lemma 5.5.**  $v_\varepsilon \longrightarrow c_1$  in  $H_1^2$  where  $c_1$  is a constant.

*Proof.* The proof is similar to this of lemma 5.3. Indeed we consider  $\varphi \in C^\infty(M)$  and the function  $\eta_\rho$  defined in this previous proof. Then

$$\int_M \langle \nabla v, \nabla \varphi \rangle = \int_M \langle \nabla v, \nabla (\eta_\rho \varphi) \rangle dv_g + \int_M \langle \nabla v, \nabla ((1 - \eta_\rho) \varphi) \rangle dv_g.$$

By the same arguments we have  $\int_M \langle \nabla v, \nabla (\eta_\rho \varphi) \rangle dv_g \longrightarrow 0$  when  $\rho \longrightarrow 0$ . Moreover

$$\begin{aligned} \left| \int_M \langle \nabla v, \nabla ((1 - \eta_\rho) \varphi) \rangle dv_g \right| &= \lim_{\varepsilon \longrightarrow 0} \left| \int_M \langle \nabla v_\varepsilon, \nabla ((1 - \eta_\rho) \varphi) \rangle dv_g \right| \\ &= \lim_{\varepsilon \longrightarrow 0} \left| \int_M (\Delta v_\varepsilon) (1 - \eta_\rho) \varphi dv_g \right| \\ &= \lim_{\varepsilon \longrightarrow 0} \left| \frac{\mu_1(g_\varepsilon)}{a_\varepsilon} \int_M f_{\alpha,\varepsilon}^2 v_\varepsilon (1 - \eta_\rho) \varphi dv_g \right|. \end{aligned}$$

Now  $\left| \frac{\mu_1(g_\varepsilon)}{a_\varepsilon} \int_M f_{\alpha,\varepsilon}^2 v_\varepsilon (1 - \eta_\rho) \varphi dv_g \right| \leq \frac{8k}{a_\varepsilon \varepsilon^2} C \varepsilon^4 \|v_\varepsilon\|_{L^2} \left( \int_M (1 - \eta_\rho) \varphi^2 dv_g \right)^{1/2}$ . Since  $\varepsilon = a_\varepsilon o(1)$ , we deduce that  $\left| \int_M \langle \nabla v, \nabla ((1 - \eta_\rho) \varphi) \rangle dv_g \right| = 0$  and then  $\int_M \langle \nabla v, \nabla \varphi \rangle = 0$ . Therefore  $\Delta v = 0$  in sense of distributions and  $v = c_1$  on  $M$ .

□

Now let  $c_\varepsilon = \bar{u}_\varepsilon + a_\varepsilon c_1$ . Then  $c_\varepsilon \rightarrow c_0$ . We denote by  $\mu(g)$  the smallest positive eigenvalue of the Laplacian with respect to the metric  $g$ . From the definition of  $a_\varepsilon$  and the definition of  $\mu(g)$ , we have

$$\begin{aligned} a_\varepsilon^2 &\leq 2 \left( \int_M |\nabla u_\varepsilon|^2 dv_g + \int_M (u_\varepsilon - \bar{u}_\varepsilon)^2 dv_g \right) \leq 2 \left( 1 + \frac{1}{\mu(g)} \right) \int_M |\nabla u_\varepsilon|^2 dv_g \\ &= 2 \left( 1 + \frac{1}{\mu(g)} \right) \int_M \Delta u_\varepsilon u_\varepsilon dv_g \\ &= 2 \left( 1 + \frac{1}{\mu(g)} \right) \mu_1(g_\varepsilon) \int_M f_{\alpha,\varepsilon}^2 u_\varepsilon^2 dv_g. \end{aligned} \quad (22)$$

Applying lemma 5.4 we get

$$\begin{aligned} \int_M f_{\alpha,\varepsilon}^2 u_\varepsilon^2 dv_g &= O(\varepsilon^2 \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \varepsilon^4) \\ &= O(\varepsilon^2 \|u_\varepsilon - \bar{u}_\varepsilon - a_\varepsilon c_1\|_{L^2}^2 + \varepsilon^4) \\ &= O \left( a_\varepsilon^2 \varepsilon^2 \left\| \frac{u_\varepsilon - \bar{u}_\varepsilon}{a_\varepsilon} - c_1 \right\|_{L^2}^2 + \varepsilon^4 \right) \\ &= O(a_\varepsilon^2 \varepsilon^2 \|v_\varepsilon - c_1\|_{L^2}^2 + \varepsilon^4) \\ &= O(\varepsilon^4) + o(a_\varepsilon^2 \varepsilon^2). \end{aligned}$$

Now reporting this in (22) with the estimate (14) we find

$$\begin{aligned} a_\varepsilon^2 &\leq C \frac{8k}{\varepsilon^2} (O(\varepsilon^4) + o(a_\varepsilon^2 \varepsilon^2)) \\ &= O(\varepsilon^2) + a_\varepsilon^2 o(1). \end{aligned}$$

But  $\varepsilon = a_\varepsilon o(1)$ . Then  $a_\varepsilon^2 \leq C a_\varepsilon^2 o(1)$  and for  $\varepsilon$  small enough  $a_\varepsilon = 0$  and  $u_\varepsilon$  is a constant which is impossible.

□

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