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G rard Henry Edmond Duchamp, Allan I. Solomon, Pawel Blasiak, Karol A. Penson, Andrzej Horzela.
A multipurpose Hopf deformation of the Algebra of Feynman-like Diagrams. 2006. hal-00095588v1

HAL Id: hal-00095588

<https://hal.science/hal-00095588v1>

Preprint submitted on 17 Sep 2006 (v1), last revised 14 Oct 2006 (v2)

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A multipurpose Hopf deformation of the Algebra of Feynman-like Diagrams

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Abstract. We construct a three parameter deformation of the Hopf algebra **LDIAG**. This new algebra is a true Hopf deformation which reduces to **LDIAG** on one hand and to **MQSym** on the other, relating **LDIAG** to other Hopf algebras of interest in contemporary physics. Further, its product law reproduces that of the algebra of polyzeta functions.

1. Introduction

The complete journey between the first appearance of a product formula by Bender et al. [1] and their related Feynman-like diagrams to the discovery of a Hopf algebra structure [8] on the diagrams themselves, goes roughly as follows.

Firstly, Bender, Brody, and Meister [1] introduced a special field theory which proved to be particularly rich in combinatorial links and by-products [11] (not to mention the link with vector fields and one-parameter groups [7, 10]).

Secondly, the Feynman-like diagrams of this theory label monomials which combine naturally in a way compatible with monomial multiplication and co-addition. This is the Hopf algebra **DIAG** [8]. The (Hopf-)subalgebra of **DIAG** generated by the primitive graphs is the Hopf algebra **BELL** described in Solomon's talk at this conference [12].

Thirdly, the natural noncommutative pull-back of this algebra, **LDIAG**, has a basis (the *labelled* diagrams) which is in one-to-one correspondence with that of the *Matrix*

Quasi-Symmetric Functions [13] (the packed matrices of **MQSym**), but their algebra and co-algebra structures are completely different. In particular, multiplication in **MQSym** involves a sort of shifted shuffle with overlappings reminiscent of Hoffmann’s shuffle used in the theory of polyzeta functions [2]. The superpositions and overlappings involved there are not present in (non-deformed) **LDIAG** and, moreover, the coproduct of **LDIAG** is co-commutative while that of **MQSym** is not.

The aim of this paper is to announce the existence of a Hopf algebra deformation which connects **LDIAG** to other Hopf algebras relevant to physics (Connes-Kreimer, Connes-Moscovici, Brouder-Frabetti, see [6]) and other fields (noncommutative symmetric functions, Euler-Zagier sums).

Acknowledgement : The authors would like to thank Christophe Tollu who checked carefully the (lengthy) computation which proves the associativity of **LDIAG**(q_c, q_s).

2. Labelled Diagrams and Diagrams

Product formula involves a summation over all diagrams of a certain type [12] a labelled version of which is described below.

Labelled diagrams can be identified with their weight functions which are mappings $\omega : \mathbb{N}^+ \times \mathbb{N}^+ \rightarrow \mathbb{N}$ such that the supporting subgraph

$$\Gamma_\omega = \{(i, j) \in \mathbb{N}^+ \times \mathbb{N}^+ \mid w(i, j) \neq 0\} \tag{1}$$

has specific projections i.e. $pr_1(\Gamma_\omega) = [1..p]$; $pr_2(\Gamma_\omega) = [1..q]$ for some $p, q \in \mathbb{N}$ (notice that when one of p, q is zero so too is the other and the diagram is empty).

These graphs are represented by labelled diagrams as follows

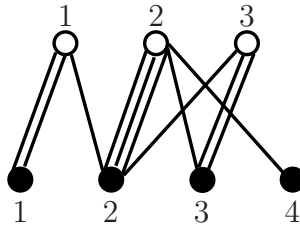


Fig 1. — Labelled diagram of format 3×4 corresponding to the weight function ω

pair $(i, j) \in \mathbb{N}^+ \times \mathbb{N}^+$	$(1, 1)$	$(1, 2)$	$(2, 2)$	$(2, 3)$	$(2, 4)$	$(3, 2)$	$(3, 3)$	otherwise
multiplicity $\omega(i, j)$	2	1	3	1	1	1	2	0

The labelled diagrams form the set **ldiag** and prescribe monomials through the formula $\mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)}$ where $\alpha(d)$ (resp. $\beta(d)$) is the “white spot type” (resp. the “black spot type”) i.e. the multi-index $(\alpha_i)_{i \in \mathbb{N}^+}$ (resp. $(\beta_i)_{i \in \mathbb{N}^+}$) such that α_i (resp. β_i) is the number of white spots (resp. black spots) of degree i (i lines connected to the spot).

There is a (graphically) natural multiplicative structure on **ldiag** such that the arrow

$$m_{(\mathbb{L}, \mathbb{V})} : d \mapsto \mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)} \tag{2}$$

is a morphism.

It is clear that one can permute black spots, or white spots, of d without changing

the monomial $\mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)}$. The classes of (labelled) diagrams up to this equivalence (permutations of white - or black - spots among themselves) are naturally represented by unlabelled diagrams and will be denoted **diag** (including the empty one).

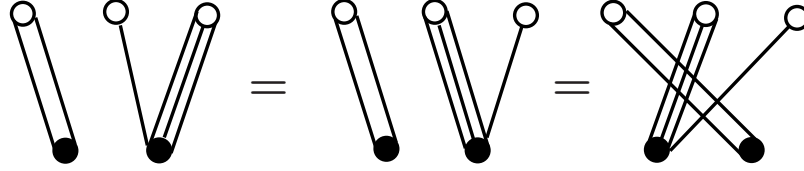


Fig 2. — *Equalities in **diag**. These diagrams, as drawn, are all different in **ldiag**.*

For both types of diagram the product consists of concatenating the diagrams i.e. placing d_2 on the right of d_1 [8] (the result, for d_1, d_2 , will be denoted $[d_1|d_2]_D$ in **diag** and $[d_1|d_2]_L$ in **ldiag**). These products endow **diag** and **ldiag** with the structure of monoids, with the empty diagram as neutral element. The corresponding commutative diagram is as follows (where X^2 means the cartesian square of the set X).

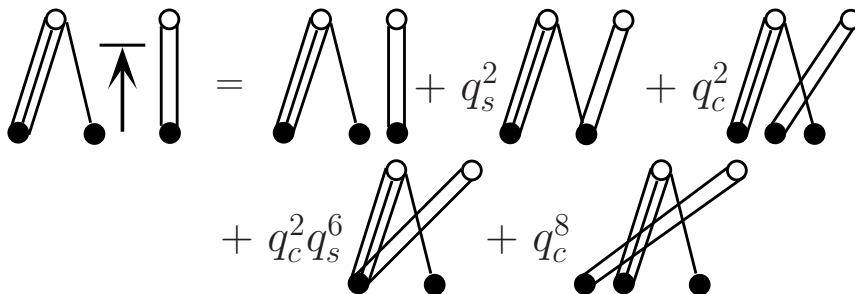
$$\begin{array}{ccccc}
 \text{Labelled diagrams}^2 & \xrightarrow{\text{Unlabelling}^2} & \text{Diagrams}^2 & \xrightarrow{m_{(L,V)} \times m_{(L,V)}} & \text{Monomials}^2 \\
 \text{product} \downarrow & & \text{product} \downarrow & & \text{product} \downarrow \\
 \text{Labelled diagrams} & \xrightarrow{\text{Unlabelling}} & \text{Diagrams} & \xrightarrow{m_{(L,V)}} & \text{Monomials}
 \end{array} \quad (3)$$

It is easy to see that the labelled diagram (resp. diagrams) form free monoids. We denote by **DIAG** and **LDIAG** the K -algebras of these monoids [8] (K is a field). One can shuffle the product in **ldiag**, counting crossing and superpositions. The definition of the deformed product is expressed by the descriptive formula and diagram below.

$$[d_1|d_2]_{L(q_c, q_s)} = \sum_{\substack{cs([d_1|d_2]_L) \text{ are all crossing and} \\ \text{superpositions of black spots}}} q_c^{nc \times \text{weight}_2} q_s^{\text{weight}_1 \times \text{weight}_2} cs([d_1|d_2]_L) \quad (4)$$

where

- q_c, q_s are coefficients in K
- the exponent of $q_c^{nc \times \text{weight}_2}$ is the number of crossings of “what crosses” times its weight
- the exponent of $q_s^{\text{weight}_1 \times \text{weight}_2}$ is the product of the weights of “what is overlapped”
- terms $cs([d_1|d_2]_L)$ are the diagrams obtained from $[d_1|d_2]_L$ by the process of crossing and superposing the black spots of d_2 on those of d_1 , the order and identity of the black spots of d_1 (resp. d_2) being preserved.



What is striking is that this law (denoted above $\bar{\uparrow}$) is associative. Moreover, it can be shown [3, 4] that this process decomposes into two transformations: twisting and shifting. In fact, specialized to certain parameters, this law is reminiscent of others [2].

Parameters	$(0, 0)$ (shifted)	$(1, 1)$ (shifted)	$(1, 1)$ (unshifted)
Laws	LDIAG	MQSym	Hoffmann & Euler-Zagier

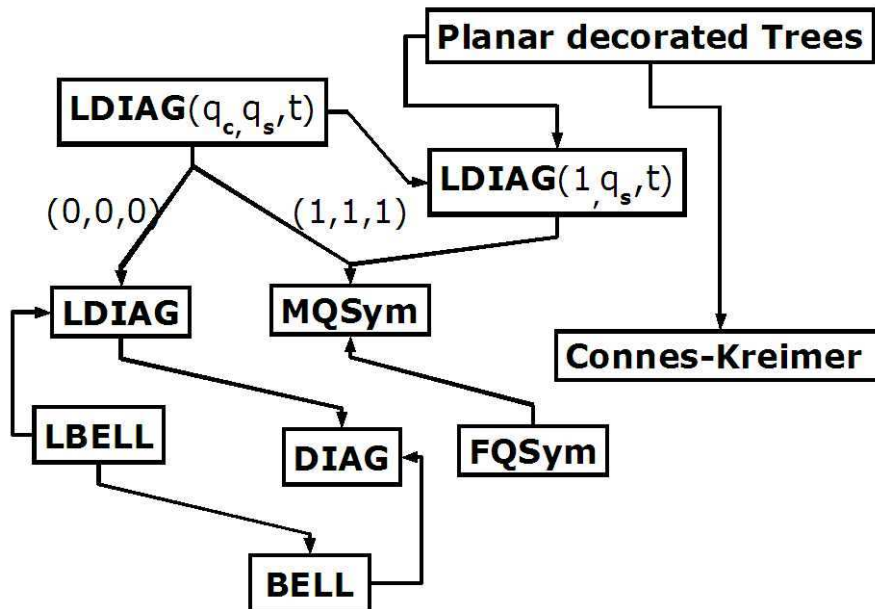
3. Hopf Deformation

Using a total order on the monomials of **ldiag**, it can be shown that the algebra **LDIAG** (q_c, q_s) is free. Thus one may construct a coproduct Δ_t such that **LDIAG** $(q_c, q_s, t) = (\mathbf{LDIAG}(q_c, q_s), 1_{\mathbf{ldiag}}, \Delta_t, \varepsilon, S)$ is a Hopf algebra. We have the following specializations

(q_c, q_s, t)	$(0, 0, 0)$	$(1, 1, 1)$
\simeq	LDIAG	MQSym

4. Conclusion

The results which we announced here in this note can be illustrated by the following picture. All details will be given in forthcoming papers [3, 4].



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