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Abdelhamid Boussejra, Khalid Koufany. Characterization of Poisson integrals on non-tube bounded symmetric domains. *Journal de Mathématiques Pures et Appliquées*, 2007, 87, pp.438-451. hal-00095481

**HAL Id: hal-00095481**

**<https://hal.science/hal-00095481>**

Submitted on 15 Sep 2006

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# CHARACTERIZATION OF POISSON INTEGRALS FOR NON-TUBE BOUNDED SYMMETRIC DOMAINS

ABDELHAMID BOUSSEJRA AND KHALID KOUFANY

ABSTRACT. We characterize the  $L^p$ -range,  $1 < p < +\infty$ , of the Poisson transform on the Shilov boundary for non-tube bounded symmetric domains. We prove that this range is a Hua-Hardy type space for harmonic functions satisfying a Hua system.

## 1. INTRODUCTION

Let  $\Omega = G/K$  be a Riemannian symmetric space of non-compact type. To each boundary  $G/P$  one can define a Poisson transform, which is an integral operator from hyperfunctions on  $G/P$  into the space of eigenfunctions on  $\Omega$  of the algebra  $\mathcal{D}(\Omega)^G$  of invariant differential operators. For the maximal boundary,  $G/P_{\min}$ , the most important result is the Helgason conjecture, proved by Kashiwara *et al.* [8] which states that a function is eigenfunction of all invariant differential operators on  $\Omega$  if and only if it is Poisson integral

$$\mathcal{P}_\lambda f(gK) = \int_K f(k) e^{-\langle \lambda + \rho, H(g^{-1}k) \rangle} dk.$$

of a hyperfunction on the maximal boundary, for a generic  $\lambda \in \mathfrak{a}_\mathbb{C}^*$ . For other function spaces such as  $L^p(G/P_{\min})$  the characterization is in connection with Fatou's theorems. We mention here the work of Helgason [5] and Michelson [14] for  $p = \infty$ , and Sjörgen [17] for  $1 \leq p < \infty$  using weak  $L^p$ -spaces. Another characterization for  $1 \leq p \leq \infty$ , using Hardy-type spaces, was done by Stoll [18] in the harmonic case and by Ben Saïd *et al.* [1] in the general case.

If  $\Omega$  is a bounded symmetric domain, one is interested in functions whose boundary values are supported on the Shilov boundary (minimal boundary)  $S := G/P_{\max}$  rather than the maximal boundary  $G/P_{\min}$ .

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1991 *Mathematics Subject Classification.* 43A85; 32A25; 32M15.

*Key words and phrases.* Bounded symmetric domains, Shilov boundary, Hua operator, Poisson transform, Fatou-type theorem.

For the Shilov boundary the Poisson transform is defined by

$$\mathcal{P}_s f(gK) = \int_K f(k) e^{-\langle s\rho_0 + \rho_1, H_1(g^{-1}k) \rangle} dk, \quad s \in \mathbb{C}.$$

In this case, Hua [6] had proved that the algebra of invariant differential operators is not necessarily the most appropriate for characterizing harmonic functions (i.e., annihilated by the algebra  $\mathcal{D}(\Omega)^G$ ). Johnson and Korányi [7], generalizing the earlier work of Hua, Korányi and Stein [10], and Korányi and Malliavin [11], introduced an invariant second order ( $\mathfrak{k}_{\mathbb{C}}$ -valued) operator  $\mathcal{H}$ , called since, second-order Hua operator (or Hua system). They showed, in the tube case, that a function is annihilated by the Hua operator if and only if it is the Poisson integral  $\mathcal{P}_{s_0} f$  ( $s_0 = n/r$ ) of a hyperfunction on the Shilov boundary. Thus, in the tube case, The Hua operator plays the same role with respect to the Shilov boundary as the algebra  $\mathcal{D}(\Omega)^G$  does with respect to the maximal boundary. In his paper [13], Lassalle showed the existence of a smaller system (a projection of the Hua operator) with the same properties.

Later Shimeno [16] generalize the result of Johnson and Korányi; namely he proved that a function is eigenfunction of  $\mathcal{H}$  if and only if it is a Poisson transform  $\mathcal{P}_s f$  of a hyperfunction on the Shilov boundary for generic  $s \in \mathbb{C}$ .

In [2], the first author gave a characterization of the Poisson transform  $\mathcal{P}_s$  on  $L^p(S)$ , which closes the tube type symmetric domains case characterization.

It thus arises the question of characterizing the range of the Poisson transform  $\mathcal{P}_s$  on  $L^p(S)$ ,  $1 < p < +\infty$ , for non-tube bounded symmetric domains on  $L^p(S)$ . The purpose of this paper is to answer this question.

For general bounded symmetric domains the Poisson integrals are not eigenfunctions of the second-order Hua operator  $\mathcal{H}$ , see for instance [3] or [12]. However for type  $\mathbf{I}_{r,r+b}$  domains of non-tube type, (see [3] and [12]) there is a variant of the second-order Hua operator,  $\mathcal{H}^{(1)}$ , by taking the first component of  $\mathcal{H}$ , since in this case  $\mathfrak{k}_{\mathbb{C}}$  is a sum of two irreducible ideals  $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}}^{(1)} \oplus \mathfrak{k}_{\mathbb{C}}^{(2)}$ . It is proved, in [12] (and in [3] for the harmonic case,  $s = (2r + b)/r$ ) that a smooth function  $f$  on  $\mathbf{I}_{r,r+b}$  is a solution of the Hua system,  $\mathcal{H}^{(1)} f = \frac{1}{4}(s^2 - (r + b)^2) f I_r$  if and only if it is the Poisson transform  $\mathcal{P}_s$  of a hyperfunction on the Shilov boundary.

For general non-tube domains, and for the harmonic case ( i.e., for  $s = n/r$  in our parametrization) the characterization of the image of the Poisson transform  $\mathcal{P}_{\frac{n}{r}}$  on hyperfunctions over the Shilov boundary was done by Berline and Vergne [3] where certain third-order Hua operator was introduced. Recently, the second author and Zhang [12] generalize the result of Berline and Vergne to any (generic)  $s$ . They introduce two third-order Hua operators  $\mathcal{U}$  and  $\mathcal{W}$  (different from the Berline and Vergne operator) and prove that an eigenfunction  $f$  of  $\mathcal{D}(\Omega)^G$  is a solution the Hua system (6) if and only if it is a Poisson transform of a hyperfunction on the Shilov boundary.

Let  $\mathcal{E}_s(\Omega)$  be the space of harmonic functions on  $\Omega$  that are solutions of the Hua system (for type  $\mathbf{I}_{r,r+b}$  domains, an eigenfunction of  $\mathcal{H}^{(1)}$  is indeed harmonic). Then the image  $\mathcal{P}_s(L^p(S))$  is a proper closed subspace of  $\mathcal{E}_s(\Omega)$ . For  $1 < p < +\infty$ , we introduce the Hua-Hardy type space,  $\mathcal{E}_s^p(\Omega)$  of functions  $f \in \mathcal{E}_s(\Omega)$  such that

$$\|f\|_{s,p} = \sup_{t>0} e^{-t(\Re(s)r-n)} \left( \int_K |f(ka_t)|^p dk \right)^{1/p} < +\infty.$$

Our main result (see Theorem 4.10) says that *if  $s \in \mathbb{C}$  is such that  $\Re(s) > \frac{a}{2}(r-1)$ , a smooth function  $F$  on  $\Omega$  is the Poisson transform  $F = \mathcal{P}_s f$  of a function  $f \in L^p(S)$  if and only if  $f \in \mathcal{E}_s^p(\Omega)$* . Our method of proving this characterization uses an  $L^2$  version of this theorem (see Theorem 4.8) and an inversion formula for the Poisson transform (see Proposition 4.9) which needs Fatou-type theorems (see Theorem 4.3 and Theorem 4.5).

## 2. PRELIMINARIES

Let  $\Omega$  be an irreducible bounded symmetric domain in a complex  $n$ -dimensional space  $V$ . Let  $G$  be the identity component of the group of biholomorphic automorphisms of  $\Omega$ , and  $K$  be the isotropy subgroup of  $G$  at the point  $0 \in \Omega$ . Then  $K$  is a maximal compact subgroup of  $G$  and as a Hermitian symmetric space,  $\Omega = G/K$ . Let  $\mathfrak{g}$  be the Lie algebra of  $G$ , and

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be its Cartan decomposition. The Lie algebra  $\mathfrak{k}$  of  $K$  has one dimensional center  $\mathfrak{z}$ . Then there exists an element  $Z_0 \in \mathfrak{z}$  such that  $\text{ad} Z_0$  defines the complex structure of  $\mathfrak{p}$ . Let

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-$$

be the corresponding eigenspaces decomposition of  $\mathfrak{g}_{\mathbb{C}}$ , the complexification of  $\mathfrak{g}$ . Let  $G_{\mathbb{C}}$  be a connected Lie group with Lie algebra  $\mathfrak{g}_{\mathbb{C}}$  and  $P^+$ ,  $K_{\mathbb{C}}$ ,  $P^-$  be the analytic subgroups of  $G_{\mathbb{C}}$  corresponding to  $\mathfrak{p}^+$ ,  $\mathfrak{k}_{\mathbb{C}}$ ,  $\mathfrak{p}^-$ . Denote by  $\sigma$  the conjugaison of  $G_{\mathbb{C}}$  with respect to  $G$ . Then we have,  $\sigma(P^{\pm}) \subset P^{\mp}$  and  $\sigma(K_{\mathbb{C}}) \subset K_{\mathbb{C}}$ .

Let  $\mathfrak{h}$  be a maximal Abelian subalgebra of  $\mathfrak{k}$ , and let  $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  be the corresponding set of roots. As  $Z_0$  belongs to  $\mathfrak{h}$ , the space  $\mathfrak{p}^+$  is stable by  $\text{ad}\mathfrak{h}$ . The roots  $\gamma \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  such that  $\mathfrak{g}^{\gamma} \subset \mathfrak{p}^+$  are said to be positive non-compact, and we denote by  $\Phi$  the set of such roots. Let  $\gamma \in \Phi$ , then one may choose elements  $H_{\gamma} \in i\mathfrak{h}$ ,  $E_{\gamma} \in \mathfrak{g}^{\gamma}$ ,  $E_{-\gamma} \in \mathfrak{g}^{-\gamma}$  such that  $[E_{\gamma}, E_{-\gamma}] = H_{\gamma}$  and  $\sigma(E_{\gamma}) = -E_{-\gamma}$ . Let  $X_{\gamma} = E_{\gamma} + E_{-\gamma}$  and  $Y_{\gamma} = i(E_{\gamma} - E_{-\gamma})$ . Then, by a classical Harish-Chandra construction, there exists a maximal set  $\Gamma = \{\gamma_1, \dots, \gamma_r\}$  of strongly orthogonal roots in  $\Phi$ . For simplicity, let us set for,  $1 \leq j \leq r$ ,

$$E_j = E_{\gamma_j}, \quad X_j = X_{\gamma_j}, \quad Y_j = Y_{\gamma_j}.$$

Then,

$$\mathfrak{a} = \sum_{j=1}^r \mathbb{R}X_j,$$

is a Cartan subspace of the pair  $(\mathfrak{g}, \mathfrak{k})$ . Let  $\mathfrak{a}^*$  denote the dual of  $\mathfrak{a}$  and let  $\{\beta_1, \beta_2, \dots, \beta_r\}$  be a basis of  $\mathfrak{a}^*$  determined by

$$\beta_j(X_k) = 2\delta_{j,k}, \quad 1 \leq j, k \leq r.$$

The restricted root system  $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$  of  $\mathfrak{g}$  relative to  $\mathfrak{a}$  is (of type  $C_r$  or  $BC_r$ ) given by

$$\pm\beta_j \quad (1 \leq j \leq r) \quad \text{each with multiplicity } 1,$$

$$\pm\frac{1}{2}(\beta_j \pm \beta_k) \quad (1 \leq j \neq k \leq r) \quad \text{each with multiplicity } a,$$

and possibly

$$\pm\frac{1}{2}\beta_j \quad (1 \leq j \leq r) \quad \text{each with multiplicity } 2b.$$

Let  $\Sigma^+ = \{\beta_j, \frac{1}{2}\beta_j, \frac{1}{2}(\beta_{\ell} \pm \beta_k); \quad 1 \leq j \leq r, \quad 1 \leq \ell \neq k \leq r\}$  the set of positive restricted roots. Then the set  $\Lambda = \{\alpha_1, \dots, \alpha_{r-1}, \alpha_r\}$  of simple roots in  $\Sigma^+$  is such that

$$\alpha_j = \frac{1}{2}(\beta_{r-j+1} - \beta_{r-j}), \quad 1 \leq j \leq r-1$$

and

$$\alpha_r = \begin{cases} \beta_1 & \text{for tube case} \\ \frac{1}{2}\beta_1 & \text{for non-tube case.} \end{cases}$$

Let  $\Lambda_1 = \{\alpha_1, \dots, \alpha_{r-1}\}$  and write  $\Sigma_1 = \Sigma \cap \mathbb{Z} \cdot \Lambda_1$ . Define

$$\mathfrak{m}_{1,1} = \mathfrak{m} + \mathfrak{a} + \sum_{\gamma \in \Sigma_1} \mathfrak{g}^\gamma, \quad \mathfrak{n}_1^+ = \sum_{\gamma \in \Sigma^+ \setminus \Sigma_1} \mathfrak{g}^\gamma.$$

Let

$$\mathfrak{a}_1 = \{H \in \mathfrak{a} : \gamma(H) = 0 \ \forall \gamma \in \Lambda_1\},$$

then  $\mathfrak{m}_{1,1}$  is the centralizer of  $\mathfrak{a}_1$  in  $\mathfrak{g}$  and  $\mathfrak{p}_1 = \mathfrak{m}_{1,1} + \mathfrak{n}_1^+$  is a standard parabolic subalgebra of  $\mathfrak{g}$  with Langlands decomposition  $\mathfrak{m}_1 + \mathfrak{a}_1 + \mathfrak{n}_1^+$ , where  $\mathfrak{m}_1$  is the orthocomplement of  $\mathfrak{a}_1$  in  $\mathfrak{m}_{1,1}$  with respect to the Killing form. Note that  $\theta(\mathfrak{n}_1^+) = \sum_{\gamma \in \Sigma^+ \setminus \Sigma_1} \mathfrak{g}^{-\gamma}$ . Let  $P_1$  be the corresponding parabolic subgroup and  $P_1 = M_1 A_1 N_1^+$  its Langlands decomposition. Obviously,  $P_1$  is a maximal parabolic subgroup of  $G$ , thus the Shilov boundary  $S$  can be viewed as  $S = G/P_1 = K/K_1$ , where  $K_1 = M_1 \cap K$ .

If we define the element  $X_0 = \sum_{j=1}^r X_j$ , Then  $\mathfrak{a}_1 = \mathbb{R}X_0$ . Let

$$\mathfrak{a}(1) = \sum_{j=1}^{r-1} \mathbb{R}(X_j - X_{j+1})$$

be the orthocomplement of  $\mathfrak{a}_1$  in  $\mathfrak{a}$  with respect to the Killing form,

$$(1) \quad \mathfrak{a} = \mathfrak{a}_1 \oplus^\perp \mathfrak{a}(1) = \mathbb{R}X_0 \oplus^\perp \sum_{j=1}^{r-1} \mathbb{R}(X_j - X_{j+1}).$$

We denote  $\rho_0$  the linear form on  $\mathfrak{a}_1$  such that,  $\rho_0(X_0) = r$ . We extend  $\rho_0$  to  $\mathfrak{a}$  via the orthogonal projection (1). If  $\rho_1$  is the restriction of  $\rho$  to  $\mathfrak{a}_1$ , then it is clear that

$$\rho_1(X_0) = rb + r + a \frac{r(r-1)}{2} = n.$$

Again, we extend  $\rho_1$  to  $\mathfrak{a}$  via the orthogonal projection (1). Then

$$\rho_1 = (b + 1 + a \frac{(r-1)}{2}) \rho_0 = \frac{n}{r} \rho_0.$$

For  $g \in G$ , define  $H(g) \in \mathfrak{a}$  as the unique element such that

$$g \in K \exp(H(g))N \subset KAN = G.$$

We also denote by  $\kappa(g) \in K$  and  $H_1(g) \in \mathfrak{a}_1$  the unique elements such that

$$g \in \kappa(g)M_1 \exp(H_1(g))N_1 \subset KM_1A_1N_1 = G.$$

The following lemma will be useful for the sequel.

**Lemma 2.1** ([15, Lemma 6.1.6]). (i) Let  $x, y \in G$ ,  $\bar{n} \in \bar{N}_1$  and  $a \in A_1$ . Then

$$(2) \quad H_1(x\kappa(y)) = H_1(xy) - H_1(y)$$

$$(3) \quad H_1(\bar{n}a^{-1}) = H_1(\bar{n}) - H_1(a)$$

(ii) Let  $t > 0$  and  $\bar{n} \in \bar{N}_1$ . Then

$$(4) \quad \rho_0(H_1(a_t \bar{n} a_{-t})) \leq \rho_0(H_1(\bar{n})).$$

### 3. THE POISSON TRANSFORM AND THE HUA OPERATORS

For any real analytic manifold  $X$ , we denote by  $\mathcal{B}(X)$  the space of all hyperfunctions on  $X$ . We will view a function on the Shilov boundary  $S = G/P_1$  as a  $P_1$ -invariant function on  $G$ . For  $s \in \mathbb{C}$ , we denote by  $\mathcal{B}(G/P_1; s)$  the space of hyperfunctions  $f$  on  $G$  satisfying

$$f(gman) = e^{(s\rho_0 - \rho_1)\log a} f(g), \quad \forall g \in G, m \in M_1, a \in A_1, n \in N_1^+,$$

The Poisson transform of a function  $f \in \mathcal{B}(G/P_1; s)$ , is defined by

$$\mathcal{P}_s f(gK) = \int_K e^{-\langle s\rho_0 + \rho_1, H_1(g^{-1}k) \rangle} f(k) dk.$$

Since  $G = KP_1$ , the restriction from  $G$  to  $K$  defines a  $G$ -isomorphism from  $\mathcal{B}(G/P_1, s)$  onto the space  $\mathcal{B}(K/K_1)$  of all hyperfunctions  $f$  on  $K$  such that  $f(kh) = f(k)$  for all  $h \in K_1$ .

We review the construction of Hua operators of the second order (see [7]) and the third order (see [3], [12]).

Let  $\{v_j\}$  be a basis of  $\mathfrak{p}^+$  and  $\{v_j^*\}$  be the dual basis of  $\mathfrak{p}^-$  with respect to the Killing form. Let  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  denote the universal enveloping algebra of  $\mathfrak{g}_{\mathbb{C}}$ . The second-order Hua operator, is the element of  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \otimes \mathfrak{k}_{\mathbb{C}}$  defined by

$$\mathcal{H} = \sum_{i,j} v_i v_j^* \otimes [v_j, v_i^*]$$

It is known that the Hua operator does not depend on basis, therefore, for computations one can choose the root vectors basis  $\{E_j\}_{j=1}^r$ .

For tube domains the Hua operator  $\mathcal{H}$  maps the Poisson kernels

$$P_s(gk) = e^{-\langle s\rho_0 + \rho_1, H_1(g^{-1}) \rangle}$$

into the center of  $\mathfrak{k}_{\mathbb{C}}$ , namely the Poisson kernels are its eigenfunctions up to an element in the center, but it is not true for non-tube domains, see [12, Theorem 5.3]. However for non-tube type **I** domains,  $\mathbf{I}_{r,r+b} \simeq SU(r, r+b)/S(U(r) \times U(r+b))$ , the situation is not quite different

from the tube case. In fact,  $\mathfrak{k}_{\mathbb{C}}$  is a sum of two irreducible ideals,  $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}}^{(1)} + \mathfrak{k}_{\mathbb{C}}^{(2)}$  where

$$\begin{aligned}\mathfrak{k}_{\mathbb{C}}^{(1)} &= \left\{ \begin{pmatrix} A & 0 \\ 0 & \frac{\text{tr}(A)}{r+b} I_{r+b} \end{pmatrix}, A \in \mathfrak{gl}(r+b, \mathbb{C}) \right\}, \\ \mathfrak{k}_{\mathbb{C}}^{(2)} &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}, D \in \mathfrak{sl}(r+b, \mathbb{C}) \right\}.\end{aligned}$$

There is a variant of the Hua operator,  $\mathcal{H}^{(1)}$  see [3], [12], by taking the projection of  $\mathcal{H}$  onto  $\mathfrak{k}_{\mathbb{C}}^{(1)}$ . In [12], the second author and Zhang showed that the operator  $\mathcal{H}^{(1)}$  has the Poisson kernels as its eigenfunctions and they found the eigenvalues. They proved further that the eigenfunctions of the Hua operator  $\mathcal{H}^{(1)}$  are harmonic functions (i.e., eigenfunctions of all invariant differential operators on  $\Omega$ ), and gave the following characterization of the range of the Poisson transform for  $\mathbf{I}_{r,r+b}$ .

**Theorem 3.1** ([12, Theorem 6.1]). *Suppose  $s \in \mathbb{C}$  satisfies the following condition*

$$-4[b+1+j+\frac{1}{2}(s-r-b)] \notin \{1, 2, 3, \dots\}, \text{ for } j = 0 \text{ and } 1.$$

*Then the Poisson transform  $\mathcal{P}_s$  is a  $G$ -isomorphism of  $\mathcal{B}(S)$  onto the space of smooth functions  $f$  on  $\Omega$  that satisfy*

$$(5) \quad \mathcal{H}^{(1)} f = \frac{1}{4}(s^2 - (r+b)^2) f I_r.$$

For the characterization of range of the Poisson transform for general non-tube domains the second author and Zhang [12] introduced new third-order Hua operators  $\mathcal{U}$  and  $\mathcal{W}$  :

$$\mathcal{U} = \sum_{i,j,k} v_i^* v_j^* v_k \otimes [v_i, [v_j, v_k^*]],$$

$$\mathcal{W} = \sum_{i,j,k} v_k v_i^* v_j \otimes [[v_k^*, v_i], v_j],$$

Similarly to  $\mathcal{H}$ , the operators  $\mathcal{U}$  and  $\mathcal{W}$  do not depend on the basis.

Denote

$$c = 2(n+1) + \frac{1}{n}(a^2 - 4) \dim(\mathcal{P}^{(1,1)}),$$

where  $\mathcal{P}^{(1,1)}$  is the dimension of the irreducible subspaces of holomorphic polynomials on  $\mathfrak{p}^+$  with lowest weight  $-\gamma_1 - \gamma_2$ . For any  $s \in \mathbb{C}$ , put  $\sigma = \frac{1}{2}(s + \frac{n}{r})$ . For general non-tube domains we have the following



**Theorem 3.2** ([12, Theorem 7.2]). *Let  $\Omega$  be a bounded symmetric non-tube domain of rank  $r$  in  $\mathbb{C}^n$ . Suppose  $s \in \mathbb{C}$  satisfies*

$$-4[b + 1 + j\frac{a}{2} + \frac{1}{2}(s - \frac{r}{n})] \notin \{1, 2, 3, \dots\}, \text{ for } j = 0 \text{ and } 1.$$

*Then the Poisson transform  $\mathcal{P}_s$  is a  $G$ -isomorphism of  $\mathcal{B}(S)$  onto the space of harmonic functions  $f$  on  $\Omega$  that satisfy*

$$(6) \quad \left( \mathcal{U} - \frac{-2\sigma^2 + 2p\sigma + c}{\sigma(2\sigma - p - b)} \mathcal{W} \right) f = 0,$$

#### 4. THE $L^p$ -RANGE OF THE POISSON TRANSFORM

For  $1 < p < +\infty$ , we will consider the space  $L^p(S) = L^p(K/K_1)$  as the space of all complex valued measurable (classes) functions  $f$  on  $K$  that are  $K_1$ -invariant and satisfying

$$\|f\|_p = \left( \int_K |f(k)|^p dk \right)^{1/p} < +\infty,$$

where  $dk$  is the Haar measure of  $K$ . Let  $d\bar{n}$  be the invariant measure on  $\bar{N}_1 = \theta(N_1)$  with the normalization

$$(7) \quad \int_{\bar{N}_1} e^{\langle -2\rho_1, H_1(\bar{n}) \rangle} d\bar{n} = 1.$$

Then for a continuous function  $f$  on  $S$  we have

$$(8) \quad \int_K f(k) dk = \int_{\bar{N}_1} f(\kappa(\bar{n})) e^{-2\langle \rho_1, H_1(\bar{n}) \rangle} d\bar{n}.$$

The space  $L^p(S)$  can be viewed as a subspace of  $\mathcal{B}(S)$ , thus its image  $\mathcal{P}_s(L^p(S))$  is a proper closed subspace of  $\mathcal{E}_s(\Omega)$ . We will now, for specific  $s$ , characterize this image. For this we need some information on the integral  $\mathbf{c}_s$  in following proposition.

**Proposition 4.1.** *For  $s \in \mathbb{C}$  such that  $\Re(s) > \frac{a}{2}(r - 1)$ , the integral*

$$\mathbf{c}_s = \int_{\bar{N}_1} e^{-\langle s\rho_0 + \rho_1, H_1(\bar{n}) \rangle} d\bar{n}$$

*converges absolutely to a constant  $\mathbf{c}_s \neq 0$ .*

*Proof.* For  $s \in \mathbb{C}$  let  $\lambda_s \in \mathfrak{a}_{\mathbb{C}}^*$  be the linear form defined by

$$\lambda_s(H) = (s\rho_0 - \rho_1)(H_1) + \rho(H), \quad H \in \mathfrak{a}$$

where  $H_1$  is the projection of  $H$  onto  $\mathfrak{a}_1$ . Then the condition  $\Re(s) > \frac{a}{2}(r - 1)$  is equivalent to

$$(9) \quad \Re(\langle \lambda_s, \alpha \rangle) > 0 \quad \forall \alpha \in \Sigma^+ \setminus \Sigma_1.$$

Moreover, we can choose (see for example [15, Lemma 6.1.4])  $\omega$  in the Weyl group  $W$  of  $\Sigma$  such that

$$\begin{aligned} (i) \quad & \omega \cdot H = H, \quad \forall H \in \mathfrak{a}_1, \\ (ii) \quad & \omega(\Sigma^+ \cap \Sigma_1) = -\Sigma^+ \cap \Sigma_1, \\ (iii) \quad & \omega(\Sigma^+ \setminus \Sigma_1) = \Sigma^+ \setminus \Sigma_1. \end{aligned}$$

Since  $\langle \omega \lambda_s, \alpha \rangle = \langle \lambda_s, \omega^{-1} \alpha \rangle$ , the condition (9) is equivalent to

$$\Re(\langle \omega \lambda_s, \alpha \rangle) > 0, \quad \forall \alpha \in \Sigma^+$$

Furthermore

$$\langle s\rho_0 + \rho_1, H_1(g) \rangle = \langle \omega \lambda_s + \rho, H(g) \rangle$$

so that

$$\int_{\bar{N}_1} e^{-\langle s\rho_0 + \rho_1, H_1(\bar{n}) \rangle} d\bar{n} = \int_{\bar{N}_1} e^{-\langle \omega \lambda_s + \rho, H(\bar{n}) \rangle} d\bar{n}$$

and the right hand side is the Harish-Chandra  $c$  function,  $c(\omega \lambda_s)$  associated with the maximal parabolic subgroup, which converges absolutely, see [15]. □

Let  $s \in \mathbb{C}$ . Let  $\mathcal{E}_s(\Omega)$  be the space of harmonic functions on  $\Omega$  that satisfy (5) in type I domains or (6) in general domains. It is clear that the image  $\mathcal{P}_s(L^p(S))$  is a proper closed subspace of the eigenspace  $\mathcal{E}_s(\Omega)$ . Hence, it is natural to look for a characterization of those  $F \in \mathcal{E}_s(\Omega)$  that are Poisson transform of some  $f \in L^p(S)$ .

For any  $1 < p < \infty$ , let  $\mathcal{E}_s^p(\Omega)$  denote the Hua-Hardy type space of functions  $f \in \mathcal{E}_s(\Omega)$  such that

$$\|f\|_{s,p} = \sup_{a \in A_1} e^{-\langle \Re(s)\rho_0 - \rho_1, \log a \rangle} \left( \int_K |f(ka)|^p dk \right)^{1/p} < +\infty.$$

Since  $\mathfrak{a}_1 = \mathbb{R}X_0$ , the above integral becomes

$$\|f\|_{s,p} = \sup_{t>0} e^{-t(\Re(s)r-n)} \left( \int_K |f(ka_t)|^p dk \right)^{1/p},$$

where  $a_t = \exp(tX_0)$ .

**4.1. Fatou type theorems.** As a preparation to Fatou-type theorems we prove the following

**Proposition 4.2.** *Let  $s \in \mathbb{C}$  be such that  $\Re(s) > \frac{a}{2}(r-1)$ . Let  $\Psi_t$  be the function defined on  $\bar{N}_1$  by*

$$\Psi_t(\bar{n}) = e^{-\langle s\rho_0 + \rho_1, H_1(\bar{n}) \rangle + \langle s\rho_0 - \rho_1, H_1(a_t \bar{n} a_{-t}) \rangle}.$$

*Then there exists a non-negative function  $\Phi \in L^1(\bar{N}_1)$  such that  $\Psi_t \leq \Phi$  for each  $t$ .*

*Proof.* It follows from (4), that for any  $t > 0$  and for any  $\bar{n} \in \bar{N}_1$ ,

$$0 \leq \rho_0(H_1(a_t \bar{n} a_{-t})) \leq \rho_0(H_1(\bar{n})).$$

Therefore,

$$|\Psi_t(\bar{n})| \leq \begin{cases} e^{-\langle \Re(s)\rho_0 + \rho_1, H_1(\bar{n}) \rangle} & \text{if } \frac{a}{2}(r-1) < \Re(s) \leq \frac{a}{2}(r-1) + b + 1 \\ e^{-2\langle \rho_1, H_1(\bar{n}) \rangle} & \text{if } \Re(s) > \frac{a}{2}(r-1) + b + 1 \end{cases}$$

and the second hand is an integrable function on  $\bar{N}_1$  by (7) and Proposition 4.1 . □

Let  $\mathcal{C}(S)$  be the space of complex-valued continuous functions on  $S$  with the topology of uniform convergence.

**Theorem 4.3.** *Let  $s \in \mathbb{C}$  be such that  $\Re(s) > \frac{a}{2}(r-1)$ . Then*

$$f(k) = \mathbf{c}_s^{-1} \lim_{t \rightarrow +\infty} e^{-(rs-n)t} \mathcal{P}_s f(ka_t)$$

*uniformly, for  $f \in \mathcal{C}(S)$ .*

*Proof.* Let  $f \in \mathcal{C}(S)$ , then

$$\mathcal{P}_s f(ka_t) = \int_K e^{-\langle s\rho_0 + \rho_1, H_1(a_{-t}h) \rangle} f(kh) dh.$$

We transform this integral using the formula (8) to an integral over  $\bar{N}_1$ ,

$$\mathcal{P}_s f(ka_t) = \int_{\bar{N}_1} e^{-\langle s\rho_0 + \rho_1, H_1(a_{-t}\kappa(\bar{n})) \rangle} f(k\kappa(\bar{n})) e^{-2\langle \rho_1, H_1(\bar{n}) \rangle} d\bar{n},$$

and by (2) we get

$$\mathcal{P}_s f(ka_t) = \int_{\bar{N}_1} e^{-\langle s\rho_0 + \rho_1, H_1(a_{-t}\bar{n}) \rangle} e^{\langle s\rho_0 - \rho_1, H_1(\bar{n}) \rangle} f(k\kappa(\bar{n})) d\bar{n}.$$

which by the substitution  $\bar{n} \mapsto a_{-t}\bar{n}a_t$  and (3), becomes

$$\begin{aligned} \mathcal{P}_s f(ka_t) &= e^{\langle s\rho_0 - \rho_1, H_1(a_t) \rangle} \times \\ &\int_{\bar{N}_1} e^{-\langle s\rho_0 + \rho_1, H_1(\bar{n}) \rangle + \langle s\rho_0 - \rho_1, H_1(a_t \bar{n} a_{-t}) \rangle} f(k\kappa(a_t \bar{n} a_{-t})) d\bar{n}. \end{aligned}$$

But  $\rho_1 = \frac{n}{r}\rho_0$ , and  $a_t \bar{n} a_{-t} \rightarrow e$  when  $t \rightarrow +\infty$ , thus, by Proposition 4.2,

$$\lim_{t \rightarrow +\infty} e^{-(rs-n)t} \mathcal{P}_s f(ka_t) = \mathbf{c}_s f(k).$$

□

Let

$$\varphi_s(a_t) := \int_K e^{-\langle s\rho_0 + \rho_1, H(a_{-t}k) \rangle} dk.$$

then, it follows from the above theorem that

$$(10) \quad \lim_{t \rightarrow \infty} e^{-(rs-n)t} \varphi_s(a_t) = \mathbf{c}_s, \quad \text{if } \Re(s) > \frac{a}{2}(r-1).$$

As a consequence we can prove the following

**Proposition 4.4.** *Let  $s \in \mathbb{C}$  be such that  $\Re(s) > \frac{a}{2}(r-1)$ . Then there exists a positive constant  $\gamma_s$  such that, for  $1 < p < \infty$  and  $f \in L^p(S)$ , we have*

$$\left( \int_K |\mathcal{P}_s f(ka_t)|^p dk \right)^{1/p} \leq \gamma_s e^{(rs-n)t} \|f\|_p.$$

*Proof.* For  $t > 0$ , we define the function  $p_s^t$  by

$$p_s^t(k) = e^{-\langle s\rho_0 + \rho_1, H_1(a_{-t}k^{-1}) \rangle}, \quad k \in K.$$

Then the Poisson transform can be written as the convolution

$$\mathcal{P}_s f(ka_t) = (f * p_s^t)(k).$$

Hence, to prove the proposition we use the Hausdorff-Young inequality,

$$\left( \int_K |\mathcal{P}_s f(ka_t)|^p dk \right)^{1/p} \leq \|p_s^t\|_1 \|f\|_p \quad (p > 1),$$

and (10). □

Let, as usual,  $\hat{K}$ , be the set of equivalence classes of finite dimensional irreducible representations of  $K$ . For  $\delta \in \hat{K}$ , let  $\mathcal{C}(S)_\delta$  be the linear span of all  $K$ -finite vectors on  $S$  of type  $\delta$ . It is well known that the space  $\mathcal{C}^K(S) := \oplus_{\delta \in \hat{K}} \mathcal{C}(S)_\delta$  is dense in  $\mathcal{C}(S)$ . Recall also, that the space  $\mathcal{C}(S)$  is dense in  $L^p(S)$  for  $1 < p < \infty$ .

**Theorem 4.5.** *Let  $s \in \mathbb{C}$  be such that  $\Re(s) > \frac{a}{2}(r-1)$ . Then*

$$f(k) = \mathbf{c}_s^{-1} \lim_{t \rightarrow +\infty} e^{-(rs-n)t} \mathcal{P}_s f(ka_t)$$

*in  $L^p(S)$ , for  $1 < p < \infty$ .*

*Proof.* Let  $f \in L^p(S)$ . By the above density arguments, for any  $\epsilon > 0$ , there exists  $\varphi \in \mathcal{C}^K(S)$  such that  $\|f - \varphi\|_p < \epsilon$ . Then we have

$$\begin{aligned} \|\mathbf{c}_s^{-1} e^{-(rs-n)t} P_s^t f - f\|_p &\leq \|\mathbf{c}_s^{-1} e^{-(rs-n)t} P_s^t (f - \varphi)\|_p + \\ &\quad + \|\mathbf{c}_s^{-1} e^{-(rs-n)t} P_s^t \varphi - \varphi\|_p + \|\varphi - f\|_p \end{aligned}$$

where the function  $P_s^t f$  is defined by

$$(11) \quad P_s^t f(k) = \mathcal{P}_s f(ka_t).$$

By Proposition 4.4,

$$\|\mathbf{c}_s^{-1} e^{-(rs-n)t} P_s^t(f - \varphi)\|_p \leq \gamma_s |\mathbf{c}_s^{-1}| \|f - \varphi\|_p,$$

and by Theorem 4.3

$$\lim_{t \rightarrow +\infty} \|\mathbf{c}_s^{-1} e^{-(rs-n)t} P_s^t \varphi - \varphi\|_p = 0.$$

Thus,  $\lim_{t \rightarrow +\infty} \|\mathbf{c}_s^{-1} e^{-(rs-n)t} P_s^t f - f\|_p \leq \epsilon(\gamma_s + 1)$  and this proves the theorem.  $\square$

We can now prove the following estimates

**Proposition 4.6.** *Let  $s \in \mathbb{C}$  be such that  $\Re(s) > \frac{a}{2}(r-1)$ . Then there exists a positive constant  $\gamma_s$  such that for  $1 < p < +\infty$  and  $f \in L^p(S)$ ,*

$$(12) \quad |\mathbf{c}_s| \|f\|_p \leq \|\mathcal{P}_s f\|_{s,p} \leq \gamma_s \|f\|_p.$$

*Proof.* In fact, the right hand side of the estimate (12) follows from Proposition 4.4. On the other hand, by Theorem 4.5 we have

$$\lim_{t \rightarrow \infty} e^{(n-rs)t} \mathcal{P}_s f(ka_t) = \mathbf{c}_s f(k)$$

in  $L^p(S)$ . Hence, there exists a sequence  $(t_j)_j$ , with  $t_j \rightarrow +\infty$  when  $j \rightarrow +\infty$  such that  $\lim_{j \rightarrow +\infty} e^{(n-rs)t_j} \mathcal{P}_s f(ka_{t_j}) = \mathbf{c}_s f(k)$ , almost every where in  $K$ . Therefore, by the classical Fatou lemma,

$$|\mathbf{c}_s| \|f\|_p \leq \sup_j e^{(n-r\Re(s))t_j} \|P_s^{t_j} f\|_p$$

and this is how we prove the left hand side of (12).  $\square$

**4.2. The  $L^2$ -Poisson transform range.** Recall that

$$L^2(S) = \oplus_{\delta \in \hat{K}} V_\delta$$

where  $V_\delta$  is the finite linear span of  $\{\varphi_\delta \circ k, \ k \in K\}$ , where  $\varphi_\delta$  is the zonal spherical function corresponding to  $\delta$ .

For  $s \in \mathbb{C}$  and  $\delta \in \hat{K}$ , define the *generalized spherical function*  $\Phi_{s,\delta}$  on  $A_1$  by

$$\Phi_{s,\delta}(a_t) = (\mathcal{P}_s \varphi_\delta)(a_t).$$

**Proposition 4.7.** *Let  $s \in \mathbb{C}$ ,  $\delta \in \hat{K}$  and  $f \in V_\delta$ . Then for any  $k \in K$  and  $a_t \in A_1$ ,*

$$(\mathcal{P}_s f)(ka_t) = \Phi_{s,\delta}(a_t) f(k).$$

*Proof.* Since  $M_1$  centralize  $A_1$ , we can view the operator (11) as a bounded operator on  $L^2(S)$ . Moreover,  $P_s^t$  commutes with the left regular representation of  $K$  in  $L^2(S)$ . Hence, by Schur's lemma,  $P_s^t = \Phi_{s,\delta}(a_t) \cdot I$  on each  $V_\delta$  and the proposition follows.  $\square$

The first main theorem of this section can now be stated as follows :

**Theorem 4.8.** *Let  $s \in \mathbb{C}$  be such that  $\Re(s) > \frac{a}{2}(r-1)$ . A smooth function  $F$  on  $\Omega$  is the Poisson transform  $F = \mathcal{P}_s f$  of a function  $f \in L^2(S)$  if and only if  $F \in \mathcal{E}_s^2(\Omega)$ .*

*Proof.* The necessary condition follows from Proposition 4.6 and [12, Theorem 6.1 and Theorem 7.2]. On the other hand, let  $F \in \mathcal{E}_s^2(\Omega)$ . We apply again [12, Theorem 6.1 and Theorem 7.2]. Then, there exists a hyperfunction  $f \in \mathcal{B}(S)$  such that  $F = \mathcal{P}_s f$ . Let  $f = \sum_{\delta \in \hat{K}} f_\delta$  be its  $K$ -type decomposition. By Proposition 4.7 we can write

$$F(ka_t) = \sum_{\delta \in \hat{K}} \Phi_{s,\delta}(a_t) f_\delta(k)$$

in  $\mathcal{C}^\infty(K \times [0, +\infty[)$ .

Now observe that

$$\|F\|_{s,2}^2 = \sup_{t>0} e^{2(n-r\Re(s))t} \sum_{\delta \in \hat{K}} |\Phi_{s,\delta}(a_t)|^2 \|f_\delta\|_2^2 < \infty.$$

Then, if  $\Lambda$  is an arbitrary finite subset of  $\hat{K}$ , we get

$$e^{2(n-r\Re(s))t} \sum_{\delta \in \Lambda} |\Phi_{s,\delta}(a_t)|^2 \|f_\delta\|_2^2 \leq \|F\|_{s,2}^2$$

for every  $t > 0$  and hence from Theorem 4.3 it follows immediately that

$$|\mathbf{c}_s|^2 \sum_{\delta \in \Lambda} \|f_\delta\|_2^2 \leq \|F\|_{s,2}^2$$

which implies that  $f = \sum_{\delta \in K} f_\delta$  in  $L^2(S)$  and that

$$|\mathbf{c}_s|^2 \|f\|_2 \leq \|F\|_{s,2}.$$

This ends the proof of the theorem.  $\square$

In the following proposition we show how to recover a function  $f \in L^2(S)$  from its Poisson transform  $\mathcal{P}_s f$ .

**Proposition 4.9.** *Let  $s \in \mathbb{C}$  be such that  $\Re(s) > \frac{a}{2}(r-1)$ . Let  $F \in \mathcal{E}_s^2(\Omega)$  and  $f \in L^2(S)$  its boundary value. Then the following inversion formula*

$$(13) \quad f(k) = |\mathbf{c}_s|^{-2} \lim_{t \rightarrow \infty} e^{2(n-r\Re(s))t} \int_K \overline{e^{-\langle s\rho_0 + \rho_1, H_1(a_{-t}k^{-1}h) \rangle}} F(ha_t) dh$$

holds in  $L^2(S)$ .

*Proof.* Let  $F \in \mathcal{E}_s^2(\Omega)$ , then it follows from Theorem 4.8 that there exists a unique  $f \in L^2(S)$  such that  $F = \mathcal{P}_s f$ . Let  $f = \sum_{\delta \in \hat{K}} f_\delta$  be its  $K$ -type expansion, then similarly to the preceding proof, we get

$$(14) \quad F(ka_t) = \sum_{\delta \in \hat{K}} \Phi_{s,\delta}(a_t) f_\delta(k).$$

For any  $t > 0$ , define the complex-valued function on  $K$  by

$$g_t(ka_t) = |\mathbf{c}_s|^{-2} e^{2(n-rs)t} \int_K \overline{e^{-\langle s\rho_0 + \rho_1, H_1(a_{-t}k^{-1}h) \rangle}} F(ha_t) dh.$$

Next, using the above series expansion we can write according to Theorem 4.5,

$$g_t(h) = |\mathbf{c}_s|^{-2} e^{2(n-rs)t} \sum_{\delta \in \hat{K}} |\Phi_{s,\delta}(a_t)|^2 f_\delta(h).$$

Thus,

$$\|g_t - f\|_2^2 = \sum_{\delta \in \hat{K}} \left| |\mathbf{c}_s|^{-2} e^{2(n-rs)t} |\Phi_{s,\delta}(a_t)|^2 - 1 \right|^2 \|f_\delta\|_2^2,$$

which shows that  $\|g_t - f\|_2 \rightarrow 0$ , since  $\lim_{t \rightarrow \infty} e^{(n-rs)t} \Phi_{s,\delta}(a_t) = \mathbf{c}_s$ .  $\square$

**4.3. The  $L^p$ -Poisson transform range,  $p \neq 2$ .** We shall now prove the second main result of this paper, more precisely, we shall characterize the  $L^p$ -range of the Poisson transform. We will need the following notation. For each function  $f$  on  $\Omega$ , define the function  $f^t$ ,  $t > 0$ , on  $K$  by

$$f^t(k) = f(ka_t).$$

**Theorem 4.10.** *Let  $s \in \mathbb{C}$  be such that  $\Re(s) > \frac{a}{2}(r-1)$ . A smooth function  $F$  on  $\Omega$  is the Poisson transform  $F = \mathcal{P}_s f$  of a function  $f \in L^p(S)$  if and only if  $F \in \mathcal{E}_s^p(\Omega)$ .*

*Proof.* We will follow the technique used by Korányi [9]. Let  $(\chi_n)_n$  be an approximation of the identity in  $\mathcal{C}(K)$ . That is  $\chi_n \geq 0$ ,  $\int_K \chi_n(k) dk = 1$  and  $\lim_{n \rightarrow +\infty} \int_{K \setminus U} \chi_n(k) dk = 0$  for every neighborhood  $U$  of  $e$  in  $K$ . Let  $F \in \mathcal{E}_s^p(\Omega)$ . For each  $n$ , define the function  $F_n$  on  $\Omega$  by

$$F_n(gK) = \int_K \chi_n(k) F(k^{-1}g) dk.$$

Then  $(F_n)_n$  converges point-wise to  $F$ , and since the set  $\mathcal{E}_s(\Omega)$  of harmonic functions satisfying the Hua system is  $G$ -invariant,  $F_n \in \mathcal{E}_s(\Omega)$ , for each  $n$ . Furthermore,

$$F_n^t(ka_tK) = (\chi_n * F^t)(k)$$

and this shows

$$(15) \quad \|F_n^t\|_2 \leq \|\chi_n\|_2 \|F^t\|_p,$$

and

$$(16) \quad \|F_n^t\|_p \leq \|F^t\|_p.$$

It follows from (15)

$$\sup_{t>0} e^{(n-rs)t} \left( \int_K |F_n(ka_t)|^2 dk \right)^{1/2} \leq \|\chi_n\|_2 \|F\|_{s,p}.$$

Thus  $F_n \in \mathcal{E}_{s,2}(\Omega)$  and by Theorem 4.8, there exists  $f_n \in L^2(S)$  such that  $F_n = \mathcal{P}_s f_n$ . Now, our goal is to prove that  $f_n$  belongs to  $L^p(S)$ . Using the inversion formula (13) we can write in  $L^2(S)$ ,

$$f_n(k) = \lim_{t \rightarrow +\infty} g_n^t(k)$$

where

$$g_n^t(h) = g_n(ha_t) = |\mathbf{c}_s|^{-2} e^{2(n-r\Re(s))t} \int_K \overline{e^{-\langle s\rho_0 + \rho_1, H_1(a_{-t}k^{-1}h) \rangle}} F_n(ka_t) dk.$$

Let  $\varphi \in \mathcal{C}(S)$  be a continuous function on  $S$ , then

$$\int_K f_n(h) \varphi(h) dh = \lim_{t \rightarrow \infty} \int_K g_n^t(h) \varphi(h) dh.$$

Moreover,

$$\begin{aligned} \int_K g_n^t(h) \varphi(h) dh &= |\mathbf{c}_s|^{-2} e^{2(n-r\Re(s))t} \times \\ &\times \int_K \int_K F_n(ka_t) \varphi(h) \overline{e^{-\langle s\rho_0 + \rho_1, H_1(a_{-t}k^{-1}h) \rangle}} dk dh \\ &= |\mathbf{c}_s|^{-2} e^{2(n-r\Re(s))t} \int_K \overline{\mathcal{P}_s \varphi(ka_t)} F_n(ka_t) dk. \end{aligned}$$

By the Holder inequality, if  $q$  is such that  $1/p + 1/q = 1$ , we get

$$\begin{aligned} \left| \int_K g_n^t(h) \varphi(h) dh \right| &\leq |\mathbf{c}_s|^{-2} e^{2(n-r\Re(s))t} \|\mathcal{P}_s \varphi\|_q \|F_n^t\|_p, \\ &\leq |\mathbf{c}_s|^{-2} e^{2(n-r\Re(s))t} \|\mathcal{P}_s \varphi\|_q \|F^t\|_p, \end{aligned}$$

where the second inequality follows from (16). But  $F \in \mathcal{E}_{s,p}(\omega)$ , then

$$\left| \int_K g_n^t(h) \varphi(h) dh \right| \leq |\mathbf{c}_s|^{-2} e^{2(n-r\Re(s))t} \|\mathcal{P}_s \varphi\|_q \|F\|_{s,p}.$$



Therefore, by Theorem 4.3,

$$\left| \int_K f_n(h) \varphi(h) dh \right| \leq |\mathbf{c}_s|^{-1} \|\varphi\|_q \|F\|_{s,p},$$

and by taking the supremum over  $\varphi \in \mathcal{C}(S)$  with  $\|\varphi\|_q = 1$ , we get

$$\|f_n\|_p \leq |\mathbf{c}_s|^{-1} \|F\|_{s,p}.$$

Now, for each  $\varphi \in L^q(S)$ , define the functional

$$T_n(\varphi) = \int f_n(h) \varphi(h) dk.$$

Then it is obvious by (4.3) that

$$|T_n(\varphi)| \leq |\mathbf{c}_s|^{-1} \|\varphi\|_q \|F\|_{s,p}$$

hence,  $T_n$  is uniformly bounded operator in  $L^q(S)$  with  $\sup_n \|T_n\| \leq |\mathbf{c}_s|^{-1} \|F\|_{s,p}$ . Thanks to Banach-Alaouglu-Bourbaki's theorem, there exists a subsequence of bounded operators  $(T_{n_j})_j$  which converges as  $n_j \rightarrow +\infty$  to a bounded operator  $T$  in  $L^q(S)$ , under the  $*$ -weak topology, with  $\|T\| \leq |\mathbf{c}_s|^{-1} \|F\|_{s,p}$ . Then, by the Riesz representation theorem, there exists a unique function  $f \in L^p(S)$  such that

$$T(\varphi) = \int_K f(h) \varphi(h) dh, \quad \forall \varphi \in L^q(S)$$

with

$$(17) \quad \|f\|_p \leq \|T_n\| \leq |\mathbf{c}_s|^{-1} \|F\|_{s,p}.$$

Now, observe that

$$F_{n_j}(g) = T_{n_j}(e^{-\langle s\rho_0 + \rho_1, H_1(g^{-1}k) \rangle}),$$

thus, by taking the limit as  $n \rightarrow +\infty$  we get

$$F(g) = T(e^{-\langle s\rho_0 + \rho_1, H_1(g^{-1}k) \rangle}) = \mathcal{P}_s f(g)$$

with  $|\mathbf{c}_s| \|f\|_p \leq \|F\|_{s,p}$ , by (17), and this finishes the proof of the theorem.  $\square$

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