



HAL
open science

Efficient estimation of the cardinality of large data sets

Philippe Chassaing, Lucas Gerin

► **To cite this version:**

Philippe Chassaing, Lucas Gerin. Efficient estimation of the cardinality of large data sets. 4th Colloquium on Mathematics and Computer Science, 2006, France. pp.419-422. hal-00095370v2

HAL Id: hal-00095370

<https://hal.science/hal-00095370v2>

Submitted on 28 Aug 2007 (v2), last revised 17 Aug 2015 (v5)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Efficient estimation of the cardinality of large data sets

Philippe Chassaing and Lucas Gerin

22 september 2006

Abstract

F.Giroire has recently proposed an algorithm which returns the *approximate* number of distinct elements in a large sequence of words, under strong constraints coming from the analysis of large data bases. His estimation is based on statistical properties of uniform random variables in $[0, 1]$. Here we propose an optimal estimation, using information and estimation theory¹.

1 Introduction

The aim of this note is to improve a solution proposed by Giroire [Gir05] to the following problem: consider a sequence $Y = (Y_1, \dots, Y_N)$ of words (one may think to a sequence of file on a disk, a list of requests, a novel from Shakespeare, *etc.*); we don't make any assumption on the structure of Y , and we want to know the number (usually denoted F_0 in the data base community) of *distinct* elements of this sequence. The motivation comes from analysis of large data sets, and especially analysis of internet traffic: certain attacks may be detected at router level, because they generate an unusual number of distinct connections (see [Fla04]). Most of algorithms use a dictionary to store every word, so that the memory needed is linear in F_0 . Here the size of data sets is huge, making it impossible to store every word, so that the algorithm should satisfy the two following constraints: it should use constant memory and do only one pass over the data. These constraints are very strong, but on the other hand we allow the algorithm to give only an *estimation* of F_0 .

The main idea used in [Gir05], introduced by Flajolet and Martin [FM85], is to transform this problem in a probabilistic one, using hash functions.

A *hash function* is a function $h : \mathcal{C} \rightarrow [0, 1]$, where \mathcal{C} is a finite set of words (say english language, $\{0, 1\}^8$, *etc.*) such that *the image of a typical sequence of words behaves as a sequence of i.i.d random variables, uniform in $[0, 1]$.*

This definition is of course somewhat informal, but we will assume, from now on, that, noting $X_i = h(Y_i)$, then $\mathbf{X} = \{X_1, \dots, X_N\}$ is the realization of F_0 i.i.d. r.v., uniform on $[0, 1]$. Existence and construction of *good* hash functions is discussed in [Knu73].

Set $\theta = F_0$ and denote as usually $X_{(1)}$ the smallest X_i , $X_{(2)}$ the second smallest, and so on. The key point is that the information on θ contained in $\{Y_1, \dots, Y_N\}$ is equivalent to that contained in $(X_{(1)}, \dots, X_{(\theta)})$.

As a consequence, we are now dealing with a classical statistical problem: given a (small) sample of (X_1, \dots, X_θ) , i.i.d. r.v., uniform on $[0, 1]$, we want to estimate the (large) parameter θ . Denote by M the memory available (how many real numbers that can be stored). One should determine:

1. A way of extracting a M -sample of \mathbf{X} (the M smallest, the M with the longest sequence of zeros in their binary representation, *etc.*).
2. A function $\hat{\xi} : [0, 1]^M \rightarrow \mathbb{R}$ which approximates θ , when applied to this M -sample.

State of the Art. Flajolet and Martin [FM85] have used these ideas to construct an algorithm based on research of patterns of 0's and 1's in the binary representation of the hashed values X_1, \dots, X_θ . It has been improved by Durand and Flajolet [DF93]. Bar-Yossef *et alii* [BYJK⁺02], have proposed 3 performant algorithms, their ideas have been generalized by Giroire [Gir05].

In a different way, Alon, Matias, and Szegedy consider estimation by *moment method*, making implementation proposed in [FM85] easier. For a nice survey about these ideas one may read [Fla04].

¹Version of August 28, 2007.

Giroire's algorithm. The starting idea in [Gir05] is to use this simple property:

$$\mathbb{E}[X_{(1)}] = \frac{1}{\theta + 1}.$$

Consequently, a naive algorithm would hash every data, compare it to the smallest hashed value already seen, and finally return $1/X_{(1)}$. Unfortunately, $\mathbb{E}[1/X_{(1)}] = \infty$. However, $1/X_{(2)}, 1/X_{(3)} \dots$ have finite expectation. This leads Giroire to propose an algorithm which return a function of $X_{(k)}$, for some k . In order to improve the precision of such an algorithm, one may wish to execute it m times with m different hashing functions, but this would cost too much time. Therefore Giroire uses *stochastic averaging*, introduced in [FM85]: the idea is to *simulate* m different experiments, by dividing $[0, 1]$ in m intervals.

Algorithm 1.

let k, m be integers. initialize $(X_{(1),i}, \dots, X_{(k),i}, i = 1, \dots, m)$ with $X_{(p),i} = \frac{i}{m}$ for all i, p .

for $j = 1$ to N

$X_j = h(Y_j)$.

let i the integer such that X_j lies in $[\frac{i-1}{m}, \frac{i}{m}[$.

update the k -dimensional vector of k smallest values $X_{(1),i}, \dots, X_{(k),i}$ lying in $[\frac{i-1}{m}, \frac{i}{m}[$.

next j .

for all p, i , renormalize $X_{(p),i} = m(X_{(p),i} - \frac{i-1}{m})$.

return an estimator $\hat{\xi} = \hat{\xi}(X_{(l),i}; i = 1, \dots, m; l = 1, \dots, k)$.

Thus we get m vectors in \mathbb{R}^k . $X_{(k),i}$ is the k -th smallest hashed value lying in $[\frac{i-1}{m}, \frac{i}{m}[$, renormalized to get a real in $[0, 1]$. If less than l values have fell in the i -th interval, then $X_{(k),i} = 1$. Obviously, Algorithm 1 makes only one pass over each data Y_i . Memory used by the algorithm is indeed M , if we have chosen $k \cdot m = M$. The estimation returned by the algorithm does not depend on any assumption on the repetitions in the sequence X_1, \dots, X_N .

Giroire [Gir05] proposes 3 estimators ξ_1, ξ_2, ξ_3 , using inverse function, square root function and log respectively. For example,

$$\xi_3 := \left(\frac{\Gamma(k-1/m)}{\Gamma(k)} \right)^{-m} \cdot e^{-\frac{1}{m} \sum_{i=1}^m \log X_{(k),i}}.$$

For each k, m these estimators are asymptotically *unbiased*, i.e. $\mathbb{E}[\xi_i] \sim \theta$ when θ goes to ∞ . Their variances are all about $1/km$. Here we give a fourth estimator, which is also asymptotically unbiased:

$$\hat{\xi} = \frac{km - 1}{\sum_{i=1}^m X_{(k),i}}.$$

Plan Using information and estimation theories, we first show that the estimator $\hat{\xi}$ is optimal under a simplified model, that we call the *independent model*. Then we discuss its actual optimality.

2 The best estimation under the *independent model*

In this section, \Rightarrow denotes the convergence in law. Recall that a real-valued random variable is said to follow the Gamma law with parameters (k, θ) if

$$\mathbb{P}(X \in [t, t + dt]) = \frac{t^{k-1}}{\Gamma(k)} \theta^k e^{-\theta t} \mathbf{1}_{t \geq 0} dt.$$

The asymptotic behavior of the minimum $X_{(1)}$ of θ random uniform variables in $[0, 1]$ is well-known (see for example [Fel70]): $X_{(1)} \Rightarrow \gamma_1$, where γ_1 follows the Gamma(1, θ) law. More generally, one can prove here the following convergence

$$(\theta X_{(k),1}, \dots, \theta X_{(k),m}) \Rightarrow [\theta \rightarrow \infty] \mathcal{L}(\gamma_1, \dots, \gamma_m), \tag{1}$$

where the γ_i are i.i.d. r.v. of law Gamma($k, 1$). Consequently, we assume in this section that the $X_{(k),i}$ are i.i.d. r.v. of law Gamma(k, θ), this is the so-called *independent model*. We set

$$\hat{\xi} = \frac{km - 1}{\sum_{i=1}^m X_{(k),i}}.$$

Remark 1. This estimator depends only on the m values $(X_{(k),i}, i = 1, \dots, m)$, not on the $km - m$ other hashed values stored by the algorithm. This follows from the fact that the knowledge of these values does not provide any information about θ . For a given i , conditionnally on $X_{(k),i}$, the r.v. $(X_{(1),i}, \dots, X_{(k-1),i})$ are distributed uniformly on $[0, X_{(k),i}]$.

A simple calculation shows that under the independent model,

$$\begin{aligned}\mathbb{E}[\hat{\theta}] &= \theta, \\ \text{Var}(\hat{\theta}) &= \frac{\theta^2}{km - 2}.\end{aligned}$$

This is indeed better than the 3 estimators proposed in [Gir05].

Recall a few definitions in Statistics (see for example [Leh83]): given a m -sample of i.i.d. random variables X_1, \dots, X_m of some law P_θ , any random variable $S = S(X_1, \dots, X_m)$ is called a *statistic*. Here we consider the statistic $S = \sum_{i=1}^m X_{(k),i}$.

Definition 1. A statistic S is sufficient for the parameter θ if and only if, conditionnally to S , the law of (X_1, \dots, X_m) does not depend on θ .

More informally, S is sufficient if, given S , the knowledge of (X_1, \dots, X_m) does not give any information on θ . \hat{S} is sufficient.

Definition 2. A statistic S is complete if whenever $h(S)$ is a function of S for which $\mathbb{E}[h(T)] = 0$ for all θ , then $h \equiv 0$, P_θ almost everywhere.

Here, the statistic $S = \sum_{i=1}^m X_{(k),i}$ is complete and sufficient. Some simple criterions to check sufficientness and completeness are given in [Leh83]. Complete sufficient statistics share the following useful property:

Theorem 1 (Lehmann-Scheffé). Let S be a sufficient and complete statistic. Let ξ^* be another unbiased (i.e. $\mathbb{E}[\xi^*] = \theta$) estimator of θ . Among all the unbiased estimators of θ , $\mathbb{E}[\xi^*|S]$ has a minimal variance. Such an estimator is said to be efficient.

Corollary 1. Let $\tilde{\xi}$ another unbiased estimator of θ . Under the independent model,

$$\mathbb{E}[(\tilde{\xi} - \theta)^2] \geq \mathbb{E}[(\hat{\xi} - \theta)^2].$$

Remark 2. Note that $\text{Var}(\hat{\xi})$ is about θ^2 . This optimal bound does not depend on the algorithm, see [PD03].

3 Optimality in the real model

From now one places oneself in the *exact model*: $X_{(p),i}$ is the p -th smallest realization of θ i.i.d. r.v. uniform on $[0, 1]$, among the values lying in $[\frac{i-1}{m}, \frac{i}{m}]$. When $i \neq j$, there is now dependency between $X_{(k),i}$ and $X_{(k),j}$. Set \mathbf{P} and \mathbf{P}_{ind} the laws corresponding respectively to the exact and independant models, \mathbb{E} and \mathbb{E}_{ind} the corresponding expectations.

Lemma 1. Let A be the event

$$A = A_{k,m,\theta} : \text{“for all } i = 1, \dots, m, X_{(k),i} < \frac{1}{m}\text{”}.$$

(i.e. at least k hashed values have fallen in each of the m intervals). From a classical inequality (see [Bol85]) we get

$$\mathbf{P}(A) \geq 1 - 2me^{-\frac{\theta}{2m^2}}.$$

Here is the main result:

Theorem 2 (Optimality in the exact model). Let $\tilde{\xi}(\mathbf{X})$, with $\mathbf{X} = (X_1, \dots, X_m)$ another estimator of θ . Let $b(\theta)$ be the bias $\mathbb{E}_\theta[\tilde{\theta} - \theta]$. We assume

1. $b(\theta) = O(\sqrt{\theta})$.
2. There exists a constant C such that $|\tilde{\xi}(\mathbf{x})| \leq \frac{C}{\|\mathbf{x}\|}$, for every \mathbf{x} in \mathbb{R}^m , $\|\mathbf{x}\|$ large enough.

Then

$$\mathbb{E}_\theta[(\tilde{\theta} - \theta)^2] \geq \mathbb{E}_\theta[(\hat{\theta} - \theta)^2] + O(\theta).$$

Proof. First write

$$\begin{aligned} \mathbb{E}[(\tilde{\theta} - \theta)^2] &= \mathbb{E}[(\tilde{\theta} - \theta)^2 \mathbf{1}_A] + \mathbb{E}[(\tilde{\theta} - \theta)^2 (1 - \mathbf{1}_A)], \\ &= \mathbb{E}[(\tilde{\theta} - \theta)^2 \mathbf{1}_A] + o(\theta), \end{aligned}$$

using Lemma 1. We now bring back ourselves to the independant model:

$$\mathbb{E}[(\tilde{\theta} - \theta)^2] = \mathbb{E}_{\text{indé}}[(\tilde{\theta} - \theta)^2] + (\mathbb{E}[(\tilde{\theta} - \theta)^2 \mathbf{1}_A] - \mathbb{E}_{\text{indé}}[(\tilde{\theta} - \theta)^2]) + o(A), \quad (2)$$

The key point is the fact that conditioned to the event A , the r.v. $(X_{(k),1}, \dots, X_{(k),m})$ admit a density toward the Lebesgue measure on \mathbb{R}_+ .

$$\begin{aligned} \mathbb{E}[(\tilde{\theta} - \theta)^2] - \mathbb{E}_{\text{indé}}[(\tilde{\theta} - \theta)^2] &= \\ &= \int_{[0, \frac{1}{m}]^m} \binom{\theta}{(k-1) \dots (k-1) \ 1 \dots 1} x_1^{k-1} \dots x_m^{k-1} (1 - x_1 - \dots - x_m)^{\theta - mk} dx_1 \dots dx_m \\ &\quad - \int_{\mathbb{R}_+^m} \frac{x_1^{k-1}}{\Gamma(k)} \dots \frac{x_m^{k-1}}{\Gamma(k)} \theta^{km} e^{-\theta(x_1 + \dots + x_m)} dx_1 \dots dx_m. \end{aligned}$$

We omit the proof of the following lemma:

Lemma 2.

$$\left| \frac{\theta(\theta-1)\dots(\theta-2k+1)}{\theta^{2k}} (1 - x_1 - \dots - x_m)^{\theta-2k} - e^{-\theta(x_1 + \dots + x_m)} \right| \leq c^{\text{ste}} \theta (x_1 + \dots + x_m)^2 e^{-\theta(x_1 + \dots + x_m)},$$

where the constant depends neither on θ nor on the x_i 's.

Hence

$$\begin{aligned} |\mathbb{E}[(\tilde{\theta} - \theta)^2] - \mathbb{E}_{\text{indé}}[(\tilde{\theta} - \theta)^2]| &\leq \\ &\leq \int_{[0, \frac{1}{m}]^m} |\tilde{\theta}(x_1, \dots, x_m) - \theta|^2 \frac{x_1^{k-1}}{\Gamma(k)} \dots \frac{x_m^{k-1}}{\Gamma(k)} \theta^{km} \left\{ c^{\text{ste}} \theta (x_1 + \dots + x_m)^2 e^{-\theta(x_1 + \dots + x_m)} \right\} dx_1 \dots dx_m + O(\theta) \end{aligned}$$

Set $y_i = \theta x_i$, $i = 1 \dots m$ in the integrand:

$$\leq \int_{[0, \frac{\theta}{m}]^m} \left| \tilde{\theta}\left(\frac{y_1}{\theta}, \dots, \frac{y_m}{\theta}\right) - \theta \right|^2 \frac{x_1^{k-1}}{\Gamma(k)} \dots \frac{x_m^{k-1}}{\Gamma(k)} \theta^{-1} \left\{ c^{\text{ste}} \theta (y_1 + \dots + y_m)^2 e^{-\theta(y_1 + \dots + y_m)} \right\} dy_1 \dots dy_m + O(\theta)$$

Here we use the hypothesis made on the estimator: $\theta(\mathbf{x}) \leq \frac{1}{\|\mathbf{x}\|}$. We also need the following arithmetico-geometric inequality:

$$(a_1 \dots a_m)^\alpha \leq \lambda (a_1^2 + \dots + a_m^2)^{m\alpha/2}.$$

Set $\mathbf{y} = (y_1, \dots, y_m)$, one gets

$$\begin{aligned} &\leq c^{\text{ste}} \int_{[0, \frac{\theta}{m}]^m} \frac{\theta^2}{\|\mathbf{y}\|^2} (\|\mathbf{y}\|)^{m(k-1)} \theta^{-1} (y_1 + \dots + y_m)^2 e^{-y_1 - \dots - y_m} dy_1 \dots dy_m + O(\theta) \\ &\leq c^{\text{ste}} \int_{[0, \frac{\theta}{m}]^m} \frac{\theta^2}{\|\mathbf{y}\|^2} (\|\mathbf{y}\|)^{m(k-1)} \theta^{-1} (\|\mathbf{y}\|)^2 e^{-\|\mathbf{y}\|} dy_1 \dots dy_m + O(\theta). \end{aligned}$$

Here we make a ‘‘polar-like’’ change of variables in \mathbb{R}^m . We get this inequality:

$$\begin{aligned} |\mathbb{E}[(\tilde{\theta} - \theta)^2] - \mathbb{E}_{\text{indé}}[(\tilde{\theta} - \theta)^2]| &\leq c^{\text{ste}} \theta \int_0^{\sqrt{m} \frac{\theta}{m}} r^\alpha e^{-r} dr + O(\theta), \quad \alpha > 0. \\ &= O(\theta). \end{aligned}$$

(2) has become

$$\begin{aligned}\mathbb{E}[(\tilde{\theta} - \theta)^2] &= \mathbb{E}_{\text{indé}}[(\tilde{\theta} - \theta)^2] + O(\theta) \\ &= \mathbb{E}_{\text{indé}}[(\hat{\theta} - b(\theta) - \theta)^2] + b^2(\theta) + O(\theta) \\ &\geq \mathbb{E}_{\text{indé}}[(\hat{\theta} - \theta)^2] + O(\theta),\end{aligned}$$

□

References

- [Bol85] B. Bollobás. *Random graphs*. Academic Press, 1985.
- [BYJK⁺02] Z. Bar-Yossef, T. Jayram, R. Kumar, D. Sivakumar, and L. Trevisan. Counting distinct elements in a data stream, 2002.
- [DF93] Marianne Durand and Philippe Flajolet. Loglog counting of large cardinalities. *11th Annual European Symposium on Algorithms (ESA03)*, 1993.
- [Fel70] William Feller. *An Introduction to Probability Theory and its Applications, Vol. I*. John Wiley & sons, 1970.
- [Fla04] Philippe Flajolet. Counting by coin tossings. *ASIAN'04*, pages 1–12, 2004.
- [FM85] Philippe Flajolet and Nigel Martin. Probabilistic counting algorithms for data base applications. *Journal of Computer and System Sciences, vol 31(2)*, pages 182–209, 1985.
- [Gir05] Frédéric Giroire. Order statistics and estimating cardinalities of massive data sets. *DMTCS proceedings, International Conference on Analysis of Algorithms:157–166*, 2005.
- [Knu73] Donald Knuth. *The Art of Computer Programming, vol. 3 : Sorting and Searching*. Addison-Wesley, 1973.
- [Leh83] E.L. Lehmann. *Theory of Point Estimation*. John Wiley & sons, 1983.
- [PD03] P. Indyk and D. Woodruff. Tight lower bounds for the distinct elements problem. *Proceedings of the 44th IEEE Symposium on Foundations of Computer Science, Boston*, pages 283–290, 2003.

INSTITUT ÉLIE CARTAN NANCY (IECN)
UNIVERSITÉ HENRI POINCARÉ NANCY I
BP 239 - 54506 VANDOEUVRE-LES-NANCY CEDEX - FRANCE

{chassain,gerin}@iecn.u-nancy.fr.