

Efficient estimation of the cardinality of large data sets Philippe Chassaing, Lucas Gerin

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Abstract

F.Giroire has recently proposed an algorithm which returns the *approximate* number of distincts elements in a large sequence of words, under strong constraints coming from the analysis of large data bases. His estimation is based on statistical properties of uniform random variables in [0, 1]. Here we propose an optimal estimation, using information and estimation theory¹.

1 Introduction

The aim of this note is to improve a solution proposed by Giroire [Gir05] to the following problem: consider a sequence $Y = (Y_1, \ldots, Y_N)$ of words (one may think to a sequence of file on a disk, a list of requests, a novel from Skakespeare, *etc...*); we don't make any assumption on the structure of Y, and we want to know the number (usually denoted F_0 in the data base community) of *distinct* elements of this sequence. The motivation comes from analysis of large data sets, and especially analysis of internet traffic: certain attacks may be detected at router level, because they generate an unusual number of dictinct connections (see [Fla04]). Most of algorithms use a dictionnary to store every word, so that the memory needed is linear in F_0 . Here the size of data sets is huge, making it impossible to store every word, so that the algorithm should satisfy the two following constraints: it should use constant memory and do only one pass over the data. These constraints are very strong, but on the other hand we allow the algorithm to give only an *estimation* of F_0 .

The main idea used in [Gir05], introduced by Flajolet and Martin [FM85], is to transform this problem in a probabilistic one, using hash fuctions.

A hash function is a function $h: \mathcal{C} \to [0, 1]$, where \mathcal{C} is a finite set of words (say english language, $\{0, 1\}^8$, etc...) such that the image of a typical sequence of words behaves as a sequence of i.i.d random variables, uniform in [0, 1].

This definition is of course somewhat informal, but we will assume, from now on, that, noting $X_i = h(Y_i)$, then $\mathbf{X} = \{X_1, \ldots, X_N\}$ is the realization of F_0 i.i.d. r.v., uniform on [0, 1]. Existence and construction of good hash functions is discussed in [Knu73].

Set $\theta = F_0$ and denote as usually $X_{(1)}$ the smallest $X_i, X_{(2)}$ the second smallest, and so on. The key point is that the information on θ contained in $\{Y_1, \ldots, Y_N\}$ is equivalent to that contained in $(X_{(1)}, \ldots, X_{(\theta)})$.

As a consequence, we are now dealing with a classical statistical problem: given a (small) sample of (X_1, \ldots, X_θ) , i.i.d. r.v., uniform on [0, 1], we want to estimate the (large) parameter θ . Denote by M the memory available (how many real numbers that can be stored). One should determine:

- 1. A way of extracting a M-sample of \mathbf{X} (the M smallest, the M with the longest sequence of zeros in their binary representation, *etc...*).
- 2. A function $\hat{\xi} : [0,1]^M \to \mathbb{R}$ which approximates θ , when applied to this *M*-sample.

State of the Art. Flajolet and Martin [FM85] have used these ideas to construct an algorithm based on research of patterns of 0's and 1's in the binary representation of the hashed values X_1, \ldots, X_{θ} . It has been improved by Durand and Flajolet [DF93]. Bar-Yossef *et alii* [BYJK⁺02], have proposed 3 performant algorithms, their ideas have been generalized by Giroire [Gir05].

In a different way, Alon, Matias, and Szegedy consider estimation by *moment method*, making implementation proposed in [FM85] easier. For a nice survey about these ideas one may read [Fla04].

¹Version of January 12, 2007.

Giroire's algorithm. The starting idea in [Gir05] is to use this simple property:

$$\mathbb{E}[X_{(1)}] = \frac{1}{\theta + 1}.$$

Consequently, a naive algorithm would hash every data, compare it to the smallest hashed value already seen, and finally return $1/X_{(1)}$. Unfortunately, $\mathbb{E}[1/X_{(1)}] = \infty$. However, $1/X_{(2)}, 1/X_{(3)} \dots$ have finite expectation. This leads Giroire to propose an algorithm which return a function of $X_{(k)}$, for some k. In order to improve the precision of such an algorithm, one may wish to execute it m times with m different hashing functions, but this would cost too much time. Therefore Giroire uses *stochastic averaging*, introduced in [FM85]: the idea is to *simulate* m different experiments, by dividing [0, 1] in m intervals.

Algorithm 1.

let k, m be integers. initialize $(X_{(1),i}, \ldots, X_{(k),i}, i = 1, \ldots, m)$ with $X_{(p),i} = \frac{i}{m}$ for all i, p. for j = 1 to N $X_j = h(Y_j)$. let i the integer such that X_j lies in $[\frac{i-1}{m}, \frac{i}{m}]$. update the k-dimensional vector of k smallest values $X_{(1),i}, \ldots, X_{(k),i}$ lying in $[\frac{i-1}{m}, \frac{i}{m}]$. next j. for all p, i, renormalize $X_{(p),i} = m(X_{(p),i} - \frac{i-1}{m})$. return an estimator $\hat{\xi} = \hat{\xi}(X_{(l),i}; i = 1, \ldots, m; l = 1, \ldots, k)$.

Thus we get m vectors in \mathbb{R}^k . $X_{(k),i}$ is the k-th smallest hashed value lying in $\left[\frac{i-1}{m}, \frac{i}{m}\right]$, renormalized to get a real in [0, 1]. If less than l values have fell in the *i*-th interval, then $X_{(k),i} = 1$. Obviously, Algorithm 1 makes only one pass over each data Y_i . Memory used by the algorithm is indeed M, if we have chosen $k \cdot m = M$. The estimation returned by the algorithm does not depend on any assumption on the repetitions in the sequence X_1, \ldots, X_N .

Giroire [Gir05] proposes 3 estimators ξ_1, ξ_2, ξ_3 , using inverse function, square root function and log respectively. For example,

$$\xi_3 := \left(\frac{\Gamma(k-1/m)}{\Gamma(k)}\right)^{-m} \cdot e^{-\frac{1}{m}\sum_{i=1}^m \log X_{(k),i}} \cdot$$

For each k, m these estimators are asymptotically *unbiased*, i.e. $\mathbb{E}[\xi_i] \sim \theta$ when θ goes to ∞ . Their variances are all about 1/km. Here we give a fourth estimator, which is also asymptotically unbiased:

$$\hat{\xi} = \frac{km - 1}{\sum_{i=1}^{m} X_{(k),i}}.$$

Plan Using information and estimation theories, we first show that the estimator $\hat{\xi}$ is optimal under a simplified model, that we call the *independent model*. Then we discuss its actual optimality.

2 The best estimation under the *independant model*

In this section, \Rightarrow denotes the convergence in law. Recall that a real-valued random variable is said to follow the Gamma law with parameters (k, θ) if

$$\mathbb{P}(X \in [t, t+dt]) = \frac{t^{k-1}}{\Gamma(k)} \theta^k e^{-\theta k} \mathbf{1}_{t \ge 0} dt.$$

The asymptotic behavior of the minimum $X_{(1)}$ of θ random uniform variables in [0, 1] is well-known (see for example [Fel70]): $X_{(1)} \Rightarrow \gamma_1$, where γ_1 follows the Gamma $(1, \theta)$ law. More generally, one can prove here the following convergence

$$(\theta X_{(k),1}, \dots, \theta X_{(k),m}) \Rightarrow [\theta \to \infty] \mathcal{L}(\gamma_1, \dots, \gamma_m), \tag{1}$$

where the γ_i are i.i.d. r.v. of law Gamma(k,1). Consequently, we assume in this section that the $X_{(k),i}$ are i.i.d. r.v. of law Gamma(k, θ), this is the so-called *independent model*. We set

$$\hat{\xi} = \frac{km - 1}{\sum_{i=1}^{m} X_{(k),i}}$$

Remark 1. This estimator depends only on the *m* values $(X_{(k),i}, i = 1, ..., m)$, not on the km - m other hashed values stored by the algorithm. This follows from the fact that the knowledge of these values does not provide any information about θ . For a given *i*, conditionnally on $X_{(k),i}$, the r.v. $(X_{(1),i}, ..., X_{(k-1),i})$ are distributed uniformally on $[0, X_{(k),i}]$.

A simple calculation shows that under the independent model,

$$\mathbb{E}[\theta] = \theta,$$

$$\operatorname{Var}(\hat{\theta}) = \frac{\theta^2}{km - 2}$$

This is indeed better than the 3 estimators proposed in [Gir05].

Recall a few definitions in Statistics (see for example [Leh83]): given a *m*-sample of i.i.d. random variables X_1, \ldots, X_m of some law P_{θ} , any random variable $S = S(X_1, \ldots, X_m)$ is called a *statistic*. Here we consider the statistic $S = \sum_{i=1}^m X_{(k),i}$.

Definition 1. A statistic S is sufficient for the parameter θ if and only if, conditionnally to S, the law of (X_1, \ldots, X_m) does not depend on θ .

More informally, S is sufficient if, given S, the knowledge of (X_1, \ldots, X_m) does not give any information on θ . \hat{S} is sufficient.

Definition 2. A statistic S is complete if whenever h(S) is a function of S for which $\mathbb{E}[h(T)] = 0$ for all θ , then $h \equiv 0$, P_{θ} almost everywhere.

Here, the statistic $S = \sum_{i=1}^{m} X_{(k),i}$ is complete and sufficient. Some simple criterions to check sufficientness and completeness are given in [Leh83]. Complete sufficient statistics share the following useful property:

Theorem 1 (Lehmann-Scheffé). Let S be a sufficient and complete statistic. Let ξ^* be another unbiased (i.e. $\mathbb{E}[\tilde{\xi}] = \theta$) estimator of θ . Among all the unbiased estimators of θ , $\mathbb{E}[\xi^*|S]$ has a minimal variance. Such an estimator is said to be efficient.

Corollary 1. Let $\tilde{\xi}$ another unbiased estimator of θ . Under the independent model,

$$\mathbb{E}[(\tilde{\xi} - \theta)^2] \ge \mathbb{E}[(\hat{\xi} - \theta)^2].$$

Remark 2. Note that $Var(\hat{\xi})$ is about θ^2 . This optimal bound does not depend on the algorithm, see [PD03].

3 Optimality in the real model

From now one places oneself in the *exact model*: $X_{(p),i}$ is the *p*-th smallest realization of θ i.i.d. r.v. uniform on [0, 1], among the values lying in $[\frac{i-1}{m}, \frac{i}{m}]$. When $i \neq j$, there is now dependancy between $X_{(k),i}$ and $X_{(k),j}$. Set **P** and **P**_{ind} the laws corresponding respectively to the exact and independant models, \mathbb{E} and \mathbb{E}_{ind} the corresponding expectations.

Lemma 1. Let A be the event

$$A = A_{k,m,\theta}$$
: "for all $i = 1, ..., m, X_{(k),i} < \frac{1}{m}$ ".

(i.e. at least k hashed values have falled in each of the m intervals). From a classical inequality (see [Bol85]) we get

$$\mathbf{P}(A) \ge 1 - 2me^{-\frac{\theta}{2m^2}}.$$

Here is the main result:

Theorem 2 (Optimality in the exact model). Let $\tilde{\xi}(\mathbf{X})$, with $\mathbf{X} = (X_1, \dots, X_m)$ another estimator of θ . Let $b(\theta)$ be the bias $\mathbb{E}_{\theta}[\tilde{\theta} - \theta]$. We assume

- 1. $b(\theta) = O(\sqrt{\theta})$.
- 2. There exists a constant C such that $|\tilde{\xi}(\mathbf{x})| \leq \frac{C}{\|\mathbf{x}\|}$, for every \mathbf{x} in $\mathbb{R}^m, \|\mathbf{x}\|$ large enough.

Then

$$\mathbb{E}_{\theta}[(\tilde{\theta} - \theta)^2] \ge \mathbb{E}_{\theta}[(\hat{\theta} - \theta)^2] + O(\theta).$$

Proof. First write

.

$$\mathbb{E}[(\tilde{\theta} - \theta)^2] = \mathbb{E}[(\tilde{\theta} - \theta)^2 \mathbf{1}_A] + \mathbb{E}[(\tilde{\theta} - \theta)^2 (1 - \mathbf{1}_A)],$$
$$= \mathbb{E}[(\tilde{\theta} - \theta)^2 \mathbf{1}_A] + o(\theta),$$

using Lemma 1. We now bring back ourselves to the independant model:

$$\mathbb{E}[(\tilde{\theta} - \theta)^2] = \mathbb{E}_{\text{ind}\acute{e}}[(\tilde{\theta} - \theta)^2] + \left(\mathbb{E}[(\tilde{\theta} - \theta)^2 \mathbf{1}_A] - \mathbb{E}_{\text{ind}\acute{e}}[(\tilde{\theta} - \theta)^2]\right) + o(A),\tag{2}$$

The key point is the fact that conditioned to the event A, the r.v. $(X_{(k),1}, \ldots, X_{(k),m})$ admit a density toward the Lebesgue measure on \mathbb{R}_+ .

$$\mathbb{E}[(\tilde{\theta}-\theta)^2] - \mathbb{E}_{\text{ind}\acute{e}}[(\tilde{\theta}-\theta)^2] = \int_{[0,\frac{1}{m}]^m} \binom{\theta}{(k-1)\dots(k-1)} x_1^{k-1}\dots x_m^{k-1}(1-x_1-\dots-x_m)^{\theta-mk} dx_1\dots dx_m \\ - \int_{\mathbb{R}^m_+} \frac{x_1^{k-1}}{\Gamma(k)}\dots \frac{x_m^{k-1}}{\Gamma(k)} \theta^{km} e^{-\theta(x_1+\dots+x_m)} dx_1\dots dx_m.$$

We omit the proof of the following lemma:

Lemma 2.

$$\left|\frac{\theta(\theta-1)\cdots(\theta-2k+1)}{\theta^{2k}}(1-x_1-\ldots-x_m)^{\theta-2k}-e^{-\theta(x_1+\ldots+x_m)}\right| \le c^{ste}\theta(x_1+\ldots+x_m)^2e^{-\theta(x_1+\ldots+x_m)},$$

where the constant depends neither on θ nor on the x_i 's.

Hence

$$\begin{aligned} |\mathbb{E}[(\tilde{\theta}-\theta)^2] - \mathbb{E}_{\text{ind}\acute{e}}[(\tilde{\theta}-\theta)^2]| \leq \\ \int_{[0,\frac{1}{m}]^m} |\tilde{\theta}(x_1,\ldots,x_m) - \theta|^2 \frac{x_1^{k-1}}{\Gamma(k)} \ldots \frac{x_m^{k-1}}{\Gamma(k)} \theta^{km} \left\{ c^{\text{ste}} \theta(x_1+\ldots+x_m)^2 e^{-\theta(x_1+\ldots+x_m)} \right\} dx_1 \ldots dx_m + \mathcal{O}(\theta) \end{aligned}$$

Set $y_i = \theta x_i, i = 1 \dots m$ in the integrand:

$$\leq \int_{[0,\frac{\theta}{m}]^m} |\tilde{\theta}(\frac{y_1}{\theta},\ldots,\frac{y_m}{\theta}) - \theta|^2 \frac{x_1^{k-1}}{\Gamma(k)} \ldots \frac{x_m^{k-1}}{\Gamma(k)} \theta^{-1} \left\{ c^{\operatorname{ste}} \theta(y_1 + \ldots + y_m)^2 e^{-\theta(y_1 + \ldots + y_m)} \right\} dy_1 \ldots dy_m + \mathcal{O}(\theta)$$

Here we use the hypothesis made on the estimator: $\theta(\mathbf{x}) \leq \frac{1}{\|\mathbf{x}\|}$. We also need the following arithmeticogeometric inequality:

$$(a_1 \dots a_m)^{\alpha} \leq \lambda (a_1^2 + \dots + a_m^2)^{m\alpha/2}.$$

Set $\mathbf{y} = (y_1, \ldots, y_m)$, one gets

$$\leq c^{\text{ste}} \int_{[0,\frac{\theta}{m}]^m} \frac{\theta^2}{\|\mathbf{y}\|^2} (\|\mathbf{y}\|)^{m(k-1)} \theta^{-1} (y_1 + \ldots + y_m)^2 e^{-y_1 - \ldots - y_m} dy_1 \ldots dy_m + \mathcal{O}(\theta) \leq c^{\text{ste}} \int_{[0,\frac{\theta}{m}]^m} \frac{\theta^2}{\|\mathbf{y}\|^2} (\|\mathbf{y}\|)^{m(k-1)} \theta^{-1} (\|\mathbf{y}\|)^2 e^{-\|\mathbf{y}\|} dy_1 \ldots dy_m + \mathcal{O}(\theta).$$

Here we make a "polar-like" change of variables in \mathbb{R}^m . We get this inequality:

$$\begin{aligned} |\mathbb{E}[(\tilde{\theta} - \theta)^2] - \mathbb{E}_{\text{ind}\acute{e}}[(\tilde{\theta} - \theta)^2]| &\leq c^{\text{ste}}\theta \int_0^{\sqrt{m}\frac{\theta}{m}} r^\alpha e^{-r} dr + \mathcal{O}(\theta), \quad \alpha > 0. \\ &= \mathcal{O}(\theta). \end{aligned}$$

(2) has become

$$\begin{split} \mathbb{E}[(\tilde{\theta} - \theta)^2] &= \mathbb{E}_{\text{ind}\acute{e}}[(\tilde{\theta} - \theta)^2] + \mathcal{O}(\theta) \\ &= \mathbb{E}_{\text{ind}\acute{e}}[(\hat{\theta} - b(\theta) - \theta)^2] + b^2(\theta) + \mathcal{O}(\theta) \\ &\geq \mathbb{E}_{\text{ind}\acute{e}}[(\hat{\theta} - \theta)^2] + \mathcal{O}(\theta), \end{split}$$

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