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# Mathematical and Numerical Analysis of a Class of Non-linear Elliptic Equations in the Two Dimensional Case.

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**Summary.** The aim of this paper is to show the existence and present a numerical analysis of weak solutions for a quasi-linear elliptic problem with Dirichlet boundary conditions in a domain  $\Omega$  and data belonging to  $L^1(\Omega)$ . A numerical algorithm to compute a numerical approximation of the weak solution is described and analyzed. Numerical examples are presented and commented.

## 1 Introduction

The principal objective of this work is to give a result of existence and present a numerical analysis of weak solutions for the following quasi-linear elliptic problem:

$$\begin{cases} -\Delta u(x) + G(x, \nabla u(x)) = F(x, u(x)) + f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega \end{cases} \quad (1)$$

where  $G, F$  are Caratheodory non negative functions. The function  $f \in L^1(\Omega)$  is given finite non negative. The domain  $\Omega \subset \mathbb{R}^N$  is open and bounded. Such problems arise from biological, chemical and physical systems.

The two essential ingredients to the analysis of this problem are the convexity of  $s \rightarrow G(x, s)$  and that  $G(x, s)$  is sub-quadratic w.r.t.  $s$  namely:

$$G(x, s) \leq C(k(x) + \|s\|^2), \quad \text{where } k(x) \in L^1(\Omega) \text{ and } C > 0. \quad (2)$$

Then the problem (1) has a solution in  $W_0^{1,q}(\Omega)$  where  $1 \leq q < N/(N-1)$ ,  $N \geq 2$ , provided that (1) has a super-solution in  $W_0^{1,1}(\Omega)$ .

In previous work [4] the authors show the existence of a weak solution in the one-dimensional case and with arbitrary growth of the non linearity and data measure.

We study a numerical method to compute the solution of the problem (1). In the first step we compute a super solution using a domain decomposition method. In the second step we compute a sequence of solutions of an intermediate problem obtained by using the Yosida approximation of  $G$ . This sequence converges to the weak solution of the problem (1).

## 2 Statement of the Main Result

Throughout this paper we suppose

$$f \in L^1(\Omega), f \geq 0. \quad (3)$$

The functions  $G : \Omega \times \mathbb{R}^N \rightarrow [0, +\infty[$  and  $F : \Omega \times \mathbb{R} \rightarrow [0, +\infty[$  are such that:

$$G, F \text{ are measurable, } r \rightarrow G(x, r) \text{ and } u \rightarrow F(x, u) \text{ are continuous.} \quad (4)$$

$$G \text{ is convex in } r \text{ and } F \text{ is nondecreasing in } u, \quad (5)$$

$$G(x, 0) = \min \{G(x, r), r \in \mathbb{R}^N\} = 0, \text{ and } F(x, 0) = 0, \quad (6)$$

$$G(x, r) \leq C(|r|^2 + K(x)), \quad (7)$$

$$F(x, u) \in L^1(\Omega) \text{ for every } u \in \mathbb{R} \quad (8)$$

with a constant  $C > 0$  and  $K \in L^1(\Omega)$ .

We introduce now the notion of weak solutions of problem (1).

**Definition 1.** *A function  $u$  is said to be a weak solution of the problem (1), if*

$$\begin{cases} u \in W_0^{1,1}(\Omega), G(x, \nabla u) \text{ and } F(x, u) \in L^1(\Omega), \\ -\Delta u + G(x, \nabla u) = F(x, u) + f \text{ in } \mathcal{D}'(\Omega). \end{cases} \quad (9)$$

We will be interested in proving the existence of weak positive solutions of problem (1).

**Theorem 1.** *Under hypotheses (3)–(7), and assuming that there exists  $w$  such that*

$$\begin{cases} w \in W_0^{1,1}(\Omega), F(x, w) \in L^1(\Omega), \\ -\Delta w = F(x, w) + f \text{ in } \mathcal{D}'(\Omega), \end{cases} \quad (10)$$

*the problem (1) has a positive weak solution.*

### 3 Proof of Theorem 1

#### 3.1 Approximation Scheme

We consider the sequence defined by  $u_0 = w$  and for  $n \geq 0$ ,  $u_{n+1}$  is the solution of the problem

$$\begin{cases} -\Delta u_{n+1} + G_{n+1}(x, \nabla u_{n+1}) = F(x, u_n) + f & \text{in } \mathcal{D}'(\Omega), \\ u_{n+1} \in W_0^{1,1}(\Omega), G_{n+1}(x, \nabla u_{n+1}) \in L^1(\Omega), \end{cases} \quad (11)$$

where  $G_n(x, r)$  denotes the Yosida approximation of  $G(x, r)$ . The function  $G_n(x, r)$  is convex in  $r$ , increases pointwise to  $G(x, r)$  as  $n$  tends to  $\infty$  and satisfies

$$G_n \leq G_{n+1} \leq G, \quad \|G_{n,r}(x, r)\|_\infty \leq n, \quad (12)$$

where  $G_{n,r}$  denotes a section of subdifferential of  $G_n$  with respect to  $r$ .

The classical works ([3], [6], [9]) combined with an induction argument can be applied to prove that (11) has a solution such that

$$0 \leq u_{n+1} \leq u_n \leq w. \quad (13)$$

#### 3.2 Estimates and Convergence

Let  $\{u_n\}_n$  be a sequence defined as above. By integrating (11) in  $\Omega$  and using (13) we obtain

$$\int_\Omega G_{n+1}(x, \nabla u_{n+1}) dx \leq \int_\Omega F(x, w) dx + \int_\Omega f(x) dx. \quad (14)$$

Therefore  $\|\Delta u_{n+1}\|_{L^1(\Omega)}$  is bounded. Then there exists a subsequence still denoted by  $u_n$  for simplicity, such that  $u_n$  converges strongly to some  $u$  in  $W_0^{1,q}(\Omega)$ ,  $1 \leq q < N/(N-1)$ , and  $(u_n, \nabla u_n)$  converges to  $(u, \nabla u)$  almost everywhere in  $\Omega$  (see [8]).

Let us prove that  $u$  is in fact a solution of problem (1). According to the definition 2.1, we only have to show that

$$-\Delta u + G(x, \nabla u) = F(x, u) + f \quad \text{in } \mathcal{D}'(\Omega). \quad (15)$$

We know that  $F(x, u_n) \rightarrow F(x, u)$  strongly in  $L^1(\Omega)$  and, for almost every  $x$  in  $\Omega$ , there holds  $G_{n+1}(x, \nabla u_{n+1}(x)) \rightarrow G(x, \nabla u(x))$ .

Then there exists a non-negative measure  $\mu$  (see [7]) such that

$$-\Delta u_n + G_{n+1}(\nabla u_{n+1}) - F(u_n) - f \rightarrow -\Delta u + G(\nabla u) - F(u) - f + \mu \quad \text{in } \mathcal{D}'(\Omega),$$

as  $n$  goes to  $\infty$ .

On the other hand

$$-\Delta u_{n+1} + G_{n+1}(x, \nabla u_{n+1}) = F(x, u_n) + f \rightarrow F(x, u) + f \quad \text{in } L^1(\Omega). \quad (16)$$

Consequently

$$-\Delta u + G(x, \nabla u) \leq F(x, u) + f \text{ in } \mathcal{D}'(\Omega). \quad (17)$$

Therefore to conclude the proof of theorem 1, we must establish the opposite inequality. To this end we introduce the following test function

$$\psi \exp(-Cu_{n+1})H\left(\frac{u_{n+1}}{k}\right), \quad (18)$$

where  $H \in C^1(\mathbb{R})$ ,  $0 \leq H(s) \leq 1$ ,  $H(s) = 0$  if  $|s| \geq 1$  and  $H(s) = 1$  if  $|s| \leq \frac{1}{2}$ ,  $C$  is given by relation (7) and  $\psi \leq 0$ ,  $\psi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ . We multiply the equation satisfied by  $u_{n+1}$  in (11) by this test function and we integrate in  $\Omega$ , to obtain

$$\int_{\Omega} (f_n + F(x, u_n))\psi \exp(-Cu_{n+1})H\left(\frac{u_{n+1}}{k}\right) dx = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \int_{\Omega} \nabla u_{n+1} \nabla \psi \exp(-Cu_{n+1})H\left(\frac{u_{n+1}}{k}\right) dx, \\ I_2 &= \frac{1}{k} \int_{\Omega} |\nabla u_{n+1}|^2 \psi \exp(-Cu_{n+1})H'\left(\frac{u_{n+1}}{k}\right) dx, \\ I_3 &= \int_{\Omega} (G_{n+1}(x, \nabla u_{n+1}) - C |\nabla u_{n+1}|^2)\psi \exp(-Cu_{n+1})H\left(\frac{u_{n+1}}{k}\right) dx. \end{aligned} \quad (19)$$

By investigating separately each term, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} I_1 &= \int_{\Omega} \nabla u \nabla \psi \exp(-Cu)H\left(\frac{u}{k}\right) dx \\ \text{and } \lim_{k \rightarrow \infty} I_2 &= 0 \text{ uniformly on } n. \end{aligned}$$

Now we investigate the remaining term  $I_3$ . Since  $G_{n+1}$  satisfies the inequality (7),  $\psi \leq 0$ , and by applying Fatou's lemma, we obtain

$$\lim_{n \rightarrow \infty} I_3 \geq \int_{\Omega} (G(x, \nabla u) - C |\nabla u|^2)\psi \exp(-Cu)H\left(\frac{u}{k}\right) dx. \quad (20)$$

Finally we have shown

$$\begin{aligned} &\int_{\Omega} \nabla u \nabla \psi \exp(-Cu)H\left(\frac{u}{k}\right) dx + \int_{\Omega} \psi (G(x, \nabla u) - C |\nabla u|^2) \exp(-Cu)H\left(\frac{u}{k}\right) dx \\ &+ \omega\left(\frac{1}{k}\right) \leq \int_{\Omega} (F(x, u) + f)\psi \exp(-Cu)H\left(\frac{u}{k}\right) dx, \end{aligned}$$

where  $\omega(\varepsilon)$  denotes a quantity that tends to 0 when  $\varepsilon$  tends to 0. Now we choose  $\psi = -\varphi \exp(Cu)H\left(\frac{u}{k}\right)$ , where  $\varphi \geq 0$ ,  $\varphi \in \mathcal{D}(\Omega)$  and we replace  $\psi$  by

this value in the previous inequality to get after appropriate calculations and using that the third term is equivalent to  $\omega(\frac{1}{k})$

$$\begin{aligned} & - \int_{\Omega} \nabla u \nabla \varphi H\left(\frac{u}{k}\right)^2 dx - \int_{\Omega} \varphi G(x, \nabla u) H\left(\frac{u}{k}\right)^2 dx + \omega\left(\frac{1}{k}\right) \\ & \leq - \int_{\Omega} (F(x, u) + f) \varphi H\left(\frac{u}{k}\right)^2 dx. \end{aligned}$$

We finally pass to the limit as  $k$  tends to infinity and we use the fact that  $\lim_{k \rightarrow \infty} H\left(\frac{u}{k}\right) = 1$ , to conclude for every  $\varphi \geq 0, \varphi \in \mathcal{D}(\Omega)$  that

$$\int_{\Omega} \nabla u \nabla \varphi dx + \int_{\Omega} \varphi G(x, \nabla u) dx \geq \int_{\Omega} (F(x, u) + f) \varphi dx.$$

This finishes the proof of the theorem 1.

## 4 Numerical Method

### 4.1 Introduction

In this section we present the numerical method to solve the equation (1) in  $\mathbb{R}^2$ . Formally the algorithm can be formulated in the following way:

1) Find  $\bar{w} \in H_0^1(\Omega)$  such that:

$$-\Delta \bar{w}(x) \geq F(x, \bar{w}) + f \text{ in } \Omega. \quad (21)$$

2) Given  $u_0 = \bar{w}$  we compute a sequence,  $\{u_n\}_n$ , solution in  $H_0^1(\Omega)$  of the non linear equation:

$$-\Delta u_{n+1}(x) + G_{n+1}(x, \nabla u_{n+1}) = F(x, u_n) + f \text{ in } \Omega. \quad (22)$$

Both problems (21) and (22) are non-linear, and if (21) has a solution, in theorem 1 we have shown that (22) then also has a solution.

### 4.2 Numerical Algorithm

This subsection summarizes the algorithm introduced in the previous subsection.

1) First step: given  $\bar{w}^0 = 0$ , iteratively for  $k = 1$  until convergence we compute  $\bar{w}^{k+1} = \bar{w}^k + \delta$  where at each iteration  $\delta$  is the solution of the linear problem:

$$\begin{cases} -\Delta \delta(x) - \frac{\partial F(x, \bar{w}^k)}{\partial r} \delta(x) = \Delta \bar{w}^k(x) + F(x, \bar{w}^k) + f \text{ in } \Omega, \\ \delta(x) = 0 \text{ on } \partial\Omega. \end{cases} \quad (23)$$

To solve at each iteration the linear problem (23) we consider the domain decomposition method which will be introduced as follows:

- a) We compute  $c_\infty = \left\| \frac{\partial F(\bar{w}^k)}{\partial r} \right\|_\infty$ . Determine an overlapping subdomain decomposition  $\Omega_i, i = 1, \dots, m$  such that  $\Omega = \cup_{i=1}^m \Omega_i$  and satisfies:

$$\max\{\text{mes}(\Omega_i), i = 1, \dots, m\} < \min\left(\frac{c_0\pi^2}{c_\infty}, \frac{\pi}{2\sqrt{c_\infty}}\right). \quad (24)$$

We denote by  $m$  the number of subdomains  $\Omega_i$  and  $\partial\Omega_i$  is the boundary of  $\Omega_i$ .

- b) Iteratively:  
for  $l = 1, \dots$  until convergence and for  $i = 1, \dots, m$  we solve the following subdomain problems:

$$\begin{cases} -\Delta\delta_i^l(x) - \frac{\partial F(x, \bar{w}^k)}{\partial r}\delta_i^l(x) = \Delta\bar{w}^k(x) + F(x, \bar{w}^k) + f \text{ in } \Omega_i, \\ \delta_i^l(x) = \delta_j^{l-1}(x), \text{ on } \partial\Omega_i \cap \Omega_j, \\ \delta_i^l(x) = 0, \text{ on } \partial\Omega \cap \partial\Omega_i. \end{cases} \quad (25)$$

On each subdomain  $\Omega_i$  we consider a finite element approximation method with  $N_i$  elements. At the end of the  $l$ -th loop we have computed an approximate discrete solution of the linear indefinite problem (23).

- 2) At this step for  $u_0 = \bar{w}$ , iteratively for  $n = 1$ , until convergence we solve the following non-linear problem

$$\begin{cases} -\Delta u_n(x) + G_n(x, \nabla u_n) = F(x, u_{n-1}) + f \text{ in } \Omega, \\ u_n(x) = 0 \text{ on } \partial\Omega. \end{cases} \quad (26)$$

At each  $n$ -th step the problem (26) is solved by using a Newton method. The discrete approximation of the solution of (1) is obtained at the end of the  $n$ -th loop.

### 4.3 Convergence of the Domain Decomposition Method

To simplify, without loss of generality, we assume that we can consider a two-domain decomposition  $\Omega = \Omega_1 \cup \Omega_2$  such that:

$$\max\{\text{mes}(\Omega_i), i = 1, 2\} < \min\left(\frac{c_0\pi^2}{c_\infty}, \frac{\pi}{2\sqrt{c_\infty}}\right). \quad (27)$$

Now to prove the convergence of the Schwarz overlapping domain decomposition algorithm applied to problem (23), we consider two problems:

$$\begin{cases} -\Delta v_1(x) + c(x)v_1(x) = h(x) \text{ in } \Omega_1, \\ v_1(x) = 0 \text{ on } \partial\Omega \cap \partial\Omega_1; v_1(x) = v_2(x) \text{ on } \partial\Omega_1 \cap \Omega_2 \end{cases} \quad (28)$$

and

$$\begin{cases} -\Delta v_2(x) + c(x)v_2(x) = h(x) \text{ in } \Omega_2, \\ v_2(x) = v_1(x) \text{ on } \partial\Omega_2 \cap \Omega_1; v_2(x) = 0 \text{ on } \partial\Omega \cap \Omega_2. \end{cases} \quad (29)$$

Let  $v$  be

$$v = \begin{cases} v_1 & \text{in } \Omega_1, \\ v_2 & \text{in } \Omega_2, \end{cases} \quad (30)$$

$v_1 = v_2$  in  $\Omega_1 \cap \Omega_2$ .

With the restriction (27) we can suppose the existence of a solution of (28) in  $W_0^{1,q}(\Omega_1)$  and a solution of (29) in  $W_0^{1,q}(\Omega_2)$ .

Then, if  $v^0$  is an initialization function defined in  $\Omega$  and vanishing in  $\partial\Omega$ , we define for  $k \geq 0$  two sequences  $v_i^k$ ,  $i = 1, 2$  solving the following problems:

$$\begin{cases} -\Delta v_1^{k+1}(x) + c(x)v_1^{k+1}(x) = h(x) & \text{in } \Omega_1, \\ v_1^{k+1}(x) = 0 & \text{on } \partial\Omega \cap \partial\Omega_1; v_1^{k+1}(x) = v_2^k(x) & \text{on } \partial\Omega_1 \cap \Omega_2 \end{cases} \quad (31)$$

and

$$\begin{cases} -\Delta v_2^{k+1}(x) + c(x)v_2^{k+1}(x) = h(x) & \text{in } \Omega_2, \\ v_2^{k+1}(x) = v_1^k(x) & \text{on } \partial\Omega_2 \cap \Omega_1; v_2^{k+1}(x) = 0 & \text{on } \partial\Omega \cap \Omega_2. \end{cases} \quad (32)$$

**Theorem 2.** *Assume  $\Omega_1$  and  $\Omega_2$  with the restriction (27). Then the sequence  $v^k$  converges to  $v$  in  $W_0^{1,q}(\Omega_1)$  and  $W_0^{1,q}(\Omega_2)$ .*

*Proof.* We give here an idea of the proof.

Let  $d^k = v_1^k - v$  in  $\Omega_1$  and  $e^k = v_2^k - v$  in  $\Omega_2$  then  $d^k \in L^\infty(\Omega_1)$  and  $e^k \in L^\infty(\Omega_2)$ .

Thanks to the maximum principle we prove the following inequalities:

$$\|d^{k+2}\|_\infty \leq \gamma \|d^k\|_\infty \quad \text{and} \quad \|e^{k+2}\|_\infty \leq \gamma \|e^k\|_\infty \quad (33)$$

where  $\gamma < 1$ .

But to be able to apply the maximum principle it will be necessary that the subdomains  $\Omega_1$  and  $\Omega_2$  verify the restriction (27).

#### 4.4 Numerical Results

The algorithm introduced in the previous section has been implemented numerically for the model problem (1) where:

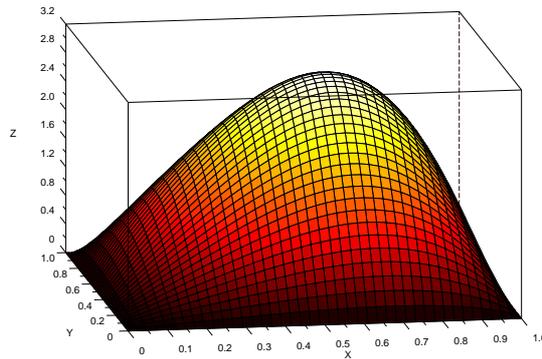
$$G(x, r) = |r|^p = (r_1^2 + r_2^2)^{\frac{p}{2}} \quad \text{and} \quad r = (r_1, r_2) \in \mathbb{R}^2 \quad \text{for } 1 < p < \infty.$$

$$F(x, s) = \eta s^q \quad \text{where } s \in \mathbb{R}^+ \quad \text{and } 1 < q < \infty.$$

$$f(x) = x_1^\alpha + x_2^\beta \quad \text{where } x = (x_1, x_2) \in \Omega \quad \text{and } -1 < \alpha, \beta < \infty.$$

The number of subdomains is not fixed, it changes at each iteration according to the criterion (27). In figure 1 we can see the solution shape when the algorithm converges with  $m = 36$  subdomains.

To study the convergence history of the numerical simulation plotted in figure 1 we consider two steps. In the first step, where we compute a super-solution, we observe the evolution of the number of subdomains: it goes from  $m = 4$  subdomains to  $m = 36$  subdomains in seven iterations according to criterion (27). Simulation stops after 34 iterations when the residual is less



**Fig. 1.**  $\eta = 45$ ,  $p = q = 3$ ,  $\alpha = \beta = 2$ ,  $m = 36$

than  $10^{-6}$ . In the second step, starting with the super-solution computed in the previous step we perform nine iterations of the Yosida approximation described in section 3 and the simulation stops when the correction computed is in uniform norm less than  $10^{-6}$ .

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