Huygens' principle for the wave equation associated with the trigonometric Dunkl-Cherednik operators

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HUYGENS' PRINCIPLE FOR THE WAVE EQUATION ASSOCIATED WITH THE TRIGONOMETRIC DUNKL-CHEREDNIK OPERATORS

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ABSTRACT. Let $\mathfrak{a}$ be an Euclidean vector space of dimension N, and let $k = (k_\alpha)_{\alpha\in\mathscr{R}}$ be a multiplicity function related to a root system $\mathscr{R}$. Let $\Delta(k)$ be the trigonometric Dunkl-Cherednik differential-difference Laplacian. For $(a,t) \in \exp(\mathfrak{a}) \times \mathbb{R}$, denote by $u_k(a,t)$ the solution to the wave equation $\Delta(k)u_k(a,t) = \partial_{tt}u_k(a,t)$, where the initial data are supported inside a ball of radius R about the origin. We prove that u_k has support in the shell $\{(a,t) \in \exp(\mathfrak{a}) \times \mathbb{R} \mid |t| - R \leq \|\log a\| \leq |t| + R\}$ if and only if the root system $\mathscr{R}$ is reduced, $k_\alpha \in \mathbb{N}$ for all $\alpha \in \mathscr{R}$, and N is odd starting from 3.

1. INTRODUCTION

It is a well known fact that propagation of waves is different in the two- and in the three-dimensional spaces. For instance, suppose we make a "noise" located near a point y at time $t = 0$. This noise can be "heard" at a point x at a later time t only if the distance $x-y$ from y to x is less then t. This is true in all dimensions, but the three-dimensional space is of a special interest. After the noise is heard, it moves away and leaves no revibration. This is what is commonly known as Huygens' principle [19]. Mathematically, a second order hyperbolic equation satisfies Huygens' principle if the solution of the corresponding Cauchy problem at some point x depends not on all the Cauchy data, but only on its part on the intersection of the characteristic conoid with vertex x with the Cauchy surface. This means that the fundamental solution of the corresponding Cauchy problem vanishes outside and inside the characteristic conoid, and thus must be located on it.

The problem of classifying all second order hyperbolic differential operators which satisfy the Huygens principle is known as the Hadamard problem [12]. This problem has received a good deal of attention and the literature is extensive [30, 20, 8, 10, 21, 31, 29, 22, 14, 2, 27, 3, 6, 1]. (Of course, this list of references is not complete.) Nevertheless, this problem is still far from being fully solved. In this paper, we will investigate a natural differential-difference operator of a similar hyperbolic nature, namely one with the same leading symbol, but with additional reflection terms.

The propagation of waves in $\mathbb{R}^N$ is governed by the wave equation

$$\textbf{(L)} \qquad \Delta^x u(x,t) = \partial_{tt}u(x,t), \qquad \text{for } (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Here Δ^x denotes the usual Laplacian operator in the x-variable, and the subscript t indicates differentiation in the t-variable. It is well known that **(L)** satisfies Huygens' principle only if N is odd starting from 3 (cf. [8]). In the present work, we will investigate the validity of Huygens' principle for **(L)** when the Laplacian Δ is replaced by the so-called trigonometric Dunkl-Cherednik Laplacian operator.

To be more precise, let W be a finite reflection group acting on an Euclidean vector space $(\mathfrak{a}, \langle\cdot,\cdot\rangle)$. Put $N := \dim(\mathfrak{a})$. Let $\mathscr{R}$ be a crystallographic root system, and let k :

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$\mathscr{R} \to \mathbb{R}^+$, $\alpha \mapsto k_\alpha$ be a multiplicity function satisfying $k_{w\alpha} = k_\alpha$ for all $w \in W$. The Dunkl-Cherednik Laplacian operator is given by

$$\begin{aligned}\Delta(k) &= \Delta_A + \sum_{\alpha\in\mathscr{R}^+} k_\alpha \coth(\alpha/2)\partial_\alpha + \langle \rho(k), \rho(k)\rangle \\ &\quad + \sum_{\alpha\in\mathscr{R}^+} k_\alpha \Big\{ \langle \alpha,\alpha\rangle \frac{e^{-\alpha}}{1-e^{-\alpha}} - 4\langle \alpha, \rho(k)\rangle \Big\} \triangle_\alpha + \sum_{\alpha,\beta\in\mathscr{R}^+} k_\alpha k_\beta \triangle_\alpha \triangle_\beta,\end{aligned}$$

where Δ_A denotes the Laplace operator on $A := \exp(\mathfrak{a})$, $\rho(k) := \frac{1}{2}\sum_{\alpha\in\mathscr{R}^+} k_\alpha \alpha$, and $\triangle_\alpha := (1-e^{-\alpha})^{-1} \otimes (1 - r_\alpha)$, where r_α is the reflection in the hyperplane $\langle\alpha\rangle^\perp$ orthogonal to the root α.

Consider the following Cauchy problem

$$\Delta(k) u_k(a,t) = \partial_{tt} u_k(a,t), \quad u_k(a,0) = f(a), \quad \partial_t u_k(a,0) = g(a),$$

where $u_k(a,t)$ is a function of $(a,t) \in A \times \mathbb{R}$, and the Cauchy data f and g are two smooth compactly supported functions on A.

For $R > 0$, let $B_R \subset \mathfrak{a}$ be the closed ball of radius R about the origin, and let $\mathscr{C}^\infty_R(A)$ be the space of smooth functions with support contained in $\exp(B_R)$. Using Opdam's Paley-Wiener theorem for the Cherednik transform [25], we prove that $u_k(a,t)$ is supported in the shell

$$\big\{(a,t) \in A \times \mathbb{R} \;\mid\; |t| - R \le \|\log a\| \le |t| + R\big\}$$

if and only if the root system $\mathscr{R}$ is reduced, $k_\alpha \in \mathbb{N}$ for all $\alpha \in \mathscr{R}^+$, and N is odd starting from 3.

As a particular case, if we consider the W-invariant situation in the above Cauchy problem, and if the parameter k corresponds to the root multiplicities of a Riemannian symmetric space G/K of the noncompact type, then we recover under the same conditions mentioned above the validity of Huygens' principle for the wave equation on G/K (cf. [3, 14, 5]).

Our approach is similar to the one used earlier by Branson, Ólafsson, and Schlichtkrull [3] for the wave equation on Riemannian symmetric spaces of the noncompact type.

2. Notations and background

Let $\mathfrak{a}$ be an Euclidean vector space of dimension N which is equipped with an inner product $\langle\cdot,\cdot\rangle$. For $x \in \mathfrak{a}$, let $\|x\| = \langle x, x\rangle^{1/2}$. Let $\mathfrak{a}_\mathbb{C}$ be the complexification of $\mathfrak{a}$, and we extend the form $\langle\cdot,\cdot\rangle$ to a bilinear form on $\mathfrak{a}_\mathbb{C} \times \mathfrak{a}_\mathbb{C}$, again denoted by $\langle\cdot,\cdot\rangle$.

For $\alpha \in \mathfrak{a}^*$, denote by $x_\alpha \in \mathfrak{a}$ the element corresponding to α, and when α is nonzero, we introduce the covector $\breve{\alpha} \in \mathfrak{a}$ of α by the formula $\breve{\alpha} = 2x_\alpha/\langle x_\alpha, x_\alpha\rangle$. Further, for $\alpha \in \mathfrak{a}^* \setminus \{0\}$, let r_α be the reflection in the hyperplane $\langle\alpha\rangle^\perp$ orthogonal to α

$$r_\alpha(\lambda) := \lambda - \lambda(\breve{\alpha})\alpha \qquad \text{for } \lambda \in \mathfrak{a}^*. \tag{2.1}$$

A crystallographic root system $\mathscr{R}$, possibly non reduced, is a finite subset in $\mathfrak{a}^* \setminus \{0\}$, such that: (i) $\mathscr{R}$ spans $\mathfrak{a}^*$, (ii) $r_\alpha(\mathscr{R}) = \mathscr{R}$, and (iii) $\alpha(\breve{\beta}) \in \mathbb{Z}$ for all $\alpha, \beta \in \mathscr{R}$.

The Weyl group W associated with $\mathscr{R}$ is the finite reflection group generated by the reflections r_α. By duality, W acts on $\mathfrak{a}_\mathbb{C}$ and on $A_\mathbb{C} := \exp(\mathfrak{a}_\mathbb{C})$.

Let $\mathscr{R}^+ \subset \mathscr{R}$ be a choice of positive roots. A multiplicity function on $\mathscr{R}$ is a W-invariant function $k : \mathscr{R} \to \mathbb{R}$. Setting $k_\alpha := k(\alpha)$, for $\alpha \in \mathscr{R}$, we have $k_{w\alpha} = k_\alpha$ for all $w \in W$. Henceforth, $\mathscr{K}$ denotes the set of all multiplicity functions on $\mathscr{R}$, and $\mathscr{K}^+$ denotes the set of $k = (k_\alpha)_{\alpha\in\mathscr{R}} \in \mathscr{K}$ such that $k_\alpha \ge 0$ for all $\alpha \in \mathscr{R}^+$.

Set $\check{\mathscr{R}} := \{\check{\alpha} \mid \alpha \in \mathscr{R}\} \subset \mathfrak{a}$. Let $\check{Q} := \mathbb{Z}\check{\mathscr{R}}$ be the coroot lattice, and let $P := (\check{Q})^*$ be the dual of $\check{Q}$. For $\alpha \in \mathscr{R}$, let $A_{\mathbb{C}}^{\alpha} := \big\{a \in A_{\mathbb{C}} \mid e^{\alpha}(a) = 1\big\}$, and define $A_{\mathbb{C}}^{\mathrm{reg}}$ to be the complementary of $\cup_{\alpha\in\mathscr{R}} A_{\mathbb{C}}^{\alpha}$ in $A_{\mathbb{C}}$. Denote by $\mathbb{C}[A_{\mathbb{C}}^{\mathrm{reg}}]$ the algebra of regular functions on $A_{\mathbb{C}}^{\mathrm{reg}}$. For $p \in \mathbb{S}(\mathfrak{a}_{\mathbb{C}})$, let $\partial(p)$ be the corresponding translation invariant differential operator on $A_{\mathbb{C}}$, so $\partial(p)e^{\mu} = p(\mu)e^{\mu}$, for $\mu \in P$.

Denote by $\mathcal{D}[A_{\mathbb{C}}^{\mathrm{reg}}]$ the algebra of differential operators on $A_{\mathbb{C}}$ with coefficients in $\mathbb{C}[A_{\mathbb{C}}^{\mathrm{reg}}]$, and by $\mathcal{D}[A_{\mathbb{C}}^{\mathrm{reg}}] \otimes \mathbb{C}[W]$ the algebra of differential-reflection operators on $A_{\mathbb{C}}^{\mathrm{reg}}$. For $\xi \in \mathfrak{a}_{\mathbb{C}}$ and $k \in \mathscr{K}$, the Dunkl-Cherednik operators $T(\xi, k) \in \mathcal{D}[A_{\mathbb{C}}^{\mathrm{reg}}] \otimes \mathbb{C}[W]$ are defined by

$$T(\xi,k) := \partial(\xi) - \rho(k)(\xi) + \sum_{\alpha\in\mathscr{R}^+} k_\alpha \alpha(\xi)\frac{1}{1-e^{-\alpha}}(1-r_\alpha),$$

where

$$\rho(k) := \frac{1}{2}\sum_{\alpha\in\mathscr{R}^+} k_\alpha \alpha \in \mathfrak{a}_{\mathbb{C}}^*.$$

In particular, for any $\xi, \eta \in \mathfrak{a}_{\mathbb{C}}$, we have $T(\xi,k)T(\eta,k) = T(\eta,k)T(\xi,k)$. We refer to [7, 25, 18] for more details on the Dunkl-Cherednik operators.

Let f and g be two elements in the space $\mathscr{C}_c^{\infty}(A)$ of smooth compactly supported functions on A, and set

$$\langle f, g\rangle_k := \int_A f(a)\overline{g(a)}d\mu(a),$$

where

$$d\mu(a) := \prod_{\alpha\in\mathscr{R}^+} |a^{\alpha/2} - a^{-\alpha/2}|^{2k_\alpha} da, \tag{2.2}$$

and da is the Lebesgue measure on A normalized by $\mathrm{Vol}(A/\exp(\check{Q})) = 1$. By [25, Lemma 7.8]

$$\langle T(\xi,k)f, g\rangle_k = \langle f, -w_0 T(w_0(\bar{\xi}), k)w_0 g\rangle_k, \tag{2.3}$$

where w_0 is the longest element in W. Fix an orthonormal basis $\{\xi_i\}_{i=1}^N$ on $\mathfrak{a}$, and set $\triangle_\alpha := (1-e^{-\alpha})^{-1} \otimes (1-r_\alpha)$. Then

$$\begin{aligned}
\Delta(k) &:= \sum_{i=1}^N T(\xi_i,k)^2 \\
&= \sum_{i=1}^N \partial(\xi_i)^2 + \sum_{\alpha\in\mathscr{R}^+} k_\alpha \coth(\alpha/2)\partial_\alpha + \langle\rho(k),\rho(k)\rangle \\
&\quad + \sum_{\alpha\in\mathscr{R}^+} k_\alpha\Big\{\langle\alpha,\alpha\rangle\frac{e^{-\alpha}}{1-e^{-\alpha}} - 4\langle\alpha,\rho(k)\rangle\Big\}\triangle_\alpha + \sum_{\alpha,\beta\in\mathscr{R}^+} k_\alpha k_\beta \triangle_\alpha\triangle_\beta.
\end{aligned}$$

This formula can be obtained by following [18], where a similar statement was proved for the Heckman-Cherednik operators.

A crucial key in Heckman-Opdam's theory of hypergeometric functions, is the existence of eigenfunctions for the Dunkl-Cherednik system of differential equations.

Theorem 1. (cf. [25]) *If $k \in \mathscr{K}^+$, then there exists an open subset U of $\exp(i\mathfrak{a})$ containing the identity element $\mathbf{e}$ such that there exists a holomorphic function $\mathbb{G}$ on $\mathfrak{a}_{\mathbb{C}}^* \times \mathscr{K}^+ \times AU$*

with the following properties

$$T^a(\xi,k)\mathbb{G}(\lambda,k,a)=\lambda(\xi)\mathbb{G}(\lambda,k,a),$$
$$\mathbb{G}(\lambda,k,\mathbf{e})=1.$$

Here the superscript denotes the relevant variable. Further, for $a\in A$ and $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$,

$$|\mathbb{G}(\lambda,k,a)|\leq|W|^{1/2}e^{\|\log a\|\,\|\mathrm{Re}(\lambda)\|}. \tag{2.4}$$

Let $f\in\mathscr{C}_c^\infty(A)$ and $\lambda\in\mathfrak{a}^*$. Following [26], we define the Cherednik transform of f by

$$\mathscr{F}_k(f)(\lambda)=\int_A f(a)\mathbb{G}(-iw_0\lambda,k,w_0a)d\mu(a),$$

where $d\mu(a)$ is given by (2.2). Using (2.3), one can check that

$$\mathscr{F}_k(T(\xi,k)f)(\lambda)=-i\lambda(\xi)\mathscr{F}_k(f)(\lambda).$$

In particular, $\mathscr{F}_k(\Delta(k)f)(\lambda)=-\|\lambda\|^2\mathscr{F}_k(f)(\lambda)$.

By [25, Theorem 9.13], the inverse transform of $\mathscr{F}_k$ is given by

$$f(a)=\int_{\mathfrak{a}^*}\mathscr{F}_k(f)(\lambda)\mathbb{G}(i\lambda,k,a)\prod_{\alpha\in\mathscr{R}_+^0}\Big(1-\frac{k_\alpha^0}{i\lambda(\breve{\alpha})}\Big)d\nu(\lambda).$$

Here $\mathscr{R}_+^0:=\mathscr{R}^0\cap\mathscr{R}^+$, where $\mathscr{R}^0$ denotes the set of un-multiplicable roots in $\mathscr{R}$, $k_\alpha^0:=\frac{1}{2}k_{\alpha/2}+k_\alpha$ for $\alpha\in\mathscr{R}_+^0$, and

$$d\nu(\lambda):=\frac{(2\pi)^{-N}|W|^{-2}\widetilde{c}(\rho(k),k)^2}{\widetilde{c}(i\lambda,k)\widetilde{c}(-i\lambda,k)}d\lambda, \tag{2.5}$$

where $\widetilde{c}$ is the meromorphic function on $\mathfrak{a}_{\mathbb{C}}^*\otimes\mathscr{K}$ defined by

$$\widetilde{c}(\lambda,k)=\prod_{\alpha\in\mathscr{R}^+}\frac{\Gamma(\lambda(\breve{\alpha})+\frac{1}{2}k_{\alpha/2})}{\Gamma(\lambda(\breve{\alpha})+\frac{1}{2}k_{\alpha/2}+k_\alpha)}. \tag{2.6}$$

In the sequel, Opdam's Paley-Wiener theorem for $\mathscr{F}_k$ is needed. Let B_R be the closed ball in $\mathfrak{a}$ with radius $R>0$ about the origin. Denote by $\mathscr{C}_R^\infty(A)$ the space of smooth functions on A with support contained in $\exp(B_R)$. Let $\mathscr{H}_R(\mathfrak{a}_{\mathbb{C}}^*)$ be the space of entire functions f on $\mathfrak{a}_{\mathbb{C}}^*$ with the property that for each integer $M>0$, there exists a constant α_M such that

$$|f(\lambda)|\leq\alpha_M(1+\|\lambda\|)^{-M}e^{R\|\mathrm{Im}(\lambda)\|},\qquad\text{for all }\ \lambda\in\mathfrak{a}_{\mathbb{C}}^*.$$

The following is a weaker version of [25, Theorem 8.6].

Theorem 2. *If $k\in\mathscr{K}^+$, then the Cherednik transform $\mathscr{F}_k$ maps $\mathscr{C}_R^\infty(A)$ bijectively onto $\mathscr{H}_R(\mathfrak{a}_{\mathbb{C}}^*)$.*

3. The wave equation for the Dunkl-Cherednik operators

For $k\in\mathscr{K}^+$, consider the following Cauchy problem for the wave equation associated with the Dunkl-Cherednik Laplacian operators

$$\begin{aligned}&\Delta(k)u_k(a,t)=\partial_{tt}u_k(a,t),\qquad(a,t)\in A\times\mathbb{R}\\&u_k(a,0)=f(a),\quad\partial_t u_k(a,0)=g(a).\end{aligned} \tag{3.1}$$

Here we assume that the functions f and g belong to the space $\mathscr{C}_R^\infty(A)$. By the principle of finite propagation speed (which holds for almost every wave equation [9]), $u_k(\cdot,t)$ is

compactly supported for every fixed $t \in \mathbb{R}$. For the time being, we assume that $k \in \mathscr{K}^+$ and $N \geq 1$.

Set $U_k(a,t) := \begin{bmatrix} u_k(a,t) \\ \partial_t u_k(a,t) \end{bmatrix}$. Thus, we may rewrite the wave equation in (3.1) as

$$\partial_t U_k(a,t) = \begin{bmatrix} 0 & 1 \\ \Delta(k) & 0 \end{bmatrix} U_k(a,t).$$

Applying the Cherednik transform $\mathscr{F}_k$ to both sides above, and using the fact that $\mathscr{F}_k(\Delta(k)f)(\lambda) = -\|\lambda\|^2 \mathscr{F}_k(f)(\lambda)$, we obtain

$$\partial_t \mathscr{F}_k(U_k(\cdot,t))(\lambda) = \begin{bmatrix} 0 & 1 \\ -\|\lambda\|^2 & 0 \end{bmatrix} \mathscr{F}_k(U_k(\cdot,t))(\lambda) := \mathbb{A}\mathscr{F}_k(U_k(\cdot,t))(\lambda).$$

Solving this ordinary differential equation, we get

$$\mathscr{F}_k(U_k(\cdot,t)) = e^{t\mathbb{A}} \mathscr{F}_k(U_k(\cdot,0))(\lambda),$$

where

$$e^{t\mathbb{A}} = \begin{bmatrix} \cos(t\|\lambda\|) & \sin(t\|\lambda\|)/\|\lambda\| \\ -\|\lambda\| \sin(t\|\lambda\|) & \cos(t\|\lambda\|) \end{bmatrix}.$$

That is[1]

$$\mathscr{F}_k(u_k(\cdot,t))(\lambda) = \mathscr{F}_k(f)(\lambda)\cos(t\|\lambda\|) + \mathscr{F}_k(g)(\lambda)\frac{\sin(t\|\lambda\|)}{\|\lambda\|}.$$

Begin with a solution $u_k(a,t)$ to the Cauchy problem (3.1), with Cauchy data (f,g). Therefore, $u_k(a,-t)$ solves (3.1) with initial data $(f,-g)$. Thus, the time is reversible, except for a minus sign that may appear when the second Cauchy datum g or its Cherednik transform are involved. So the past is determined by the present as well as the future. We shall present arguments valid for $t > 0$, and formulate the suitably altered statements for $t \in \mathbb{R}$ without further proof.

By the inversion formula of $\mathscr{F}_k$, we have

$$\begin{aligned} u_k(a,t) &= (2\pi)^{-N}|W|^{-2} \int_{\mathfrak{a}^*} \Big\{ \mathscr{F}_k(f)(\lambda)\cos(t\|\lambda\|) + \mathscr{F}_k(g)(\lambda)\frac{\sin(t\|\lambda\|)}{\|\lambda\|} \Big\} \times \\ &\qquad \mathbb{G}(i\lambda,k,a) \prod_{\alpha \in \mathscr{R}_+^0} \Big(1 - \frac{k_\alpha^0}{i\lambda(\breve{\alpha})}\Big) \frac{d\lambda}{|c(\lambda,k)|^2}, \end{aligned}$$

where

$$c(\lambda,k) := \frac{\widetilde{c}(i\lambda,k)}{\widetilde{c}(\rho(k),k)}.$$

Using the spherical coordinates, we obtain

$$\begin{aligned} u_k(a,t) &= (2\pi)^{-N}|W|^{-2} \int_0^\infty \int_S \Big\{ \mathscr{F}_k(f)(r\omega)\cos(tr) + \mathscr{F}_k(g)(r\omega)\frac{\sin(tr)}{r} \Big\} \times \\ &\qquad \mathbb{G}(ir\omega,k,a) \prod_{\alpha \in \mathscr{R}_+^0} \Big(1 - \frac{k_\alpha^0}{ir\omega(\breve{\alpha})}\Big) \frac{d\omega}{|c(r\omega,k)|^2}\, r^{N-1}\, dr, \end{aligned} \tag{3.2}$$

where $S \ni \omega$ is the unit sphere in $\mathfrak{a}^*$, and $d\omega$ is the $O(\mathfrak{a})$-invariant measure on S.

[1]Using the Plancherel formula for the transform $\mathscr{F}_k$ [25, Theorem 3.19], we can show that the total energy of the solution u_k is a conserved quantity, i.e. the total energy of u_k is independent of the variable t. As a consequence, we deduce that the solutions to the wave equation are uniquely determined by the initial Cauchy data.

On the other hand, by means of $\Gamma(2z)=\pi^{-1/2}2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})$, we may rewrite (2.6) as

$$\begin{aligned}\widetilde{c}(\lambda,k) &= \prod_{\alpha\in\mathscr{R}_+^0}\frac{2^{-k_{\alpha/2}}\Gamma\big(\lambda(\breve{\alpha})\big)\Gamma\big(\lambda(\breve{\alpha})+\frac{1}{2}\big)}{\Gamma\big(\lambda(\breve{\alpha})+\frac{1}{2}(k_{\alpha/2}+1)\big)\Gamma\big(\lambda(\breve{\alpha})+k_\alpha^0\big)}\\ &= \prod_{\alpha\in\mathscr{R}_+^0}\lambda(\breve{\alpha})^{-1}\prod_{\alpha\in\mathscr{R}_+^0}\frac{2^{-k_{\alpha/2}}\Gamma\big(\lambda(\breve{\alpha})+1\big)\Gamma\big(\lambda(\breve{\alpha})+\frac{1}{2}\big)}{\Gamma\big(\lambda(\breve{\alpha})+\frac{1}{2}(k_{\alpha/2}+1)\big)\Gamma\big(\lambda(\breve{\alpha})+k_\alpha^0\big)}.\end{aligned}$$

Thus $|c(\lambda,k)|^{-2}=\mathscr{P}(\lambda,k)\eta(\lambda,k)$, where $\lambda\mapsto\mathscr{P}(\lambda,k)$ is a homogeneous holomorphic polynomial of degree $2|\mathscr{R}_+^0|$, and $\lambda\mapsto\eta(\lambda,k)$ is a meromorphic function with the same poles as $|c(\lambda,k)|^{-2}$, such that $\eta(0,k)\neq 0$. Notice that, when the root system $\mathscr{R}$ is reduced, we have

$$\widetilde{c}(\lambda,k)=\prod_{\alpha\in\mathscr{R}^+}\frac{\Gamma\big(\lambda(\breve{\alpha})\big)}{\Gamma\big(\lambda(\breve{\alpha})+k_\alpha\big)},$$

and if in addition $k_\alpha\in\mathbb{N}$, then we obtain

$$\widetilde{c}(\lambda,k)=\prod_{\alpha\in\mathscr{R}^+}\prod_{m=0}^{k_\alpha-1}(\lambda(\breve{\alpha})+m)^{-1},$$

and

$$\begin{aligned}|c(\lambda,k)|^{-2} &= c(\rho(k),k)^2\prod_{\alpha\in\mathscr{R}^+}\lambda(\breve{\alpha})^2\prod_{\alpha\in\mathscr{R}^+}\prod_{m=1}^{k_\alpha-1}(\lambda(\breve{\alpha})^2+m^2)\\ &= \mathscr{P}(\lambda,k)\eta(\lambda,k),\end{aligned}$$

where now η is a holomorphic function. Thus, in general, (3.2) can be written as

$$u_k(a,t)=(2\pi)^{-N}|W|^{-2}\int_0^\infty\Big\{\Phi_k(r,a)\cos(tr)+\Psi_k(r,a)\frac{\sin(tr)}{r}\Big\}dr, \tag{3.3}$$

where

$$\Phi_k(r,a):=r^{N-1}\int_S\mathscr{F}_k(f)(r\omega)\mathbb{G}(ir\omega,k,a)\prod_{\alpha\in\mathscr{R}_+^0}\Big(1-\frac{k_\alpha^0}{ir\omega(\breve{\alpha})}\Big)\mathscr{P}(r\omega,k)\eta(r\omega,k)d\omega,$$

and

$$\Psi_k(r,a):=r^{N-1}\int_S\mathscr{F}_k(g)(r\omega)\mathbb{G}(ir\omega,k,a)\prod_{\alpha\in\mathscr{R}_+^0}\Big(1-\frac{k_\alpha^0}{ir\omega(\breve{\alpha})}\Big)\mathscr{P}(r\omega,k)\eta(r\omega,k)d\omega.$$

Let $\widetilde{\mathbb{S}}\subset\mathfrak{a}_{\mathbb{C}}^*$ be the singular set of $\eta(\cdot,k)$, and let $\mathbb{S}=\big\{\|\lambda\|\mid\lambda\in\widetilde{\mathbb{S}}\big\}$. If $c^{-1}(\cdot,k)$ is entire, then $\mathbb{S}=\emptyset$. Otherwise $\mathbb{S}$ attains a minimum $\gamma_0\neq 0$. If $\|\lambda\|\in\mathbb{S}_0:=\big\{\|\lambda\|\in\mathbb{S}\mid\|\lambda\|=\gamma_0\big\}$, then

$$\lambda=\pm i\,\min\Big(\frac{1}{2}(k_{\alpha/2}+1),k_\alpha^0\Big)\frac{\alpha}{2},$$

for some $\alpha\in\mathscr{R}_+^0$, and $\gamma_0=\min\big(\frac{1}{2}(k_{\alpha/2}+1),k_\alpha^0\big)\frac{|\alpha|}{2}>0$. (We exclude the case where $k_\alpha=k_{\alpha/2}=0$ for all $\alpha\in\mathscr{R}_+^0$, since this case corresponds to the classical setting where $\widetilde{c}\equiv 1$.) Recall that when $\mathscr{R}$ is reduced and $k_\alpha\in\mathbb{N}$, the function $\lambda\mapsto\eta(\lambda,k)$ is entire. We refer to this case by writing $\gamma_0=\infty$.

For $(z,\omega)\in\mathbb{C}\times S$, denote by $\eta(z,\omega,k):=\eta(z\omega,k)$ the meromorphic function in the variable $z\in\mathbb{C}$, for fixed $\omega\in S$. Here $\eta(z\omega,k)$ is the function obtained from $\eta(r\omega,k)$ by replacing r by z.

From the above discussion it follows that, for fixed $\omega\in S$, the singularities of $\eta(z,\omega,k)$ are poles, and

$$\gamma_0=\min\big\{|z|\ \big|\ \eta(\cdot,\omega,k)\text{ has poles at }z\text{ for some }\omega\in S\big\}\in\big]0,\infty\big],$$

(recall that $\gamma_0=\infty$ if and only if $\mathscr{R}$ is reduced and $k_\alpha\in\mathbb{N}$ for all α).

For $\gamma>0$, set

$$L(\gamma):=\{\tau+i\sigma\mid\ -\gamma<\sigma<\gamma\}.$$

Observe that $\eta(\cdot,\omega,k)$ is holomorphic on $L(\gamma_0)$, for each $\omega\in S$. As the measure $d\omega$ on S is $O(\mathfrak{a})$-invariant, we have:

Lemma 3. *If $N-1$ is even, then for fixed $a\in A$, the integral formulas for Φ_k and Ψ_k continue analytically to even functions for $r\in L(\gamma_0)$.*

Henceforth, we will assume that $N-1$ is even. So we can rewrite (3.3) as

$$u_k(a,t)=\frac{1}{2}(2\pi)^{-N}|W|^{-2}\int_{\mathbb{R}}\Big\{\Phi_k(r,a)+\frac{\Psi_k(r,a)}{ir}\Big\}e^{irt}dr. \tag{3.4}$$

By [3, Equation (9)], if $\gamma<\gamma_0\neq\infty$, there exists a constant c_k and $M\in\mathbb{N}$, both independent of γ, such that

$$|\eta(z,\omega,k)|\leq c_k(\gamma_0-\gamma)^{-1}(1+|z|)^M,\qquad\text{for all } z\in\overline{L(\gamma)}\text{ and }\omega\in S. \tag{3.5}$$

Clearly, if $\gamma_0=\infty$, i.e. $\mathscr{R}$ is reduced and $k_\alpha\in\mathbb{N}$ for all α, then

$$|\eta(z,\omega,k)|\leq c_k(1+|z|)^M,\qquad\text{for all } z\in\mathbb{C}\text{ and }\omega\in S.$$

Now let $z\in\overline{L(\gamma)}$. In view of Lemma 3, and since S is compact, we have

$$\begin{aligned}
&|\Phi_k(z,a)|\\
&\leq|z|^{N-1}\int_S|\mathscr{F}_k(f)(z\omega)||\mathbb{G}(iz\omega,k,a)|\prod_{\alpha\in\mathscr{R}^0_+}\Big|1-\frac{k^0_\alpha}{iz\omega(\check{\alpha})}\Big||\mathscr{P}(z\omega,k)||\eta(z,\omega,k)|d\omega\\
&\leq c'_k|z|^{N-1+|\mathscr{R}^0_+|}\sup_{\omega\in S}\Big\{|\mathscr{F}_k(f)(z\omega)||\mathbb{G}(iz\omega,k,a)|\prod_{\alpha\in\mathscr{R}^0_+}|iz\omega(\check{\alpha})-k^0_\alpha||\eta(z,\omega,k)|\Big\},
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\frac{\Psi_k(z,a)}{z}\Big|\\
&\leq c'_k|z|^{N-2+|\mathscr{R}^0_+|}\sup_{\omega\in S}\Big\{|\mathscr{F}_k(g)(z\omega)||\mathbb{G}(iz\omega,k,a)|\prod_{\alpha\in\mathscr{R}^0_+}|iz\omega(\check{\alpha})-k^0_\alpha||\eta(z,\omega,k)|\Big\}.
\end{aligned}$$

If $z=0$, the last estimate gives a problem if $N=1$ and $|\mathscr{R}^0_+|=0$. Henceforth, we add the assumption $N>1$ if $|\mathscr{R}^0_+|=0$. In view of the Paley-Wiener Theorem 2 applied to the Cauchy data (f,g), and in the light of (2.4) and (3.5), we conclude that for any $M\in\mathbb{N}$ there exist two constants $\alpha_M(k)$ and $\beta_M(k)$, depending only on (f,g) and k but not on z, such that

$$|\Phi_k(z,a)|\leq\alpha_M(k)(\gamma_0-\gamma)^{-1}(1+|z|)^{-M}e^{|\mathrm{Im}(z)|\left[R+\|\log a\|\right]},$$

and

$$\Big|\frac{\Psi_k(z,a)}{z}\Big|\leq\beta_M(k)(\gamma_0-\gamma)^{-1}(1+|z|)^{-M}e^{|\mathrm{Im}(z)|\left[R+\|\log a\|\right]},$$

for $z \in \overline{L(\gamma)}$ and $\gamma < \gamma_0 \neq \infty$. If $\gamma_0 = \infty$, the above two inequalities hold without the term $(\gamma_0 - \gamma)^{-1}$.

Next, we will shift the contour in the integral (3.4) from $\mathbb{R}$ to $\mathbb{R} + i\gamma$, so that, by means of the above inequalities, we have

$$\begin{aligned} |u_k(a,t)| &= \frac{1}{2}(2\pi)^{-N}|W|^{-2}\left|\int_{\mathbb{R}}\Big\{\Phi_k(z,a)+\frac{\Psi_k(z,a)}{iz}\Big\}e^{izt}dz\right| \\ &= \frac{1}{2}(2\pi)^{-N}|W|^{-2}e^{-\gamma t}\left|\int_{\mathbb{R}}\Big\{\Phi_k(\tau+i\gamma,a)+\frac{\Psi_k(\tau+i\gamma,a)}{i(\tau+i\gamma)}\Big\}e^{it\tau}d\tau\right| \\ &\leq \begin{cases} c_M(k)(\gamma_0-\gamma)^{-1}e^{-\gamma(t-R-\|\log a\|)}\displaystyle\int_{\mathbb{R}}(1+|\tau|)^{-M}d\tau & \text{if } \gamma<\gamma_0\neq\infty \\ c_M(k)e^{-\gamma(t-R-\|\log a\|)}\displaystyle\int_{\mathbb{R}}(1+|\tau|)^{-M}d\tau & \text{if } \gamma<\gamma_0=\infty. \end{cases} \end{aligned}$$

The following theorem collects all of the above computations and discussions.

Theorem 4. *Assume that $N-1$ is even, with the extra condition $N>1$ if $|\mathscr{R}_+^0|=0$. Let u_k be a solution to the Cauchy problem (3.1) where the Cauchy data (f,g) belong to $\mathscr{C}_R^\infty(A)$. Denote by $\gamma_0 \neq 0$ the minimal distance from a singular point of $|c(\lambda,k)|^{-2}$ to the origin in $\mathfrak{a}_{\mathbb{C}}^*$, and suppose that $0<\gamma<\gamma_0$.*

(i) *If $\gamma_0 \lneqq \infty$, then there exists a constant depending only on k and (f,g), such that*

$$|u_k(a,t)| \leq c(k)(\gamma_0-\gamma)^{-1}e^{-\gamma(|t|-R-\|\log a\|)}, \qquad \forall\,(a,t)\in A\times\mathbb{R}.$$

(ii) *If $\gamma_0=\infty$, then*

$$|u_k(a,t)| \leq c(k)\,e^{-\gamma(|t|-R-\|\log a\|)}, \qquad \forall\,(a,t)\in A\times\mathbb{R}.$$

Before we state the main result of the paper, let us make the following observation regarding the support of $u_k(a,t)$: recall that

$$\mathscr{F}_k(u_k(\cdot,t))(\lambda) = \mathscr{F}_k(f)(\lambda)\cos(t\|\lambda\|) + \mathscr{F}_k(g)(\lambda)\frac{\sin(t\|\lambda\|)}{\|\lambda\|},$$

where f and g belong to $\mathscr{C}_R^\infty(A)$. Observe that $\cos(t\|\lambda\|)$ and $\sin(t\|\lambda\|)/\|\lambda\|$ can be extended to entire functions on $\mathfrak{a}_{\mathbb{C}}^*$. Indeed, for $z\in\mathbb{C}$, the functions $\cos z$ and $\sin z/z$ are both even, and thus we may consider the functions $\cos(\sqrt{z})$ and $\sin(\sqrt{z})/\sqrt{z}$ which are entire analytic functions of z (even though $\sqrt{z}$ it is not single-valued). Thus, the analytic extensions of $\cos(t\|\lambda\|)$ and $\sin(t\|\lambda\|)/\|\lambda\|$, respectively, are

$$\cos(t\langle\lambda,\lambda\rangle^{1/2}), \qquad \frac{\sin(t\langle\lambda,\lambda\rangle^{1/2})}{\langle\lambda,\lambda\rangle^{1/2}}.$$

Further, if we write $\langle\lambda,\lambda\rangle^{1/2}=u+iv$ and use the fact that $|\cos(u+iv)|$ and $\left|\frac{\sin(u+iv)}{u+iv}\right|$ are both bounded by $e^{|v|}$, up to a constant, we obtain

$$|\cos(t\langle\lambda,\lambda\rangle^{1/2})|\ \left|\frac{\sin(t\langle\lambda,\lambda\rangle^{1/2})}{\langle\lambda,\lambda\rangle^{1/2}}\right| \leq ce^{|t||v|}.$$

We claim that $|v|\leq\|\mathrm{Im}(\lambda)\|$. One can see this as follows: since $\langle\lambda,\lambda\rangle=(u+iv)^2$, we have $u^2-v^2=\|\mathrm{Re}(\lambda)\|^2-\|\mathrm{Im}(\lambda)\|^2$ and $uv=\langle\mathrm{Re}(\lambda),\mathrm{Im}(\lambda)\rangle$. Thus, by Cauchy-Schwartz-Buniakowsly inequality, we have $u^2v^2\leq\|\mathrm{Re}(\lambda)\|^2\|\mathrm{Im}(\lambda)\|^2$, which is equivalent

to $v^2(v^2+\|\mathrm{Re}(\lambda)\|^2-\|\mathrm{Im}(\lambda)\|^2)\leq\|\mathrm{Re}(\lambda)\|^2\|\mathrm{Im}(\lambda)\|^2$. This amounts to

$$\Big(v^2+\frac{\|\mathrm{Re}(\lambda)\|^2-\|\mathrm{Im}(\lambda)\|^2}{2}\Big)^2\leq\Big(\frac{\|\mathrm{Re}(\lambda)\|^2+\|\mathrm{Im}(\lambda)\|^2}{2}\Big)^2,$$

which yields $v^2\leq\|\mathrm{Im}(\lambda)\|^2$, and the claim above holds. Now, by means of the Paley-Wiener Theorem 2 one can conclude that for all $N\geq 1$ and for every $k\in\mathscr{K}^+$, $u_k(a,t)$ is supported in the set

$$\{(a,t)\in A\times\mathbb{R}\mid\|\log a\|\leq R+|t|\}.$$

As an immediate consequence of Theorem 4 and in the light of the above observation, we obtain below our main result.

Theorem 5. (Huygens' Principle) *The solution u_k is supported in the shell*

$$\mathscr{C}=\big\{(a,t)\in A\times\mathbb{R}\ \mid|t|-R\leq\|\log a\|\leq|t|+R\big\}$$

if and only if the root system $\mathscr{R}$ is reduced, $k_\alpha\in\mathbb{N}$ for all $\alpha\in\mathscr{R}$, and N is odd starting from 3. The set $\mathscr{C}$ is the union

$$\bigcup_{\|\log b\|\leq R}\mathscr{C}_b$$

where $\mathscr{C}_b$ is the light cone

$$\mathscr{C}_b=\big\{(a,t)\in A\times\mathbb{R}\mid\|\log a-\log b\|=|t|\big\}.$$

For the remainder of this section, we will restrict ourselves to the W-invariant situation. For $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$, $k\in\mathscr{K}^+$, and $a\in A$, denote by $F(\lambda,k,a)$ the Heckman-Opdam hypergeometric function, which is, by construction, an eigenfunction for the hypergeometric system of differential equations. In particular

$$\begin{aligned}&\Delta(k)^WF(\lambda,k,a)=\langle\lambda,\lambda\rangle F(\lambda,k,a),\\&F(\lambda,k,\mathbf{e})=1,\end{aligned}$$

where $\Delta(k)^W$ is the W-invariant part of the Dunkl-Cherednik Laplacian $\Delta(k)$, i.e.

$$\Delta(k)^W=\sum_{j=1}^N\partial(\xi_j)^2+\sum_{\alpha\in\mathscr{R}^+}k_\alpha\coth(\alpha/2)\partial_\alpha+\langle\rho(k),\rho(k)\rangle.$$

Furthermore, in terms of the kernel $\mathbb{G}(\lambda,k,a)$ introduced in Theorem 1, we have

$$\begin{aligned}F(\lambda,k,a)&=\frac{1}{|W|}\sum_{w\in W}\mathbb{G}(\lambda,k,wa)\\&=\frac{1}{|W|}\sum_{w\in W}\prod_{\alpha\in\mathscr{R}_+^0}\Big(1-\frac{k_\alpha^0}{w\lambda(\breve{\alpha})}\Big)\mathbb{G}(w\lambda,k,a).\end{aligned}$$

We refer to [16, 17, 23, 24, 18] for further details on Heckman-Opdam's theory of hypergeometric functions. Moreover, if $f\in\mathscr{C}_c^\infty(A)$ is a W-invariant function, then we may rewrite the Cherednik transform of f as

$$\mathscr{F}_k(f)(\lambda)=\int_A f(a)F(-i\lambda,k,a)d\mu(a),$$

where $d\mu(a)$ is given by (2.2). When the parameter k corresponds to the root multiplicities of a Riemannian symmetric space of the noncompact type, the transform $\mathscr{F}_k$ coincides

with the spherical Harish-Chandra transform [11]. The W-invariance of f implies the W-invariance of $\mathscr{F}_k(f)$. Thus, the inverse transform of $\mathscr{F}_k(f)$ is given by

$$f(a) = |W| \int_{\mathfrak{a}^*} \mathscr{F}_k(f)(\lambda) F(i\lambda, k, a) d\nu(\lambda),$$

where $d\nu(\lambda)$ is given by (2.5).

In this setting, the Cauchy problem (3.1) becomes

$$\Delta(k)^W u_k^W(a,t) = \partial_{tt} u_k^W(a,t), \quad u_k^W(a,0) = f^W(a),\ \partial_t u_k^W(a,0) = g^W(a), \tag{3.6}$$

where f^W and g^W are two W-invariant elements in $\mathscr{C}_R^\infty(A)$. Clearly

$$\begin{aligned} u_k^W(a,t) &= \sum_{w\in W} u_k(wa,t) \\ &= |W| \int_{\mathfrak{a}^*} \Big\{ \mathscr{F}_k(f^W)(\lambda) \cos(t\|\lambda\|) + \mathscr{F}_k(g^W)(\lambda) \frac{\sin(t\|\lambda\|)}{\|\lambda\|} \Big\} F(i\lambda, k, a) d\nu(\lambda), \end{aligned}$$

where u_k is the solution to the Cauchy problem (3.1). As an immediate consequence of Theorem 5 we have:

Theorem 6. *Let $\mathscr{R}$ be a reduced root system with integral-valued multiplicity functions, and let N be odd starting from 3. Then the support of the solution u_k^W to (3.6) is contained in the shell $\Gamma = \big\{(a,t) \in A \times \mathbb{R} \mid R - |t| \le \|\log a\| \le R + |t|\big\}$.*

In these circumstances, and if in particular k corresponds to the root multiplicities of a Riemannian symmetric space G/K of the noncompact type, the above theorem implies the validity of Huygens' principle for the wave equation on G/K. The latter is a well known fact due to Solomatina [32], Helgason [14], Branson, Ólafsson, and Schlichtkrull [3], and to Chalykh and Veselov [5]. More precisely, let G be a connected noncompact semisimple Lie group with finite center, and let K be a maximal compact subgroup of G. The quotient manifold G/K can be endowed with the structure of a Riemannian symmetric space of the noncompact type. Let θ be a Cartan involution on G corresponding to K, i.e. $K = \big\{k \in G \mid \theta(k) = k\big\}$. Denote by the same letter the derived involution θ on $\mathfrak{g} := \mathrm{Lie}(G)$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to θ. Assume that $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$, and let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ be the restricted root system of $\mathfrak{a}$ in $\mathfrak{g}$. For $\alpha \in \Sigma$, set $\mathfrak{g}_\alpha := \big\{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\big\}$ to be the corresponding root space, and let $m_\alpha := \dim(\mathfrak{g}_\alpha)$. Now if we put $\mathscr{R} := \big\{2\alpha \mid \alpha \in \Sigma\big\}$ and $k_{2\alpha} := m_\alpha/2$, for $\alpha \in \Sigma$, then $\Delta(k)^W$ coincides with the radial part of the Laplace-Beltrami operator on the symmetric space G/K, up to the term $\langle \rho(k), \rho(k) \rangle$ (cf. [13]). In this setting, the condition $\mathscr{R}$ is reduced with $k_{2\alpha} \in \mathbb{N}$ amounts to the property that all Cartan subalgebras in $\mathfrak{g}$ are conjugate under the adjoint group of $\mathfrak{g}$. For completeness, we present the list of all Riemannian symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ which satisfy the latter property. We refer to [13] for more details on Riemannian symmetric spaces.

Riemannian symmetric pairs with even multiplicity

$\mathfrak{g}$	$\mathfrak{k}$	Σ	m_α	Comments
$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{su}(n)$	A_{n-1}	2	$n \geq 2$
$\mathfrak{so}(2n+1,\mathbb{C})$	$\mathfrak{so}(2n+1)$	B_n	2	$n \geq 2$
$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sp}(n)$	C_n	2	$n \geq 3$
$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{so}(2n)$	D_n	2	$n \geq 4$
$\mathfrak{so}(2n+1,1)$	$\mathfrak{so}(2n+1)$	A_1	$2n$	$n \geq 3$
$\mathfrak{su}^*(2n)$	$\mathfrak{sp}(n)$	A_{n-1}	4	$n \geq 2$
$(\mathfrak{e}_6)_{\mathbb{C}}$	$\mathfrak{e}_6$	E_6	2	–
$(\mathfrak{e}_7)_{\mathbb{C}}$	$\mathfrak{e}_7$	E_7	2	–
$(\mathfrak{e}_8)_{\mathbb{C}}$	$\mathfrak{e}_8$	E_8	2	–
$(\mathfrak{f}_4)_{\mathbb{C}}$	$\mathfrak{f}_4$	F_4	2	–
$(\mathfrak{g}_2)_{\mathbb{C}}$	$\mathfrak{g}_2$	G_2	2	–
$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_{4(-20)}$	A_2	8	–

Special isomorphisms of Riemannian symmetric pairs with even multiplicity

$\mathfrak{g}$	$\mathfrak{k}$
$\mathfrak{sp}(1,\mathbb{C}) \approx \mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{sp}(1) \approx \mathfrak{su}(2)$
$\mathfrak{so}(3,\mathbb{C}) \approx \mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{so}(3) \approx \mathfrak{su}(2)$
$\mathfrak{sp}(2,\mathbb{C}) \approx \mathfrak{so}(5,\mathbb{C})$	$\mathfrak{sp}(2) \approx \mathfrak{so}(5)$
$\mathfrak{so}(6,\mathbb{C}) \approx \mathfrak{sl}(4,\mathbb{C})$	$\mathfrak{so}(6) \approx \mathfrak{su}(4)$
$\mathfrak{so}(3,1) \approx \mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{so}(3) \approx \mathfrak{su}(2)$
$\mathfrak{so}(5,1) \approx \mathfrak{su}^*(4)$	$\mathfrak{so}(5) \approx \mathfrak{sp}(2)$

Finally, we will give explicit expressions for the solution u_k^W to (3.6). These formulas are obtained by means of the well known expressions of the solution to the classical Euclidean wave equation. This approach was used earlier in [4] for the solution to the wave equation associated with compact Riemannian symmetric spaces.

In [28, Theorem 5.1], the authors were able to give an explicit formula for the Heckman-Opdam hypergeometric functions, when $\mathscr{R}$ is reduced and $k_\alpha \in \mathbb{N}$ for all $\alpha \in \mathscr{R}$. More precisely, if

$$\Delta(k,a) := \prod_{\alpha\in\mathscr{R}^+} (e^{\alpha(\log a)} - e^{-\alpha(\log a)})^{k_\alpha}, \qquad a \in A,$$

then, for a reduced root system $\mathscr{R}$ with $k_\alpha \in \mathbb{N}$ for all $\alpha \in \mathscr{R}$, there exists a W-invariant differential operator $\mathbb{D}(k) \in \mathbb{C}[A_{\mathbb{C}}] \otimes \mathbb{S}(\mathfrak{a}_{\mathbb{C}})$ such that

$$F(\lambda,k,a) = \Delta(-k,a) \prod_{\alpha\in\mathscr{R}^+} \prod_{m=0}^{k_\alpha-1} (m^2 - \lambda(\breve{\alpha})^2)^{-1} \mathbb{D}(k)\Big(\sum_{w\in W} e^{w\lambda(X)} \Big),$$

where $a = \exp(X) \in A$. In [28, Theorem 4.10] the operator $\mathbb{D}(k)$ is expressed in terms of the Opdam shift operators. Using the W-invariance of the functions

$$\lambda \mapsto \mathscr{F}_k(f^W)(\lambda), \quad \lambda \mapsto \mathscr{F}_k(g^W)(\lambda), \quad \lambda \mapsto \cos(t\|\lambda\|), \quad \lambda \mapsto \sin(t\|\lambda\|)/\|\lambda\|,$$

we obtain

$$\begin{aligned} (3.7)\quad \mathscr{H}_k^W(a,t) &= (2\pi)^{-N}|W|\widetilde{c}(\rho(k),k)^2\Delta(-k,a) \times \\ &\quad \mathbb{D}(k)\Big[\int_{\mathfrak{a}^*} \Big\{ \mathscr{F}_k(f^W)(\lambda)\cos(t\|\lambda\|) + \mathscr{F}_k(g^W)(\lambda)\frac{\sin(t\|\lambda\|)}{\|\lambda\|} \Big\} e^{i\lambda(X)} d\lambda \Big], \end{aligned}$$

where $a = \exp(X) \in A$.

On the other hand, if $\Delta_{\mathfrak{a}}$ denotes the usual Laplacian operator on $\mathfrak{a} \cong \mathbb{R}^N$, and if we denote the Euclidean Fourier transform of a function ψ on $\mathfrak{a}$ by $\widehat{\psi}$, then, for $X \in \mathfrak{a} \cong \mathbb{R}^N$, the function

$$v(X,t) := \int_{\mathfrak{a}^*} \Big\{ \mathscr{F}_k(f^W)(\lambda)\cos(t\|\lambda\|) + \mathscr{F}_k(g^W)(\lambda)\frac{\sin(t\|\lambda\|)}{\|\lambda\|} \Big\} e^{i\lambda(X)} d\lambda$$

satisfies

$$\Delta_{\mathfrak{a}} v(X,t) = \partial_{tt} v(X,t), \quad v(X,0) = \psi(X), \quad \partial_t v(X,0) = \phi(X), \tag{3.8}$$

where $\psi, \phi \in \mathscr{C}^\infty(\mathfrak{a})^W$ such that $\widehat{\psi} = \mathscr{F}_k(f^W)$ and $\widehat{\phi} = \mathscr{F}_k(g^W)$.

For $r > 0$, let $S_r(X) \subset \mathfrak{a}$ be the Euclidean sphere of radius r and center X, and let $\Omega_N(r)$ be the surface area of $S_r(X)$. For a complex-valued function f on $\mathfrak{a}$, set

$$M^r(f)(X) = \frac{1}{\Omega_N(r)} \int_{S_r(X)} f(Y) d\sigma(Y)$$

to be the mean value of f on $S_r(X)$, where $d\sigma$ is the $O(N)$-invariant measure on $S_r(X)$. The following two facts can be found in [15, p. 481]:

(i) Put

$$I_r(f)(X) = \Big(\frac{\partial}{\partial(r^2)}\Big)^{(N-3)/2} \big(r^{N-2} M^r(f)(X)\big).$$

If $N \geq 2$ is odd, then the solution to (3.8) is given by

$$v(X,t) = \frac{\Omega_N(1)}{(N-3)!\Omega_{N-1}(1)} \Big[\frac{\partial}{\partial t}\Big(I_t(\psi)(X) \Big) + I_t(\phi)(X) \Big], \qquad (t > 0),$$

where $\Omega_N(1) = 2\pi^{N/2}\Gamma\left(\frac{N}{2}\right)^{-1}$.

(ii) Put

$$J_r(f)(X) = \Big(\frac{\partial}{\partial(r^2)}\Big)^{(N-2)/2}\left(r^{N-2}M^r(f)(X)\right).$$

If $N \geq 2$ is even, then, for $t > 0$, the solution to (3.8) is given by

$$v(X,t) = \frac{1}{(N-2)!}\Big[\frac{\partial}{\partial t}\int_0^t r(t^2-r^2)^{-1/2}J_r(\psi)(X)dr + \int_0^t r(t^2-r^2)^{-1/2}J_r(\phi)(X)dr\Big].$$

As an immediate consequence of (3.7) and the above two facts, we obtain the following explicit expressions for the solution $u_k^W(a,t)$.

Corollary 7. *Let $\mathscr{R}$ be a reduced root system such that $k_\alpha \in \mathbb{N}$ for all $\alpha \in \mathscr{R}$, and assume that $N \geq 2$. There exists a differential operator $\mathbb{D}(k) \in \mathbb{C}[A_\mathbb{C}] \otimes \mathbb{S}(\mathfrak{a}_\mathbb{C})$ such that, for all $t > 0$, the solution $u_k^W(a,t)$ to the Cauchy problem (3.6) is given by:*

(i) *If N is odd*

$$\begin{aligned} u_k^W(a,t) &= \frac{(2\pi)^{-N}|W|\Omega_N(1)}{(N-3)!\Omega_{N-1}(1)}\widetilde{c}(\rho(k),k)^2\Delta(-k,a)\times \\ &\qquad \mathbb{D}(k)\Big[\frac{\partial}{\partial t}\Big(I_t(\mathscr{F}_k(f^W)^\vee)(X)\Big) + I_t(\mathscr{F}_k(g^W)^\vee)(X)\Big], \end{aligned}$$

where $a = \exp(X) \in A$. Here $\vee$ denotes the inverse of the Euclidean Fourier transform.

(ii) *If N is even*

$$\begin{aligned} u_k^W(a,t) &= \frac{(2\pi)^{-N}|W|}{(N-2)!}\widetilde{c}(\rho(k),k)^2\Delta(-k,a)\times \\ &\qquad \mathbb{D}(k)\Big[\frac{\partial}{\partial t}\int_0^t r(t^2-r^2)^{-1/2}J_r(\mathscr{F}_k(f^W)^\vee)(X)dr \\ &\qquad\qquad + \int_0^t r(t^2-r^2)^{-1/2}J_r(\mathscr{F}_k(g^W)^\vee)(X)dr\Big], \end{aligned}$$

where $a = \exp(X) \in A$.

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