Bessel functions for root systems via the trigonometric setting

Salem Ben Said, Bent Orsted

BESSEL FUNCTIONS FOR ROOT SYSTEMS VIA THE TRIGONOMETRIC SETTING

SALEM BEN SAÏD AND BENT ØRSTED

ABSTRACT. By taking an appropriate limit, we obtain the Bessel functions related to root systems as limits of Heckman-Opdam hypergeometric functions. A more general class of Bessel functions is also investigated, which we shall call the Θ-Bessel functions. Explicit formulas for the Θ-Bessel functions are obtained when the multiplicity functions are even and positive integer-valued. This class encloses the Bessel functions on the tangent space at the origin of non-compact causal symmetric spaces, were an integral representation for these special functions is shown.

1. INTRODUCTION

In conjunction with the study of representation theory of non-compact semi-simple Lie groups, Harish-Chandra developed the theory of spherical functions on Riemannian symmetric spaces of the non-compact type in the late 1950's. Based on Harish-Chandra's point of view, the theory of spherical functions is the theory of eigenfunction expansions associated with the invariant differential operators on semi-simple symmetric spaces. In the 80s, Heckman and Opdam generalized Harish-Chandra's theory of spherical functions to a theory of multi-variables hypergeometric functions associated with root systems [17, 14, 26, 27]. Another direction has been attempted of extending the theory of Harish-Chandra to non-compact causal symmetric spaces. This was done in 1994 by Faraut-Hilgert-Ólafsson in [11]. Recently, in [32], Pasquale presented an extension of the theory of Heckman-Opdam, which also includes the non-compact causal symmetric spaces situation, by introducing the so-called Θ-spherical functions.

In this paper we study the Bessel functions related to root systems, which satisfy the Bessel system of differential equations. Our point of view is to see the Bessel functions as limit of Heckman-Opdam hypergeometric functions by analyzing a deformation of Opdam's shift operators. This leads to new shift operators. Using this approach we shall see that one may derive the same amount of results for Bessel functions as for Heckman-Opdam hypergeometric functions by a limit analysis. We should note here that in [29] Opdam also investigated the Bessel functions associated with a finite Coxeter group from another point of view (see also [15]). By using the Θ-spherical functions, we investigate a more general class of Bessel functions, which we will call the Θ-Bessel functions, and we obtain explicit formulas for the Θ-Bessel functions in the case of even multiplicity functions. This is the main new result of the paper. As a particular case, we investigate the Bessel functions on non-compact causal symmetric spaces. This limit transition was used earlier by Stanton [34], Clerc [3], and by Meaney [22], in connection with Mehler-Heine type formula.

2000 Mathematics Subject Classification. 33C80, 39A70, 43A85.

A similar limit approach was employed by Dooley and Rice [8] to investigate the principle series on flat symmetric spaces.

To be more specific about our results, let $\mathfrak{a}$ be a n-dimensional Euclidean real vector space with complexification $\mathfrak{a}_{\mathbb{C}}$, and let $\mathfrak{a}^*$ and $\mathfrak{a}^*_{\mathbb{C}}$ respectively denote the real and complex dual vector spaces of $\mathfrak{a}$. Set $\mathcal{R}$ to be a root system in $\mathfrak{a}^*$, $\mathcal{R}^+$ to be a subsystem of positive roots, and $k = (k_\alpha)_{\alpha\in\mathcal{R}}$ to be a fixed multiplicity function on $\mathcal{R}$. The Heckman-Opdam hypergeometric function $F(\lambda, k, a)$ is a two variable function: a spectral parameter $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and a space parameter $a \in A = \exp(\mathfrak{a})$. These functions are eigenvalues for the hypergeometric system of differential equations.

A key tool in the theory of Heckman and Opdam are the shift operators $\mathbb{G}_\pm(\pm\ell, k)$, where ℓ is a positive integer-valued multiplicity function. Here $\pm$ reflects the property of $\mathbb{G}_\pm$ of relating $F(\lambda, k, X)$ corresponding to multiplicity functions differing by the shift $\pm\ell$.

Set $\epsilon > 0$. After a deformation process on $\mathbb{G}_\pm$, we define the differential operators $\mathbb{G}^{(\epsilon)}_\pm(\pm\ell, k)$, and we study the limit of $\mathbb{G}^{(\epsilon)}_\pm$ as $\epsilon \to 0$. In particular, we prove that

$$\mathbb{G}^\circ_\pm(\pm\ell, k) := \lim_{\epsilon\to 0} \mathbb{G}^{(\epsilon)}_\pm(\pm\ell, k)$$

are shift operators. More properties about $\mathbb{G}^\circ_\pm$ are also obtained using Heckman-Opdam's results on the shift operators $\mathbb{G}_\pm$ by studying the limit behavior.

Next, we shall assume that $\mathcal{R}$ is reduced, and $k_\alpha \in \mathbb{Z}^+$ for all $\alpha \in \mathcal{R}$. To obtain the Bessel functions, we study the limit of $F(\frac{\lambda}{\epsilon}, k, \exp(\epsilon X))$, with $X \in \mathfrak{a}$, as ϵ goes to zero. We prove that the limit

$$F^\circ(\lambda, k, X) := \lim_{\epsilon\to 0} F(\frac{\lambda}{\epsilon}, k, \exp(\epsilon X)) \tag{1.1}$$

exists, and it satisfies the following Bessel system of differential equations (in the standard notations)

$$T^\circ(p, k)\Big|_{\mathbb{C}[\mathfrak{a}_{\mathbb{C}}]^W} F^\circ(\lambda, k, X) = p(\lambda) F^\circ(\lambda, k, X), \qquad p \in S(\mathfrak{a}_{\mathbb{C}})^W,\ \ \lambda \in \mathfrak{a}^*_{\mathbb{C}}.$$

Here $T^\circ(p, k)$ is the limit of the Dunkl-Cherednik operator, after a deformation process. The operator $T^\circ(\xi, k)$, $\xi \in \mathfrak{a}_{\mathbb{C}}$, coincides with the Dunkl differential-difference operator [5]. The limit (1.1) is obtained by induction on the multiplicity function k. The Bessel functions F° are known in the theory of Dunkl operators (see for instance [6, 17, 20, 30, 33]). However, here we obtain them via the trigonometric setting, which in particular allows us to obtain, when $k_\alpha \in \mathbb{Z}^+$, the following explicit formula

$$F^\circ(\lambda, k, X) = c_0(k) \frac{\mathbb{D}(k)\Big(\sum_{w\in W} \epsilon(w) e^{w\lambda(X)}\Big)}{\prod_{\alpha\in\mathcal{R}^+} \langle\alpha, X\rangle^{2k_\alpha} \prod_{\alpha\in\mathcal{R}^+} \langle\alpha, \lambda\rangle^{2k_\alpha - 1}},$$

for some constant $c_0(k)$ depending only on k. Here $\mathbb{D}(k)$ is a differential operator given in terms of $\mathbb{G}^\circ_+$.

This limit transition has some interest in itself: for instance, using the Plancherel formula for the Heckman-Opdam transform, proved by Opdam in [30], we give, when $k_\alpha \in \mathbb{Z}^+$ for all α, an alternative proof of de Jeu's theorem [20] on the Plancherel formula for the Dunkl transform on the space $C^\infty_c(\mathfrak{a})$. Moreover, one can use the same approach to prove a Paley-Wiener theorem for the Dunkl transform by means of Opdam's Paley-Wiener theorem in the trigonometric setting [30]. (We shall not carry out the proof of the latter claim, see Remark 3.16 below.)

The second part of the paper deals with a more general class of Bessel functions. Let Θ denote a subset in a given fundamental system Π of simple roots associated with $\mathcal{R}^+$. As mentioned above, we also treat the case of Θ-Bessel functions by studying the limit behavior of the Θ-spherical functions. In this setting, the Bessel functions F° become a special case where $\Theta = \Pi$. In particular, when the multiplicity function k is a positive integer-valued function, explicit formulas for the Θ-Bessel functions can be obtained by means of the shift operators $\mathbb{G}_+^\circ(\ell, k)$. This is done by using Ólafsson-Pasquale's results on the Θ-spherical functions (cf. [24]).

Now, let G/H be a non-compact causal symmetric space and $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ be the (appropriate) associated restricted root system. This case corresponds to $\Theta = \Pi_0$, where Π_0 is the set of simple positive compact roots in Σ^+. In this setting, we prove that for all non-compact causal symmetric spaces G/H, the Π_0-Bessel functions admit an integral representation given by (in the standard notation)

$$\int_H e^{B(A_\lambda, \mathrm{Ad}(h)X)} dh, \qquad \lambda \in \mathfrak{a}_{\mathbb{C}}^*,$$

up to a positive constant. As H is a non-compact subgroup of G, one needs to restrict both λ and X for the above integral to converge. Further, a linear convexity theorem proved in [10] is also needed here. Therefore, one can see the Π_0-Bessel functions as the Fourier transforms of orbits of H in the tangent space at the origin of the symmetric space G/H. Moreover, if the multiplicity of each root $\alpha \in \Sigma$ is even, explicit evaluation of the above integral is found by means of the explicit formulas of the Θ-Bessel functions. For instance, when G is a connected semi-simple Lie group such that $G_{\mathbb{C}}/G$ is ordered, we obtain

$$\int_G e^{B(A_\lambda, \mathrm{Ad}(g)X)} dg = c_0 \frac{\sum_{w \in W_0} \epsilon(w) e^{-w\lambda(X)}}{\prod_{\alpha \in \Sigma^+} \langle \alpha, \lambda \rangle \prod_{\alpha \in \Sigma^+} \langle \alpha, X \rangle},$$

for some positive constant c_0. Here W_0 is a subgroup of the Weyl group W. The latter integral formula could also be proved using the method of stationary phase (cf. [4, Theorem 4.1]).

The organization of this paper is as follows: In section 2 we introduce some notations and known results on the Dunkl-Cherednik operators. In the third section, first we establish the deformation principle and use it to identify the shift operators $\mathbb{G}_\pm^\circ$ with the limit of Opdam's shift operators; second we obtain the Bessel functions as the limit of Heckman-Opdam hypergeometric functions. Section 4 is devoted to studying the Θ-Bessel functions and their explicit expressions. We close the forth section by restricting ourselves to the case of Bessel functions on non-compact causal symmetric spaces.

We thank the anonymous referee for the expert comments and suggestions.

2. Notations and background

Let $\mathfrak{a}$ be a Euclidean vector space of dimension n with inner product $\langle \cdot, \cdot \rangle$. The real dual $\mathfrak{a}^*$ of $\mathfrak{a}$, consists of all $\mathbb{R}$-linear functionals on $\mathfrak{a}$. For $\alpha \in \mathfrak{a}^*$ there exists an element $X_\alpha \in \mathfrak{a}$ such that $\alpha(Y) = \langle X_\alpha, Y \rangle$ for all $Y \in \mathfrak{a}$. The formula $\langle \alpha, \beta \rangle = \langle X_\alpha, X_\beta \rangle$, for α and β in $\mathfrak{a}^*$, defines an inner product in $\mathfrak{a}^*$. For every $\alpha \in \mathfrak{a}^*$, we denote by r_α the reflection in $\mathfrak{a}^*$ defined by

$$r_\alpha(\lambda) := \lambda - \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha, \qquad \lambda \in \mathfrak{a}^*.$$

A root system in $\mathfrak{a}^*$ is a finite set $\mathcal{R}$ of non-zero elements which span $\mathfrak{a}^*$ and which satisfy $2\langle\alpha,\beta\rangle/\langle\beta,\beta\rangle \in \mathbb{Z}$, with $r_\alpha(\beta)\in\mathcal{R}$ for all $\alpha,\beta\in\mathcal{R}$. The Weyl group W of $\mathcal{R}$ is the finite group of orthogonal transformations of $\mathfrak{a}^*$ generated by $\{r_\alpha \mid \alpha\in\mathcal{R}\}$.

Let $\mathcal{R}^+$ be a choice of positive roots in $\mathcal{R}$. Define

$$\mathcal{R}_0^+ = \left\{ \alpha\in\mathcal{R}^+ \;\mid\; 2\alpha\notin\mathcal{R} \right\}.$$

We indicate by $\mathfrak{a}^+$ the open Weyl chamber in $\mathfrak{a}$ on which all elements of $\mathcal{R}^+$ are strictly positive. We set $A=\exp(\mathfrak{a})$ and $A^+=\exp(\mathfrak{a}^+)$. The polar decomposition of the complexification $A_\mathbb{C}$ of A is given by $A_\mathbb{C}=AT$, where $T:=\exp(i\mathfrak{a})$ is a compact torus in $A_\mathbb{C}$ with Lie algebra $i\mathfrak{a}$. Let $\mathfrak{a}_\mathbb{C}^*$ be the space of all $\mathbb{C}$-linear functionals on $\mathfrak{a}$. The action of W extends to $\mathfrak{a}$ by duality, to $\mathfrak{a}_\mathbb{C}^*$ and $\mathfrak{a}_\mathbb{C}$ by $\mathbb{C}$-linearity, and to $A_\mathbb{C}$ and A by the exponential map.

A multiplicity function on $\mathcal{R}$ is a W-invariant function $k:\mathcal{R}\to\mathbb{C}$. Setting $k_\alpha := k(\alpha)$ for $\alpha\in\mathcal{R}$, we have $k_{w\alpha}=k_\alpha$ for all $w\in W$. We say that a multiplicity function k is geometric if there is a Riemannian symmetric space of the non-compact type G/K with restricted root system $\mathcal{R}$, such that k_α is the multiplicity of the root α. Otherwise, k is said to be non-geometric. Let $\mathcal{K}$ be the set of all multiplicity functions on $\mathcal{R}$. If $m=\sharp\{W\text{-orbits in } \mathcal{R}\}$, then $\mathcal{K}\cong\mathbb{C}^m$.

For $k\in\mathcal{K}$, we set

$$\rho(k) := \frac{1}{2}\sum_{\alpha\in\mathcal{R}^+} k_\alpha\alpha \in \mathfrak{a}_\mathbb{C}^*,$$

and for $\alpha\in\mathcal{R}$ and $\lambda\in\mathfrak{a}_\mathbb{C}^*$, we define

$$\lambda(\check{\alpha}) = \frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}.$$

Put

$$P \;=\; \{\lambda\in\mathfrak{a}^* \mid \lambda(\check{\alpha})\in\mathbb{Z} \text{ for all } \alpha\in\mathcal{R}\}.$$

Let $\mathbb{C}[A_\mathbb{C}]$ be the algebra of regular functions (Laurent polynomials) on $A_\mathbb{C}$. It has a $\mathbb{C}$-basis e^μ indexed by $\mu\in P$, and the multiplication is given by $e^\mu e^\nu = e^{\mu+\nu}$, $e^0=1$.

The set $A_\mathbb{C}^{\mathrm{reg}} := \{h\in A_\mathbb{C} \mid e^{\alpha(\log h)}\neq 1 \text{ for all } \alpha\in\mathcal{R}\ \}$ consists of the regular elements of $A_\mathbb{C}$ for the W-action. Let $\mathbb{C}[A_\mathbb{C}^{\mathrm{reg}}]$ be the algebra of regular functions on $A_\mathbb{C}^{\mathrm{reg}}$, and let $\mathbb{C}[A_\mathbb{C}]^W$ be its W-invariant elements. Let $S(\mathfrak{a}_\mathbb{C})$ be the symmetric algebra over $\mathfrak{a}_\mathbb{C}$ considered as the space of polynomial functions on $\mathfrak{a}_\mathbb{C}$, and let $S(\mathfrak{a}_\mathbb{C})^W$ be the subalgebra of W-invariant elements. For $p\in S(\mathfrak{a}_\mathbb{C})$, let $\partial(p)$ denote the corresponding translation invariant differential operator on $A_\mathbb{C}$, so that $\partial(p)e^\mu = p(\mu)e^\mu$ for $\mu\in P$.

Set $\mathbb{D}(A_\mathbb{C}^{\mathrm{reg}}) := \mathbb{C}[A_\mathbb{C}^{\mathrm{reg}}]\otimes S(\mathfrak{a}_\mathbb{C})$ to be the algebra of differential operators on $A_\mathbb{C}$ with coefficients in $\mathbb{C}[A_\mathbb{C}^{\mathrm{reg}}]$, and let $\mathbb{D}(A_\mathbb{C}^{\mathrm{reg}})\otimes\mathbb{C}[W]$ be the algebra of differential-reflection operators on $A_\mathbb{C}^{\mathrm{reg}}$. A differential-reflection operator acts on a function f on $A_\mathbb{C}$ by $(D\otimes w)f = D(wf)$.

For $k\in\mathcal{K}$ and $\xi\in\mathfrak{a}_\mathbb{C}$, the Dunkl-Cherednik operator $T(\xi,k)\in\mathbb{D}(A_\mathbb{C}^{\mathrm{reg}})\otimes\mathbb{C}[W]$ is defined by

$$T(\xi,k) := \partial_\xi + \sum_{\alpha\in\mathcal{R}^+} k_\alpha\alpha(\xi)(1-e^{-\alpha})^{-1}\otimes(1-r_\alpha) - \rho(k)(\xi). \tag{2.1}$$

Here ∂_ξ denotes the invariant vector field on $A_\mathbb{C}$ corresponding to $\xi\in\mathfrak{a}_\mathbb{C}$. In particular, the operators $T(\xi,k)$ commute on $\mathbb{C}[A_\mathbb{C}]$, i.e. $T(\xi,k)T(\nu,k)=T(\nu,k)T(\xi,k)$

for all $\xi,\nu\in\mathfrak{a}_{\mathbb{C}}$, and for all $k\in\mathcal{K}$ (cf. [2, 5, 16]). Due to the commutativity of the Dunkl-Cherednik operators, the map $\mathfrak{a}_{\mathbb{C}}\to\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}})\otimes\mathbb{C}[W]$, $\xi\mapsto T(\xi,k)$, can be extended in a unique way to an algebra homomorphism $S(\mathfrak{a}_{\mathbb{C}})\to\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}})\otimes\mathbb{C}[W]$. The image of $p\in S(\mathfrak{a}_{\mathbb{C}})$ will be denoted by $T(p,k)$. Notice that $T(pq,\xi)=T(p,k)T(q,\xi)$ for $p,q\in S(\mathfrak{a}_{\mathbb{C}})^W$. Suppose $p\in S(\mathfrak{a}_{\mathbb{C}})^W$, then by [16]

$$T(p,k)=\sum_{w\in W}D(w,p,k)\otimes w\in\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}})\otimes\mathbb{C}[W].$$

Moreover, if $\mathbb{P}:\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}})\otimes\mathbb{C}[W]\to\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}})$ is defined by $\mathbb{P}(\sum_j D_j\otimes w)=\sum_j D_j$, then

$$D(p,k):=\mathbb{P}(T(p,k))=\sum_{w\in W}D(w,p,k)\in\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}})^W.$$

The operator $D(p,k)$ is the unique element in $\mathbb{D}(A_{\mathbb{C}}^{\mathrm{reg}})^W$ which has the same restriction to $\mathbb{C}[A_{\mathbb{C}}]^W$ as $T(p,k)$. By [16], the element $D(p,k)$ preserves $\mathbb{C}[A_{\mathbb{C}}]^W$, and $D(p,k)D(q,k)=D(pq,k)$ for $p,q\in S(\mathfrak{a}_{\mathbb{C}})^W$. Hence the set $\{D(p,k)\ \mid\ p\in S(\mathfrak{a}_{\mathbb{C}})^W\}$ is a commutative algebra of differential operators. For instance, fix an orthonormal basis $\{\xi_1,\ldots,\xi_n\}$ on $\mathfrak{a}$. If $p=\sum_{j=1}^n\xi_j^2$, then $D(p,k)=\Delta(k)+\langle\rho(k),\rho(k)\rangle$ with

$$\Delta(k):=\sum_{j=1}^n\partial_{\xi_j}^2+\sum_{\alpha\in\mathcal{R}^+}k_\alpha\coth\left(\frac{\alpha}{2}\right)\partial_\alpha,\tag{2.2}$$

where ∂_α is the first order differential operator on A (or on $\mathfrak{a}$) associated with the root α.

Example 2.1. Let $\mathfrak{g}$ be a real semisimple Lie algebra with Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$. Let $\mathfrak{a}\subset\mathfrak{p}$ be a maximal abelian subspace in $\mathfrak{p}$, and $\Sigma(\mathfrak{g},\mathfrak{a})$ be the restricted root system associated with $\mathfrak{a}$. If we put $\mathcal{R}=2\Sigma(\mathfrak{g},\mathfrak{a})$ and $k_\alpha=\frac{1}{2}m_\alpha$, where m_α is the multiplicity of the root α, then $\Delta(k)$ coincides with the radial part of the Laplace operator on the symmetric space G/K. The commuting algebra $\{D(p,k)\mid p\in S(\mathfrak{a}_{\mathbb{C}})^W\}$ represents the radial parts of the algebra $\mathbb{D}(G/K)$ of all invariant differential operators on G/K.

3. Bessel functions via Heckman-Opdam theory

3.1. **Deformation of the Opdam shift operators.** For $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$, the following system of differential equations

$$D(p,k)F=p(\lambda)F,\qquad p\in S(\mathfrak{a}_{\mathbb{C}})^W,\tag{3.1}$$

is the so-called hypergeometric system of differential equations associated with the root system $\mathcal{R}$. If $\{\xi_j\}_{j=1}^n$ is an orthonormal basis on $\mathfrak{a}$, and if $p=\sum_{j=1}^n\xi_j^2$, then (3.1) becomes

$$\Delta(k)F=(\langle\lambda,\lambda\rangle-\langle\rho(k),\rho(k)\rangle)F.\tag{3.2}$$

For geometric multiplicity k, the hypergeometric system (3.2) is the well known system of differential equations on A defining Harish-Chandra's spherical functions [12].

By the explicit expression of the differential equation (3.2), Heckman and Opdam searched for solutions for the hypergeometric system on $A^+=\exp(\mathfrak{a}^+)$ of the form

$$\Phi(\lambda,k,a)=\sum_{\kappa>0}\Gamma_\kappa(\lambda,k)e^{\lambda-\rho(k)-\kappa(\log a)},\qquad a\in A^+,$$

where $\Gamma_0(\lambda,k)=1$ and $\Gamma_\kappa(\lambda,k)\in\mathbb{C}$ satisfying some recurrence relations [17]. Using $\Phi(\lambda,k,\cdot)$, Heckman and Opdam were able to build a basis for the solutions space of the entire hypergeometric system with spectral parameter λ. This is possible if λ is generic, i.e. $\lambda(\breve{\alpha})\notin\mathbb{Z}$ for all $\alpha\in\mathcal{R}$.

Theorem 3.1. (cf. [17, 18]) *Let $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$ be a generic element. There is a connected and simply connected open subset U of T containing the identity element, such that $\{\Phi(w\lambda,k,a):w\in W\}$ forms a basis of the solution space on A^+U of the hypergeometric system (3.1).*

Let $\mathcal{Z}$ be the set of $k\in\mathcal{K}$ such that $k_\alpha\in\mathbb{Z}$ for $\alpha\in\mathcal{R}\setminus\frac{1}{2}\mathcal{R}$, and $k_\alpha\in 2\mathbb{Z}$ for $\alpha\in\mathcal{R}\cap\frac{1}{2}\mathcal{R}$. Further, let $\mathcal{Z}^+$ denote the set of multiplicity functions $k\in\mathcal{K}$ such that (i) $k_\alpha\in\mathbb{Z}^+$ for $\alpha\in\mathcal{R}\setminus\frac{1}{2}\mathcal{R}$ (ii) $k_\alpha\in2\mathbb{Z}^+$ for $\alpha\in\mathcal{R}\cap\frac{1}{2}\mathcal{R}$, and (iii) $\frac{1}{2}k_{\frac{\alpha}{2}}+k_\alpha\in\mathbb{Z}^+$ with the convention $k_{\frac{\alpha}{2}}=0$ if $\frac{\alpha}{2}\notin\mathcal{R}$. Finally we let $\mathcal{Z}^-:=-\mathcal{Z}^+$.

A key tool in the theory of Heckman and Opdam are the shift operators [26, 27]. For $k\in\mathcal{Z}$, put $\pi^\pm(k):=\prod_{\alpha\in\mathcal{R}_0^+}(\breve{\alpha}\pm(\frac{1}{2}k_{\frac{\alpha}{2}}+k_\alpha))\in S(\mathfrak{a}_{\mathbb{C}})$. Following [31], define the fundamental shift operators by

$$\begin{aligned}
\mathbb{G}_+(1,k) &:= \prod_{\alpha\in\mathcal{R}_0^+}(e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}})^{-1}T(\pi^+(k),k)\Big|_{\mathbb{C}[\mathfrak{a}_{\mathbb{C}}]^W},\\
\mathbb{G}_-(-1,k+1) &:= T(\pi^-(k),k)\circ\prod_{\alpha\in\mathcal{R}_0^+}(e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}})\Big|_{\mathbb{C}[\mathfrak{a}_{\mathbb{C}}]^W}.
\end{aligned}$$

For $\ell,\ell'\in\mathcal{Z}^+$, we have

$$\mathbb{G}_-(-\ell-\ell',k) = \mathbb{G}_-(-\ell,k-\ell')\circ\mathbb{G}_-(-\ell',k), \tag{3.3}$$

$$\mathbb{G}_+(\ell+\ell',k) = \mathbb{G}_+(\ell,k+\ell')\circ\mathbb{G}_+(\ell',k). \tag{3.4}$$

Theorem 3.2. (cf. [26, 27]) *Let $\ell\in\mathcal{Z}^+$.*

(i) *The shift operator $\mathbb{G}_-(-\ell,k)$ of shift $-\ell$ satisfies*

$$\mathbb{G}_-(-\ell,k)\Phi(\lambda,k,a)=\frac{\tilde{c}(\lambda,k-\ell)}{\tilde{c}(\lambda,k)}\Phi(\lambda,k-\ell,a)$$

for all $(\lambda,k,a)\in(\mathfrak{a}_{\mathbb{C}}^*\setminus P)\times\mathcal{K}\times A^+$, with

$$\tilde{c}(\lambda,k)=\prod_{\alpha\in\mathcal{R}^+}\frac{\Gamma(\lambda(\breve{\alpha})+\frac{1}{2}k_{\frac{\alpha}{2}})}{\Gamma(\lambda(\breve{\alpha})+\frac{1}{2}k_{\frac{\alpha}{2}}+k_\alpha)}. \tag{3.5}$$

(ii) *For $a\in A_{\mathbb{C}}$, let $\delta(k)(a):=\prod_{\alpha\in\mathcal{R}^+}(a^{\frac{\alpha}{2}}-a^{-\frac{\alpha}{2}})^{2k_\alpha}$. We have*

$$\mathbb{G}_+(\ell,k)=\delta(-\ell-k)\circ\mathbb{G}_-^*(-\ell,k+\ell)\circ\delta(k),$$

and

$$\mathbb{G}_+(\ell,k)\Phi(\lambda,k,a)=\frac{\tilde{c}(-\lambda,k)}{\tilde{c}(-\lambda,k+\ell)}\Phi(\lambda,k+\ell,a)$$

for all $(\lambda,k,a)\in(\mathfrak{a}_{\mathbb{C}}^*\setminus P)\times\mathcal{K}\times A^+$. Here $\mathbb{G}_-^*$ denotes the formal transpose of $\mathbb{G}_-$ with respect to the Haar measure on A.

For more details on the shift operators we refer to [26, 18].

Let $\mathbb{C}_\pi[\mathfrak{a}]$ denote the localization of $\mathbb{C}[\mathfrak{a}]$ along the polynomial $\pi:=\prod_{\alpha\in\mathcal{R}_0^+}\langle\alpha,\cdot\rangle\in\mathbb{C}[\mathfrak{a}]$. Denote by $\mathbb{C}_\Delta[P]$ the localization of $\mathbb{C}[P]$ along the Weyl denominator $\Delta=\prod_{\alpha\in\mathcal{R}_0^+}(e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}})$. For $D\in\mathbb{C}_\Delta[P]\otimes S(\mathfrak{a}_{\mathbb{C}})$ write $D=\sum_{m\in\mathbb{Z},m\geq m_0}D_m$ for some

$m_0 \in \mathbb{Z}$ and $D_m \in \mathbb{C}_\pi[\mathfrak{a}] \otimes S(\mathfrak{a}_\mathbb{C})$, with $[E, D_m] = mD_m$ and $E := \sum_{j=1}^n \langle \xi_j, \cdot \rangle \partial_{\xi_j}$. If $D_{m_0} \neq 0$ we call m_0 the lowest homogeneous degree of D, and we write $\mathrm{LHD}(D) = m_0$.

Lemma 3.3. (cf. [28]) *For* $\ell \in \mathcal{Z}^+$
(i) $\mathrm{LHD}(\mathbb{G}_-(-\ell, k)) = 0$,
(ii) $\mathrm{LHD}(\mathbb{G}_+(\ell, k)) = -2\sum_{\alpha\in\mathcal{R}^+} \ell_\alpha$.

Proof. The first statement is [26, Therorem 4.4]. The second statement follows directly from (i) and the fact that $\mathrm{LHD}(\delta(\ell)) = 2\sum_{\alpha\in\mathcal{R}^+} \ell_\alpha$ by using Theorem 3.2(ii). ☐

Henceforth, ϵ denotes a strictly positive small real number. For $\epsilon > 0$, define

$$T^{(\epsilon)}(\xi, k) \quad := \quad \frac{1}{\epsilon}\partial_\xi + \sum_{\alpha\in\mathcal{R}^+} k_\alpha \frac{\alpha(\xi)}{\epsilon\alpha} \sum_{m=0}^{\infty} \frac{B_m(1)\epsilon^m \alpha^m}{m!}(1 - r_\alpha) - \rho(k)(\xi),$$

where $B_m(x)$ is the n-th Bernoulli polynomial. Therefore, the following limit exists

$$\lim_{\epsilon\to 0} \epsilon\, T^{(\epsilon)}(\xi, k) = T^\circ(\xi, k),$$

with

$$T^\circ(\xi, k) = \partial_\xi + \sum_{\alpha\in\mathcal{R}^+} k_\alpha \frac{\alpha(\xi)}{\alpha}(1 - r_\alpha).$$

The differential operator $T^\circ(\xi, k)$ is the so-called Dunkl operator [5]. We are not aware of any previous work mentioning a connection of this type between the Dunkl-Cherednik operators $T(\xi, k)$ and the Dunkl operators $T^\circ(\xi, k)$. A similar relation between $T(\xi, k)$ and $T^\circ(\xi, k)$ is however given in [35].

Remark 3.4. Using the following principle

$$(\mathcal{P}) \qquad \begin{array}{l} \text{- substitute } \alpha \text{ by } \epsilon\alpha \\ \text{- substitute } \langle\cdot,\cdot\rangle \text{ by } \epsilon^2\langle\cdot,\cdot\rangle \text{ on } \mathfrak{a}, \end{array}$$

one can see that $T^{(\epsilon)}(\xi, k) = \epsilon^* \circ T(\xi, k) \circ (\epsilon^*)^{-1}$, where $\epsilon^* f(X) := f(\epsilon X)$ and $T(\xi, k)$ is the Dunkl-Cherednik operator (2.1).

For $\epsilon > 0$, set

$$\mathbb{G}_+^{(\epsilon)}(1, k) := \prod_{\alpha\in\mathcal{R}_0^+} (e^{\frac{\epsilon\alpha}{2}} - e^{-\frac{\epsilon\alpha}{2}})^{-1} T^{(\epsilon)}(\pi^+(k), k)\Big|_{\mathbb{C}[\mathfrak{a}_\mathbb{C}]^W},$$

$$\mathbb{G}_-^{(\epsilon)}(-1, k+1) := T^{(\epsilon)}(\pi^-(k), k) \circ \prod_{\alpha\in\mathcal{R}_0^+} (e^{\frac{\epsilon\alpha}{2}} - e^{-\frac{\epsilon\alpha}{2}})\Big|_{\mathbb{C}[\mathfrak{a}_\mathbb{C}]^W}.$$

Proposition 3.5. *The following limits exist*

$$\mathbb{G}_+^\circ(1, k) := \lim_{\epsilon\to 0} \epsilon^{2|\mathcal{R}_0^+|}\mathbb{G}_+^{(\epsilon)}(1, k),$$

$$\mathbb{G}_-^\circ(-1, k+1) := \lim_{\epsilon\to 0} \mathbb{G}_-^{(\epsilon)}(-1, k+1).$$

Moreover

$$\mathbb{G}_+^\circ(1,k) = \left(\prod_{\alpha\in\mathcal{R}_0^+}\alpha\right)^{-1}\left(\prod_{\alpha\in\mathcal{R}_0^+}T^\circ(\check{\alpha},k)\right)_{\Big|\mathbb{C}[\mathfrak{a}_\mathbb{C}]^W},$$

$$\mathbb{G}_-^\circ(-1,k+1) = \left(\prod_{\alpha\in\mathcal{R}_0^+}T^\circ(\check{\alpha},k)\right)\circ\left(\prod_{\alpha\in\mathcal{R}_0^+}\alpha\right)_{\Big|\mathbb{C}[\mathfrak{a}_\mathbb{C}]^W}.$$

Proof. The proposition holds by using the fact that $\mathrm{LHD}\big[\prod_{\alpha\in\mathcal{R}_0^+}(e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}})^{-1}\big] = -|\mathcal{R}_0^+|$, and that $\epsilon\, T^{(\epsilon)}(\xi,k)$ converges to $T^\circ(\xi,k)$ as $\epsilon\to 0$. $\square$

Remark 3.6. The shift operators $\mathbb{G}_\pm^\circ(\pm 1,k)$ appeared earlier in [28] and in [15], from a different point of view.

By (3.3) and (3.4), for $\ell\in\mathcal{Z}^+$ one can define the shift operators $\mathbb{G}_\pm^{(\epsilon)}(\pm\ell,k)$ as composition of the deformed fundamental shift operators $\mathbb{G}_\pm^{(\epsilon)}(\pm 1,k)$. In particular, by Lemma 3.3 the following limits hold

$$\lim_{\epsilon\to 0}\epsilon^{2\sum_{\alpha>0}\ell_\alpha}\mathbb{G}_+^{(\epsilon)}(\ell,k) = \mathbb{G}_+^\circ(\ell,k), \tag{3.6}$$

$$\lim_{\epsilon\to 0}\mathbb{G}_-^{(\epsilon)}(-\ell,k) = \mathbb{G}_-^\circ(-\ell,k), \tag{3.7}$$

where $\mathbb{G}_\pm^\circ(\pm\ell,k)$ are the shift operators obtained as composition of the shift operators $\mathbb{G}_\pm^\circ(\pm 1,k)$. For $\ell,\ell'\in\mathcal{Z}^+$ and for $k\in\mathcal{Z}$

$$\mathbb{G}_-^\circ(-\ell-\ell',k) = \mathbb{G}_-^\circ(-\ell,k-\ell')\circ\mathbb{G}_-^\circ(-\ell',k), \tag{3.8}$$

$$\mathbb{G}_+^\circ(\ell+\ell',k) = \mathbb{G}_+^\circ(\ell,k+\ell')\circ\mathbb{G}_+^\circ(\ell',k). \tag{3.9}$$

Set $(\cdot,\cdot)_k$ to be the inner product on $\mathbb{C}[\mathfrak{a}_\mathbb{C}]$ defined by

$$(f,g)_k := \int_{i\mathfrak{a}} f(X)\overline{g(X)}\prod_{\alpha\in\mathcal{R}^+}|\langle\alpha,X\rangle|^{2k_\alpha}dX.$$

Here we collect some properties about $\mathbb{G}_\pm^\circ(\ell,k)$ with $\ell\in\mathcal{Z}^\pm$, respectively.

Lemma 3.7. (i) *The differential operators $\mathbb{G}_\pm^\circ(\ell,k)$ transform $\mathbb{C}[\mathfrak{a}_\mathbb{C}]^W$ to $\mathbb{C}[\mathfrak{a}_\mathbb{C}]^W$.*

(ii) *For all $f,g\in\mathbb{C}[\mathfrak{a}_\mathbb{C}]^W$ and $\ell\in\mathcal{Z}^+$*

$$\big(\mathbb{G}_+^\circ(\ell,k-\ell)f,g\big)_k = (-1)^{\sum_{\alpha>0}\ell_\alpha}\big(f,\mathbb{G}_-^\circ(-\ell,k)g\big)_{k-\ell}.$$

(iii) *For all $p\in S(\mathfrak{a}_\mathbb{C})^W$ and $\ell\in\mathcal{Z}^+$*

$$T^\circ(p,k\pm\ell)\mathbb{G}_\pm^\circ(\pm\ell,k) = \mathbb{G}_\pm^\circ(\pm\ell,k)T^\circ(p,k).$$

(iv) *For any W-invariant holomorphic germ f at 0 we have*

$$(\mathbb{G}_-^\circ(\ell,k)f)(0) = \frac{\tilde{c}(\rho(k+\ell),k+\ell)}{\tilde{c}(\rho(k),k)}f(0), \qquad \ell\in\mathcal{Z}^-$$

where $\tilde{c}$ is given by (3.5).

Proof. Using the fact that $\mathrm{LHD}(\delta(k)) = 2\sum_{\alpha\in\mathcal{R}^+} k_\alpha$, we can deduce that

$$\lim_{\epsilon\to 0} \epsilon^{-2\sum_{\alpha\in\mathcal{R}^+} k_\alpha} \prod_{\alpha\in\mathcal{R}^+} |e^{\frac{\epsilon\alpha}{2}} - e^{-\frac{\epsilon\alpha}{2}}|^{2k_\alpha} = \prod_{\alpha\in\mathcal{R}^+} |\langle\alpha,\cdot\rangle|^{2k_\alpha}.$$

Now the lemma holds from Lemma 3.3, Proposition 3.5, and from the fact that the same statements hold also for the shift operators $\mathbb{G}_\pm(\pm\ell, k)$ (cf. [18, 31]). □

3.2. **Limit transition of Heckman-Opdam hypergeometric functions.** For $(\lambda, k) \in \mathfrak{a}_\mathbb{C}^* \times \mathcal{K}$, define the following meromorphic c-function

$$c(\lambda, k) = \frac{\tilde{c}(\lambda, k)}{\tilde{c}(\rho(k), k)},$$

where $\tilde{c}$ is given by (3.5). The function

$$F(\lambda, k, a) := \sum_{w\in W} c(w\lambda, k)\Phi(w\lambda, k, a),$$

is the so-called Heckman-Opdam hypergeometric function associated with the root system $\mathcal{R}$.

Theorem 3.8. (cf. [17, 18]) *Let $S = \{$ zeros of the entire function $\tilde{c}(\rho(k), k)$ $\}$. The hypergeometric function $F(\lambda, k, a)$ is a holomorphic function on $\mathfrak{a}_\mathbb{C}^* \times (\mathcal{K}\setminus S) \times U$, with U a W-invariant tubular neighborhood of A in $A_\mathbb{C}$, and it satisfies*

$$F(w\lambda, k, a) = F(\lambda, k, wa) = F(\lambda, k, a), \tag{3.10}$$

for all $w \in W$ and $(\lambda, k, a) \in \mathfrak{a}_\mathbb{C}^* \times (\mathcal{K}\setminus S) \times U$.

Heckman-Opdam's hypergeometric function is by construction a solution to (3.1), and coincides with Harish-Chandra's spherical function in the geometric case.

Remark 3.9. The singular parameter set S is explicitly determined in [7]. The results in [7, Section 4] show by inspection that integral k's are not among the singular set. However, for simplicity, we shall always assume that $k \in \mathcal{Z}^+$.

Fix an orthonormal basis $\{\xi_j\}_{j=1}^n$ on $\mathfrak{a}$.

Lemma 3.10. *If $\tilde{p} = \sum_{j=1}^n \xi_j^2$, then*

$$T^\circ(\tilde{p}, k)\Big|_{\mathbb{C}[\mathfrak{a}_\mathbb{C}]^W} = \sum_{j=1}^n \partial_{\xi_j}^2 + \sum_{\alpha\in\mathcal{R}^+} k_\alpha \left(\frac{2}{\alpha}\right)\partial_\alpha. \tag{3.11}$$

We shall denote the right hand side by $\Delta^\circ(k)$.

Proof. Recall that $\mathbb{P}(T(\tilde{p}, k)) = D(\tilde{p}, k)$ where $D(\tilde{p}, k) = \Delta(k) + \langle\rho(k), \rho(k)\rangle$ (see the end of Section 2). Using the principle $(\mathcal{P})$ for $\Delta(k)$, we get

$$\Delta^{(\epsilon)}(k) := \frac{1}{\epsilon^2}\sum_{j=1}^n \partial_{\xi_j}^2 + \frac{1}{\epsilon^2}\sum_{\alpha\in\mathcal{R}^+} \frac{2k_\alpha}{\alpha}\partial_\alpha + \sum_{\alpha\in\mathcal{R}^+}\sum_{m=1}^\infty \frac{k_\alpha B_{2m}(0)}{(2m!)!}\left(\frac{\alpha}{2}\right)^{2m-1}\epsilon^{2(m-1)}\partial_\alpha,$$

where $B_{2m}(0)$ is the Bernoulli number. Therefore

$$\lim_{\epsilon\to 0}\epsilon^2\Delta^{(\epsilon)}(k) = \Delta^\circ(k),$$

and the lemma holds. □

From the definition of the $\tilde{c}$-function, one can see that $\tilde{c}(\lambda, 0) = 1$. By [18, (3.5.14)] $\lim_{k\to 0} \tilde{c}(\rho(k), k) = |W|$ and therefore

$$F(\lambda, 0, a) = \frac{1}{|W|} \sum_{w\in W} e^{w\lambda(\log a)}, \qquad a \in A.$$

In particular, the following limit formula holds

$$\lim_{\epsilon\to 0} F(\frac{\lambda}{\epsilon}, 0, \exp(\epsilon X)) = \frac{1}{|W|} \sum_{w\in W} e^{w\lambda(X)}, \qquad X \in \mathfrak{a}.$$

Next, for $X \in \mathfrak{a}$, we will denote by $F^\circ(\lambda, k, X)$ the limit of $F(\frac{\lambda}{\epsilon}, k, \exp(\epsilon X))$ as $\epsilon \to 0$, if it exists.

Since the main applications of our results to symmetric spaces are cases where the associated root systems are reduced, we shall restrict ourselves to the case where $\mathcal{R}$ is reduced. After this article was finished, de Jeu proved in [21, Theorem 4.13] that the following limit transition still holds for general positive multiplicities.

Theorem 3.11. *Assume that $\mathcal{R}$ is reduced. For all $k \in \mathcal{Z}^+$, the following limit and its derivatives exist*

$$F^\circ(\lambda, k, X) := \lim_{\epsilon\to 0} F(\frac{\lambda}{\epsilon}, k, \exp(\epsilon X)),$$

and it satisfies the following Bessel system of differential equations on $W \setminus \mathfrak{a}_{\mathbb{C}}$

$$T^\circ(p, k)\Big|_{\mathbb{C}[\mathfrak{a}_{\mathbb{C}}]^W} \Psi = p(\lambda)\Psi, \qquad p \in S(\mathfrak{a}_{\mathbb{C}})^W. \tag{3.12}$$

Moreover, for $\ell \in \mathcal{Z}^+$ such that $k \pm \ell \in \mathcal{Z}^+$

$$\mathbb{G}_+^\circ(\ell, k)F^\circ(\lambda, k, X) = \lambda^{2\sum_{\alpha\in\mathcal{R}^+} \ell_\alpha} \frac{\tilde{c}(\rho(k+\ell), k+\ell)}{\tilde{c}(\rho(k), k)} F^\circ(\lambda, k+\ell, X),$$

$$\mathbb{G}_-^\circ(-\ell, k)F^\circ(\lambda, k, X) = \frac{\tilde{c}(\rho(k-\ell), k-\ell)}{\tilde{c}(\rho(k), k)} F^\circ(\lambda, k-\ell, X).$$

We shall call F° the Bessel function associated with the root system $\mathcal{R}$.

Proof. For $k \equiv 0$ the statement holds from the above discussion. We will use induction on the multiplicity function k. Assume the theorem holds for $k_0 \in \mathcal{Z}^+$ and prove that the statement holds for $k_0 + \ell$ with $\ell \in \mathcal{Z}$. Recall that $k_0 + \ell \notin S$ for $k_0, \ell \in \mathcal{Z}$.

First assume that $\ell \in \mathcal{Z}^+$. Then

$$\mathbb{G}_+(\ell, k_0)F(\lambda, k_0, a) = \frac{\tilde{c}(\rho(k_0+\ell), k_0+\ell)}{\tilde{c}(\rho(k_0), k_0)} \frac{\tilde{c}(-\lambda, k_0)}{\tilde{c}(-\lambda, k_0+\ell)} \frac{\tilde{c}(\lambda, k_0)}{\tilde{c}(\lambda, k_0+\ell)} F(\lambda, k_0+\ell, a).$$

Using the well known formula

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left\{1 + \mathcal{O}(z^{-1})\right\}, \qquad \text{if } z \to \infty,$$

we obtain the following asymptotic expression as $\epsilon \to 0$

$$\frac{\tilde{c}(-\lambda/\epsilon, k_0)}{\tilde{c}(-\lambda/\epsilon, k_0+\ell)} \frac{\tilde{c}(\lambda/\epsilon, k_0)}{\tilde{c}(\lambda/\epsilon, k_0+\ell)} \sim \prod_{\alpha\in\mathcal{R}^+} \left(\frac{\lambda}{\epsilon}\right)^{2\sum_{\alpha\in\mathcal{R}^+} \ell_\alpha}.$$

By the induction assumption and (3.6), we can deduce that

$$\begin{aligned}
&\lim_{\epsilon\to\infty} F(\frac{\lambda}{\epsilon}, k_0+\ell, \exp(\epsilon X))\\
&= \frac{\tilde{c}(\rho(k_0),k_0)}{\tilde{c}(\rho(k_0+\ell),k_0+\ell)} \lim_{\epsilon\to\infty} \frac{\tilde{c}(-\frac{\lambda}{\epsilon},k_0+\ell)\tilde{c}(\frac{\lambda}{\epsilon},k_0+\ell)}{\tilde{c}(-\frac{\lambda}{\epsilon},k_0)\tilde{c}(\frac{\lambda}{\epsilon},k_0)} \mathbb{G}_+^{(\epsilon)}(\ell,k_0) F(\frac{\lambda}{\epsilon},k_0,\exp(\epsilon X))\\
&= \lambda^{-2\sum_{\alpha\in\mathcal{R}^+}\ell_\alpha} \frac{\tilde{c}(\rho(k_0),k_0)}{\tilde{c}(\rho(k_0+\ell),k_0+\ell)} \mathbb{G}_+^\circ(\ell,k_0) F^\circ(\lambda,k_0,X).
\end{aligned}$$

Hence

$$\mathbb{G}_+^\circ(\ell,k)F^\circ(\lambda,k,X) = \lambda^{2\sum_{\alpha\in\mathcal{R}^+}\ell_\alpha} \frac{\tilde{c}(\rho(k+\ell),k+\ell)}{\tilde{c}(\rho(k),k)} F^\circ(\lambda,k+\ell,X).$$

For $\ell\in\mathcal{Z}^-$, we have

$$\mathbb{G}_-(\ell,k_0)F(\lambda,k_0,a) = \frac{\tilde{c}(\rho(k_0+\ell),k_0+\ell)}{\tilde{c}(\rho(k_0),k_0)} F(\lambda,k_0+\ell,a).$$

Since $\lim_{\epsilon\to 0}\mathbb{G}_-^{(\epsilon)}(\ell,k_0) = \mathbb{G}_-^\circ(\ell,k_0)$, and by the induction assumption, we obtain

$$\mathbb{G}_-^\circ(\ell,k)F^\circ(\lambda,k,X) = \frac{\tilde{c}(\rho(k+\ell),k+\ell)}{\tilde{c}(\rho(k),k)} F^\circ(\lambda,k+\ell,X).$$

In order to show that F° satisfies the Bessel system of differential equations, by [15, Propostion 3.4] it is enough to prove the statement for $\tilde{p} = \sum_{j=1}^n \xi_j^2$, which turns out to be true. Indeed, using the fact that $F(\lambda,,k,\exp(X))$, with $X\in\mathfrak{a}$, is an eigenfunction for $D(\widetilde{p},k) = \Delta(k) + \langle\rho(k),\rho(k)\rangle$ with eigenvalue $\langle\lambda,\lambda\rangle$, and applying the principle $(\mathcal{P})$, we obtain

$$\Delta^{(\epsilon)}(k)F(\frac{\lambda}{\epsilon},k,\exp(\epsilon X)) = \Big\{\langle\frac{\lambda}{\epsilon},\frac{\lambda}{\epsilon}\rangle - \langle\rho(k),\rho(k)\rangle\Big\} F(\frac{\lambda}{\epsilon},k,\exp(\epsilon X)).$$

Now Lemma 3.10 yields

$$T^\circ(\tilde{p},k)\Big|_{\mathbb{C}[\mathfrak{a}_\mathbb{C}]^W} F^\circ(\lambda,k,X) = \tilde{p}(\lambda)F^\circ(\lambda,k,X).$$

□

As one can notice, we may derive at least the same amount of explicit information for the Bessel functions F° by a limit analysis of the hypergeometric functions F. Indeed, in [24], the authors were able to give an explicit formula for Heckman-Opdam hypergeometric functions when the root system $\mathcal{R}$ is reduced and $k\in\mathcal{Z}^+$, which we use to prove the following theorem. See the next section for more details.

Theorem 3.12. *Assume that $k\in\mathcal{Z}^+$, and $\mathcal{R}$ is reduced. There exists a differential operator*

$$\mathbb{D}(k) := \Big\{\prod_{\alpha\in\mathcal{R}^+}\langle\alpha,\cdot\rangle^{2k_\alpha}\Big\}\mathbb{G}_+^\circ(k-1,1)\circ\prod_{\alpha\in\mathcal{R}^+}\langle\alpha,\cdot\rangle^{-1} \in \mathbb{C}[\mathfrak{a}_\mathbb{C}]\otimes S(\mathfrak{a}_\mathbb{C}),$$

and a W-invariant tubular neighborhood $\mathfrak{u}$ of $\mathfrak{a}$ in $\mathfrak{a}_\mathbb{C}$ such that

(3.13)

$$F^\circ(\lambda,k,X) = \frac{(-1)^{\sum_{\alpha>0}1+k_\alpha}2^{\sum_{\alpha>0}1-2k_\alpha}}{\tilde{c}(\rho(k),k)} \frac{\mathbb{D}(k)\Big(\sum_{w\in W}\epsilon(w)e^{w\lambda(X)}\Big)}{\prod_{\alpha\in\mathcal{R}^+}\langle\alpha,X\rangle^{2k_\alpha}\prod_{\alpha\in\mathcal{R}^+}\langle\alpha,\lambda\rangle^{2k_\alpha-1}},$$

for all $(\lambda,X)\in\mathfrak{a}_\mathbb{C}^*\times\mathfrak{u}$.

As we mentioned above, this class of Bessel functions encloses the spherical functions on the tangent space at the origin of non-compact Riemannian symmetric spaces. More precisely, let $\mathfrak{g}$ be a real semi-simple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace in $\mathfrak{p}$, and $\Sigma(\mathfrak{g}, \mathfrak{a})$ be the restricted root system associated with $\mathfrak{a}$. If we put $\mathcal{R} = 2\Sigma(\mathfrak{g}, \mathfrak{a})$ and $k_\alpha = \frac{1}{2} m_\alpha$, where m_α is the multiplicity of the root α, then F° is known as the spherical function on the flat symmetric space $\mathfrak{p}$. Its integral representation is given by

$$F^\circ(\lambda, m/2, X) = \int_K e^{B(A_\lambda, \mathrm{Ad}(\kappa)X)} d\kappa, \qquad \lambda \in \mathfrak{a}_{\mathbb{C}}^*, \;\; X \in \mathfrak{p}, \tag{3.14}$$

where B is the Killing form of $\mathfrak{g}$, and A_λ is defined by $B(A_\lambda, H) = \lambda(H)$ for $H \in \mathfrak{a}$. Thus, replacing k_α by $m_\alpha/2$ in Theorem 3.12 gives the explicit expressions of the integral (3.14). Bellow we give the list of all possible symmetric spaces G/K where $m_\alpha \in 2\mathbb{N}$, so that $k_\alpha \in \mathbb{N}$. The list has been extracted from [13].

Riemannian symmetric pairs with even multiplicity

$\mathfrak{g}$	$\mathfrak{k}$	Σ	m_α	Comments
$\mathfrak{sl}(n, \mathbb{C})$	$\mathfrak{su}(n)$	A_{n-1}	2	$n \geq 2$
$\mathfrak{so}(2n+1, \mathbb{C})$	$\mathfrak{so}(2n+1)$	B_n	2	$n \geq 2$
$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{sp}(n)$	C_n	2	$n \geq 3$
$\mathfrak{so}(2n, \mathbb{C})$	$\mathfrak{so}(2n)$	D_n	2	$n \geq 4$
$\mathfrak{so}(2n+1, 1)$	$\mathfrak{so}(2n+1)$	A_1	$2n$	$n \geq 3$
$\mathfrak{su}^*(2n)$	$\mathfrak{sp}(n)$	A_{n-1}	4	$n \geq 2$
$(\mathfrak{e}_6)_{\mathbb{C}}$	$\mathfrak{e}_6$	E_6	2	
$(\mathfrak{e}_7)_{\mathbb{C}}$	$\mathfrak{e}_7$	E_7	2	
$(\mathfrak{e}_8)_{\mathbb{C}}$	$\mathfrak{e}_8$	E_8	2	
$(\mathfrak{f}_4)_{\mathbb{C}}$	$\mathfrak{f}_4$	F_4	2	
$(\mathfrak{g}_2)_{\mathbb{C}}$	$\mathfrak{g}_2$	G_2	2	
$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_{4(-20)}$	A_2	8	

Special isomorphisms of Riemannian symmetric pairs with even multiplicities are

$\mathfrak{g}$	$\mathfrak{k}$
$\mathfrak{sp}(1, \mathbb{C}) \approx \mathfrak{sl}(2, \mathbb{C})$	$\mathfrak{sp}(1) \approx \mathfrak{su}(2)$
$\mathfrak{so}(3, \mathbb{C}) \approx \mathfrak{sl}(2, \mathbb{C})$	$\mathfrak{so}(3) \approx \mathfrak{su}(2)$
$\mathfrak{sp}(2, \mathbb{C}) \approx \mathfrak{so}(5, \mathbb{C})$	$\mathfrak{sp}(2) \approx \mathfrak{so}(5)$
$\mathfrak{so}(6, \mathbb{C}) \approx \mathfrak{sl}(4, \mathbb{C})$	$\mathfrak{so}(6) \approx \mathfrak{su}(4)$
$\mathfrak{so}(3, 1) \approx \mathfrak{sl}(2, \mathbb{C})$	$\mathfrak{so}(3) \approx \mathfrak{su}(2)$
$\mathfrak{so}(5, 1) \approx \mathfrak{su}^*(4)$	$\mathfrak{so}(5) \approx \mathfrak{sp}(2)$

Next we will make a connection between the Bessel functions $F^\circ(\lambda,k,X)$ and the eigenfunctions of $T^\circ(\xi,k)$ with spectral parameter λ. Recall that $\mathcal{R}$ is supposed to be reduced.

Theorem 3.13. *Let $k \in \mathcal{Z}^+$.*

(i) *There exists a unique holomorphic function $G^\circ(\lambda,k,\,\cdot\,)$ in a tubular neighborhood $\mathfrak{u}$ of $\mathfrak{a}$ in $\mathfrak{a}_\mathbb{C}$ such that*

$$T^\circ(\xi,k)G^\circ(\lambda,k,X) = \lambda(\xi)G^\circ(\lambda,k,X), \qquad \xi \in \mathfrak{a}_\mathbb{C},$$
$$G^\circ(\lambda,k,0) = 1.$$

(ii) *The Bessel functions can be written as*

$$F^\circ(\lambda,k,X) = \frac{1}{|W|}\sum_{w\in W} G^\circ(w\lambda,k,X).$$

(iii) *For all $w \in W$, $G^\circ(w\lambda,k,wX) = G^\circ(\lambda,k,X)$, and $G^\circ(\lambda,k,0) = 1$.*

Proof. Assume that $\lambda(\check\alpha) \neq 0, \pm k_\alpha$ for all $\alpha \in \mathcal{R}^+$. Put

$$G(\lambda,k,\exp(X)) := |W|T(p_*,k)F(\lambda,k,\exp(X))$$

with

$$p_* := \prod_{\alpha\in\mathcal{R}^+}\left(\frac{\lambda(\check\alpha)}{\lambda(\check\alpha)-k_\alpha}\right)\prod_{w\in W, w\neq e}\frac{\xi - w\lambda(\xi)}{\lambda(\xi)-w\lambda(\xi)},$$

where ξ is any element in $\mathfrak{a}_\mathbb{C}$ such that $\lambda(\xi) \neq w\lambda(\xi)$ for $w \neq e$. By [30, Theorem 3.15] the function $G(\lambda,k,\exp(X))$ is an eigenfunction for the Dunkl-Cherednik operator $T(\xi,k)$, with eigenvalue $\lambda(\xi)$. Using the explicit expression of p_* and Theorem 3.11, we can deduce that the following limit exists

$$G^\circ(\lambda,k,X) := \lim_{\epsilon\to 0} G\left(\frac{\lambda}{\epsilon},k,\exp(\epsilon X)\right),$$

where $G^\circ(\lambda,k,X) = |W|T^\circ(p_*^\circ,k)F^\circ(\lambda,k,X)$, and

$$p_*^\circ = \prod_{w\in W, w\neq e}\frac{w\lambda(\xi)}{w\lambda(\xi)-\lambda(\xi)},$$

for all λ.

Since for $a \in A$, $G(\lambda,k,a)$ satisfies

$$F(\lambda,k,a) = \frac{1}{|W|}\sum_{w\in W} G(w\lambda,k,a)$$

(cf. [18]), and since $F^\circ(\lambda,k,0) = 1$, then $G^\circ(\lambda,k,\cdot)$ satisfies

$$F^\circ(\lambda,k,X) = \frac{1}{|W|}\sum_{w\in W} G^\circ(w\lambda,k,X)$$

with

$$G^\circ(\lambda,k,0) = 1.$$

The last statement follows as the following: Since the Dunkl-Cherednik operator $T(\xi,k)$ is not W-equivariant, $G(w\lambda,k,a)$ and $G(\lambda,k,w^{-1}a)$ do not coincide. The relation between them is given by

$$G(\lambda,k,r_\alpha^{-1}a) = \left(\frac{\lambda(\check\alpha)+k_\alpha}{\lambda(\check\alpha)}\right)G(r_\alpha\lambda,k,a) - \frac{k_\alpha}{\lambda(\check\alpha)}G(\lambda,k,a).$$

Thus

$$G^\circ(\lambda, k, r_\alpha^{-1}X) = G^\circ(r_\alpha\lambda, k, X),$$

which is equivalent to $G^\circ(w\lambda, k, wX) = G^\circ(\lambda, k, X)$ for all $w \in W$. □

Remark 3.14. (i) We should note that the above theorem (other than (ii)) is also proved in [29] for $\mathrm{Re}(k) \geq 0$, where the author uses a different approach. In [29], the statement (ii) appears as the definition of the Bessel functions.

(ii) The eigenfunction $G^\circ(\,\cdot, k, \cdot\,)$ is also called the Dunkl-kernel associated with W and k. The literature is rich by its applications mainly in the theory of orthogonal polynomials and special functions (cf. [6, 15, 20, 29, 33]). (Of course, this list of references is not complete.)

For $\lambda \in \mathfrak{a}_\mathbb{C}^*$, and for a smooth compactly supported function $f \in C_c^\infty(\mathfrak{a})$, define

$$\mathcal{F}f(\lambda, k) := \int_\mathfrak{a} f(X) G^\circ(-\lambda, k, X) w(k)(X) dX,$$

where

$$w(k)(X) = \prod_{\alpha \in \mathcal{R}^+} \langle \alpha, X\rangle^{2k_\alpha},$$

and dX denotes the fixed normalization of the Haar measure on $\mathfrak{a}$. Using the limit transition approach, we shall give an alternative way of deriving the Plancherel formula for $\mathcal{F}$ when $k \in \mathcal{Z}^+$. Note however, that we will use the Plancherel formula by Opdam [30] – hardly a trivial result. The same statement, with real-valued k's, was proved earlier in [20] using a different approach.

Theorem 3.15. (Plancherel formula) *Let $k \in \mathcal{Z}^+$.*

(i) *For $f, g \in C_c^\infty(\mathfrak{a})$, the following Parseval-type formula holds*

$$\int_\mathfrak{a} f(X)\overline{g(X)} |w(k)(X)| dX = c_0 \int_{i\mathfrak{a}^*} \mathcal{F}f(\lambda, k)\overline{\mathcal{F}g(\lambda, k)} |w(k)(\lambda)| d\lambda,$$

where $d\lambda$ is a suitable normalization of the Lebesgue measure on $i\mathfrak{a}^*$, and c_0 is a constant depending only on the normalization of the measures.

(ii) *If $f, g \in C_c^\infty(\mathfrak{a})^W$, then*

$$\int_\mathfrak{a} f(X)\overline{g(X)} |w(k)(X)| dX = c_0 |W| \int_{i\mathfrak{a}^*_+} \mathcal{F}_W f(\lambda, k)\overline{\mathcal{F}_W g(\lambda, k)} |w(k)(\lambda)| d\lambda,$$

with

$$\mathcal{F}_W f(\lambda, k) := \int_\mathfrak{a} f(X) F^\circ(-\lambda, k, X) |w(k)(X)| dX.$$

(iii) *If $f \in C_c^\infty(\mathfrak{a})^W$*

$$f(X) = c_0 |W| \int_{i\mathfrak{a}^*_+} \mathcal{F}_W f(\lambda, k) F^\circ(\lambda, k, X) |w(k)(\lambda)| d\lambda.$$

Proof. Let w_0 be the longest element of W. By [30, Theorem 9.13], for all $f, g \in C_c^\infty(\mathfrak{a})$

$$\int_\mathfrak{a} f(X)\overline{g(X)} d\mu(X) = \int_{i\mathfrak{a}^*} \mathbb{F}f(\lambda, k)\overline{(w_0 \cdot \mathbb{F}g)(w_0\lambda, k)} d\nu(\lambda),$$

where

$$d\mu(X) = \prod_{\alpha\in\mathcal{R}^+} |2\,\mathrm{sh}(\frac{\alpha(X)}{2})|^{2k_\alpha} dX,$$

$$\mathbb{F}f(\lambda,k) = \int_{\mathfrak{a}} f(X)G(-w_0\lambda,k,\exp w_0X)d\mu(X),$$

$$d\nu(\lambda) = (2\pi)^{-n}\frac{\tilde{c}^2_{w_0}(\rho(k),k)}{\tilde{c}(\lambda,k)\tilde{c}_{w_0}(w_0\lambda,k)}\prod_{\alpha\in\mathcal{R}^+}(1-\frac{k_\alpha}{\lambda(\check{\alpha})})^{-1}d\lambda.$$

For $\epsilon>0$, set $f_\epsilon(X)=f(\epsilon^{-1}X)$. Thus

$$\int_{\mathfrak{a}} f_\epsilon(X)\overline{g_\epsilon(X)}d\mu(X) = \int_{i\mathfrak{a}^*}\mathbb{F}f_\epsilon(\lambda,k)\overline{(w_0\cdot\mathbb{F}g_\epsilon)(w_0\lambda,k)}d\nu(\lambda). \tag{3.15}$$

Using the definition of the Fourier transform $\mathbb{F}$, (3.15) becomes

$$\begin{aligned}\int_{\mathfrak{a}} f(X)\overline{g(X)}d\mu(\epsilon X) &= \int_{i\mathfrak{a}^*}\left[\int_{\mathfrak{a}} f(a)G(-w_0\frac{\lambda}{\epsilon},k,\exp w_0\epsilon a)d\mu(\epsilon a)\right]\\ &\quad \left[\int_{\mathfrak{a}} g(b)G(-\frac{\lambda}{\epsilon},k,\exp\epsilon b)d\mu(\epsilon b)\right]d\nu(\frac{\lambda}{\epsilon}).\end{aligned}$$

By the following facts

$$\lim_{\epsilon\to 0} G(\frac{\lambda}{\epsilon},k,\exp\epsilon X) = G^\circ(\lambda,k,X),$$

$$G^\circ(w\lambda,k,wX) = G^\circ(\lambda,k,X),$$

$$\lim_{\epsilon\to 0}\epsilon^{-2\sum_{\alpha>0}k_\alpha}\prod_{\alpha\in\mathcal{R}^+}|2\,\mathrm{sh}(\frac{\alpha(\epsilon X)}{2})|^{2k_\alpha} = \prod_{\alpha\in\mathcal{R}^+}|\alpha(X)|^{2k_\alpha},$$

$$\begin{aligned}\lim_{\epsilon\to 0}\epsilon^{2\sum_{\alpha>0}k_\alpha}(2\pi)^{-n}\frac{\tilde{c}^2_{w_0}(\rho(k),k)}{\tilde{c}(\frac{\lambda}{\epsilon},k)\tilde{c}_{w_0}(w_0\frac{\lambda}{\epsilon},k)} &= (2\pi)^{-n}\tilde{c}^2_{w_0}(\rho(k),k)\\ &\quad \prod_{\alpha\in\mathcal{R}^+}\lambda(\check{\alpha})^{k_\alpha}\prod_{\alpha\in\mathcal{R}^+}(w_0\lambda(\check{\alpha}))^{k_\alpha},\end{aligned}$$

the first statement holds. The statement (ii) follows directly form the fact that $F^\circ(\lambda,k,X)=\frac{1}{|W|}\sum_{w\in W}G^\circ(w\lambda,k,X)$, and therefore the inversion formula holds. □

Remark 3.16. After this manuscript was finished, de Jeu used the limit transition argument to prove a Paley-Wiener theorem for the Bessel transform $\mathcal{F}$ for functions with compact support in the convex hull of a W-orbit [21]. de Jeu's proof relies on the Paley-Wiener theorem proved by Opdam in the trigonometric setting [30]. As suggested by the referee, such an approach is very natural in the framework here.

Example 3.17. Let us investigate the rank one case, and in order to be even more convincing, we consider the non-reduced root system BC_1, i.e. $\mathcal{R}=\{\pm\alpha,\pm2\alpha\}$. In this example we have $A_{\mathbb{C}}\simeq\mathbb{C}^*$ and $\mathbb{C}[A_{\mathbb{C}}]=\mathbb{C}[x^{-1},x]$ where $x=e^\alpha$. The nontrivial Weyl group element acts by $x\mapsto x^{-1}$ on $A_{\mathbb{C}}$.

If $\xi = (2\alpha)^v$, then $\partial_\xi = x\partial_x$. We will normalize the inner product on $\mathfrak{a}_{\mathbb{C}}$ and $\mathfrak{a}_{\mathbb{C}}^*$ by $\langle \alpha, \alpha \rangle = 1$. In the x coordinate, the differential operators (2.1) and (2.2) become

$$T(\xi, k) = x\partial_x + \left(\frac{k_\alpha}{1 - x^{-1}} + \frac{2k_{2\alpha}}{1 - x^{-2}} \right)(1 - r) - \left(\frac{1}{2}k_\alpha + k_{2\alpha} \right),$$

$$\Delta(k) = (x\partial_x)^2 - \left(k_\alpha \frac{1 + x}{1 - x} + 2k_{2\alpha}\frac{1 + x^2}{1 - x^2} \right) x\partial_x,$$

where $r(x^m) = x^{-m}$. The shift operators $\mathbb{G}_-(-1, k)$ and $\mathbb{G}_+(+1, k)$ are given by [18]

$$\mathbb{G}_+(+1, k) = \frac{x\partial_x}{x - x^{-1}},$$
$$\mathbb{G}_-(-1, k) = (x^2 - 1)\partial_x + (k_\alpha + 2k_{2\alpha} - 1)(x + x^{-1}) + 2k_\alpha.$$

Let $z = -\frac{1}{4}x^{-1}(1 - x)^2$ be a coordinate on $W \setminus A_{\mathbb{C}}$. Put $\gamma_1 = \lambda + \frac{1}{2}k_\alpha + k_{2\alpha}$, $\gamma_2 = -\lambda + \frac{1}{2}k_\alpha + k_{2\alpha}$, and $\gamma_3 = \frac{1}{2} + k_\alpha + k_{2\alpha}$. In this notations, the functions $F(\lambda, k, a)$ and $G(\lambda, k, a)$ are given by

$$F(\lambda, k, a) = {}_2F_1(\gamma_1, \gamma_2; \gamma_3; z),$$
$$G(\lambda, k, a) = {}_2F_1(\gamma_1, \gamma_2; \gamma_3; z) + \frac{\gamma_1}{4\gamma_3}\big(x - x^{-1}\big)\,{}_2F_1(\gamma_1 + 1, \gamma_2 + 1; \gamma_3 + 1; z),$$

where ${}_2F_1(\gamma_1, \gamma_2; \gamma_3; z)$ is the Gauss hypergeometric function.

On $\mathfrak{a}_{\mathbb{C}}$, the infinitesimal operator $T^\circ(\xi, k)$ associated with $T(\xi, k)$ is given by

$$T^\circ(\xi, k) = \partial_X + \frac{k_\alpha + k_{2\alpha}}{X}(1 - r).$$

The shift operators $\mathbb{G}^\circ_\pm$ are given by

$$\mathbb{G}^\circ_+(+1, k) = \frac{1}{2X}\partial_X, \qquad \mathbb{G}^\circ_-(-1, k) = (2X)\partial_X + 2(2k_\alpha + 2k_{2\alpha} - 1).$$

Using the fact that

$$\frac{\Gamma(z + a)}{\Gamma(z + b)} = z^{a-b}\left\{1 + \mathcal{O}(z^{-1})\right\}, \qquad \text{if } z \to \infty,$$

one can see that

$$\begin{aligned} F^\circ(\lambda, k, X) &= \lim_{\epsilon \to 0} F\Big(\frac{\lambda}{\epsilon}, k, \exp(\epsilon X)\Big) \\ &= \Gamma\Big(\frac{1}{2} + k_\alpha + k_{2\alpha}\Big)\Big(\frac{\lambda X}{2}\Big)^{\frac{1}{2} - k_\alpha - k_{2\alpha}} I_{k_\alpha + k_{2\alpha} - \frac{1}{2}}(\lambda X), \end{aligned}$$

where $I_\nu(z) = e^{-i\pi\nu/2}J_\nu(iz)$, with $J_\nu(z)$ is the Bessel function of the first kind

$$J_\nu(z) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \left(\frac{z}{2}\right)^{2\ell+\nu}}{\Gamma(1 + \nu + \ell)\ell!}.$$

From classical analysis on special functions, it is a well known fact that for fixed $\lambda \in \mathbb{C}$, the function $\zeta(X) = \Gamma(\frac{1}{2} + k_\alpha + k_{2\alpha})\left(\frac{\lambda X}{2}\right)^{\frac{1}{2} - k_\alpha - k_{2\alpha}} I_{k_\alpha + k_{2\alpha} - \frac{1}{2}}(\lambda X)$ is the unique analytic solution of the differential equation

$$\zeta'' + \frac{2(k_\alpha + k_{2\alpha})}{X}\zeta' = \lambda^2\zeta,$$

which is even and normalized by $\zeta(0) = 1$.

The eigenfunction $G^\circ(\lambda,k,X)$ is given by

$$\begin{aligned} G^\circ(\lambda,k,X) &= \lim_{\epsilon\to\infty} G(\frac{\lambda}{\epsilon},k,\exp(\epsilon X)) \\ &= \Gamma(\frac{1}{2}+k_\alpha+k_{2\alpha})\left(\frac{\lambda X}{2}\right)^{\frac{1}{2}-k_\alpha-k_{2\alpha}} \left\{I_{k_\alpha+k_{2\alpha}-\frac{1}{2}}(\lambda X)+I_{k_\alpha+k_{2\alpha}+\frac{1}{2}}(\lambda X)\right\}. \end{aligned}$$

4. A more general class of Bessel functions

4.1. **The Θ-Bessel functions.** Recall that $\mathcal{R}$ is reduced, and let $\Pi=\{\alpha_1,\dots,\alpha_n\}$ be the system of simple roots associated with $\mathcal{R}^+$. Let $\Theta\subseteq\Pi$ be an arbitrary subset of Π. The set $\langle\Theta\rangle$ of elements in $\mathcal{R}$, which can be written as linear combinations of elements from Θ, is a subsystem of $\mathcal{R}$. Its Weyl group W_Θ is the parabolic subgroup of W generated by the reflections r_{α_j} with $\alpha_j\in\Theta$.

For a multiplicity function $k\in\mathcal{K}$ and $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$, we set

$$\begin{aligned} c^+_\Theta(\lambda,k) &= \prod_{\alpha\in\langle\Theta\rangle^+}\frac{\Gamma(\lambda(\check\alpha))}{\Gamma(\lambda(\check\alpha)+k_\alpha)}, \\ c^-_\Theta(\lambda,k) &= \prod_{\alpha\in\mathcal{R}^+\setminus\langle\Theta\rangle^+}\frac{\Gamma(-\lambda(\check\alpha)-k_\alpha+1)}{\Gamma(-\lambda(\check\alpha)+1)}, \\ c^{+,c}_\Theta(\lambda,k) &= \prod_{\alpha\in\mathcal{R}^+\setminus\langle\Theta\rangle^+}\frac{\Gamma(\lambda(\check\alpha))}{\Gamma(\lambda(\check\alpha)+k_\alpha)}, \end{aligned}$$

with the conventions

$$c^+_\emptyset = c^{+,c}_\Pi = 1, \qquad \text{and} \qquad c^-_\Pi = 1.$$

If $\Theta=\Pi$, the function $c^+_\Pi(\lambda,k)$ coincides with the $\tilde c$-function (3.5).

Let U be a connected and simply connected open subset of T containing the identity (the same U as in Theorem 3.8). The function on A^+U defined for generic $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$ by

$$F_\Theta(\lambda,k,h) = c^-_\Theta(\lambda,k)\sum_{w\in W_\Theta} c^+_\Theta(w\lambda,k)\Phi(w\lambda,k,h), \qquad h\in A^+U,$$

is called the Θ-spherical function of spectral parameter λ (see the beginning of Section 3 for the definition of $\Phi(\lambda,k,X)$). We refer to [32] for more details on Θ-spherical functions. As a linear combination of the Harish-Chandra series $\Phi(w\lambda,k,h)$, the Θ-spherical function is by construction a solution of the hypergeometric system (3.1).

Example 4.1. When $\Theta=\Pi$, the ratio $F_\Pi(\lambda,k,h)/c^+_\Pi(\rho(k),k)$ coincides with the Heckman-Opdam's hypergeometric function $F(\lambda,k,h)$ used in Section 3. For geometric multiplicity, the ratio coincides with Harish-Chandra's spherical function.

Example 4.2. If $\mathcal{R}$ is the restricted root system of a non-compact causal symmetric space G/H, then $\mathcal{R}$ is reduced. Let Π_0 be the fundamental system for the positive compact roots Σ_0^+ and set $\Theta=\Pi_0$. See the next part for more details. In this case, $\langle\Theta\rangle^+=\Sigma_0^+$ and $W_\Theta=W_0$ is the so-called small Weyl group. The ratio $F_{\Pi_0}(\lambda,k,h)/c^+_{\Pi_0}(\rho(k),k)c^-_{\Pi_0}(\rho(k),k)$ coincides with the spherical function on G/H with spectral parameter λ as defined in [11]. See the next subsection for more details on the non-compact causal symmetric spaces case.

Example 4.3. If $\Theta=\emptyset$, $W_\emptyset=\{\mathrm{id}\}$ and $F_\emptyset(\lambda,k,h)=c_\emptyset^-(\lambda,k)\Phi(\lambda,k,h)$.

Set
$$\mathfrak{a}_{\Theta,+}=\left\{\ H\in\mathfrak{a}\mid \alpha(H)>0\ \text{ for all } \alpha\in\mathcal{R}^+\setminus\langle\Theta\rangle^+\ \right\},$$
and $A_{\Theta,+}=\exp(\mathfrak{a}_{\Theta,+})$.

The definition of the Θ-spherical functions, together with analytic continuation, yields the following transition relations linking the Θ-spherical functions for arbitrary Θ to Heckman-Opdam hypergeometric functions.

Lemma 4.4. (cf. [32, Lemma 2.1.5]) *There is a W_Θ-invariant tubular neighborhood U_Θ of $A_{\Theta,+}$ in $A_{\mathbb{C}}$ so that for all $(\lambda,k,h)\in\mathfrak{a}_{\mathbb{C}}^*\times\mathcal{Z}^+\times U_\Theta$ the following equality of meromorphic functions holds*
$$F_\Pi(\lambda,k,h)=\sum_{w\in W_\Theta\backslash W}\frac{c_\Theta^{+,c}(w\lambda,k)}{c_\Theta^-(w\lambda,k)}F_\Theta(w\lambda,k,h).$$

Let $k\in\mathcal{Z}^+$, and recall that $\delta(k)(a)=\prod_{\alpha\in\mathcal{R}^+}(a^{\frac{\alpha}{2}}-a^{-\frac{\alpha}{2}})^{2k_\alpha}$, for $a\in A_{\mathbb{C}}$. The differential operator
$$D(k):=\delta(k)\mathbb{G}_+(k,0)\ \ (=\mathbb{G}_-^*(-k,k)),$$
is a W-invariant element of $\mathbb{C}[A_{\mathbb{C}}]\otimes S(\mathfrak{a}_{\mathbb{C}})$ (cf. [24, Theorem 4.10]).

In [24], Ólafsson and Pasquale give an explicit global formula for the Θ-spherical functions, for positive integer-valued multiplicity and reduced root system, by means of Opdam's shift operators. Put $d(\Theta,k)=\sum_{\alpha\in\mathcal{R}^+\setminus\langle\Theta\rangle^+}k_\alpha$.

Theorem 4.5. (cf. [24, Theorem 5.1]) *Let $k\in\mathcal{Z}^+$ and let $\mathcal{R}$ be a reduced root system. As meromorphic function of λ, the Θ-spherical functions are given by*
$$F_\Theta(\lambda,k,a)=(-1)^{d(\Theta,k)}\delta(-k)(a)\Big\{\prod_{\alpha\in\mathcal{R}^+}\prod_{j=0}^{k_\alpha-1}j^2-\lambda(\check{\alpha})^2\Big\}^{-1}D(k)\Big(\sum_{w\in W_\Theta}e^{w\lambda(\log a)}\Big),$$
for all $(\lambda,a)\in\mathfrak{a}_{\mathbb{C}}^*\times U_\Theta$.

Using this theorem, together with the previous results on $\mathbb{G}_+^\circ(\ell,k)$, we can see that, when $\mathcal{R}$ is reduced, the following limit exists
$$\tilde{F}_\Theta^\circ(\lambda,k,X):=\lim_{\epsilon\to0}F_\Theta(\frac{\lambda}{\epsilon},k,\exp(\epsilon X)),\qquad X\in\mathfrak{a}_{\Theta,+}.$$
Next we shall call $\tilde{F}_\Theta^\circ$ the Θ-Bessel function, which is a solution of the Bessel system (3.12). Notice that $F^\circ(\lambda,k,X)=\tilde{F}_\Pi^\circ(\lambda,k,X)/c_\Pi^+(\rho(k),k)$ for $X\in\mathfrak{a}_{\Pi,+}$ and $k\in\mathcal{Z}^+$. The above transition relation, linking F° to $\tilde{F}_\Pi^\circ$, can be generalized by linking the Θ-Bessel functions for arbitrary Θ to the Bessel function F°.

Lemma 4.6. *There exists a W_Θ-invariant tubular neighborhood $\mathfrak{u}_\Theta$ of $\mathfrak{a}_{\Theta,+}$ in $\mathfrak{a}_{\mathbb{C}}$ such that for $(\lambda,k,X)\in\mathfrak{a}_{\mathbb{C}}^*\times\mathcal{Z}^+\times\mathfrak{u}_\Theta$*
$$F^\circ(\lambda,k,X)=\frac{(-1)^{d(\Theta,k)}}{c_\Pi^+(\rho(k),k)}\sum_{w\in W_\Theta\backslash W}\tilde{F}_\Theta^\circ(w\lambda,k,X).$$

Next we will give explicit formulas for the Θ-Bessel functions by means of Theorem 4.5.

For $X \in \mathfrak{a}$, recall that $w(k)(X) = \prod_{\alpha\in\mathcal{R}^+}\langle\alpha, X\rangle^{2k_\alpha}$. Let $k \in \mathcal{Z}^+$, and set

$$D^\circ(k) := w(k)\mathbb{G}_+^\circ(k, 0).$$

Example 4.7. If $k_\alpha = 1$ for all $\alpha \in \mathcal{R}^+$, then $D^\circ(k) = (-1)^{|\mathcal{R}^+|}\prod_{\alpha\in\mathcal{R}^+}\check{\alpha}\partial_\alpha$.

Remark 4.8. The differential operator $D^\circ(k)$ is obtained as the limit transition of $D(k) = \prod_{\alpha\in\mathcal{R}^+}(a^{\frac{\alpha}{2}} - a^{-\frac{\alpha}{2}})^{2k_\alpha}\mathbb{G}_+(k, 0)$, after the deformation process. Further, $D^\circ(k) \in \mathbb{C}[\mathfrak{a}_\mathbb{C}] \otimes S(\mathfrak{a}_\mathbb{C})$. This follows from [24, Theorem 4.10], where the authors proved that $D(k) \in \mathbb{C}[A_\mathbb{C}] \otimes S(\mathfrak{a}_\mathbb{C})$.

Theorem 4.9. *Let $k \in \mathcal{Z}^+$, and assume that $\mathcal{R}$ is a reduced root system. The Θ-Bessel functions are given by*

$$\tilde{F}_\Theta^\circ(\lambda, k, X) = (-1)^{d'(\Theta,k)}\frac{D^\circ(k)\Big(\sum_{w\in W_\Theta} e^{w\lambda(X)}\Big)}{\prod_{\alpha\in\mathcal{R}^+}\langle\alpha, X\rangle^{2k_\alpha}\prod_{\alpha\in\mathcal{R}^+}\langle\check{\alpha}, \lambda\rangle^{2k_\alpha}},$$

for all $(\lambda, X) \in \mathfrak{a}_\mathbb{C}^* \times \mathfrak{u}_\Theta$. Here $d'(\Theta, k) = \sum_{\alpha\in\langle\Theta\rangle^+} k_\alpha$.

We may express $\tilde{F}_\Theta^\circ$ in terms of an alternate series.

Corollary 4.10. *Assume that $k \in \mathcal{Z}^+$, and let $\mathcal{R}$ be a reduced root system. There exists a differential operator $\mathbb{D}^\circ(k) \in \mathbb{C}[\mathfrak{a}_\mathbb{C}] \otimes S(\mathfrak{a}_\mathbb{C})$ such that*

$$\tilde{F}_\Theta^\circ(\lambda, k, X) = (-1)^{d'(\Theta,k)+|\mathcal{R}^+|}\frac{\mathbb{D}^\circ(k)\Big(\sum_{w\in W_\Theta}\epsilon(w)e^{w\lambda(X)}\Big)}{\prod_{\alpha\in\mathcal{R}^+}\langle\alpha, X\rangle^{2k_\alpha}\prod_{\alpha\in\mathcal{R}^+}\langle\check{\alpha}, \lambda\rangle^{2k_\alpha-1}},$$

for all $(\lambda, X) \in \mathfrak{a}_\mathbb{C}^* \times \mathfrak{u}_\Theta$.

Proof. For $k \in \mathcal{Z}^+$, set $\mathbb{D}^\circ(k) := w(k)\mathbb{G}_+^\circ(k-1, 1) \circ w(-1/2)$, and rewrite $D^\circ(k)$ as $D^\circ(k) = w(k)\mathbb{G}_+^\circ(k-1, 1) \circ \mathbb{G}_+^\circ(1, 0)$ where

$$\mathbb{G}_+^\circ(1, 0) = (-1)^{|\mathcal{R}^+|}\prod_{\alpha\in\mathcal{R}^+}\frac{2}{\langle\alpha, \alpha\rangle}w(-1/2)\prod_{\alpha\in\mathcal{R}^+}\partial_\alpha.$$

Therefore

$$\begin{aligned}
D^\circ(k)e^{w\lambda(X)} &= w(k)\mathbb{G}_+^\circ(k-1, 1) \circ \mathbb{G}_+^\circ(1, 0)e^{w\lambda(X)}\\
&= w(k)\mathbb{G}_+^\circ(k-1, 1)\left[(-1)^{|\mathcal{R}^+|}w(-1/2)\prod_{\alpha\in\mathcal{R}^+}\frac{2\partial_\alpha}{\langle\alpha, \alpha\rangle}e^{w\lambda(X)}\right]\\
&= (-1)^{|\mathcal{R}^+|}\prod_{\alpha\in\mathcal{R}^+}\lambda(\check{\alpha})\left\{w(k)\mathbb{G}_+^\circ(k-1, 1) \circ w(-1/2)\right\}\epsilon(w)e^{w\lambda(X)}\\
&= (-1)^{|\mathcal{R}^+|}\prod_{\alpha\in\mathcal{R}^+}\lambda(\check{\alpha})\mathbb{D}^\circ(k)(\epsilon(w)e^{w\lambda(X)}).
\end{aligned}$$

Now we prove $\mathbb{D}^\circ(k) \in \mathbb{C}[\mathfrak{a}_\mathbb{C}] \otimes S(\mathfrak{a}_\mathbb{C})$ just like in Remark 4.8 □

Remark 4.11. (i) For $k = 0$, $\tilde{F}_\Theta^\circ(\lambda, 0, X) = \sum_{w\in W_\Theta} e^{w\lambda(X)}$.

(ii) In the particular case where $k_\alpha = 1$ for all $\alpha \in \mathcal{R}$, we have $d'(\Theta, k) = |\langle\Theta\rangle^+|$ and the differential operator $\mathbb{D}^\circ(1) = w(\frac{1}{2})$. Therefore

$$\tilde{F}^\circ_\Theta(\lambda, 1, X) = (-1)^{|\mathcal{R}^+\setminus\langle\Theta\rangle^+|} \frac{\sum_{w\in W_\Theta} \epsilon(w) e^{w\lambda(X)}}{\prod_{\alpha\in\mathcal{R}^+} \langle\check{\alpha}, \lambda\rangle \prod_{\alpha\in\mathcal{R}^+} \langle\alpha, X\rangle}. \tag{4.1}$$

(iii) Let G/K be a Riemannian symmetric space, where G is a complex Lie group and K its maximal compact subgroup (here $\Theta = \Pi$). In this case $\mathcal{R} = 2\Sigma(\mathfrak{g}, \mathfrak{a})$ and $k_\alpha = \frac{1}{2} m_\alpha = 1$, where $\Sigma(\mathfrak{g}, \mathfrak{a})$ is the (reduced) restricted root system associated with $\mathfrak{a}$, and $m_\alpha (= 2)$ is the multiplicity of the root α. In this setting, formula (4.1) coincides with the so-called Harish-Chandra's formula [12, 19]. Thus, for arbitrary Θ, one can think of (4.1) as a generalization of Harish-Chandra's formula.

(iv) Recall that for non-compact causal symmetric spaces, the root system is always reduced. The only irreducible spaces with $m_\alpha \in 2\mathbb{N}$, so that $k_\alpha = m_\alpha/2 \in \mathbb{N}$, are, up to coverings, $SO_0(2n+1, 1)/SO_0(2n, 1)$, $SU^*(2(p+q))/Sp(p, q)$, $E_{6(-26)}/F_{4(-20)}$ and all the irreducible spaces $G_\mathbb{C}/G$, where G is Hermitian, i.e. semi-simple, non-compact, with G/K a bounded symmetric domain. See the next part and the tables bellow for a complete investigation.

4.2. **Bessel functions related to non-compact causal symmetric spaces.** Let (G, H) be a symmetric pair, i.e. G is a connected semisimple Lie group with finite center, H is a closed subgroup, and there exists an involutive automorphism τ of G such that

$$(G^\tau)_0 \subset H \subset G^\tau,$$

where $G^\tau := \{\, g \in G \mid \tau(g) = g \,\}$, and $(G^\tau)_0$ is the identity component in G^τ. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H and denote the differential of τ also by the same letter. Therefore $\mathfrak{h} = \mathfrak{g}^\tau$. Set $\mathfrak{q} = \mathfrak{g}^{-\tau}$. Let $x_0 = \mathbf{1}H$ where $\mathbf{1}$ is the unit element in G. The tangent space at x_0 can be identified with $\mathfrak{q}$. Let θ be a Cartan involution of G commuting with τ, and K be the corresponding maximal compact subgroup of G with Lie algebra $\mathfrak{k} = \mathfrak{g}^\theta$. Put $\mathfrak{p} = \mathfrak{g}^{-\theta}$.

Assume that there exists in $\mathfrak{p} \cap \mathfrak{q}$ a non-zero vector X_0 which is invariant under $\mathrm{Ad}(H \cap K)$ and such that the projection on every irreducible component is non zero. Thus if $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$, then $X_0 \in \mathfrak{a}$ and $\mathfrak{a}$ is a maximal abelian in $\mathfrak{p}$. We refer to [10] for more details on causal symmetric spaces.

Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ be the restricted root system and choose a positive system Σ^+ in Σ. Note that Σ is always reduced. Therefore one obtains in $\mathfrak{q}$ a closed convex H-invariant cone $C_{\max}$ such that $C_{\max} \cap \mathfrak{a} \neq \emptyset$. Put as usual $\mathfrak{n} = \oplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$, and $\rho = \frac{1}{2} \sum_{\alpha\in\Sigma^+} m_\alpha \alpha$ where $m_\alpha = \dim(\mathfrak{g}_\alpha)$. On the Lie algebra level, $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{h}$ and the map $N \times A \times H \ni (n, a, h) \mapsto nah \in G$ is a diffeomorphism onto an open subset of G. From this it follows that the map $N \times \mathfrak{a} \to G/H$, $\quad (n, X) \mapsto n \exp(X) \cdot x_0$ is a diffeomorphism of $N \times \mathfrak{a}$ onto the open set $NA \cdot x_0$. For x in this set, $x = n \exp(X) \cdot x_0$, we set $A(x) = X$. Note that the map A is right H-invariant.

Denote by $\mathcal{S}$ the semigroup given by $\mathcal{S} = \exp(C_{\max})H$, and denote by $\mathcal{E} = \{\, \lambda \in \mathfrak{a}_\mathbb{C} \mid \mathrm{Rel}\langle\lambda + \rho, \alpha\rangle < 0 \;\forall \alpha \in \Sigma^+ \,\}$. For $\lambda \in \mathcal{E}$, the spherical function φ_λ is defined on the interior $\mathcal{S}^0$ of $\mathcal{S}$ by

$$\varphi_\lambda(x) := \int_H e^{\langle\rho-\lambda, A(hx)\rangle} dh,$$

(cf. [11]). (The measures are normalized via the Killing form.)

Put $\Sigma_0^+ = \Sigma^+(\mathfrak{k}\cap\mathfrak{h}\oplus\mathfrak{p}\cap\mathfrak{q},\mathfrak{a})$. Using the notations of the previous part, let $\Theta = \Pi_0$ be the fundamental system for the system Σ_0^+ of positive compact roots. Therefore $\langle\Theta\rangle^+ = \Sigma_0^+$ and $W_{\Pi_0} = W_0$ is the Weyl group associated with the root system Σ_0. Moreover, for $X \in \mathfrak{a}\cap C^0_{\max}$

$$\varphi_\lambda(\exp(X)) = \frac{F_{\Pi_0}(\lambda, m, \exp(X))}{c^+_{\Pi_0}(\rho(m), m)c^-_{\Pi_0}(\rho(m), m)},$$

with $m = (m_\alpha)_{\alpha\in\Sigma^+}$ being the multiplicity function.

For $\epsilon > 0$, write $\gamma_\epsilon = \exp(\epsilon X)h$ with $h \in H$ and $X \in C^0_{\max}$. For $\lambda \in \mathcal{E}$, denote by

$$\Psi(\lambda, X) := \lim_{\epsilon\to 0}\varphi_{\frac{\lambda}{\epsilon}}(\gamma_\epsilon).$$

Theorem 4.12. *Let G/H be a non-compact causal symmetric space, $\lambda \in \mathcal{E}$, and $X \in C^0_{\max}$. The limit $\Psi(\lambda, X)$ and its derivatives exist. Its integral representation is given by*

$$\Psi(\lambda, X) = \int_H e^{-B(A_\lambda,\mathrm{Ad}(h)X)}dh.$$

Proof. Denote by $\mathcal{P} : \mathfrak{q} \to \mathfrak{a}$ the orthogonal projection on $\mathfrak{a}$. Since $\gamma \mapsto \varphi_\lambda(\gamma)$ is H-bi-invariant, it is enough to prove the statement for $\gamma_\epsilon = \exp(\epsilon X)$ where $X \in C^0_{\max}$. Note that $A(h\exp(\epsilon X)) = A(\exp(\mathrm{Ad}(h)X))$. Write $\mathrm{Ad}(h)X = \mathcal{P}(\mathrm{Ad}(h)X) + Y \in \mathfrak{a}\oplus\mathfrak{a}^\perp$, where $\mathfrak{a}^\perp$ is the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{q}$. Since each $Y \in \mathfrak{a}^\perp$ can be written as

$$\begin{aligned} Y &= \sum_{\alpha\in\Sigma^+}(Y_\alpha - \tau(Y_\alpha)) = \sum_{\alpha\in\Sigma^+}(2Y_\alpha - (Y_\alpha + \tau(Y_\alpha))) \\ &= Y_{\mathfrak{n}} + Y_{\mathfrak{h}} \in \mathfrak{n}\oplus\mathfrak{h} \end{aligned}$$

(cf. [10, p. 106]), then $\mathrm{Ad}(h)X = \mathcal{P}(\mathrm{Ad}(h)X) + Y_{\mathfrak{h}} + Y_{\mathfrak{n}}$. Using the fact that

$$\begin{aligned} &\exp(\epsilon Y_{\mathfrak{n}})\exp(\epsilon\mathcal{P}(\mathrm{Ad}(h)X))\exp(\epsilon Y_{\mathfrak{h}}) \\ &= \exp(\epsilon\mathcal{P}(\mathrm{Ad}(h)X) + \epsilon(Y_{\mathfrak{h}} + Y_{\mathfrak{n}}) + \mathcal{O}(\epsilon^2)) \\ &= \exp(\epsilon\,\mathrm{Ad}(h)X + \mathcal{O}(\epsilon^2)), \end{aligned}$$

one can deduce that both functions $\epsilon \mapsto A(\exp(\epsilon Y_{\mathfrak{n}})\exp(\epsilon\mathcal{P}(\mathrm{Ad}(h)X))\exp(\epsilon Y_{\mathfrak{h}}))$ and $\epsilon \mapsto A(\exp(\epsilon\,\mathrm{Ad}(h)X))$ have the same derivative at $\epsilon = 0$. Using the definition of $A(\cdot)$ for the first function, we can see that

$$\frac{d}{d\epsilon}A(\exp(\epsilon\,\mathrm{Ad}(h)X))_{|\epsilon=0} = \mathcal{P}(\mathrm{Ad}(h)X)$$

uniformly for $h \in H$ and X in a bounded set in $C^0_{\max}$. Hence

$$\lim_{\epsilon\to 0}\frac{\lambda}{\epsilon}A(\exp(\epsilon\,\mathrm{Ad}(h)X)) = \lambda\mathcal{P}(\mathrm{Ad}(h)X),$$

and the integral representation of $\Psi(\lambda, X)$ holds. To finish the proof for the derivatives, one needs to apply the dominated convergence theorem of Lebesgue to $\Psi(\lambda,\cdot)$, which is valid by the linear convexity theorem due to Hilgert and Ólafsson [10, Theorem 4.3.1]. $\square$

Example 4.13. Let $G = SO_0(1,n)$ and let $H = SO_0(1,n-1)$, $n \geq 2$. Let $\mathfrak{a} = \mathbb{R}X_0$ where $X_0 = E_{1,n+1} + E_{n+1,1}$. Here we use the standard notations for the matrix element $E_{i,j}$. We choose the positive roots such that $\alpha(X_0) = 1$ and identify $\mathfrak{a}^*_{\mathbb{C}}$ with $\mathbb{C}$ via $z \mapsto -z\alpha$. Then $\rho = -(n-1)/2$. For $t > 0$ and $\mathrm{Rel}(\lambda) < -(n-3)/2$, we have

$$\begin{aligned}\varphi_\lambda(\exp(tX_0)) &= \pi^{-1/2}2^{n/2-1}e^{-i\pi(n/2-1)}\Gamma(\frac{n}{2}-\frac{1}{2})(\operatorname{sh} t)^{-(n/2-1)}\\ &\quad \frac{\Gamma(\lambda-\frac{n}{2}+\frac{3}{2})}{\Gamma(\lambda+\frac{n}{2}-\frac{1}{2})}Q^{n/2-1}_{\lambda-1/2}(\operatorname{ch} t),\end{aligned}$$

where Q^μ_ν is the Legendre function of the second kind (cf. [11]). Using [9, 3.2 (10)], we can write

$$\begin{aligned}e^{-i\pi(n/2-1)}Q^{n/2-1}_{\lambda-1/2}(\operatorname{ch} t) &= 2^{\frac{n}{2}-2}\Gamma(\frac{n}{2}-1)(i)^{n/2-1}(\operatorname{sh} t)^{-(n/2-1)}\\ &\quad {}_2F_1(\frac{\lambda}{2}-\frac{n}{4}+\frac{3}{4},-\frac{\lambda}{2}-\frac{n}{4}+\frac{3}{4},-\frac{n}{2}+2;-\operatorname{sh}^2 t)\\ &+ 2^{-\frac{n}{2}}\Gamma(-\frac{n}{2}+1)(i)^{n/2-1}(\operatorname{sh} t)^{(n/2-1)}\frac{\Gamma(\lambda+\frac{n}{2}-\frac{1}{2})}{\Gamma(\lambda-\frac{n}{2}+\frac{3}{2})}\\ &\quad {}_2F_1(\frac{\lambda}{2}+\frac{n}{4}-\frac{1}{4},-\frac{\lambda}{2}+\frac{n}{4}-\frac{1}{4},\frac{n}{2};-\operatorname{sh}^2 t).\end{aligned}$$

Therefore we can rewrite the spherical function φ_λ as $\varphi_\lambda(\exp(tX_0)) = \varphi^{(1)}_\lambda(\exp(tX_0)) + \varphi^{(2)}_\lambda(\exp(tX_0))$, where

$$\begin{aligned}\varphi^{(1)}_\lambda(\exp(tX_0)) &= (i)^{\frac{n}{2}-1}\pi^{\frac{-1}{2}}2^{n-3}\Gamma(\frac{n}{2}-\frac{1}{2})\Gamma(\frac{n}{2}-1)(\operatorname{sh} t)^{-(n-2)}\frac{\Gamma(\lambda-\frac{n}{2}+\frac{3}{2})}{\Gamma(\lambda+\frac{n}{2}-\frac{1}{2})}\\ &\quad {}_2F_1(\frac{\lambda}{2}-\frac{n}{4}+\frac{3}{4},-\frac{\lambda}{2}-\frac{n}{4}+\frac{3}{4},-\frac{n}{2}+2;-\operatorname{sh}^2 t),\end{aligned}$$

and

$$\begin{aligned}\varphi^{(2)}_\lambda(\exp(tX_0)) &= (i)^{\frac{n}{2}-1}\pi^{\frac{-1}{2}}2^{-1}\Gamma(\frac{n}{2}-\frac{1}{2})\Gamma(-\frac{n}{2}+1)\\ &\quad {}_2F_1(\frac{\lambda}{2}+\frac{n}{4}-\frac{1}{4},-\frac{\lambda}{2}+\frac{n}{4}-\frac{1}{4},\frac{n}{2};-\operatorname{sh}^2 t).\end{aligned}$$

Using the well known formula

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left\{1+\mathcal{O}(z^{-1})\right\}, \qquad \text{if } \ z \to \infty,$$

one can prove that

$$\lim_{\epsilon\to 0} {}_2F_1(\frac{\lambda}{2\epsilon}-\frac{n}{4}+\frac{3}{4},-\frac{\lambda}{2\epsilon}-\frac{n}{4}+\frac{3}{4},-\frac{n}{2}+2;-\operatorname{sh}^2(\epsilon t)) = \Gamma(2-\frac{n}{2})\left(\frac{\lambda t}{2}\right)^{\frac{n}{2}-1} I_{1-\frac{n}{2}}(\lambda t),$$

with

$$I_\nu(z) = \sum_{k=0}^{\infty}\frac{(z/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)},$$

and

$$\lim_{\epsilon\to 0} {}_2F_1(\frac{\lambda}{2\epsilon}+\frac{n}{4}-\frac{1}{4},-\frac{\lambda}{2\epsilon}+\frac{n}{4}-\frac{1}{4},\frac{n}{2};-\operatorname{sh}^2(\epsilon t)) = \Gamma(\frac{n}{2})\left(\frac{\lambda t}{2}\right)^{-\frac{n}{2}+1} I_{\frac{n}{2}-1}(\lambda t).$$

Note that $I_\nu(z) = e^{-i\pi\nu/2}J_\nu(iz)$, where J_ν is the Bessel function of the first type. Using the same techniques to prove that

$$\lim_{\epsilon\to 0}\frac{\Gamma(\frac{\lambda}{\epsilon}-\frac{n}{2}+\frac{3}{2})}{\Gamma(\frac{\lambda}{\epsilon}+\frac{n}{2}-\frac{1}{2})}(\mathrm{sh}(\epsilon t))^{-(n-2)} = (\lambda t)^{-(n-2)}.$$

Therefore, by $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, we get

$$\begin{aligned}
&\lim_{\epsilon\to 0}\varphi^{(1)}_{\frac{\lambda}{\epsilon}}(\exp(\epsilon t X_0))\\
=\ & (i)^{\frac{n}{2}-1}\pi^{-\frac{1}{2}}2^{\frac{n}{2}-2}\Gamma(\frac{n}{2}-\frac{1}{2})\Gamma(\frac{n}{2}-1)\Gamma(2-\frac{n}{2})(\lambda t)^{-\frac{n}{2}+1}I_{1-\frac{n}{2}}(\lambda t)\\
=\ & -(i)^{\frac{n}{2}-1}\pi^{\frac{1}{2}}2^{\frac{n}{2}-2}\Gamma(\frac{n}{2}-\frac{1}{2})(\lambda t)^{-\frac{n}{2}+1}\frac{I_{1-\frac{n}{2}}(\lambda t)}{\sin(\pi(1-\frac{n}{2}))},
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\epsilon\to 0}\varphi^{(2)}_{\frac{\lambda}{\epsilon}}(\exp(\epsilon t X_0))\\
=\ & (i)^{\frac{n}{2}-1}\pi^{-\frac{1}{2}}2^{\frac{n}{2}-2}\Gamma(\frac{n}{2}-\frac{1}{2})\Gamma(\frac{n}{2})\Gamma(1-\frac{n}{2})(\lambda t)^{-\frac{n}{2}+1}I_{\frac{n}{2}-1}(\lambda t)\\
=\ & (i)^{\frac{n}{2}-1}\pi^{\frac{1}{2}}2^{\frac{n}{2}-2}\Gamma(\frac{n}{2}-\frac{1}{2})(\lambda t)^{-\frac{n}{2}+1}\frac{I_{\frac{n}{2}-1}(\lambda t)}{\sin(\pi(1-\frac{n}{2}))}.
\end{aligned}$$

In conclusion

$$\lim_{\epsilon\to 0}\varphi_{\frac{\lambda}{\epsilon}}(\exp(\epsilon t X_0)) = (i)^{\frac{n}{2}-1}\pi^{-\frac{1}{2}}2^{\frac{n}{2}-1}\Gamma(\frac{n}{2}-\frac{1}{2})(\lambda t)^{-\frac{n}{2}+1}K_{1-\frac{n}{2}}(\lambda t),$$

where the function

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z)-I_\nu(z)}{\sin(\pi\nu)}$$

is known as Macdonald's function.

When Corollary 4.10 is combined with the previous results on the Π_0-Bessel functions, with $k_\alpha = m_\alpha/2 \in \mathbb{N}$, one obtains the following result for all non-compact causal symmetric spaces with even multiplicity. Recall that for all non-compact causal symmetric spaces, Σ is always reduced.

Theorem 4.14. *Let G/H be a non-compact causal symmetric space where the multiplicity function m is even. For $(\lambda, X) \in \mathcal{E}\times(\mathfrak{a}\cap C^0_{\max})$*

$$\int_H e^{-B(A_\lambda,\mathrm{Ad}(h)X)}dh = \frac{(-1)^{d'(\Pi_0,m/2)}}{c^+_{\Pi_0}(\rho(m/2),m/2)c^-_{\Pi_0}(\rho(m/2),m/2)}\times \frac{\mathbb{D}^\circ(m)\Big(\sum_{w\in W_0}\epsilon(w)e^{-w\lambda(X)}\Big)}{\prod_{\alpha\in\Sigma^+}\langle\alpha,X\rangle^{m_\alpha}\prod_{\alpha\in\Sigma^+}\langle\check{\alpha},\lambda\rangle^{m_\alpha-1}},$$

where $\mathbb{D}^\circ(m) = \prod_{\alpha\in\Sigma^+}\alpha^{m_\alpha}\mathbb{G}^\circ_+(m/2-1,1)\circ\prod_{\alpha\in\Sigma^+}\alpha^{-1}$.

In the remaining part of this section, we assume that G is a connected semisimple Lie group such that $G_{\mathbb{C}}/G$ is ordered. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{q} = i\mathfrak{g}$ contained in $\mathfrak{p}\oplus i\mathfrak{k}$. Note that $i\mathfrak{a}$ is a compact Cartan subalgebra of $\mathfrak{g}$. The Weyl group W_0 can be identified with $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$, which is the Weyl group of $\mathfrak{a}$ in K. In particular, $m_\alpha = 2$ for all $\alpha\in\Sigma^+$.

Using Theorem 4.14 we can write the Fourier transforms of orbits of G in the tangent space at the origin of $G_{\mathbb{C}}/G$ as a character formula, i.e. similar to the formula for the character of discrete series representations of G.

Corollary 4.15. *For $\lambda \in \mathcal{E}$ and $X \in C^0_{\max}$*

$$\int_G e^{-B(A_\lambda, \mathrm{Ad}(g)X)} dg = 2^{-|\Sigma^+|} \prod_{\alpha\in\Sigma^+} \rho(\check{\alpha}) \frac{\sum_{w\in W_0} \epsilon(w) e^{-w\lambda(X)}}{\prod_{\alpha\in\Sigma^+}\langle\check{\alpha},\lambda\rangle \prod_{\alpha\in\Sigma^+}\langle\alpha, X\rangle}.$$

The following tables represent the list of all non-compact causal symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ with even multiplicity. The list has been extracted from [10, 23].

Non-compactly causal symmetric pairs with even multiplicities

$\mathfrak{g}$	$\mathfrak{h}$	Σ	m_α	Comments
$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{su}(n-j,j)$	A_{n-1}	2	$n\geq 2,\ 1\leq j\leq [n/2]$
$\mathfrak{so}(2n+1,\mathbb{C})$	$\mathfrak{so}(2n-1,2)$	B_n	2	$n\geq 2$
$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sp}(n,\mathbb{R})$	C_n	2	$n\geq 3$
$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{so}(2n-2,2)$	D_n	2	$n\geq 4$
$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{so}^*(2n)$	D_n	2	$n\geq 5$
$(\mathfrak{e}_6)_{\mathbb{C}}$	$\mathfrak{e}_{6(-14)}$	E_6	2	
$(\mathfrak{e}_7)_{\mathbb{C}}$	$\mathfrak{e}_{7(-25)}$	E_7	2	
$\mathfrak{su}^*(2n)$	$\mathfrak{sp}(n-j,j)$	A_{n-1}	4	$n\geq 2,\ 1\leq j\leq [n/2]$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_{4(-20)}$	A_2	8	
$\mathfrak{so}(2n+1,1)$	$\mathfrak{so}(2n,1)$	A_1	$2n$	$n\geq 3$

Special isomorphisms of non-compactly causal symmetric pairs with even multiplicities are

$\mathfrak{g}$	$\mathfrak{h}$
$\mathfrak{sp}(1,\mathbb{C})\approx\mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{sp}(1,\mathbb{R})\approx\mathfrak{su}(1,1)$
$\mathfrak{so}(3,\mathbb{C})\approx\mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{so}(1,2)\approx\mathfrak{su}(1,1)$
$\mathfrak{sp}(2,\mathbb{C})\approx\mathfrak{so}(5,\mathbb{C})$	$\mathfrak{sp}(2,\mathbb{R})\approx\mathfrak{so}(3,2)$
$\mathfrak{so}(6,\mathbb{C})\approx\mathfrak{sl}(4,\mathbb{C})$	$\mathfrak{so}(4,2)\approx\mathfrak{su}(2,2)$
$\mathfrak{so}(6,\mathbb{C})\approx\mathfrak{sl}(4,\mathbb{C})$	$\mathfrak{so}^*(6)\approx\mathfrak{su}(3,1)$
$\mathfrak{so}(8,\mathbb{C})=\mathfrak{so}(8,\mathbb{C})$	$\mathfrak{so}^*(8)\approx\mathfrak{so}(2,6)$
$\mathfrak{so}(3,1)\approx\mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{so}(2,1)\approx\mathfrak{su}(1,1)$
$\mathfrak{so}(5,1)\approx\mathfrak{su}^*(4)$	$\mathfrak{so}(4,1)\approx\mathfrak{sp}(1,1)$

References

[1] F. Calogero, *Solution of a three-body problem in one dimension,* J. Math. Phys. **10** (1969), 2191–2196

[2] I. Cherednik, *A unification of Knizhnik-Zamolodchikov equations and Dunkl operators via affine Hecke algebras,* Inv. Math. **106** (1991), 411–432

[3] J.L. Clerc, *Une formule asymptotique du type Mehler-Heine pour les zonoles d'un espace riemannien symtrique*, Studia Math. **57** (1976), 27–32.

[4] J.J. Duistermaat and G.J. Heckman, *On the variation in the cohomology of the symplectic form on the reduced phase space*, Invent. Math. **69** (1982), 259–268

[5] C.F. Dunkl, *Differential-difference operators associated to reflection groups,* Tran. Amer. Math. soc. **311** (1989), 167–183

[6] C.F. Dunkl and Y. Xu, *Orthogonal polynomials of several variables,* Cambridge Univ. Press, Cambridge, UK (2001)

[7] C.F. Dunkl, M. de Jeu, and E. Opdam, *Singular polynomials for finite reflection groups*, Trans. Amer. Math. Soc. **346** (1994), 237–256.

[8] A.H. Dooley and J.W. Rice, *On contractions of semisimple Lie groups,* Trans. Amer. Math. Soc. **289** (1985), 185–202

[9] A. Erdelyi, *et. al.*, *Higher transcendental functions I,* Mc Graw-Hill, New York (1953)

[10] J. Hilgert and G. Ólafsson, *Causal symmetric spaces. Geometry and harmonic analysis,* Prespectives in Mathematics, **18**. Academic Press, Inc. San Diego, CA (1997)

[11] J. Faraut, J. Hilgert, and G. Ólafsson, *Spherical functions on ordered symmetric spaces,* Anna. Inst. Fourier **44** (1994), 927–965

[12] Harish-Chandra, *Spherical functions on a semisimple Lie groups. I and II*, Amer. J. Math. **80** (1958), 241–310, 553–613

[13] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces,* Academic Press, INC., New York-London (1978)

[14] G.J. Heckman, *Root systems and hypergeometric functions. II,* Compositio Math. **64** (1987), 353–373

[15] ———, *A remark on the Dunkl differential-difference operators,* Harmonic analysis on reductive groups (Brunswick, ME, 1989), 181–191, Progr. Math., 102, Birkhauser Boston, Boston, MA, 1991

[16] ———, *Dunkl operators,* Séminaire Bourbaki, Asterisque **245** (1997), 223–146

[17] G.J. Heckman and E.M. Opdam, *Root systems and hypergeometric functions. I,* Compositio Math. **64** (1987), 329–352

[18] G.J. Heckman and H. Schlichtkrull, *Harmonic analysis and special functions on symmetric spaces,* Academic Press (1994)

[19] C. Itzykson and J. B. Zuber, *The planer approximation. II.*, J. Math. Phys. **21** (1980), 411–421

[20] M.E. de Jeu, *The Dunkl transform,* Invent. Math. **113** (1993), 147–162

[21] ———, *Paley–Wiener theorems for the Dunkl transform*, Preprint (2004). ArXiv: math.CA/0404439

[22] C. Meaney, *The inverse Abel transform for $SU(p,q)$*, Ark. Matematik **24** (1985), 131–140

[23] T. Oshima and J. Sekiguchi, *Eigenspaces of invariant differential operators on an affine symmetric space,* Invent. Math. **57** (1980), 1–81

[24] G. Ólafsson and A. Pasquale, *A Paley-Winer theorem for the Θ-spherical transform: The even multiplicity case,* Preprint (2003). ArXiv: math.FA/0304361

[25] M. A. Olshanetsky and A. M. Perelomov, *Completely integrable Hamiltonian systems connected with semisimple Lie algebras,* Invent. Math. **37** (1976), 93–108

[26] E.M. Opdam, *Root systems and hypergeometric functions. III,* Compositio Math. **67** (1988), 21–49

[27] ———, *Root systems and hypergeometric functions. IV,* Compositio Math. **67** (1988), 191–209

[28] ———, *Some applications of hypergeometric shift operators*, Invent. Math. **98** (1989), 1–18

[29] ———, *Dunkl operators, Bessel functions and the discriminiant of a finite Coxeter group,* Compositio Math. **85** (1993), 333–373

[30] ———, *Harmonic analysis for certain representations of graded Hecke algebras,* Acta. Math. **175** (1995), 75–121

[31] ______, *Lecture notes on Dunkl operators for real and complex reflection groups,* AMS Memoire **8** (2000).

[32] A. Pasquale, *A theory of Θ-spherical functions,* Habilitatiosschrift, Technische Universität Clausthal (2002).

[33] M. Rösler, *Generalized Hermite polynomials and the heat equation for Dunkl operators,* Comm. Math. Phys. **192** (1998), 519–542

[34] R.J. Stanton, *Mean convergence of Fourier series on compact Lie groups*, Trans. Amer. Math. Soc. **218** (1976), 61–87.

[35] C. Torossian, *Une application des opérateurs de Cherednik l'isomorphisme d'Harish-Chandra pour les espaces symétriques,* C. R. Acad. Sci. Paris Sér. I Math. **320** (1995), 139–144.

Authors' address: University of Southern Denmark, Department of Mathematics and Computer Sciences, Campusvej 55, DK-5230, Odense M, Denmark

E-mail address: ssaid@imada.sdu.dk

E-mail address: orsted@imada.sdu.dk