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GLOBAL SOLUTIONS FOR THE ONE-DIMENSIONAL VLASOV–MAXWELL SYSTEM FOR LASER-PLASMA INTERACTION

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We analyse a reduced 1D Vlasov–Maxwell system introduced recently in the physical literature for studying laser-plasma interaction. This system can be seen as a standard Vlasov equation in which the field is split in two terms: an electrostatic field obtained from Poisson's equation and a vector potential term satisfying a nonlinear wave equation. Both nonlinearities in the Poisson and wave equations are due to the coupling with the Vlasov equation through the charge density. We show global existence of weak solutions in the non-relativistic case, and global existence of characteristic solutions in the quasi-relativistic case. Moreover, these solutions are uniquely characterised as fixed points of a certain operator. We also find a global energy functional for the system allowing us to obtain L^p-nonlinear stability of some particular equilibria in the periodic setting.

Keywords: Kinetic equations; Vlasov–Maxwell system; existence and uniqueness of solutions; nonlinear stability.

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1. Introduction

Given a population of electrons, with mass m and charge $-e$, assumed to be relativistic, we denote

$$\mathbf{v}(\mathbf{p}) := \frac{\mathbf{p}}{m\sqrt{1+\dfrac{|\mathbf{p}|^2}{m^2\,c^2}}} := \frac{\mathbf{p}}{m\,\gamma},$$

the velocity corresponding to a given momentum $\mathbf{p}$. The electrons move under the effect of an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$. Then, their distribution function $\mathfrak{f}(t, \mathbf{q}, \mathbf{p})$, where $\mathbf{q}$ denotes the position variable, is solution to the Vlasov equation:

$$\frac{\partial \mathfrak{f}}{\partial t} + \mathbf{v}(\mathbf{p}) \cdot \frac{\partial \mathfrak{f}}{\partial \mathbf{q}} - e\ (\mathbf{E} + \mathbf{v}(\mathbf{p}) \times \mathbf{B}) \cdot \frac{\partial \mathfrak{f}}{\partial \mathbf{p}} = 0. \tag{1.1}$$

The fields $\mathbf{E}$ and $\mathbf{B}$ are the sum of three parts:

(1) the self-consistent fields created by the electrons;
(2) the electromagnetic field of a laser wave which is sent into the medium (called the *pump* wave);
(3) the electrostatic field $\mathbf{E}_{\rm ext}(\mathbf{q})$ generated by a background of ions which are considered immobile during the time scale of the wave, and/or by an external, static confinement potential.

In all cases, we denote by $n_{\rm ext} := \varepsilon_0 \operatorname{div} \mathbf{E}_{\rm ext}/e$. Without this term, the population of electrons could not be dynamically stable. Then, the Maxwell system is written:

$$\frac{\partial \mathbf{E}}{\partial t} = c^2\, \mathbf{curl}\, \mathbf{B} + \frac{e}{\varepsilon_0}\, \mathbf{j}, \tag{1.2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\mathbf{curl}\, \mathbf{E}, \tag{1.3}$$

$$\operatorname{div} \mathbf{E} = \frac{e}{\varepsilon_0}\ (n_{\rm ext} - n)\,, \tag{1.4}$$

$$\operatorname{div} \mathbf{B} = 0, \tag{1.5}$$

where c and ε_0 are the speed of light and the dielectric permittivity of vacuum, and the electron density and flux n and $\mathbf{j}$ are the first two moments of the distribution function $\mathfrak{f}$:

$$\{n, \mathbf{j}\}(t, \mathbf{q}) := \int \{1, \mathbf{v}(\mathbf{p})\}\, \mathfrak{f}(t, \mathbf{q}, \mathbf{p})\, \mathrm{d}\mathbf{p}.$$

It is well known that Eqs. (1.5, 1.3) amount to the existence of vector and scalar potentials such that

$$\mathbf{B} = \mathbf{curl}\, \mathbf{A}, \quad \mathbf{E} = -\partial_t \mathbf{A} - \mathbf{grad}\, \Phi. \tag{1.6}$$

The assumptions below the 1D model are the following: all variables depend on only one space variable, denoted x, and the electrons are monokinetic in the directions transversal to x. This is physically justifed by the fact that all the phenomena, especially heating, are much more rapid along the direction of propagation of the laser wave than in the transversal directions. So, the distribution function becomes:

$$\mathfrak{f}(t, \mathbf{q}, \mathbf{p}) = f(t, x, p_x)\, \delta\left(\mathbf{p}_\perp - \mathbf{p}_0(t, x)\right). \tag{1.7}$$

The function $\mathbf{p}_0(t, x)$ can be determined by Hamiltonian-mechanical considerations[16]. The Hamiltonian for one particle is $H := \gamma\, m\, c^2 - e\, \Phi$, and (1.1) reads:

$$\frac{\partial \mathfrak{f}}{\partial t} + [H, \mathfrak{f}] = 0,$$

where $[\cdot,\cdot]$ is the Poisson bracket. Then, by Hamilton's equation, the transversal component of the canonical conjugate momentum $\mathbf{P_c} := \mathbf{p} - e\,\mathbf{A}$ is conserved:

$$\frac{\mathrm{d}\mathbf{P_c}}{\mathrm{d}t} = -\frac{\partial H}{\partial \mathbf{q}} \implies \frac{\mathrm{d}\mathbf{P_{c\perp}}}{\mathrm{d}t} = -\frac{\partial H}{\partial \mathbf{q}_\perp} = 0.$$

By a suitable change of referential, we can suppose that $\mathbf{P_{c\perp}} = 0$; and by imposing the Coulomb gauge $\operatorname{div}\mathbf{A} = 0$, that $A_x = 0$. Hence, $\mathbf{p}_0(t,x) = e\,\mathbf{A}(t,x)$.

For the sake of simplicity, we shall assume in this work that the pump wave is linearly polarised in a direction which we call y; however, the forthcoming computations can be easily generalised to an arbitrary polarisation. Under these circumstances, Eq. (1.6) becomes:

$$B_z = \partial_x A_y, \quad E_y = -\partial_t A_y, \quad E_x = -\partial_x \Phi, \quad E_z = B_y = 0,$$

which allows to recast the Vlasov equation (1.1) and the two remaining Maxwell equations (1.2, 1.4) as the following system:

$$\frac{\partial f}{\partial t} + \frac{p_x}{m\gamma}\frac{\partial f}{\partial x} - e\left(E_x + \frac{e\,A_y}{m\,\gamma}\frac{\partial A_y}{\partial x}\right)\frac{\partial f}{\partial p_x} = 0, \tag{1.8}$$

$$\frac{\partial^2 A_y}{\partial t^2} - c^2\frac{\partial^2 A_y}{\partial x^2} = -\frac{e}{m\varepsilon_0}\,n_\gamma\,A_y \tag{1.9}$$

$$\frac{\partial E_x}{\partial t} = \frac{e}{\varepsilon_0}\,j_x \tag{1.10}$$

$$\frac{\partial E_x}{\partial x} = \frac{e}{\varepsilon_0}\,(n_{\mathrm{ext}} - n)\,. \tag{1.11}$$

The Lorentz factor γ and the density, *quasi-density* and flux are now given by:

$$\gamma = \sqrt{1 + \frac{p_x^2}{m^2\,c^2} + \frac{e^2\,A_y^2}{m^2\,c^2}}\,; \tag{1.12}$$

$$\{n, n_\gamma, j_x\} = \int \left\{1, \frac{1}{\gamma}, \frac{p_x}{m\gamma}\right\} f\,\mathrm{d}p_x. \tag{1.13}$$

The equations (1.10) and (1.11), which are relative to the same variable E_x, are redundant under regularity conditions. Eq. (1.10) is a (simple) evolution equation, while (1.11) is interpreted as a constraint. As usual, the satisfaction of this constraint at $t = 0$ implies its satisfaction at any time, thanks to (1.10) and to the continuity equation $\partial_t n + \partial_x j_x = 0$.

For a general polarisation, (1.9) would be duplicated, with a similar equation for A_z. Quadratic terms in A_y and A_z would be added in (1.12) as well as in the third term in (1.8). The reader will convince himself that the study of this slightly more complicated system is completely similar to that of (1.8–1.13). From now on, we shall omit the subscripts in $p_x,\ E_x,\ j_x,\ A_y$.

Let us notice that the model (1.8–1.13), as well as its extended version for a general polarisation, are classes of *exact* solutions to the relativistic Vlasov–Maxwell model, without any approximation. They are, as far as we know, the simplest exact solutions beyond 1D Vlasov–Poisson models.

1.1. *Discussion of the relativistic character*

The model (1.8–1.13) features a strongly non-linear coupling between the kinetic and electromagnetic variables, through the Lorentz factor (1.12). This phenomenon makes this system difficult to study on the theoretical level, but also to solve numerically: no splitting between the variables is possible.[16] This is why two reduced models have been defined by physicists:

(1) The *non-relativistic model* (hereafter denoted NR) approximates the relativistic dynamic by the Newtonian one by setting $\gamma = 1$ everywhere. It is physically justified when the temperature is low enough, so that the proportion of relativistic electrons is negligible, *and* the intensity of the pump wave is small.

(2) The *quasi-relativistic*[a] *model* (QR) consists in approximating γ by $\sqrt{1 + \frac{p^2}{m^2\,c^2}}$ in the second term in (1.8) and in the definition of j, and setting $\gamma = 1$ in the third term in (1.8) and in the definition of n_γ, which amounts to setting $n_\gamma = n$. It is acceptable when the proportion of ultra-relativistic ($v \simeq c$) electrons is negligible and the pump intensity is moderate.

The original model, with γ defined by (1.12) will be referred to as *fully relativistic* (FR). We remark that the NR model is a class of exact solutions to the non-relativistic Vlasov–Maxwell system, i.e. (1.1–1.5) with $\mathbf{v}(\mathbf{p}) = \mathbf{p}/m$. By contrast, the QR model is only an approximation to the FR one. What makes its interest for the applied mathematician — besides its widespread use for simulation — is that it already contains certain features of higher-dimensional relativistic Vlasov–Maxwell systems, while being simpler to study.

1.2. *Rescaled equations and Cauchy problem*

The set of equations (1.8–1.13) can be simplified by introducing some rescaled variables. Let $\overline{n}$ be the unit of density; we choose the units for the independent variables as:

$$\overline{x} = \frac{c}{e}\sqrt{\frac{m\,\varepsilon_0}{\overline{n}}}, \quad \overline{t} = \frac{\overline{x}}{c}, \quad \overline{p} = m\,c\,;$$

and for the dependent variables as:

$$\overline{E} = \sqrt{\frac{m\,c^2\,\overline{n}}{\varepsilon_0}}, \quad \overline{A} = \frac{m\,c}{e}, \quad \overline{f} = \frac{\overline{n}}{m\,c}, \quad \overline{j} = \overline{n}\,c.$$

Keeping the same notations for the rescaled variables, we obtain the rescaled system:

$$\frac{\partial f}{\partial t} + \frac{p}{\gamma_1}\frac{\partial f}{\partial x} - \left(E + \frac{A}{\gamma_2}\frac{\partial A}{\partial x}\right)\frac{\partial f}{\partial p} = 0, \tag{1.14}$$

$$\frac{\partial E}{\partial t} = j \tag{1.15}$$

[a]Also called *semi-relativistic* by some authors.

$$\frac{\partial^2 A}{\partial t^2} - \frac{\partial^2 A}{\partial x^2} = -n_\gamma\, A \tag{1.16}$$

where the flux j and the quasi-density n_γ are defined as

$$j(t,x) := \int \frac{p}{\gamma_1}\, f(t,x,p)\, \mathrm{d}p, \qquad n_\gamma(t,x) := \int \frac{1}{\gamma_2}\, f(t,x,p)\, \mathrm{d}p.$$

Of course, suitable initial conditions are supplied, namely

$$f(0,x,p) = f_0(x,p),\ A(0,x) = A_0(x),\ \partial_t A(0,x) = \dot{A}_0(x). \tag{1.17}$$

For coherence, the initial electrostatic field must be given by:

$$E_0(x) = E_0(0) - \int_0^x (n_0(y) - n_{\mathrm{ext}}(y))\ \mathrm{d}y, \quad \text{where:} \quad n_0(x) := \int f_0(x,p)\, \mathrm{d}p. \tag{1.18}$$

This guarantees that the Poisson (or Gauss) equation

$$\frac{\partial E}{\partial x} = n_{\mathrm{ext}} - n, \quad \text{where:} \quad n(t,x) := \int f(t,x,p)\, \mathrm{d}p, \tag{1.19}$$

will hold at any time provided the continuity equation is satisfied.

As far as the relativistic character of the particles is concerned, the three versions of the model are respectively:

- NR: $\gamma_1 = \gamma_2 = 1$.
- QR: $\gamma_1 = \sqrt{1+p^2},\ \gamma_2 = 1$.
- FR: $\gamma_1 = \gamma_2 = \sqrt{1+p^2+A^2}$.

Let us note that we shall investigate the existence of two classes of solutions for the system (1.14–1.16): periodic solutions, corresponding to initial data that are periodic in space, with a given period L, and "open-space" solutions, i.e. solutions of finite mass and energy. In both cases, we always assume that n_{ext} is at least bounded. Moreover, in the periodic setting, we assume that it is periodic and

$$\int_0^L (n_{\mathrm{ext}}(x) - n_0(x))\ \mathrm{d}x = 0,$$

so that E_0 is indeed periodic.

To the best of our knowledge, this is the first mathematical work on this particular Vlasov–Maxwell system (1.14–1.16): most of the previous mathematical works on reduced Vlasov–Maxwell models deal with systems living in two or one-and-a-half dimensions.[15,14,13,2] In our case, the lower dimension is, to some extent, compensated by a stronger nonlinearity. This system shares some common features with the Nordström–Vlasov system recently studied in [7,8]. Both systems present a Vlasov equation coupled to a wave equation whose right-hand side depends on the charge density. The main differences between them are the gravitational character of the Nordström–Vlasov system, our coupling with the Poisson equation and our more complicated right-hand side in the wave equation.

In this article, we shall only investigate the existence and uniqueness of solutions to the NR and QR models. The differences between them can be bridged easily by using the shorthand notation

$$\widehat{p} := p \text{ (NR case)}, \quad \widehat{p} := \frac{p}{\sqrt{1+p^2}} \text{ (QR case)}.$$

On the other hand, the method presented in this article cannot apply directly to the FR model, because of its much stronger and more non-linear coupling between the kinetic and electromagnetic variables.

The next Section is devoted to reviewing some basic estimates for the Vlasov equation, and the different notions of solutions we will deal with. In Section 3 we use the procedure of [9] to prove the global existence of weak solutions in the NR and QR cases, which are characterised as unique fixed points of a certain operator.

This result is improved in two ways in Sections 4 and 5 respectively. We first prove the local-in-time existence of characteristic solutions in the NR case. The main difficulty in proving global existence of characteristic solutions consists in controlling the second space derivative of the vector potential A or, equivalently, the first space derivative of the density n. These difficulties are analogous to those found in the two and one-half dimensional Vlasov–Maxwell systems studied in [15,14,13,2]. In the QR case, we obtain the existence result thanks to a good integral representation of the second derivative of A reminiscent of similar ideas in [14], used recently in [8].

Finally, the last section is devoted to obtaining a global energy functional for the three cases. It is conserved in time by characteristic solutions in the NR and QR cases. This fact, together with now standard relative entropy arguments, leads to the L^p-nonlinear stability of a family of steady states in the periodic setting.

2. Solutions to the forced Vlasov equation

In this Section, we assume that the fields (E, A) are given and we introduce several notions of solution to (1.14) and summarise their associated regularity properties.

Whatever their regularity, we are mainly interested in *global* solutions, i.e. which exist for any time. However, our estimates will generally not be uniform in time and thus, we fix an *arbitrary* target time T and we look for solutions defined on $[0, T]$, for all $T > 0$. We use the following notations for functional spaces:

- $\mathcal{C}^m(0, T; \mathcal{X})$: the space of m times continuously differentiable functions from $[0, T]$ to the Banach space $\mathcal{X}$;
- $W^{k,\infty}(\mathbb{R})$: the space of functions from $\mathbb{R}$ to $\mathbb{R}$ with all derivatives (in the sense of distributions) bounded, up to order k;
- $\mathcal{C}^k_b(\mathbb{R}) = \mathcal{C}^k(\mathbb{R}) \cap W^{k,\infty}(\mathbb{R})$: the space of k times continuously differentiable functions with all derivatives bounded;
- $W^{k,\infty}_L(\mathbb{R})$, $\mathcal{C}^k_L(\mathbb{R})$: the subspaces of functions which have the space period L.

We will denote by $\|\cdot\|_t$ the norm in $L^\infty((0, t) \times \mathbb{R})$, for $t \leq T$.

2.1. *The characteristic system*

From now on, we denote

$$F(t,x) := E(t,x) + A(t,x)\,\partial_x A(t,x) \tag{2.1}$$

the force generated by the fields (E, A). The characteristic system associated to the transport equation (1.14) reads:

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}s} = \widehat{P}(s), & \dfrac{\mathrm{d}P}{\mathrm{d}s} = -F(s, X(s)), \\ X(t) = x, & P(t) = p. \end{cases} \tag{2.2}$$

Global existence and uniqueness of solution to the above system is ensured by assuming that the force field is continuous in time and globally Lipschitz in space; in turn, a sufficient condition for this is:

$$E \in \mathcal{C}^0\left(0,T;W^{1,\infty}(\mathbb{R})\right), \quad A \in \mathcal{C}^0\left(0,T;W^{2,\infty}(\mathbb{R})\right). \tag{2.3}$$

Under this assumption, the unique solution to the characteristic system (2.2) denoted by $(X(s;t,x,p), P(s;t,x,p))$ becomes (at least) a continuous function in all its variables. We shall also consider a stronger regularity condition

$$E \in \mathcal{C}^0\left(0,T;\mathcal{C}^1_b(\mathbb{R})\right), \quad A \in \mathcal{C}^0\left(0,T;\mathcal{C}^2_b(\mathbb{R})\right). \tag{2.4}$$

From the uniqueness of the solution, we deduce a periodicity result:

Lemma 2.1. *If the force field is periodic in space, i.e.:* $\forall (t,x),\ F(t,x+L) = F(t,x)$, *then the following identity holds for all* $s,\ t,\ x,\ p$*:*

$$X(s;t,x+L,p) = X(s;t,x,p) + L, \quad P(s;t,x+L,p) = P(s;t,x,p). \tag{2.5}$$

The divergence of characteristics generated by different force fields is measured in a classical way (see for instance [9] or [2] and references therein).

Lemma 2.2. *Let* (X^1,P^1) *and* (X^2,P^2) *be the characteristics associated to the respective forces* $F_1,\ F_2 \in \mathcal{C}^0\left(0,T;W^{1,\infty}(\mathbb{R})\right)$. *Then, the following inequalities hold for all* (t,x,p)*:*

$$\left|X^1(0;t,x,p) - X^2(0;t,x,p)\right| \le t \int_0^t \|F_1 - F_2\|_s \,\mathrm{d}s, \tag{2.6}$$

$$\left|P^1(0;t,x,p) - P^2(0;t,x,p)\right| \le \int_0^t \|F_1 - F_2\|_s \,\mathrm{d}s. \tag{2.7}$$

Proof. Eq. (2.7) clearly stems from the integration of the equation for P in (2.2). In the NR case, Eq. (2.6) immediately follows. In the QR case, we notice that

$$\left|\frac{\mathrm{d}}{\mathrm{d}p}\left[\frac{p}{\sqrt{1+p^2}}\right]\right| = \frac{1}{(1+p^2)^{3/2}} \le 1,$$

so $\left|\widehat{P^1} - \widehat{P^2}\right| \le \left|P^1 - P^2\right|$, hence (2.6) by integration. □

In a very similar fashion, the derivatives of the characteristics can be estimated in terms of derivatives of the force field.

Lemma 2.3. *The solution (X,P) to (2.2), with $F \in \mathcal{C}^0\left(0,T;W^{1,\infty}(\mathbb{R})\right)$, is Lipschitz w.r.t. the variables (x,p), and thus a.e. differentiable. The derivatives satisfy the following bound for a.e. (x,p) and all $0 \le \tau \le t$:*

$$\max\left\{\left|\frac{\partial X}{\partial x}(\tau)\right|,\ \left|\frac{\partial X}{\partial p}(\tau)\right|,\ \left|\frac{\partial P}{\partial x}(\tau)\right|,\ \left|\frac{\partial P}{\partial p}(\tau)\right|\right\} \le \mathrm{e}^{(t-\tau)\left(1+\|\partial_x F\|_t\right)}. \tag{2.8}$$

Proof. Consider two final conditions (x_1,p_1) and (x_2,p_2). We use the shorthand:

$$(X_i(s),P_i(s)) := (X(s;t,x_i,p_i),P(s;t,x_i,p_i))\,,\quad i=1,\ 2\,;$$

and we denote by $\Lambda F(s)$ the Lipschitz constant of $F(s,\cdot)$, i.e. $\sup_x |\partial_x F(s,x)|$.

By integrating the characteristic system (2.2), we get:

$$\begin{aligned}
&|X_1(\tau)-X_1(\tau)| + |P_1(\tau)-P_2(\tau)| \\
&\le |x_1-x_2| + |p_1-p_2| + \int_\tau^t \left\{\left|\widehat{P}_1(s)-\widehat{P}_2(s)\right| + |F(s,X_1(s))-F(s,X_2(s))|\right\}\,\mathrm{d}s \\
&\le |x_1-x_2| + |p_1-p_2| + \int_\tau^t \left\{|P_1(s)-P_2(s)| + \Lambda F(s)\ |X_1(s)-X_2(s)|\right\}\,\mathrm{d}s \\
&\le |x_1-x_2| + |p_1-p_2| + (1+\|\partial_x F\|_t)\int_\tau^t \left\{|X_1(s)-X_2(s)| + |P_1(s)-P_2(s)|\right\}\,\mathrm{d}s.
\end{aligned}$$

Hence, by Gronwall's lemma:

$$|X_1(\tau)-X_1(\tau)| + |P_1(\tau)-P_2(\tau)| \le (|x_1-x_2|+|p_1-p_2|)\ \mathrm{e}^{(t-\tau)\left(1+\|\partial_x F\|_t\right)},$$

which proves the Lipschitz character of the functions $X,\ P$ and the quantitative estimate (2.8). □

2.2. *Characteristic, mild and weak solutions*

In this work, we shall always assume the following two hypotheses about the initial distribution function f_0.

Hypothesis 1. *In the open-space case, $f_0 \in L^1(\mathbb{R}^2)$; in the periodic case,*

$$f_0 \in L^1_L(\mathbb{R}^2) := \Big\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^2) :\ f(x+L,p) = f(x,p) \text{ a.e. in } (x,p)\in\mathbb{R}^2$$
$$\text{and }\ f|_{(0,L)\times\mathbb{R}} \in L^1((0,L)\times\mathbb{R})\Big\}.$$

In both cases, $(\partial_x f_0, \partial_p f_0) \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$.

Hypothesis 2. *There exists a continuous, positive, even function $g(p)$, which moreover is decreasing in $|p|$ and satisfies*

$$\int_{\mathbb{R}} |p|\, g(p)\,\mathrm{d}p < \infty,$$

such that

$$f_0(x,p) \le g(p), \quad |\partial_x f_0(x,p)| \le g(p), \quad |\partial_p f_0(x,p)| \le g(p).$$

Let us notice that these hypotheses imply, in the open-space case, that $f_0 \in W^{1,1}_{\rm loc}(\mathbb{R}^2) \cap W^{1,\infty}(\mathbb{R}^2)$; the initial density $n_0 \in L^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$; and the initial electrostatic field, given by (1.11), $E_0 \in W^{1,\infty}(\mathbb{R})$ if $n_{\rm ext} \in L^\infty(\mathbb{R})$, $E_0 \in \mathcal{C}^1_b(\mathbb{R})$ if $n_{\rm ext} \in \mathcal{C}^0_b(\mathbb{R})$. In the periodic case, one has similar properties in terms of periodic spaces; indeed, electrical neutrality ensures that $n - n_{\rm ext}$ admits a periodic primitive.

For further reference, let us note that the function $g_r(p)$ defined as

$$g_r(p) = g(0) \text{ for } |p| \le r, \quad g_r(p) = g(|p| - r) \text{ for } |p| \ge r,$$

satisfies

$$\int_{\mathbb{R}} g_r(p)\,\mathrm{d}p = 2r\,g(0) + \int_{\mathbb{R}} g(p)\,\mathrm{d}p. \tag{2.9}$$

The first notion of solutions which we consider is the one given by the usual characteristic method.

Definition 2.1. Given a force field $F \in \mathcal{C}^0\left(0,T;W^{1,\infty}(\mathbb{R})\right)$ and f_0 satisfying Hypotheses 1–2, we define the *characteristic* solution of the Vlasov equation (1.14) as:

$$f(t,x,p) := f_0(X(0;t,x,p), P(0;t,x,p)), \tag{2.10}$$

where $(X(s;t,x,p), P(s;t,x,p))$ is the unique solution to (2.2). Moreover, assuming $F \in \mathcal{C}^0\left(0,T;\mathcal{C}^1_b(\mathbb{R})\right)$ and $f_0 \in \mathcal{C}^1(\mathbb{R}^2)$, then we shall often refer to the characteristic solution as a *classical* solution.

The previous definition gets clarified by the following result.

Lemma 2.4. *Under the assumption (2.3), the characteristic solution f belongs to $W^{1,\infty}\left(0,T;W^{1,\infty}(\mathbb{R}^2)\right)$. If, moreover, $f_0 \in \mathcal{C}^1(\mathbb{R}^2)$ and (2.4) holds, then $f \in \mathcal{C}^1\left(0,T;\mathcal{C}^1_b(\mathbb{R}^2)\right)$ and (1.14) is satisfied in the classical sense.*

Proof. According to classical dynamical system theory, $(X(s;t,x,p), P(s;t,x,p))$ is $\mathcal{C}^1$ w.r.t. s, and also w.r.t. t given the symmetry of these variables. The Lipschitz character in (x,p) was obtained in Lemma 2.3. The first part of the conclusion then follows from Hypothesis 2, and the fact that the composition of Lipschitz functions is Lipschitz.

Moreover, under (2.4), the force is $\mathcal{C}^1$; as $p \mapsto \widehat{p}$ is $\mathcal{C}^\infty$, we deduce that the solution to (2.2) is $\mathcal{C}^1$. The last statement then follows from the chain rule. □

Mild solutions are introduced for relaxing the assumption of differentiability of the characteristics but keeping the fact that they define a family of Lipschitz homeomorphisms in phase space (see [1] and references therein). In fact, this definition

can be rephrased using the concept of push-forward of a density through a map, which is quite well-known in mass transport theory.[18] The pushed-forward measure $T_{\#}\rho$ of a given measure ρ in $\mathbb{R}^n$ assigns mass

$$T_{\#}\rho[K] := \rho[T^{-1}(K)]$$

to each Borel set $K \subset \mathbb{R}^n$. By this property, it satisfies that

$$\int_{\mathbb{R}^n} \psi \, \mathrm{d}(T_{\#}\rho) = \int_{\mathbb{R}^n} (\psi \circ T) \, \mathrm{d}\rho$$

for all test functions $\psi \in \mathcal{C}_c^0(\mathbb{R}^n)$.

Definition 2.2. Given a force field $F \in \mathcal{C}^0\left(0,T;W^{1,\infty}(\mathbb{R})\right)$ and $f_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$, we say that a weakly continuous function $f(t,x,p) \in \mathcal{C}_w\left([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^2)\right)$ is a *mild* solution of the Vlasov equation (1.14) if it satisfies

$$\int_{\mathbb{R}^2} \psi(x,p)\, f(t,x,p)\, \mathrm{d}x\, \mathrm{d}p = \int_{\mathbb{R}^2} \psi(X(t;0,x,p),P(t;0,x,p))\, f_0(x,p)\, \mathrm{d}x\, \mathrm{d}p \quad (2.11)$$

for all test functions $\psi \in \mathcal{C}_c^0(\mathbb{R}^2)$ and all $t \geq 0$, i.e.,

$$f(t,x,p) = (X(t;0,x,p),P(t;0,x,p))_{\#} f_0$$

for all $t \geq 0$.

Taking into account the change of variables formula for Lipschitz functions,[11] we deduce that a characteristic solution is a mild solution of the Vlasov equation (1.14).

We can relax even more the assumptions on the force field and talk about distributional solutions for the Vlasov equation (1.14).

Definition 2.3. Given a force field $F \in L^\infty((0,T)\times\mathbb{R})$ and $f_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$, we say that $f(t,x,p) \in L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}^2)$ is a *distributional* solution of the Vlasov equation (1.14) if it satisfies

$$-\int_0^T\!\!\int_{\mathbb{R}^2} \left(\frac{\partial\psi}{\partial t} + \widehat{p}\,\frac{\partial\psi}{\partial x} - F\,\frac{\partial\psi}{\partial p}\right) f \, \mathrm{d}x\, \mathrm{d}p\, \mathrm{d}t = \int_{\mathbb{R}^2} \psi(0,x,p)\, f_0 \, \mathrm{d}x\, \mathrm{d}p \quad (2.12)$$

for all test functions $\psi \in \mathcal{C}_c^\infty([0,T)\times\mathbb{R}^2)$.

It is easy to check that any mild solution is a distributional solution. Moreover, in the particular case of characteristic solutions, in which $f_0 \in W^{1,\infty}(\mathbb{R}^2)$, f belongs to $W^{1,\infty}\left(0,T;W^{1,\infty}(\mathbb{R}^2)\right)$ by Lemma 2.4. Therefore, one can check from the weak formulation (2.12) that f satisfies the Vlasov equation (1.14) as an equality almost everywhere of locally bounded functions on $(0,\infty)\times\mathbb{R}^2$.

2.3. A priori *estimates*

Since bounds for functions in $\mathcal{C}_b^1(\mathbb{R})$ or $W^{1,\infty}(\mathbb{R})$ can often be obtained in the same manner, we shall treat in the following the two types of characteristic solutions (classical and mild) together.

Thanks again to Hypothesis 2, one can define the density and flux:

$$\{n, j\}(t,x) := \int_{\mathbb{R}} \{1, \widehat{p}\}\, f(t,x,p)\, \mathrm{d}p. \tag{2.13}$$

Indeed, we have the following more general lemma for moments of characteristic solutions of (1.14). Let us denote by $m_k(t,x)$ the moment of order $k \in \mathbb{N}$ of a solution $f(t,x,p)$ with a given force field F.

Lemma 2.5. *Let f be a characteristic solution of the Vlasov equation (1.14) with force field $F \in \mathcal{C}^0\left(0,T; W^{1,\infty}(\mathbb{R})\right)$. Assume that the function $g(p)$ from Hypothesis 2 has the moment of order k bounded, i.e.,*

$$M_k := \int_{\mathbb{R}} |p|^k\, g(p)\, \mathrm{d}p < \infty,$$

then $m_k(t,x)$ is well-defined for all (t,x) and

$$|m_k(t,x)| \leq M_k + R_k\left(M_0 + M_k, t\, \|F\|_t\right) \tag{2.14}$$

where $R_k(a,b)$ will be defined below. In particular, for $k=0$ we obtain

$$\|n\|_t \leq M_0 + 2\, g(0)\, t\, \|F\|_t. \tag{2.15}$$

Proof. From (2.10) and Hypothesis 2, we deduce:

$$|p|^k f(t,x,p) \leq |p|^k g\left(|P(0;t,x,p)|\right).$$

Now, Eq. (2.7) with one of the force fields replaced by 0 yields $|P(0;t,x,p) - p| \leq t\, \|F\|_t$; as g is decreasing, this gives: $g\left(|P(0;t,x,p)|\right) \leq g_{t\,\|F\|_t}(p)$, and thus

$$|p|^k f(t,x,p) \leq |p|^k g_{t\,\|F\|_t}(p)$$

which is clearly integrable in p. This proves that $m_k(t,x)$ is well-defined and

$$|m_k(t,x)| \leq \int |p|^k g_{t\,\|F\|_t}(p)\, \mathrm{d}p.$$

Moreover, we notice that for $k \geq 1$ we have

$$\begin{aligned}
\int_{\mathbb{R}} |p|^k\, g_r(p)\, \mathrm{d}p &= \frac{2\, g(0)}{k+1}\, r^{k+1} + \sum_{i=1}^{k} C_k^i\, r^{k-i} \int_{\mathbb{R}} |p|^i\, g(p)\, \mathrm{d}p \\
&\leq \frac{2\, g(0)}{k+1}\, r^{k+1} + \left(\sum_{i=1}^{k} C_k^i\, r^{k-i}\right) \int_{\mathbb{R}} \left(1 + |p|^k\right) g(p)\, \mathrm{d}p \\
&\leq \frac{2\, g(0)}{k+1}\, r^{k+1} + \left(\sum_{i=1}^{k} C_k^i\, r^{k-i}\right) \left(M_0 + M_k\right) \\
&:= R_k(M_0 + M_k, r).
\end{aligned} \tag{2.16}$$

Finally, combining previous inequalities, we obtain (2.14). The estimate on the density (2.15) follows directly from previous arguments and (2.9) which defines the function R_0. □

We can also estimate the divergence of moments corresponding to two different solutions of the Vlasov equation (1.14).

Lemma 2.6. *Let $f_1,\ f_2$ be characteristic solutions of the Vlasov equation (1.14) with forces $F_1,\ F_2 \in \mathcal{C}^0\left(0,T;W^{1,\infty}(\mathbb{R})\right)$ respectively. Assume that the function $g(p)$ from Hypothesis 2 has the moment of order k bounded, then*

$$|m_{1,k}-m_{2,k}|\,(t,x) \le R_k\left(M_0+M_k,\max_i \|F_i\|_t\right)\int_0^t \|F_1-F_2\|_s\,\mathrm{d}s. \tag{2.17}$$

Proof. To estimate $m_{1,k}-m_{2,k}$, we consider the characteristics (X^1,P^1) and (X^2,P^2) associated to F_1 and F_2 respectively. Using the shorthand notation for the characteristics $\left(X_0^i,P_0^i\right)=\left(X^i(0;t,x,p),P^i(0;t,x,p)\right)$, we write:

$$(m_{1,k}-m_{2,k})\,(t,x)=\int_{\mathbb{R}}|p|^k\left[f_0\left(X_0^1,P_0^1\right)-f_0\left(X_0^2,P_0^2\right)\right]\,\mathrm{d}p$$

and thus,

$$\begin{aligned}|m_{1,k}-m_{2,k}|\,(t,x)\le\int_{\mathbb{R}}|p|^k\ &\left\{\left|\frac{\partial f_0}{\partial x_0}\left(\tilde X,P_0^1\right)\right|\,\left|X_0^1-X_0^2\right|\right.\\ &\left.+\left|\frac{\partial f_0}{\partial p_0}\left(X_0^2,\tilde P\right)\right|\,\left|P_0^1-P_0^2\right|\right\}\,\mathrm{d}p,\end{aligned}$$

where we have made use twice of the one-dimensional Taylor–Lagrange formula; $\tilde X$, respectively $\tilde P$, lie between X_0^1 and X_0^2, resp. P_0^1 and P_0^2. Then we invoke (2.6–2.7) to bound

$$|m_{1,k}-m_{2,k}|\,(t,x)\le\int_{\mathbb{R}}|p|^k\left\{g\left(\left|P_0^1\right|\right)+t\,g\left(\left|\tilde P\right|\right)\right\}\,\mathrm{d}p\times\int_0^t\|F_1-F_2\|_s\,\mathrm{d}s. \tag{2.18}$$

Applying again Eq. (2.7) with one of the force fields replaced by 0 yields $\left|P_0^i-p\right|\le t\,\|F_i\|_t$; as g is decreasing, this gives: $g\left(\left|P_0^i\right|\right)\le g_{t\,\|F_i\|_t}(p)$, and then:

$$g\left(\left|\tilde P\right|\right)\le\max\left\{g\left(\left|P_0^1\right|\right),g\left(\left|P_0^2\right|\right)\right\}\le\max_i g_{t\,\|F_i\|_t}(p).$$

Moreover, $r\mapsto g_r(p)$ is an increasing function of r, for all p; this implies:

$$\begin{aligned}\int_{\mathbb{R}}|p|^k\left\{g\left(\left|P_0^1\right|\right)+t\,g\left(\left|\tilde P\right|\right)\right\}\,\mathrm{d}p&\le(1+t)\int_{\mathbb{R}}|p|^k\max_i g_{t\,\|F_i\|_t}(p)\,\mathrm{d}p\\&\le(1+t)\max_i\int_{\mathbb{R}}|p|^k g_{t\,\|F_i\|_t}(p)\,\mathrm{d}p.\end{aligned}$$

Inequality (2.17) is obtained by using the last lines in Lemma 2.5. □

A small variation on the above arguments allows us to prove that in fact, the density and flux are regular enough to satisfy the continuity equation. This is important in order to be able to say that the description via the Poisson equation (1.11) is equivalent to the Ampère equation (1.10).

Corollary 2.1. *Under Hypotheses 1–2 and the assumption (2.3), resp. (2.4), n and j are Lipschitz, resp. continuously differentiable in space and time and they satisfy*

$$\frac{\partial n}{\partial t} + \frac{\partial j}{\partial x} = 0. \tag{2.19}$$

Moreover, there is a constant C, depending only on the majorising function $g(p)$, such that:

$$\left\| \frac{\partial n}{\partial x} \right\|_t \leq C\, \mathrm{e}^{t\left(1+\|\partial_x F\|_t\right)}. \tag{2.20}$$

Proof. When the solution is classical, we can apply the chain rule and estimate directly the derivative in x of the solution $f(t,x,p)$ to get

$$\left|\frac{\partial f}{\partial x}\right| \leq \left(\sup_x \left|\frac{\partial X}{\partial x}(0)\right|\right) \left|\frac{\partial f_0}{\partial x_0}\right| + \left(\sup_x \left|\frac{\partial P}{\partial x}(0)\right|\right) \left|\frac{\partial f_0}{\partial p_0}\right|.$$

Lemma 2.3 and Hypothesis 2 imply that (see previous Lemma):

$$\left|\frac{\partial f}{\partial x}\right| \leq \mathrm{e}^{t\left(1+\|\partial_x F\|_t\right)}\, g_{t\,\|F\|_t}(p),$$

and thus $\partial_x f$ is integrable. Moreover, Lebesgue's dominated convergence theorem implies that n is Lipschitz or differentiable with respect to x and hence (2.20). Similar arguments apply to $\partial_t n$ and $\partial_x j$ and the continuity equation (2.19) becomes an easy consequence of Eq. (1.14) upon integration on p and using Hypotheses 1–2 on the initial data. In case the solution is only mild under assumption (2.3), one can reproduce the Lipschitz bounds by estimating the difference $n(t,x_1)-n(t,x_2)$ using analogous arguments to previous Lemma 2.6, we leave the details to the reader. □

And it stems from Lemma 2.1 that:

Lemma 2.7. *Under the hypotheses:*

$$\forall (x,p),\ f_0(x+L,p) = f_0(x,p)\,; \quad \forall (t,x),\ \{E,A\}(t,x+L) = \{E,A\}(t,x)\,;$$

there holds:

$$\forall (t,x,p),\ f(t,x+L,p) = f(t,x,p),\ \{n,j\}(t,x+L) = \{n,j\}(t,x).$$

Let us finally remark that thanks to the mass conservation property of the Vlasov equation, we estimate the integrals of n. In the open-space case, we have

$$\int_a^b n(x)\,\mathrm{d}x \leq M, \tag{2.21}$$

where M is the total mass of f_0. In the periodic case,

$$\int_a^b n(x)\,\mathrm{d}x \leq M \left\lceil \frac{b-a}{L} \right\rceil, \tag{2.22}$$

where M is now the mass of f_0 over one period, and $\lceil r \rceil$ is the smallest integer larger or equal to r.

Moreover, in the open-space case we can estimate moments in x under suitable additional assumptions on the initial data. In fact, let us consider the following

Hypothesis 3. *$f_0(x,p) \le g(x)\, g(p)$ in $\mathbb{R}^2$.*

By following the same lines of argument as in Lemma 2.5, we can prove:

Lemma 2.8. *Let f be a characteristic solution of the Vlasov equation (1.14) in the open-space case with force field $F \in \mathcal{C}^0\left(0,T;W^{1,\infty}(\mathbb{R})\right)$ and assume that Hypothesis 3 is satisfied. Then, there exists a constant Z, depending polynomially on the first two moments of $g(p)$, t and $\|F\|_t$, such that*

$$\int_{\mathbb{R}^2} (|x|+|p|)\, f(t,x,p)\,\mathrm{d}x\,\mathrm{d}p \le Z\left(M_0, M_1, t, \|F\|_t\right). \tag{2.23}$$

This can be generalised to all moments of f and its derivatives, under

Hypothesis 4. *$|\partial_x f_0(x,p)| \le g(x)\, g(p)$ and $|\partial_p f_0(x,p)| \le g(x)\, g(p)$ in $\mathbb{R}^2$.*

Lemma 2.9. *Assume that Hypotheses 3 and 4 hold, and that g has its moment of order m bounded. Then, for all k, $\ell \le m$, there exists a constant $Z_{k,\ell}$ such that, for all $t \in [0,T]$:*

$$\iint \left(|x|^k + |p|^\ell\right)\, f(t,x,p)\,\mathrm{d}x\,\mathrm{d}p \le Z_{k,\ell},$$
$$\iint \left(|x|^k + |p|^\ell\right)\, |\partial_x f(t,x,p)|\,\mathrm{d}x\,\mathrm{d}p \le Z_{k,\ell}\, \mathrm{e}^{t\,(1+\|\partial_x F\|_t)},$$

and a similar bound holds for $\partial_p f$.

3. Iterative procedure and global weak solutions

In this section, we present an iterative procedure to solve the 1D Vlasov–Maxwell system (1.14–1.16) for the NR and QR cases.

First, we define the iterative procedure based on the Cooper–Klimas[9] approach. Then, we will derive estimates on the fields that allow us to obtain a limit by telescopic series. However, these estimates will not allow us to get a global characteristic solution, due to the lack of a global-in-time estimate on the space derivative of the density, which is needed to control that of the right-hand side in (1.16), and thus the second space derivative of the vector potential. Therefore, at this level of generality we are only able to obtain global weak solutions. Improvements of this basic result, namely local (in the NR case) and global (in the QR case) existence of classical and mild solutions will be postponed to the next two sections.

We now fix an initial condition f_0 for the distribution function, as well as initial data $(A_0, \dot{A}_0)$ for the vector potential. n_0 and E_0 are the density and electrostatic field given by f_0, as defined in (1.18). Let us remind that we always assume that the external density verifies $n_{\text{ext}} \in L^\infty(\mathbb{R})$. We also fix a target time T, and, if we are interested in periodic solutions, a space period L.

3.1. *Definition of the recurrence operator*

Given the field pair $(E, A) \in \mathcal{C}^0\left(0, T; W^{1,\infty}(\mathbb{R}) \times W^{2,\infty}(\mathbb{R})\right)$, one constructs $(E', A') = \mathcal{L}(E, A)$ as follows.

(1) The characteristic system (2.2) is solved, with the force F given by (2.1).
(2) One computes the characteristic solution to Vlasov's equation by (2.10) and its density and flux (n, j) by (2.13).
(3) Finally, E' and A' are computed as:

$$E'(t,x) := E_0(x) + \int_0^t j(s,x)\,\mathrm{d}s, \tag{3.1}$$

$$A'(t,x) := \frac{1}{2}\left\{ A_0(x+t) + A_0(x-t) + \int_{x-t}^{x+t} \dot{A}_0(y)\,\mathrm{d}y - \int_0^t \int_{x+s-t}^{x+t-s} (n\,A)(s,y)\,\mathrm{d}y\,\mathrm{d}s \right\}. \tag{3.2}$$

From Corollary 2.1 and the Duhamel formulae (A.2–A.4), we immediately deduce:

Theorem 3.1. *If $(A_0, \dot{A}_0) \in W^{2,\infty}(\mathbb{R}) \times W^{1,\infty}(\mathbb{R})$, the operator $\mathcal{L}$ maps $\mathcal{C}^0\left(0, T; W^{1,\infty}(\mathbb{R}) \times W^{2,\infty}(\mathbb{R})\right)$ to itself. If $(A_0, \dot{A}_0) \in \mathcal{C}^2_b(\mathbb{R}) \times \mathcal{C}^1_b(\mathbb{R})$, $f_0 \in \mathcal{C}^1(\mathbb{R}^2)$, and $n_{\text{ext}} \in \mathcal{C}^0_b(\mathbb{R})$, then $\mathcal{L}$ maps $\mathcal{C}^0\left(0, T; \mathcal{C}^1_b(\mathbb{R}) \times \mathcal{C}^2_b(\mathbb{R})\right)$ to itself.*

Moreover, the Poisson equation (1.19) is satisfied for the pair (E', n); and the time derivative of A' is bounded in x, i.e. $A' \in \mathcal{C}^1\left(0, T; L^\infty(\mathbb{R})\right)$ or $\mathcal{C}^1\left(0, T; \mathcal{C}^0_b(\mathbb{R})\right)$, even if A does not belong a priori *to such a space.*

This ensures that the operator $\mathcal{L}$ can be iterated. Of course, we shall need some quantitative estimates. To establish them is the goal of the next Subsection.

Moreover, we deduce from Lemma 2.7 the following

Corollary 3.1. *If $(A_0, \dot{A}_0) \in W^{2,\infty}_L(\mathbb{R}) \times W^{1,\infty}_L(\mathbb{R})$, the operator $\mathcal{L}$ maps $\mathcal{C}^0\left(0, T; W^{1,\infty}_L(\mathbb{R}) \times W^{2,\infty}_L(\mathbb{R})\right)$ to itself. If $(A_0, \dot{A}_0) \in \mathcal{C}^2_L(\mathbb{R}) \times \mathcal{C}^1_L(\mathbb{R})$, $f_0 \in \mathcal{C}^1_L(\mathbb{R}^2)$, and $n_{\text{ext}} \in \mathcal{C}^0_L(\mathbb{R})$, then $\mathcal{L}$ maps $\mathcal{C}^0\left(0, T; \mathcal{C}^1_L(\mathbb{R}) \times \mathcal{C}^2_L(\mathbb{R})\right)$ to itself.*

3.2. A priori *estimates*

In the sequel, we shall always assume *at least* that $(A_0, \dot{A}_0) \in W^{2,\infty}(\mathbb{R}) \times W^{1,\infty}(\mathbb{R})$. The constants denoted C or C_i may vary from one line to the next, and depend on the initial conditions, T and L (but on nothing else).

First, we fix $(E,A) \in \mathcal{C}^0\left(0,T;W^{1,\infty}(\mathbb{R}) \times W^{2,\infty}(\mathbb{R})\right)$, and $(E',A') := \mathcal{L}(E,A)$. F is the force field corresponding to (E,A).

For the electrostatic field we have the following properties:

Lemma 3.1. *The following estimates hold:*

$$\|E'\|_t \leq C_0 + C_1 \int_0^t \|F\|_s \,\mathrm{d}s\,; \tag{3.3}$$

$$\left\| \frac{\partial E'}{\partial x} \right\|_t \leq C_0 + C_1 \|F\|_t. \tag{3.4}$$

Proof. To obtain the first estimate, we proceed as in [1] by duality. Given $\varphi \in L^1(\mathbb{R})$, we apply changes of variables to get

$$\begin{aligned}\int_{\mathbb{R}} \int_0^t j(s,x)\,\varphi(x)\,\mathrm{d}s\,\mathrm{d}x &= \int_{\mathbb{R}^2} f_0(x,p) \int_0^t P(s;0,x,p)\,\varphi(X(s;0,x,p))\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}p \\ &= \int_{\mathbb{R}^2} f_0(x,p) \int_x^{X(t;0,x,p)} \varphi(u)\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}p,\end{aligned}$$

and thus, using Hypothesis 2

$$\begin{aligned}\left| \int_{\mathbb{R}} \int_0^t j(s,x)\varphi(x)\,\mathrm{d}s\,\mathrm{d}x \right| &\leq \int_{\mathbb{R}^2} f_0(x,p) \left| \int_x^{X(t;0,x,p)} \varphi(u)\,\mathrm{d}u \right| \mathrm{d}x\,\mathrm{d}p \\ &\leq \int_{\mathbb{R}} g(p) \int_{\mathbb{R}} |\varphi(u)| \int_{\mathbb{R}} \chi(u;S(t,x,p))\,\mathrm{d}x\,\mathrm{d}u\,\mathrm{d}p\end{aligned}$$

where $\chi(u;S(t,x,p))$ is the characteristic function of the set $S(t,x,p) = \{|u-x| \leq |X(t;0,x,p)-x|\}$ as a function of u. Using the bound on the divergence of forward-in-time characteristics as in Lemma 2.6, we obtain

$$|X(t;0,x,p)-x| \leq t \int_0^t \|F\|_s \,\mathrm{d}s \quad \text{and} \quad \int_{\mathbb{R}} \chi(u;S(t,x,p))\,\mathrm{d}x \leq 2t \int_0^t \|F\|_s \,\mathrm{d}s\,;$$

and therefore, we deduce

$$\left\| \int_0^t j(s,x)\,\mathrm{d}s \right\|_t \leq 2t \int_{\mathbb{R}} g(p)\,\mathrm{d}p \int_0^t \|F\|_s \,\mathrm{d}s.$$

Finally, (3.3) is deduced directly from (3.1).

Now, Corollary 2.1 assures that the continuity equation (2.19) is satisfied. Taking into account that equation and the Ampère equation (3.1), then

$$\frac{\partial E'}{\partial x} = n_{\text{ext}} - n,$$

holds. Estimate (3.4) follows from (2.15) and $n_{\text{ext}} \in L^\infty(\mathbb{R})$. □

For the vector potential, we have a similar result.

Lemma 3.2. *The following estimates hold:*

$$\|A'\|_t \leq C_0 + C_1 \int_0^t \|A\|_s \,\mathrm{d}s, \tag{3.5}$$

$$\left\| \frac{\partial A'}{\partial x} \right\|_t \leq C_0 + \|A\|_t \left(C_1 + C_2 \int_0^t \|F\|_s \,\mathrm{d}s \right), \tag{3.6}$$

$$\left\| \frac{\partial A'}{\partial t} \right\|_t \leq C_0 + \|A\|_t \left(C_1 + C_2 \int_0^t \|F\|_s \,\mathrm{d}s \right). \tag{3.7}$$

Proof. From (3.2), we bound

$$|A'(t,x)| \leq C_0 + \int_0^t \|A\|_s \left(\int_{x+s-t}^{x+t-s} n(s,y) \,\mathrm{d}y \right) \mathrm{d}s.$$

Using (2.21) or (2.22), we then bound the integral of n by M or $M \lceil 2\,T/L \rceil$. Hence (3.5).

Let us estimate the derivatives of A. $\partial_x A'$ is given by the Duhamel formula (A.3), with f replaced by $-n\,A$. Hence the majoration:

$$2 \left| \frac{\partial A'}{\partial x}(t,x) \right| \leq C_0 + \|A\|_t \int_0^t (n(s, x+s-t) + n(s, x-s+t)) \,\mathrm{d}s.$$

Now, using the uniform bound on the density (2.15), we obtain

$$\int_0^t (n(s, x+s-t) + n(s, x-s+t)) \,\mathrm{d}s \leq C_1 + C_2 \int_0^t \|F\|_s \,\mathrm{d}s.$$

This gives (3.6). As the Duhamel formula (A.2) for the time derivative is very similar to (A.3), one can establish (3.7) by the same reasoning. □

Now, we consider two field pairs (E_1, A_1) and (E_2, A_2), with the corresponding forces F_1 and F_2, and we set $(E_1', A_1') := \mathcal{L}(E_1, A_1)$, $(E_2', A_2') := \mathcal{L}(E_2, A_2)$.

Lemma 3.3. *The following estimates hold for all field pairs (E_1, A_1) and (E_2, A_2):*

$$\|E_1' - E_2'\|_t \leq R_1 \left(M_0 + M_1, \max_i \|F_i\|_t \right) \int_0^t \|F_1 - F_2\|_s \,\mathrm{d}s, \tag{3.8}$$

$$\begin{aligned} \|A_1' - A_2'\|_t \leq \|A_2\|_t \left(C_0 + C_1 \max_i \|F_i\|_t \right) \int_0^t \|F_1 - F_2\|_s \,\mathrm{d}s \\ + C_2 \int_0^t \|A_2 - A_1\|_s \,\mathrm{d}s. \end{aligned} \tag{3.9}$$

Proof. From (3.1) we deduce

$$\|E_1' - E_2'\|_t \leq \int_0^t |j_1(s,x) - j_2(s,x)| \,\mathrm{d}s,$$

and thus (3.8) is a simple consequence of Lemma 2.6.

From (3.2), we have

$$2\,(A_1' - A_2')(t,x) = \int_0^t \int_{x+s-t}^{x+t-s} (n_2\,A_2 - n_1\,A_1)(s,y)\,\mathrm{d}y\,\mathrm{d}s.$$

Writing: $n_2\,A_2 - n_1\,A_1 = (n_2 - n_1)\,A_2 - (A_2 - A_1)\,n_1$, we bound:

$$2\,\|A_1' - A_2'\|_t \le \int_0^t \|A_2\|_s \int_{x+s-t}^{x+t-s} |n_2(s,y) - n_1(s,y)|\,\mathrm{d}y\,\mathrm{d}s + \int_0^t \|A_2 - A_1\|_s \int_{x+s-t}^{x+t-s} n_1(s,y)\,\mathrm{d}y\,\mathrm{d}s.$$

Using (2.21) or (2.22), the second line of this inequality is easily bounded as $\lambda\,M\int_0^t \|A_2 - A_1\|_s\,\mathrm{d}s$, with $\lambda = 1$ or $\lceil 2\,T/L\rceil$. Then, the first line is bounded thanks to Lemma 2.6:

$$|n_1 - n_2|\,(t,x) \le \Big(C_0 + C_1\,\max_i \|F_i\|_t\Big) \int_0^t \|F_1 - F_2\|_s\,\mathrm{d}s, \tag{3.10}$$

and finally:

$$\|A_1' - A_2'\|_t \le \|A_2\|_t\,\Big(C_0 + C_1\,\max_i \|F_i\|_t\Big) \int_0^t \|F_1 - F_2\|_s\,\mathrm{d}s + \frac{\lambda\,M}{2} \int_0^t \|A_2 - A_1\|_s\,\mathrm{d}s,$$

which is (3.9). □

Now, we estimate the difference of the derivatives.

Lemma 3.4. *There holds, for any (E_1, A_1) and (E_2, A_2):*

$$\left\|\frac{\partial E_1'}{\partial x} - \frac{\partial E_2'}{\partial x}\right\|_t \le \Big(C_0 + C_1\,\max_i \|F_i\|_t\Big) \int_0^t \|F_2 - F_1\|_s\,\mathrm{d}s, \tag{3.11}$$

$$\left\|\frac{\partial A_1'}{\partial x} - \frac{\partial A_2'}{\partial x}\right\|_t \le \Big(C_0 + C_1\,\max_i \|F_i\|_t\Big) \left\{\int_0^t \|A_2 - A_1\|_s\,\mathrm{d}s + \|A_2\|_t \int_0^t \|F_1 - F_2\|_s\,\mathrm{d}s\right\}, \tag{3.12}$$

$$\left\|\frac{\partial A_1'}{\partial t} - \frac{\partial A_2'}{\partial t}\right\|_t \le \Big(C_0 + C_1\,\max_i \|F_i\|_t\Big) \left\{\int_0^t \|A_2 - A_1\|_s\,\mathrm{d}s + \|A_2\|_t \int_0^t \|F_1 - F_2\|_s\,\mathrm{d}s\right\}. \tag{3.13}$$

Proof. The Poisson equation and (3.10) imply (3.11). From the Duhamel formula (A.3), with f replaced successively with $-n_1\,A_1$ and $-n_2\,A_2$, we derive:

$$2\left(\frac{\partial A_1'}{\partial x} - \frac{\partial A_2'}{\partial x}\right)(t,x) = \int_0^t (n_2\,A_2 - n_1\,A_1)(s, x+s-t)\,\mathrm{d}s - \int_0^t (n_2\,A_2 - n_1\,A_1)(s, x+t-s)\,\mathrm{d}s\,;$$

once more, we write $n_2\,A_2 - n_1\,A_1 = (n_2 - n_1)\,A_2 - (A_2 - A_1)\,n_1$, which gives

$$\left\| \frac{\partial A_1'}{\partial x} - \frac{\partial A_2'}{\partial x} \right\|_t \leq \|A_2\|_t \int_0^t \|n_1 - n_2\|_s \,\mathrm{d}s + \|n_1\|_t \int_0^t \|A_1 - A_2\|_s \,\mathrm{d}s.$$

Using the bounds (3.10) for the first term, and (2.15) for the second yields

$$\left\| \frac{\partial A_1'}{\partial x} - \frac{\partial A_2'}{\partial x} \right\|_t \leq \|A_2\|_t \left(C_0 + C_1 \max_i \|F_i\|_t \right) \int_0^t \|F_1 - F_2\|_s \,\mathrm{d}s + (C_2 \|F_1\|_t + C_3) \int_0^t \|A_1 - A_2\|_s \,\mathrm{d}s,$$

which implies (3.12). Once again, the similarity of the formulae (A.2) and (A.3) allows to deduce (3.13). □

3.3. *Convergence of successive approximations*

We start from the initial data $(E_0(x), A_0(x))$, by extending them to constant-in-time functions over $[0,T] \times \mathbb{R}$. Then, we construct a sequence $(E_k, A_k)_{k\in\mathbb{N}}$ by the recurrence formula:

$$(E_{k+1}, A_{k+1}) := \mathcal{L}(E_k, A_k), \quad \forall k \geq 0.$$

Of course, we set $F_k := E_k + A_k\,\partial_x A_k$.

To establish the convergence of $(E_k, A_k)_{k\in\mathbb{N}}$, the following result will be useful. It is easily proved by induction.

Lemma 3.5. *Let $(u_k)_{k\in\mathbb{N}}$ be a sequence of positive functions, $u_k : [0,T] \to \mathbb{R}^+$, satisfying*

$$\begin{aligned} &(i) \quad \forall t \in [0,T], \quad u_0(t) \leq c, \\ &(ii) \quad \forall k \in \mathbb{N},\ \forall t \in [0,T], \quad u_{k+1}(t) \leq a + b \int_0^t u_k(s)\,\mathrm{d}s, \end{aligned}$$

for some constants $a,\ b,\ c \in \mathbb{R}^+$. Then the following estimate holds:

$$\forall k \in \mathbb{N},\ \forall t \in [0,T], \quad u_k(t) \leq a \sum_{i=1}^{k-1} \frac{b^i\,t^i}{i!} + c\,\frac{b^k\,t^k}{k!}\ ;$$

hence the whole sequence is uniformly bounded by a constant u_* which depends on a, b, c and T.

Theorem 3.2. *The sequence $(E_k, A_k)_{k\in\mathbb{N}}$ converges uniformly in t and x towards a limit (E, A).*

(1) If $(A_0, \dot{A}_0) \in W^{2,\infty}(\mathbb{R}) \times W^{1,\infty}(\mathbb{R})$, the sequences (E_k) and (A_k) converge respectively within $\mathcal{C}^0\left(0,T; W^{1,\infty}(\mathbb{R})\right)$ and $\mathcal{C}^1\left(0,T; L^\infty(\mathbb{R})\right) \cap \mathcal{C}^0\left(0,T; W^{1,\infty}(\mathbb{R})\right)$.

(2) If $(A_0, \dot{A}_0) \in \mathcal{C}_b^2(\mathbb{R}) \times \mathcal{C}_b^1(\mathbb{R})$, $f_0 \in \mathcal{C}^1(\mathbb{R}^2)$, and $n_{\mathrm{ext}} \in \mathcal{C}_b^0(\mathbb{R})$, then the convergences take place within $\mathcal{C}^0\left(0,T; \mathcal{C}_b^1(\mathbb{R})\right)$ and $\mathcal{C}^1\left(0,T; \mathcal{C}_b^0(\mathbb{R})\right) \cap \mathcal{C}^0\left(0,T; \mathcal{C}_b^1(\mathbb{R})\right)$.

(3) The above conclusions are valid when all spaces are replaced by their periodic counterparts.

Proof. First, Eq. (3.5) gives:

$$\|A_{k+1}\|_t \le C_0 + C_1 \int_0^t \|A_k\|_s \, \mathrm{d}s$$

and thus, the previous lemma ensures that $\|A_k\|_t \le A_*$ uniformly in k and t. Then, Eqs. (3.3) and (3.6) imply:

$$\|F_{k+1}\|_t \le \|E_{k+1}\|_t + A_* \, \|\partial_x A_{k+1}\|_t \le C_2 + C_3 \int_0^t \|F_k\|_s \, \mathrm{d}s,$$

so, once more, $\|F_k\|_t \le F_*$. Using again (3.3) and (3.6) shows that the sequences $\|E_k\|_t$ and $\|\partial_x A_k\|_t$ are uniformly bounded in k and t.

All these bounds allow to rewrite (3.9) and (3.12) as

$$\|A_{k+1} - A_k\|_t \le C_1 \int_0^t \|F_k - F_{k-1}\|_s \, \mathrm{d}s + C_2 \int_0^t \|A_k - A_{k-1}\|_s \, \mathrm{d}s \, ; \quad (3.14)$$

$$\left\| \frac{\partial A_{k+1}}{\partial x} - \frac{\partial A_k}{\partial x} \right\|_t \le C_1 \int_0^t \|F_k - F_{k-1}\|_s \, \mathrm{d}s + C_2 \int_0^t \|A_k - A_{k-1}\|_s \, \mathrm{d}s. \quad (3.15)$$

Then, writing

$$F_{k+1} - F_k = E_{k+1} - E_k + A_{k+1} \, (\partial_x A_{k+1} - \partial_x A_k) + \partial_x A_k \, (A_{k+1} - A_k) \, ,$$

one deduces from (3.8), (3.14) and (3.15)

$$\|F_{k+1} - F_k\|_t \le C_1 \int_0^t \|F_k - F_{k-1}\|_s \, \mathrm{d}s + C_2 \int_0^t \|A_k - A_{k-1}\|_s \, \mathrm{d}s. \quad (3.16)$$

Hence, the sequence $u_k(t) := \|A_k - A_{k-1}\|_t + \|F_k - F_{k-1}\|_t$ satisfies: $u_{k+1}(t) \le b \int_0^t u_k(s) \, \mathrm{d}s$, for some constant b. Lemma 3.5 with $a = 0$ then implies $u_k(t) \le b^k \, T^k / k!$, i.e.:

$$\|A_{k+1} - A_k\|_t \le \frac{b^k \, T^k}{k!}, \quad \|F_{k+1} - F_k\|_t \le \frac{b^k \, T^k}{k!},$$

so by (3.8) and (3.15)

$$\|E_{k+1} - E_k\|_t \le C \, \frac{b^k \, T^k}{k!}, \quad \left\| \frac{\partial A_{k+1}}{\partial x} - \frac{\partial A_k}{\partial x} \right\|_t \le C \, \frac{b^k \, T^k}{k!}.$$

As a consequence, the four sequences $(E_k)_{k\in\mathbb{N}}$, $(A_k)_{k\in\mathbb{N}}$, $(\partial_x A_k)_{k\in\mathbb{N}}$, $(F_k)_{k\in\mathbb{N}}$, all converge uniformly on $[0,T] \times \mathbb{R}$. Let E, A, B, F, be the limits; clearly $B = \partial_x A$ in the sense of distributions, and $F = E + A \, \partial_x A$. Under the hypothesis $(A_0, \dot{A}_0) \in W^{2,\infty}(\mathbb{R}) \times W^{1,\infty}(\mathbb{R})$, each term of the four sequences is in $\mathcal{C}^0 \left(0, T; L^\infty(\mathbb{R})\right)$, and so are the limits. So: $E \in \mathcal{C}^0 \left(0, T; L^\infty(\mathbb{R})\right)$ and $A \in \mathcal{C}^0 \left(0, T; W^{1,\infty}(\mathbb{R})\right)$. If $(A_0, \dot{A}_0) \in \mathcal{C}_b^2(\mathbb{R}) \times \mathcal{C}_b^1(\mathbb{R})$, $f_0 \in \mathcal{C}^1(\mathbb{R}^2)$,

and $n_{\text{ext}} \in \mathcal{C}_b^0(\mathbb{R})$, the terms and the limits are in $\mathcal{C}^0\left(0,T;\mathcal{C}_b^0(\mathbb{R})\right)$, i.e. $E \in \mathcal{C}^0\left(0,T;\mathcal{C}_b^0(\mathbb{R})\right)$ and $A \in \mathcal{C}^0\left(0,T;\mathcal{C}_b^1(\mathbb{R})\right)$.

With all these results, Eq. (3.13) shows that: $\|\partial_t A_{k+1} - \partial_t A_k\| \leq C\, b^k\, T^k/k!$; since A_0 is independent of time, the sequence $(\partial_t A_k)_{k\in\mathbb{N}}$ also converges uniformly in $\mathcal{C}^0\left(0,T;L^\infty(\mathbb{R})\right)$, resp. $\mathcal{C}^0\left(0,T;\mathcal{C}_b^0(\mathbb{R})\right)$, towards a limit which is necessarily equal to $\partial_t A$. Thus, $A \in \mathcal{C}^1\left(0,T;L^\infty(\mathbb{R})\right)$, resp. $\mathcal{C}^1\left(0,T;\mathcal{C}_b^0(\mathbb{R})\right)$.

Similarly, Eq. (3.11) implies that $(\partial_x E_k)_{k\in\mathbb{N}}$ is a Cauchy sequence and thus converges uniformly toward a limit which is necessarily equal to $\partial_x E$. Hence, $E \in \mathcal{C}^0\left(0,T;W^{1,\infty}(\mathbb{R})\right)$ or $\mathcal{C}^0\left(0,T;\mathcal{C}_b^1(\mathbb{R})\right)$.

The last point easily follows from Corollary 3.1. □

3.4. *Global existence and properties of weak solutions*

If $(E,A) \in \mathcal{C}^0\left(0,T;W^{1,\infty}(\mathbb{R})\right) \times \mathcal{C}^0\left(0,T;W^{2,\infty}(\mathbb{R})\right)$, then it is a fixed point of $\mathcal{L}$ within this space. Thus, the triple (f,E,A), where the function f is defined by (2.10), is a characteristic solution to (1.14–1.19).

This fixed point and its associated characteristic solution, if they exist, are unique. Indeed, let (f_1,E_1,A_1) and (f_2,E_2,A_2) be two such solutions, with respective force fields F_1 and F_2. Reasoning as in the proof of the above theorem shows:

$$\|A_1 - A_2\|_t + \|F_1 - F_2\|_t \leq C \int_0^t \{\|A_1 - A_2\|_s + \|F_1 - F_2\|_s\}\, \mathrm{d}s\,;$$

hence $A_1 = A_2$ and $F_1 = F_2$ by Gronwall's lemma, then $E_1 = E_2$ by (3.8), and finally $f_1 = f_2$ by the Cauchy–Lipschitz theorem.

Without further *a priori* estimates on the second derivatives of the vector potential A, we will not be able to obtain global existence of characteristic solutions. However, the bounds in Subsection 3.2 show that the norm of $\mathcal{L}(E,A)$ in $\mathcal{C}^0\left(0,T;L^\infty(\mathbb{R}) \times W^{1,\infty}(\mathbb{R})\right)$, resp. $\mathcal{C}^0\left(0,T;\mathcal{C}_b^0(\mathbb{R}) \times \mathcal{C}_b^1(\mathbb{R})\right)$, is controlled by the norm of (E,A) in the same space. These considerations suggest to extend the operator $\mathcal{L}$ to a bigger space in order to obtain weak solutions, which would also enjoy a *uniqueness property*. This programme cannot be achieved within the framework of $\mathcal{C}^0\left(0,T;L^\infty(\mathbb{R}) \times W^{1,\infty}(\mathbb{R})\right)$. The reason is the well-known lack of density of smooth functions within $L^\infty(\mathbb{R})$ or $W^{1,\infty}(\mathbb{R})$. But this obstruction is removed if one works within $\mathcal{C}^0\left(0,T;\mathcal{C}_b^0(\mathbb{R}) \times \mathcal{C}_b^1(\mathbb{R})\right)$.

Lemma 3.6. *Assume* $(A_0,\dot{A}_0) \in \mathcal{C}_b^2(\mathbb{R}) \times \mathcal{C}_b^1(\mathbb{R})$, $f_0 \in \mathcal{C}^1(\mathbb{R}^2)$, *and* $n_{\text{ext}} \in \mathcal{C}_b^0(\mathbb{R})$. *The operator* $\mathcal{L}$ *can be extended to a continous operator from* $\mathcal{C}^0\left(0,T;\mathcal{C}_b^0(\mathbb{R}) \times \mathcal{C}_b^1(\mathbb{R})\right)$ *to itself, which satisfies the estimates of Subsection 3.2. If the initial conditions are periodic, then* $\mathcal{L}$ *maps* $\mathcal{C}^0\left(0,T;\mathcal{C}_L^0(\mathbb{R}) \times \mathcal{C}_L^1(\mathbb{R})\right)$ *to itself.*

Proof. $\mathcal{L}$ is defined on the dense subspace $Y := \mathcal{C}^0\left(0,T;\mathcal{C}_b^1(\mathbb{R}) \times \mathcal{C}_b^2(\mathbb{R})\right)$ of $X := \mathcal{C}^0\left(0,T;\mathcal{C}_b^0(\mathbb{R}) \times \mathcal{C}_b^1(\mathbb{R})\right)$, with values in X. From (3.8, 3.9, 3.12), we see that it is uniformly continuous, in the norm of X, on any set $K \cap Y$, where K is a bounded set of X. Hence, it admits a unique continuous extension from X to itself.

The extension procedure preserves the estimates of Subsection 3.2, as their r.h.s. are uniformly continuous in the norm of X on any $K \cap Y$. Finally, the closedness of $\mathcal{C}^0\left(0,T;\mathcal{C}^0_L(\mathbb{R})\times\mathcal{C}^1_L(\mathbb{R})\right)$ within X guarantees the invariance of this space by the extended operator. □

Hence, the following theorem, whose proof rephrases that of Theorem 3.2.

Theorem 3.3. *Assume $(A_0,\dot{A}_0)\in\mathcal{C}^2_b(\mathbb{R})\times\mathcal{C}^1_b(\mathbb{R})$, $f_0\in\mathcal{C}^1(\mathbb{R}^2)$, and $n_{\rm ext}\in\mathcal{C}^0_b(\mathbb{R})$. The operator $\mathcal{L}$ admits a unique fixed point $(E,A)\in\mathcal{C}^0\left(0,T;\mathcal{C}^0_b(\mathbb{R})\times\mathcal{C}^1_b(\mathbb{R})\right)$, which moreover belongs to $\mathcal{C}^0\left(0,T;\mathcal{C}^1_b(\mathbb{R})\right)\times\mathcal{C}^1\left(0,T;\mathcal{C}^0_b(\mathbb{R})\right)$. If the initial conditions are periodic, so is the fixed point.*

Let us now check that this unique fixed point defines a distribution function f in such a way that the triple (f,E,A) is a global weak solution of the Vlasov–Maxwell system (1.14–1.19).

Theorem 3.4. *Under the hypotheses of Theorem 3.3, let $(E_k,A_k)_k$ a sequence converging to the fixed point (E,A); without loss of generality, we can assume that its terms belong to $\mathcal{C}^0\left(0,T;\mathcal{C}^1_b(\mathbb{R})\times\mathcal{C}^2_b(\mathbb{R})\right)$. Let $(f_k)_k$ be the associated sequence of distribution functions obtained by the method of characteristics. Then, $(f_k)_k$ converges uniformly in all its variables toward a function $f\in\mathcal{C}^0\left(0,T;\mathcal{C}^0_b(\mathbb{R}^2)\right)$, which does not depend on the sequence $(E_k,A_k)_k$, and the triple (f,E,A) satisfies (1.14)–(1.19) in the sense of distributions.*

Proof. Repeating the argument of Lemma 2.6, we easily obtain the estimate

$$|f_k(t,x,p)-f_\ell(t,x,p)|\le(1+t)\max_{i=k,\ell}g_{t\,\|F_i\|_t}(p)\int_0^t\|F_k-F_\ell\|_s\,\mathrm{d}s, \tag{3.17}$$

from which follows that $(f_k(t))_k$ is a Cauchy sequence in L^∞ norm, uniformly in t. A similar argument shows that, if we consider another approximating sequence denoted by tildes, $f_k(t)-\tilde{f}_k(t)$ will converge to zero, uniformly in x, p and t. Let us then call f the common limit of all the sequences $(f_k)_k$; its continuity in all variables follows from that of the f_k.

As f_k is a classical solution to the Vlasov equation, it is also a distributional solution, i.e. an equation similar to (2.12) holds, with f and F replaced resp. with f_k and F_k. For any test function ψ, the integrals in this formula are taken over a compact subset of $[0,T)\times\mathbb{R}^2$. Thus, the uniform convergence in x, p and t of the sequences $(f_k)_k$ and $(F_k)_k$ ensures that (2.12) will also hold at the limit: the limiting triple (f,E,A) satisfies (1.14–1.19) in the sense of distributions. □

The above result can be improved in the following two ways. First, we show that $f(t)$ will retain the mass of the initial distribution f_0.

Proposition 3.1. *In the periodic case, $f_k(t)$ converges toward $f(t)$ in $L^1_L(\mathbb{R}^2)$, uniformly in t, and the mass of $f(t)$ over one space period is equal to that of f_0.*

In the open-space case, under Hypothesis 3, $f_k(t)$ converges weakly toward $f(t)$ in $L^1(\mathbb{R}^2)$, and the mass of $f(t)$ is equal to that of f_0.

Proof. Inequality (3.17) shows that $f_k(t,x,\cdot)$ converges to $f(t,x,\cdot)$ in $L^1(\mathbb{R})$, uniformly in x and t. Hence the convergence of $f_k(t)$ toward $f(t)$ in $L^1((a,b)\times\mathbb{R})$ for any $a<b$.

Yet, the f_k are *classical* solutions to the Vlasov equation. In the periodic case, they conserve the mass of f_0 over one space period. So, $f(t)$ retains this mass.

In the open-space case, taking into account the L^1 convergence in bounded intervals in space, it suffices to show that mass does not escape at infinity, uniformly in t. Using Lemma 2.8, we obtain:

$$\int_{|x|+|p|>R} |f_k(t,x,p)|\,\mathrm{d}x\,\mathrm{d}p \le \frac{1}{R}\int_{|x|+|p|>R} (|x|+|p|)\,f_k(t,x,p)\,\mathrm{d}x\,\mathrm{d}p \le \frac{Z}{R}\to 0,$$

uniformly in k and t when $R\to\infty$. Thus, it is a standard argument to check that $\int_{\mathbb{R}^2} f(t,x,p)\,\mathrm{d}x\,\mathrm{d}p = \lim_k \int_{\mathbb{R}^2} f_k(t,x,p)\,\mathrm{d}x\,\mathrm{d}p = M$. □

The second useful precision is that $f(t)$ is "almost a mild solution" to (1.14).

Theorem 3.5. *The characteristics given by the F_k converge uniformly in all their variables toward a limit (X,P), and Eq. (2.11) holds for this (X,P) and $f(t)$ given by Theorem 3.4.*

Proof. Let (X^k,P^k) be the characteristics associated to F_k. Their uniform convergence follows from (2.6–2.7). As the f_k are classical solutions to the Vlasov equation, they are also mild solutions, i.e. they satisfy

$$\int_{\mathbb{R}^2} \psi(x,p)\,f_k(t,x,p)\,\mathrm{d}x\,\mathrm{d}p = \int_{\mathbb{R}^2} \psi(X^k(t;0,x,p),P^k(t;0,x,p))\,f_0(x,p)\,\mathrm{d}x\,\mathrm{d}p \quad (3.18)$$

for all test functions $\psi\in\mathcal{C}_c(\mathbb{R}^2)$ and all $t>0$.

The sequence $\psi(X^k(t;0,x,p),P^k(t;0,x,p))\,f_0(x,p)$, resp. $\psi(x,p)\,f_k(t,x,p)$, converge uniformly on the support of ψ towards $\psi(X(t;0,x,p),P(t;0,x,p))\,f_0(x,p)$, resp. $\psi(x,p)\,f(t,x,p)$. Hence, the integrals on both sides of (3.18) converge toward the two sides of (2.11), and the latter equality is obtained at the limit. □

4. Local-in-time existence of characteristic solutions

In order to have stronger solutions, i.e. such that f satisfies (1.14) in the characteristic sense, we need estimates on the Lipschitz constant of the force, which in turn amounts to bounding the second derivative of the vector potential. In this Section we show that, in both the NR and QR cases, the limiting vector potential given by Theorem 3.2 does satisfy such a bound, at least for a short time.

In the sequel, the operator $\mathcal{L}$ will be that of Subsection 3.1. We no longer consider the extended version, so that we can obtain estimates based on characteristics.

Lemma 4.1. *Let* $(E,A) \in \mathcal{C}^0\left(0,T;W^{1,\infty}(\mathbb{R})\times W^{2,\infty}(\mathbb{R})\right)$, *and* $(E',A') = \mathcal{L}(E,A)$. *Then, the second derivative of* A' *is bounded as:*

$$\left\|\frac{\partial^2 A'}{\partial x^2}\right\|_t \leq C_0 + t\left\{\left\|\frac{\partial A}{\partial x}\right\|_t (C_1 + C_2\|F\|_t) + C_3\|A\|_t \,\mathrm{e}^{t\left(1+\|\partial_x F\|_t\right)}\right\}. \tag{4.1}$$

Proof. We use the formula (A.4), with f replaced by $-n\,A$, and the bounds (2.15) and (2.20) for n:

$$\begin{aligned} 2\left|\frac{\partial^2 A'}{\partial x^2}(t,x)\right| &\leq C_0 + \int_0^t \left(n\left|\frac{\partial A}{\partial x}\right| + |A|\left|\frac{\partial n}{\partial x}\right|\right)(s,x+s-t)\,\mathrm{d}s \\ &\quad + \int_0^t \left(n\left|\frac{\partial A}{\partial x}\right| + |A|\left|\frac{\partial n}{\partial x}\right|\right)(s,x-s+t)\,\mathrm{d}s \\ &\leq C_0 + t\left\|\frac{\partial A}{\partial x}\right\|_t (C_1 + C_2\|F\|_t) + t\|A\|_t C_3\,\mathrm{e}^{t\left(1+\|\partial_x F\|_t\right)}. \end{aligned}$$

□

Theorem 4.1. *Let* $(E_k,A_k)_{k\in\mathbb{N}}$ *and* (E,A) *be as in Theorem 3.2. There exists* $0<T^*\leq T$ *such that, for* $0\leq t<T^*$, *the sequence* $\left(\partial_x^2 A_k\right)_{k\in\mathbb{N}}$ *is uniformly bounded in* x *and* t. *As a consequence, the couple* (E,A) *allows to define a characteristic solution* f *to (1.14) on the interval* $[0,T^*)$.

Proof. We have:

$$\partial_x F_k = \partial_x E_k + (\partial_x A_k)^2 + A_k\,\partial_x^2 A_k.$$

Thus, the boundedness results of Theorem 3.2 and (4.1) yield:

$$\|\partial_x F_{k+1}\|_t \leq \alpha + \beta\, t\,\mathrm{e}^{t\left(1+\|\partial_x F_k\|_t\right)},$$

for some positive constants α, β depending only on the initial conditions, T and (possibly) L. Hence, $\|\partial_x F_k\|_t \leq v_k(t)$, where $v_k(t)$ is defined by the recurrence formula:

$$v_0(t) = \|\partial_x F_0\|_t, \quad v_{k+1}(t) = \alpha + \beta\, t\,\mathrm{e}^{t\left(1+v_k(t)\right)} := \varphi_t(v_k(t)). \tag{4.2}$$

In Appendix B we show that, for $t<T^*$, the sequence $(v_k(t))_{k\in\mathbb{N}}$ is convergent and hence bounded. This gives the uniform boundedness of $(\partial_x F_k(t))_{k\in\mathbb{N}}$ and, by (4.1), that of $(\partial_x^2 A_k(t))_{k\in\mathbb{N}}$, on $[0,T^*)\times\mathbb{R}$. Hence, the latter sequence admits a subsequence which converges weakly-$*$ in $L^\infty([0,T^*]\times\mathbb{R})$ towards a limit which is necessarily equal to $\partial_x^2 A$. This gives the last part of the conclusion. □

5. Global characteristic solutions in the quasi-relativistic case

In this section, *we only consider the QR case*. We show that, in this framework, characteristic solutions exist for any time. This result rests on a subtler treatment of the formula (A.4). On the other hand, we need to introduce another assumption:

Hypothesis 5. *The majorising function g of Hypothesis 2 moreover satisfies*

$$\int |p|^2 \, g(p) \, \mathrm{d}p < \infty.$$

With this hypothesis, we can derive a better bound on $\partial_x^2 A$.

Lemma 5.1. *Let $E \in \mathcal{C}^0\left(0,T;W^{1,\infty}(\mathbb{R})\right)$, $A \in \mathcal{C}^0\left(0,T;W^{1,\infty}(\mathbb{R})\right) \cap \mathcal{C}^1\left(0,T;L^\infty(\mathbb{R})\right)$, and $(E',A') = \mathcal{L}(E,A)$. Then, the second derivative of A' is bounded as:*

$$\left\| \frac{\partial^2 A'}{\partial x^2} \right\|_t \leq C_0 + C_1 \, \Pi_3 \left(t \, \|F\|_t \right) \left\{ \left\| \frac{\partial A}{\partial x} \right\|_s + \left\| \frac{\partial A}{\partial t} \right\|_s + \|A\|_s \left(1 + \|F\|_s \right) \right\}, \quad (5.1)$$

where Π_3 is a polynomial of third degree, whose coefficients are positive and depend only on the function g.

Proof. We use the formula (A.4) with f replaced by $-n\,A$:

$$\begin{aligned} 2 \, \frac{\partial^2 A'}{\partial x^2}(t,x) \leq D_0(t,x) &+ \int_0^t \left\{ \left(n \, \frac{\partial A}{\partial x} \right)(s, x+s-t) - \left(n \, \frac{\partial A}{\partial x} \right)(s, x+t-s) \right\} \mathrm{d}s \\ &- \int_0^t \left\{ \left(A \, \frac{\partial n}{\partial x} \right)(s, x+s-t) + \left(A \, \frac{\partial n}{\partial x} \right)(s, x+t-s) \right\} \mathrm{d}s \end{aligned} \quad (5.2)$$

where D_0 depends on the initial data only. The first line is bounded by (2.15) as:

$$C_0 + (C_1 + C_2 \, t \, \|F\|_t) \int_0^t \left\| \frac{\partial A}{\partial x} \right\|_s \mathrm{d}s.$$

In order to bound the second line in (5.2), we use (1.14) to rewrite

$$(\widehat{p}-1) \, \frac{\partial f}{\partial x}(s, x+s-t, p) = -\frac{\partial}{\partial s}[f(s, x+s-t, p)] + F(s, x+s-t) \, \frac{\partial f}{\partial p}(s, x+s-t, p).$$

We can integrate this w.r.t. p, since $\widehat{p} - 1$ never vanishes. Hence, the first part of the integral which appears on the second line in (5.2) becomes:

$$\begin{aligned} I_1 &:= \int_0^t \left(A \, \frac{\partial n}{\partial x} \right)(s, x+s-t) \, \mathrm{d}s \\ &= \int_0^t A(s, x+s-t) \int_{\mathbb{R}} \frac{1}{\widehat{p}-1} \left[-\frac{\partial}{\partial s}[f(s, x+s-t, p)] \right. \\ &\qquad \left. + F(s, x+s-t) \, \frac{\partial f}{\partial p}(s, x+s-t, p) \right] \mathrm{d}p \, \mathrm{d}s \, ; \end{aligned}$$

then, performing integration by parts in s and p:

$$\begin{aligned} I_1 = -A(t,x) \int_{\mathbb{R}} \frac{f(t,x,p)}{\widehat{p}-1} \, \mathrm{d}p + A_0(x-t) \int_{\mathbb{R}} \frac{f_0(x-t,p)}{\widehat{p}-1} \, \mathrm{d}p \\ + \int_0^t \frac{\partial}{\partial s}[A(s, x+s-t)] \int_{\mathbb{R}} \frac{f(s, x+s-t, p)}{\widehat{p}-1} \, \mathrm{d}p \, \mathrm{d}s \end{aligned}$$

$$+\int_0^t (A\,F)(s,x+s-t)\int_{\mathbb{R}} f(s,x+s-t,p)\,\frac{\mathrm{d}}{\mathrm{d}p}\left[\frac{1}{\widehat{p}-1}\right]\mathrm{d}p\,\mathrm{d}s.$$

When p tends to $+\infty$, there holds:

$$\frac{1}{\widehat{p}-1}\sim -2\,p^2,\quad \frac{\mathrm{d}}{\mathrm{d}p}\left[\frac{1}{\widehat{p}-1}\right]=\frac{p+\sqrt{1+p^2}}{\sqrt{1+p^2}\,\left(p-\sqrt{1+p^2}\right)}\sim -4\,p,$$

so that the integrations by parts mentioned above are fully justified using Hypotheses 1, 2, 5 and moreover,

$$|I_1|\le C_0+C_1\,\|A\|_t\int_{\mathbb{R}}(1+p^2)\,f(t,x,p)\,\mathrm{d}p$$
$$+C_2\int_0^t\left\{\|\partial_x A\|_s+\|\partial_t A\|_s+\|A\|_s\,\|F\|_s\right\}\int_{\mathbb{R}}(1+p^2)\,f(s,x+s-t,p)\,\mathrm{d}p\,\mathrm{d}s.$$

Using Lemma 2.5, we obtain a bound on the integrals in p, so that:

$$|I_1|\le C_0+C_1\,\Pi_3\left(t\,\|F\|_t\right)\left\{\left\|\frac{\partial A}{\partial x}\right\|_s+\left\|\frac{\partial A}{\partial t}\right\|_s+\|A\|_s\,(1+\|F\|_s)\right\}.$$

Of course, $I_2:=\int_0^t\left(A\,\frac{\partial n}{\partial x}\right)(s,x+s-t)\,\mathrm{d}s$ is bounded in the same manner. Hence (5.1). □

In the case of two solutions corresponding to two pairs of fields (E_1,A_1) and (E_2,A_2), we have similarly:

Lemma 5.2. *Let (E_1,A_1) and (E_2,A_2) be as in Lemma 5.1; and $(E_i',A_i')=\mathcal{L}(E_i,A_i)$. We have:*

$$\left\|\frac{\partial^2 A_1'}{\partial x^2}-\frac{\partial^2 A_2'}{\partial x^2}\right\|_t\le K\left\{\left\|\frac{\partial A_1}{\partial x}-\frac{\partial A_2}{\partial x}\right\|_t+\left\|\frac{\partial A_1}{\partial t}-\frac{\partial A_2}{\partial t}\right\|_t\right.$$
$$\left.+\|A_1-A_2\|_t+\|F_1-F_2\|_t\right\}\tag{5.3}$$

where the constant K is polynomial in $(t,\|A_i\|_t,\|\partial_x A_i\|_t,\|\partial_t A_i\|_t,\|F_i\|_t)$.

Proof. This time, we have to bound the integral

$$I=\int_0^t\left(\frac{\partial(n_1\,A_1)}{\partial x}-\frac{\partial(n_2\,A_2)}{\partial x}\right)(s,x+s-t)\,\mathrm{d}s,$$

as well as a similar one in which the current point is $(s,x-s+t)$, and which will be handled in exactly the same manner. Writing:

$$\frac{\partial(n_1\,A_1)}{\partial x}-\frac{\partial(n_2\,A_2)}{\partial x}=A_1\left(\frac{\partial n_1}{\partial x}-\frac{\partial n_2}{\partial x}\right)+\frac{\partial n_2}{\partial x}\,(A_1-A_2)$$
$$+n_1\left(\frac{\partial A_1}{\partial x}-\frac{\partial A_2}{\partial x}\right)+\frac{\partial A_2}{\partial x}\,(n_1-n_2),$$

we split I into four parts $I_1,\ I_2,\ I_3,\ I_4$, which are the integrals of the four terms above. By (2.15) and (3.10), we obtain:

$$|I_3| \leq (C_0 + C_1\ \|F_2\|_t)\ t\ \|\partial_x A_2 - \partial_x A_1\|_t\ ;$$
$$|I_4| \leq \left(C_0 + C_1\ \max_i \|F_i\|_t\right)\ \|\partial_x A_2\|_t\ t\ \|F_2 - F_1\|_t\,.$$

Then I_2 is bounded as in the previous Lemma:

$$|I_2| \leq C\,\Pi_3\,(t\,\|F_2\|_t)\ \{\|\partial_x A_1 - \partial_x A_2\|_t + \|\partial_t A_1 - \partial_t A_2\|_t + \|A_1 - A_2\|_t\,(1 + \|F_2\|_t)\}\,.$$

There remains to bound I_1. Performing the same computations as in Lemma 5.1, we obtain:

$$\begin{aligned} I_1 = &-A_1(t,x)\int_{\mathbb{R}} \frac{(f_1 - f_2)(t,x,p)}{\widehat{p} - 1}\,\mathrm{d}p \\ &+ \int_0^t \frac{\partial}{\partial s}[A_1(s, x+s-t)]\int_{\mathbb{R}} \frac{(f_1 - f_2)(s, x+s-t, p)}{\widehat{p} - 1}\,\mathrm{d}p\,\mathrm{d}s \\ &+ \int_0^t (A_1\,F_1)(s, x+s-t)\int_{\mathbb{R}} f_1(s, x+s-t, p)\,\frac{\mathrm{d}}{\mathrm{d}p}\left[\frac{1}{\widehat{p} - 1}\right]\mathrm{d}p\,\mathrm{d}s \\ &- \int_0^t (A_1\,F_2)(s, x+s-t)\int_{\mathbb{R}} f_2(s, x+s-t, p)\,\frac{\mathrm{d}}{\mathrm{d}p}\left[\frac{1}{\widehat{p} - 1}\right]\mathrm{d}p\,\mathrm{d}s. \end{aligned}$$

We rearrange the last two integrals by writing, as usual, $F_1\,f_1 - F_2\,f_2 = F_1\,(f_1 - f_2) + (F_1 - F_2)\,f_2$. Hence:

$$\begin{aligned} |I_1| \leq C\Big\{ &\|A_1\|_t \int_{\mathbb{R}} (1+p^2)\,(f_1 - f_2)(t,x,p)\,\mathrm{d}p \\ &+ \int_0^t \{\|\partial_x A_1\|_s + \|\partial_t A_1\|_s + \|A_1\|_s \|F_1\|_s\}\int_{\mathbb{R}} (1+p^2)(f_1 - f_2)(s, x+s-t, p)\,\mathrm{d}p\,\mathrm{d}s \\ &+ \int_0^t \|A_1\|_s\,\|F_1 - F_2\|_s \int_{\mathbb{R}} (1+p^2)\,f_2(s, x+s-t, p)\,\mathrm{d}p\,\mathrm{d}s\Big\}, \end{aligned}$$

which is bounded with the help of Lemmas 2.6 and 2.5 as:

$$\begin{aligned} |I_1| \leq C\Big\{ &\{\|\partial_x A_1\|_t + \|\partial_t A_1\|_t + \|A_1\|_t\,(1 + \|F_1\|_t)\}\,\Pi_3\left(t\ \max_i \|F_i\|_t\right)\int_0^t \|F_1 - F_2\|_s \\ &+ \|A_1\|_t\,\Pi_3\,(t\,\|F_2\|_t)\int_0^t \|F_1 - F_2\|_s.\Big\} \end{aligned}$$

Putting all these bounds together, we obtain (5.3). □

Such a bound as (5.3) cannot be derived in the NR case, basically because the velocity $\widehat{p} = p$ is not bounded: singularity formation could happen at any speed. The proof of the above two Lemmas does not apply, because neither $1/(p-1)$ nor its derivative $-1/(p-1)^2$ are locally integrable near $p = 1$, and the integrations by parts would be unjustified.

Nevertheless, in the QR case we immediately deduce the following theorem:

Theorem 5.1. *Let $(E_k, A_k)_{k\in\mathbb{N}}$ and (E, A) be as in Theorem 3.2. The sequence $\left(\partial_x^2 A_k\right)_{k\in\mathbb{N}}$ converges uniformly in x and t. As a consequence, the triple (f, E, A) defines a characteristic solution to (1.14–1.16) on the interval $[0, T]$.*

Proof. The convergence of $\left(\partial_x^2 A_k\right)_{k\in\mathbb{N}}$ immediately follows from (5.3) and the convergence results of Theorem 3.2, and the limit is necessarily equal to $\partial_x^2 A$. So, $\partial_x F_k$ converges uniformly towards the x-derivative of the force given by (E, A); and we get the conclusion. □

6. Stability of equilibria

In this section, we restrict ourselves to classical solutions to the system (1.14–1.19). Thus, in the NR case, we take $T > 0$ small enough to have a classical solution on $[0, T]$.

Furthermore, we only consider solutions of finite mass, energy and entropy — the latter concepts being precised in the following Subsection — in order to rule out some unphysical or pathological behaviours.

6.1. *Energies and entropy for the system (1.14–1.19)*

The physical energy of the system naturally splits in two parts, the *transversal* energy made up of the terms containing the vector potential A, and the *longitudinal* energy for the other terms.

The transversal energy is written $WT[f(t), A(t), \partial_t A(t)]$, where:

$$WT[f, A, \dot{A}] := \frac{1}{2}\int \left\{ n[f](x)\, A(x)^2 + |\partial_x A(x)|^2 + \dot{A}(x)^2 \right\} \mathrm{d}x \tag{6.1}$$

and, of course, $n[f]$ denotes the density of f. In the periodic case, the integral is to be taken over one space period; in the open-space case, over the whole space.

Let $\mathcal{V} = H^1(\mathbb{R})$ or $H^1_L(\mathbb{R})$ and $\mathcal{H} = L^2(\mathbb{R})$ or $L^2_L(\mathbb{R})$. The norm denoted $\|\cdot\|$, without any subscript, will be that of $\mathcal{H}$; and $(\cdot \mid \cdot)$ is its scalar product. For any function $w \in \mathcal{C}^0(0, T; \mathcal{H})$, we set

$$\langle w \rangle_t = \sup_{0\le s\le t} \|w(s)\|^2.$$

We have the following result:

Proposition 6.1. *Assume that $A_0 \in \mathcal{V}$ and $\dot{A}_0 \in \mathcal{H}$. Then $A \in \mathcal{C}^0(0, T; \mathcal{V}) \cap \mathcal{C}^1(0, T; \mathcal{H})$. As a consequence, the transversal energy is finite-valued and differentiable on $[0, T]$, and there holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t} WT[f(t), A(t), \partial_t A(t)] = \frac{1}{2}\int \partial_t n(t, x)\, A(t, x)^2 \,\mathrm{d}x. \tag{6.2}$$

Proof. Consider the approximating sequence $(A_k)_{k\in\mathbb{N}}$ from Theorem 3.2. Clearly, $A_0 \in \mathcal{C}^0(0,T;\mathcal{V}) \cap \mathcal{C}^1(0,T;\mathcal{H})$. Assuming that $A_k \in \mathcal{C}^0(0,T;\mathcal{V}) \cap \mathcal{C}^1(0,T;\mathcal{H})$, we see that A_{k+1} can be identified with the variational solution to:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(A_{k+1}(t) \mid B\right) + \left(\partial_x A_{k+1}(t) \mid \partial_x B\right) = -\left(n_k(t)\, A_k(t) \mid B\right), \quad \forall B \in \mathcal{V},$$
$$A_{k+1}(0) = A_0 \in \mathcal{V}, \quad \partial_t A_{k+1}(0) = \dot{A}_0 \in \mathcal{H},$$

where, of course, $n_k = n[f_k]$; recall that, thanks to (2.15), $\|n_k\|_t$ is uniformly bounded by a constant n_* on the interval $[0,T]$. Hence $A_{k+1} \in \mathcal{C}^0(0,T;\mathcal{V}) \cap \mathcal{C}^1(0,T;\mathcal{H})$. By induction, the whole sequence belongs to this space.

Moreover, we have the classical energy estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\left(\left\|\frac{\partial A_{k+1}}{\partial t}\right\|^2 + \left\|\frac{\partial A_{k+1}}{\partial x}\right\|^2\right) = -\left(n_k(t)\, A_k(t) \,\middle|\, \frac{\partial A_{k+1}}{\partial t}\right), \tag{6.3}$$

from which we deduce, by Young's inequality:

$$\frac{1}{2}\left\langle \frac{\partial A_{k+1}}{\partial t}\right\rangle_t + \left\langle \frac{\partial A_{k+1}}{\partial x}\right\rangle_t \leq C_0 + C_1\, \langle A_k\rangle_t\,. \tag{6.4}$$

On the other hand,

$$\|A_k(t)\|^2 = \|A_0\|^2 + 2\int_0^t \left(A_k(t) \,\middle|\, \frac{\partial A_k}{\partial t}\right) \mathrm{d}s\,;$$

applying once more Young's inequality, we get:

$$\langle A_k(t)\rangle_t \leq C_0 + C_1 \int_0^t \left\langle \frac{\partial A_k}{\partial t}\right\rangle_s \mathrm{d}s. \tag{6.5}$$

This inequality, together with (6.4), shows that

$$\frac{1}{2}\left\langle \frac{\partial A_{k+1}}{\partial t}\right\rangle_t + \left\langle \frac{\partial A_{k+1}}{\partial x}\right\rangle_t \leq C_0 + C_1 \int_0^t \left\{\frac{1}{2}\left\langle \frac{\partial A_k}{\partial t}\right\rangle_s + \left\langle \frac{\partial A_k}{\partial x}\right\rangle_s\right\} \mathrm{d}s,$$

which gives the uniform boundedness of $\langle \partial_t A_k\rangle_t$ and $\langle \partial_x A_k\rangle_t$ by Lemma 3.5, and that of $\langle A_k\rangle_t$ by (6.5).

As a consequence, for any t the sequences $(A_k(t))_{k\in\mathbb{N}}$ $(\partial_x A_k(t))_{k\in\mathbb{N}}$, $(\partial_t A_k(t))_{k\in\mathbb{N}}$ admit weakly convergent subsequences in $\mathcal{H}$, hence $(A(t), \partial_x A(t), \partial_t A(t)) \in \mathcal{H}^3$. Then, similar computations show that the sequence $(A_k)_{k\in\mathbb{N}}$ is indeed strongly convergent in $\mathcal{C}^0(0,T;\mathcal{V}) \cap \mathcal{C}^1(0,T;\mathcal{H})$; and, at the limit (6.3) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\left(\left\|\frac{\partial A}{\partial t}\right\|^2 + \left\|\frac{\partial A}{\partial x}\right\|^2\right) = -\left(n(t)\, A(t) \,\middle|\, \frac{\partial A}{\partial t}\right),$$

from which we deduce (6.2) by integration by parts thanks to Corollary 2.1. □

We now study the longitudinal energy. To this end, we first define the electrostatic field and potential functionals. In the open-space case, we set

$$E[f] = -\partial_x \Phi[f] := \partial_x\left(\phi_{\mathrm{ext}} - \phi[f]\right),$$

where $\phi_{\rm ext}$ is (the opposite of) the external confining potential, satisfying $\phi''_{\rm ext} = n_{\rm ext}$ and $\phi[f] \geq 0$ is the second primitive of the density $n[f]$ of f, given by the formula:

$$\phi[f](x) = \frac{1}{2} \int_{\mathbb{R}} |x-y|\, n[f](y)\, \mathrm{d}y \tag{6.6}$$

due to Lemma 2.8. We remark that ϕ is a linear, self-adjoint and positive operator. As for the potential $\phi_{\rm ext}$, we assume that it satisfies the following hypotheses.

Hypothesis 6. *There exist $C \geq 0$ and $m \in \mathbb{N}$ s.t.*

$$|\phi_{\rm ext}(x)| \leq C\ (1 + |x|^m) \quad \textit{and} \quad \int_{\mathbb{R}} |p|^m\, g(p)\, \mathrm{d}p < \infty.$$

(g is the majorising function in Hypothesis 2.)

Let us remark that, if $0 < n_{\min} \leq n_{\rm ext} \leq n_{\max}$, the above hypotheses are satisfied with $m = 2$, provided Hypothesis 5 holds.

In the periodic case, we do not use the above decomposition of the potential, since the external and the self potential would not be periodic. We set:

$$E[f] = -\partial_x \Phi[f], \quad -\partial_x^2 \Phi[f] = n_{\rm ext} - n[f],$$

the uniqueness of $\Phi[f]$ being ensured by imposing periodicity and

$$\int_0^L \Phi[f](x)\, n_{\rm ext}(x)\, \mathrm{d}x = 0.$$

This, in turn, ensures the self-adjointness of the operator Φ.

As for the kinetic part, we denote by $\kappa(p)$ the primitive of $\widehat{p}$, viz.

$$\kappa(p) = \frac{p^2}{2}\ \text{(NR case)}, \quad \kappa(p) = \sqrt{1+p^2}\ \text{(QR case)}.$$

We recall that, in the QR case, we always assume Hypothesis 5. In the NR case, the following assumption will be needed to ensure the differentiability of the energy.

Hypothesis 7. *The majorising function g satisfies*

$$\int_{\mathbb{R}} |p|^3\, g(p)\, \mathrm{d}p < \infty.$$

We are now ready to state the

Definition 6.1. In the periodic case, the longitudinal energy functional is given by the two equivalent formulae:

$$WL[f] = \int_0^L \int_{\mathbb{R}} \left\{ \kappa(p) - \frac{1}{2}\, \Phi[f](x) \right\} f(x,p)\, \mathrm{d}p\, \mathrm{d}x, \tag{6.7}$$

$$WL[f] = \int_0^L \int_{\mathbb{R}} \kappa(p)\, f(x,p)\, \mathrm{d}p\, \mathrm{d}x + \frac{1}{2} \int_0^L |\partial_x \Phi[f](x)|^2\ \mathrm{d}x\,; \tag{6.8}$$

while in the open-space case, one sets:

$$WL[f] = \int_{\mathbb{R}} \int_{\mathbb{R}} \left\{ \kappa(p) + \phi_{\rm ext}(x) - \frac{1}{2}\,\phi[f](x) \right\} f(x,p)\,\mathrm{d}p\,\mathrm{d}x. \tag{6.9}$$

Let us remark that the definition (6.9) becomes formally equivalent to (6.7) or (6.8). However, in the open-space case $|\partial_x \phi[f]|^2$ is clearly not integrable and the integration by parts to arrive to a formula like (6.8) are not justified. We will comment on this in the next subsection.

Now, we can analyse the evolution of the longitudinal energy.

Proposition 6.2. *Let (f, E, A) be a classical solution to (1.14–1.19). Assume Hypotheses 1, 2 and 7 if the system is NR, Hypotheses 1, 2 and 5 if it is QR. In the open-space setting, assume moreover that Hypotheses 3, 4 and 6 hold. Then $t \mapsto WL[f(t)]$ is finite-valued and differentiable for all $t \in [0,T]$. Namely:*

$$\frac{\mathrm{d}}{\mathrm{d}t} WL[f(t)] = \int j(t,x)\,\frac{\partial}{\partial x}\frac{A^2}{2}(t,x)\,\mathrm{d}x. \tag{6.10}$$

Proof. First, we consider the case of periodic solutions. The finiteness of the energy defined by (6.8) follows from Lemma 2.5 and the various boundedness results of Theorem 3.2. By the way, the boundedness of E yields that of Φ; and integrating by parts gives the equivalent expression (6.7). Then, as in Corollary 2.1, we bound

$$\frac{\partial}{\partial t}[\kappa(p)\,f(t,x,p)] = -\widehat{p}\,\kappa(p)\,\frac{\partial f}{\partial x} + \kappa(p)\,F(t,x)\,\frac{\partial f}{\partial p},$$

$$\left|\frac{\partial}{\partial t}[\kappa(p)\,f(t,x,p)]\right| \leq \kappa(p)\;(F_* + |\widehat{p}|)\;g_{t\,\|F\|_t}(p)\,\mathrm{e}^{t\,(1+\|\partial_x F\|_t)}.$$

The highest power of p is p^3 (NR) or p (QR): the above function is integrable in p, and in x given the finite length of $(0,L)$. Finally, Eq. (1.15) gives $\partial_t\left(\frac{1}{2}\,E^2\right) = E\,j$; these functions are continuous on $(0,L)$, so the second term in (6.8) is also differentiable.

Let us now consider the open-space setting. The finiteness and differentiablity of the integral $\iint \{\kappa(p) + \phi_{\rm ext}(x)\}\; f(t,x,p)\,\mathrm{d}x\,\mathrm{d}p$ follows from Lemma 2.9. Then, the self potential defined by (6.6) satisfies:

$$\begin{aligned} 0 \leq \phi[f(t)](x) &\leq \frac{1}{2}\left(|x|\int n[f](y)\,\mathrm{d}y + \int |y|\,n[f](y)\,\mathrm{d}y\right) \\ &\leq \frac{M\,|x|}{2} + \frac{1}{2}\iint (|y|+|p|)\,f(t,y,p)\,\mathrm{d}y \leq \frac{1}{2}\,(M\,|x| + Z), \end{aligned} \tag{6.11}$$

Z being defined in Lemma 2.8. By the same token,

$$0 \leq \iint \phi[f(t)](x)\,f(t,x,p)\,\mathrm{d}x\,\mathrm{d}p \leq \frac{1}{2}\int (M\,|x| + Z)\,f(t,x,p)\,\mathrm{d}x\,\mathrm{d}p \leq M\,Z. \tag{6.12}$$

There remains to check the differentiability of the self potential energy. Using the Vlasov equation (1.14) and Corollary 2.1, we find:

$$|\partial_t f(t,x,p)| \leq (F_* + |\widehat{p}|)\;g_{t\,\|F\|_t}(p)\,g_{t^2\,\|F\|_t}(x - t\,p)\,\mathrm{e}^{t\,(1+\|\partial_x F\|_t)}. \tag{6.13}$$

But the linearity and self-adjointness of ϕ imply: $\partial_t(\phi[f]\,f) = 2\,\partial_t f\,\phi[f]$, which allows to bound

$$\left|\frac{\partial}{\partial t}[\phi[f(t)]\,f(t)](x,p)\right| \leq (C_0 + C_1\,|x|)\;(1 + F_* + |\widehat{p}|)\;g_{t\,\|F\|_t}(p)\,g_{t^2\,\|F\|_t}(x - t\,p)$$

by (6.11) and (6.13). This proves the integrability of this function, and the differentiability of the self potential energy.

Finally, Eq. (6.10) is obtained through tedious but straightforward computations (cf. Propositions 1.5 and 1.6 in [2]), all the integrations by parts being justified by the above arguments. □

From (6.2), (6.10) and the continuity equation (2.19), we immediately deduce:

Theorem 6.1. *Under the hypotheses of Propositions 6.1 and 6.2, the energy* $W(t) := WT\,[f(t), A(t), \partial_t A(t)] + WL[f(t)]$ *is constant.*

For reference, we give the formulae for the energy in the FR case. There is no splitting in transversal and longitudinal parts. The total energy is given by:

$$W(t) := \int_0^L \int \left[\sqrt{1 + p^2 + A(t,x)^2} - \frac{1}{2}\,\Phi[f(t)](x)\right]\,f(t,x,p)\,\mathrm{d}p\,\mathrm{d}x + \frac{1}{2}\int_0^L \left[\left|\frac{\partial A}{\partial t}(t,x)\right|^2 + \left|\frac{\partial A}{\partial x}(t,x)\right|^2\right]\,\mathrm{d}x \tag{6.14}$$

in the periodic case, and

$$W(t) := \int_{\mathbb{R}} \int \left[\sqrt{1 + p^2 + A(t,x)^2} + \phi_{\mathrm{ext}}(x) - \frac{1}{2}\,\phi[f(t)](x)\right]\,f(t,x,p)\,\mathrm{d}p\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} \left[\left|\frac{\partial A}{\partial t}(t,x)\right|^2 + \left|\frac{\partial A}{\partial x}(t,x)\right|^2\right]\,\mathrm{d}x \tag{6.15}$$

in the open-space case. The reader may check that these expressions are formally constant, even though the mere existence of classical solutions is an open problem, and subtler arguments are probably necessary to justify the calculations.

Definition 6.2. Let $\sigma \in \mathcal{C}^2((0,+\infty)) \cap \mathcal{C}^0([0,+\infty))$ be a strictly convex and bounded-from-below function, which satisfies

$$\lim_{s\to+\infty} \frac{\sigma(s)}{s} = +\infty.$$

Let γ denote the generalised inverse of $-\sigma'$ (extended by 0 if necessary): it is a decreasing function in its support.

The *entropy* associated to a distribution function f is defined as:

$$S_\sigma[f] := \iint \sigma(f(x,p))\,\mathrm{d}p\,\mathrm{d}x.$$

The most classical example is $\sigma(s) = s\,\ln s - s$, i.e. $\gamma(s) = \mathrm{e}^{-s}$, associated to Maxwellian distribution functions.

Clearly, S_σ is a convex, bounded-from-below, weakly lower semicontinuous functional on its domain of definition. From [5], we know that for any $f_* \in L^1(\mathbb{R}^d)$, there exists a function σ as above s.t. $S_\sigma[f_*]$ is finite. Thus, we shall choose σ according to the following

Hypothesis 8. *In the periodic case,* $\int \sigma(g(p))\,\mathrm{d}p < \infty$. *In the open-space case,* $\int \sigma(g(0)\,g(p))\,\mathrm{d}p < \infty$ *and* $\iint \sigma(g(x)\,g(p))\,\mathrm{d}x\,\mathrm{d}p < \infty$.

In this case, the following identities hold:

$$\int \sigma\left(g_r(p)\right)\,\mathrm{d}p = 2\,r\,\sigma(g(0)) + \int \sigma(g(p))\,\mathrm{d}p\,;$$

$$\iint \sigma\left(g_\rho(x)\,g_r(p)\right)\,\mathrm{d}x\,\mathrm{d}p = 4\,\rho\,r\,g(0) + 2\,(\rho + r)\int \sigma(g(0)\,g(s))\,\mathrm{d}s + \iint \sigma(g(x)\,g(p))\,\mathrm{d}x\,\mathrm{d}p.$$

Then, the finiteness and differentiability of $S_\sigma[f(t)]$ stem from arguments very similar to the proof of Proposition 6.2, and one can easily deduce.

Theorem 6.2. *Under Hypothesis 8, the function* $S_\sigma[f(t)]$ *is constant on* $[0, T]$.

6.2. *Equilibria of (1.14–1.19)*

We are looking for solutions to (1.14–1.19) which do not depend on time, i.e., solutions to the coupled problem in $(f(x,p), A(x))$:

$$\widehat{p}\,\frac{\partial f}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}x}\left[\Phi[f] - \frac{A^2}{2}\right]\frac{\partial f}{\partial p} = 0, \tag{6.16}$$

$$-\frac{\mathrm{d}^2 A}{\mathrm{d}x^2} + n[f]\,A = 0 \tag{6.17}$$

with the potential satisfying the Poisson equation $-\frac{\partial^2}{\partial x^2}\Phi[f] = n_{\mathrm{ext}} - n[f]$.

We have the following result

Lemma 6.1. *Any classical solution to (6.16–6.17) s.t.* $M > 0$ *and* $A \in \mathcal{V}$ *is a Vlasov–Poisson equilibrium, i.e. it has the form* $(f_\infty, 0)$, *where* f_∞ *solves*

$$\widehat{p}\,\frac{\partial f_\infty}{\partial x} + \frac{\mathrm{d}\Phi[f_\infty]}{\mathrm{d}x}\,\frac{\partial f_\infty}{\partial p} = 0, \tag{6.18}$$

with $-\frac{\partial^2}{\partial x^2}\Phi[f_\infty] = n_{\mathrm{ext}} - n[f_\infty]$ *and consequently satisfies:*

$$f_\infty(x,p) = \mathcal{F}\left(\kappa(p) - \Phi[f_\infty](x)\right), \tag{6.19}$$

for some function $\mathcal{F}$.

Proof. If $A \in \mathcal{V}$, then for any $B \in \mathcal{V}$,

$$a_f(A,B) := \int \left(A'(x)\,B'(x) + n[f](x)\,A(x)\,B(x)\right)\,\mathrm{d}x = 0. \tag{6.20}$$

Now, take $B = A$ to deduce that $a_f(A, A) = 0$. Since both terms in $a_f(A, A)$ are non-negative, we deduce that

$$\int |A'(x)|^2 \, \mathrm{d}x = \int n[f](x) \, A(x)^2 \, \mathrm{d}x = 0.$$

From the first identity we conclude A is constant, while from the second we conclude $A = 0$ since $n[f] > 0$ on some non-negligible subset of $\mathbb{R}$ or $(0, L)$. We are left with (6.18), whose solution is well-known to be of the form (6.19). □

The function $\mathcal{F}$ can be precised by demanding that the solution should minimise a "free energy" functional. In other words, the choice of the entropy function σ is determined by the particular equilibrium one is interested in.

Definition 6.3. Let $f \in L^1(\mathbb{R}^2)$, resp. $f \in L^1_L(\mathbb{R}^2)$, the *free energy* of f is:

$$K_\sigma[f] := WL[f] + S_\sigma[f].$$

Let then $A \in \mathcal{V}$ and $\dot{A} \in \mathcal{H}$; we set

$$KT_\sigma[f, A, \dot{A}] := K_\sigma[f] + WT[f, A, \dot{A}].$$

We consider the set of suitable distribution functions with fixed mass, i.e.

$$\mathcal{K}(L, M) := \left\{ f \in L^1_L(\mathbb{R}^2) : f \geq 0 \text{ a.e. and } \|f\|_{L^1_L(\mathbb{R}^2)} = M \right\}.$$

Lemma 6.2. *In the periodic case, KT_σ is a strictly convex and bounded-from-below functional on $\mathcal{K}(L, M) \times \mathcal{V} \times \mathcal{H}$. It has a unique global minimum which takes the form $(f_{\infty,\sigma}, 0, 0)$, where*

$$f_{\infty,\sigma} = \gamma \left(\kappa(p) - \Phi[f_{\infty,\sigma}](x) - \alpha \right) \tag{6.21}$$

and is therefore a stationary solution of the Vlasov–Poisson system. The constant α is uniquely determined by M and L.

Proof. Eq. (6.7) shows that WL is convex in this case; it is clearly lower semicontinuous and on $\mathcal{K}(L, M)$ it is bounded from below. Since S_σ enjoys the same property and is strictly convex, we deduce that K_σ has a unique global minimum $f_{\infty,\sigma}$. Writing the Lagrange equation expressing the minimisation under the constraint $\int f = M$ (cf. [3,5]), yields the formula (6.21), where α is the Lagrange multiplier. Then, it is clear that for any $(f, A, \dot{A}) \neq (f_{\infty,\sigma}, 0, 0)$,

$$KT_\sigma[f, A, \dot{A}] > KT_\sigma[f, 0, 0] > KT_\sigma\left[f_{\infty,\sigma}, 0, 0\right],$$

where the last inequality is a consequence of the results in [3,5]. □

A similar result in the open-space case does not hold in our case, in contrast to the situation studied for nonlinear stability of the Vlasov–Poisson system in higher dimensions. [17,4,3,5] The main difference in 1D being that $\partial_x \phi[f] \notin L^2(\mathbb{R})$ as pointed out before. Sobolev embeddings in $d \geq 2$ allow, under confining conditions on the external potential ϕ_{ext}, to deduce a result similar to previous Lemma. [10,3,5] Moreover, the functional KT_σ ceases to be convex in our 1D open-space case.

6.3. *L^p-nonlinear stability of equilibria in the periodic case*

In this subsection, we just collect the known results in several references and applied to the particular case we deal with. Like in [17,4,5], one can rewrite:

$$KT_\sigma[f,A,\dot{A}] - KT_\sigma\left[f_{\infty,\sigma},0,0\right] = \Sigma_\sigma\left[f|f_{\infty,\sigma}\right] + WT[f,A,\dot{A}], \tag{6.22}$$

where the *relative entropy* of the distribution function f w.r.t. g is defined as:

$$\Sigma_\sigma[f|g] := \iint [\sigma(f) - \sigma(g) - \sigma'(g)(f-g)]\,\mathrm{d}x\,\mathrm{d}p + \frac{1}{2}\int |\partial_x \Phi[f-g]|^2\,\mathrm{d}x. \tag{6.23}$$

Yet, as a consequence of Theorems 6.1 and 6.2, $KT_\sigma\left[f(t),A(t),\partial_t A(t)\right]$ is constant for any classical solution to (1.14–1.19). This implies (see [17,4,5]) that:

(1) The L^p norm of $f - f_{\infty,\sigma}$ is bounded for $1 \le p \le 2$, if $\inf \sigma''(s)/s^{p-2} > 0$.
(2) The L^2 norm of $f - f_{\infty,\sigma}$ is bounded, if $f_{\infty,\sigma}$ is a Maxwellian, i.e. $\sigma(s) = s\,\ln s - s$.
(3) The H^1 norm of $\Phi[f] - \Phi[f_{\infty,\sigma}]$ is bounded.
(4) The transversal energy is bounded — indeed, we already knew this, and even a little bit more, from Proposition 6.1.

Here, all the norms are taken for $x \in [0,L]$, $p \in \mathbb{R}$.

The first three points follow from §§3 and 4 of [5] and references therein. Indeed, the arguments in these passages are independent of the dimension.

In other words, the one-dimensional periodic Vlasov–Poisson equilibria are L^p-nonlinearly stable under one-dimensional Vlasov–Maxwell perturbations.

Let us finally mention that Landau damping was proved in [6] in the case of the Vlasov–Poisson system in the periodic case. As a consequence, it was proved in [6] that some Vlasov–Poisson equilibria which are L^1-nonlinearly stable, are unstable for a weak topology. This is not known to happen in our 1D Vlasov–Maxwell system, although numerical computations seem to indicate that nonlinear Landau damping should occur in this model. Let us point out that there is no contradiction between these two stability assertions, since weak topology neighbourhoods of the equilibria are much larger than L^1 neighbourhoods.

Appendix A. The Duhamel formulae

The following representation formulae are capital in the various estimations of Sections 3–5.

The unique temperate solution u to the wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = f \in L^1\left(0,T;\mathcal{X}\right), \\ u(0,x) = u_0(x) \in \mathcal{X}, \quad \partial_t u(0,x) = v_0(x) \in \mathcal{X}, \end{cases}$$

where $\mathcal{X} = L^1_{\mathrm{loc}}(\mathbb{R}) \cap \mathcal{S}'(\mathbb{R})$, is explicitly given by the formula:

$$u(t,x) = \frac{1}{2}\left\{ u_0(x+t) + u_0(x-t) + \int_{x-t}^{x+t} v_0(y)\,\mathrm{d}y \right.$$

$$+\int_0^t \int_{x+s-t}^{x+t-s} f(s,y)\,\mathrm{d}y\,\mathrm{d}s\Big\}. \tag{A.1}$$

Hence, after some computations, the formulae for the various derivatives of u:

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Big\{u_0'(x+t) - u_0'(x-t) + v_0(x+t) + v_0(x-t) + \int_0^t [f(s,x+s-t) + f(s,x+t-s)]\,\mathrm{d}s\Big\}, \tag{A.2}$$

$$\frac{\partial u}{\partial x}(t,x) = \frac{1}{2}\Big\{u_0'(x+t) + u_0'(x-t) + v_0(x+t) - v_0(x-t) + \int_0^t [-f(s,x+s-t) + f(s,x+t-s)]\,\mathrm{d}s\Big\}, \tag{A.3}$$

$$\frac{\partial^2 u}{\partial x^2}(t,x) = \frac{1}{2}\Big\{u_0''(x+t) + u_0''(x-t) + v_0'(x+t) - v_0'(x-t) + \int_0^t \left[-\frac{\partial f}{\partial x}(s,x+s-t) + \frac{\partial f}{\partial x}(s,x+t-s)\right]\,\mathrm{d}s\Big\}, \tag{A.4}$$

which are valid e.g. if $u_0',\ u_0'',\ v_0' \in \mathcal{X}$ and $f \in \mathcal{C}^1\,(0,T;\mathcal{X})$.

Appendix B. Convergence of the sequence defined by (4.2)

Here is the technical lemma announced in the proof of Theorem 4.1.

Lemma Appendix B.1. *Let $(v_k(t))_{k\in\mathbb{N}}$ and φ_t be defined by (4.2).*

(1) There exists $T_1 < +\infty$ such that, for $0 < t < T_1$, φ_t admits two fixed points $v_t < v^t$. If $v_0(t) < v^t$, the unstable fixed point, then the sequence $v_k(t)$ converges toward the stable point v_t.
(2) For $t > T_1$, there is no fixed point, and $(v_k(t))_{k\in\mathbb{N}}$ diverges to $+\infty$.
(3) When t tends to zero, v^t goes to infinity, while v_t remains bounded.

Consequently, there exists $0 < T^* \le T_1$ s.t. $(v_k(t))_{k\in\mathbb{N}}$ is convergent for $0 \le t < T^*$.

Proof. As φ_t is convex and increasing, there are only three possibilities:

(1) The equation $\varphi_t(x) = x$ admits two solutions $v_t < v^t$, which are respectively stable and unstable fixed points: $0 < \varphi_t'(v_t) < 1$ and $\varphi_t'(v^t) > 1$.
(2) The two solutions merge in a unique fixed point v, which satifies both $\varphi_t(v) = v$ and $\varphi_t'(v) = 1$; it is stable on the left side, unstable on the right side.
(3) There is no fixed point at all, and $\varphi_t(x) > x$ for all $x \in \mathbb{R}$.

The case 1 is achieved for t small enough. Indeed:

$$\lim_{t\to 0} \beta\, t\, \mathrm{e}^{t\,(1+x)} = 0, \quad \forall x \in \mathbb{R}^+.$$

Hence, for any $\mu > 1$, there exists $t_\mu > 0$ s.t. $\varphi_t(\mu\,\alpha) < \mu\,\alpha$ for $t < t_\mu$. On the other hand, $\varphi_t(\alpha) > \alpha$. Hence, φ_t has (at least) one fixed point in the interval $(\alpha, \mu\,\alpha)$

when $t < t_\mu$. Given that $\varphi_t(x) \gg x$ when $x \to +\infty$, there is another fixed point in $(\mu\,\alpha, +\infty)$.

Then, we notice that, when x is fixed, $\varphi_t(x)$ is a decreasing function of t. This has two consequences. Firstly, if $s < t$ and φ_t has fixed points, then φ_s also has fixed points. This proves the existence of T_1, which may be finite or not. Clearly, if T_1 is finite, it achieves the case 2. Eliminating v between $\varphi_t(v) = v$ and $\varphi_t'(v) = 1$ gives the following equation for t:

$$\beta\, t^2\, \mathrm{e}^{\alpha\, t^2 + 2\, t} = 1.$$

As the l.h.s. is zero for $t = 0$, infinite for $t = +\infty$ and strictly increasing in t, the equation admits a unique solution $T_1 \in (0, +\infty)$.

The behaviour of $v_{k+1}(t) = \varphi_t(v_k(t))$ then follows from the elementary theory of sequences (Figure 1), and the claims 1 and 2 are obtained.

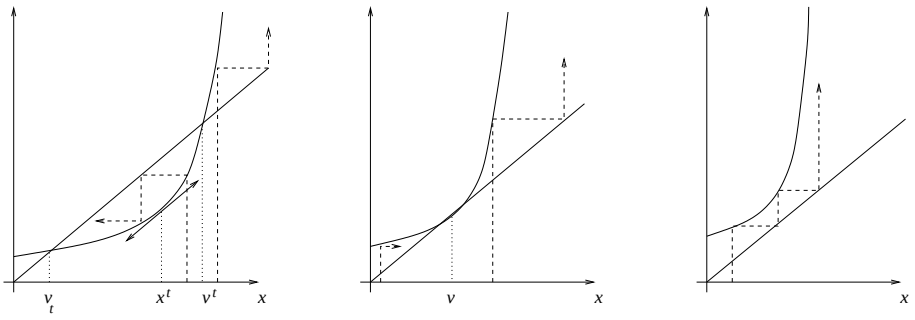

Fig. 1. Dynamic of the sequence $v_{k+1}(t) = \varphi_t\,(v_k(t))$.

The second consequence is that v_t and v^t are resp. decreasing and increasing functions of t. Hence, v_t is bounded when $t \to 0$. On the other hand, $v^t > x^t$, where x^t is defined by $\varphi_t'(x^t) = 1$. The latter equation gives

$$\beta\, t^2\, \mathrm{e}^{t\,(1+x^t)} = 1 \iff x^t = \frac{1}{t}\,\ln\left(\frac{1}{\beta\, t^2}\right) - 1 \to +\infty \quad \text{when } t \to 0.$$

This proves the third claim. Finally, let us remark that $v_0(t)$ is indeed independent of t: E_0 and A_0, hence F_0 are constant in time. Consequently, for t small enough, $v^t > v_0(t)$. This gives the last part of the conclusion. □

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