The Fourier Singular Complement Method for the Poisson problem. Part I: prismatic domains

Patrick Ciarlet, Beate Jung, Samir Kaddouri, Simon Labrunie, Jun Zou

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P. Ciarlet, Jr, [1] B. Jung, [2] S. Kaddouri, [3] S. Labrunie, [4] J. Zou [5]

Abstract

This is the first part of a threefold article, aimed at solving numerically the Poisson problem in three-dimensional prismatic or axisymmetric domains. In this first part, the Fourier Singular Complement Method is introduced and analysed, in prismatic domains. In the second part, the FSCM is studied in axisymmetric domains with conical vertices, whereas, in the third part, implementation issues, numerical tests and comparisons with other methods are carried out. The method is based on a Fourier expansion in the direction parallel to the reentrant edges of the domain, and on an improved variant of the Singular Complement Method in the 2D section perpendicular to those edges. Neither refinements near the reentrant edges of the domain nor cut-off functions are required in the computations to achieve an optimal convergence order in terms of the mesh size and the number of Fourier modes used.

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1 Introduction

The *Singular Complement Method* (SCM) was originally introduced by Assous *et al* [8, 7], for the 2D static or instationary Maxwell equations without charges. The cases with charges have been recently solved by Garcia *et al* [6, 19], including the numerical solution to the 2D Vlasov-Maxwell system of equations. The SCM has been extended in [13] to the 2D Poisson problem. Further extensions to the 2D heat or wave equations, or to similar problems with piecewise constant coefficients, can be obtained easily. As a matter of fact, this stems from the analysis which is performed hereafter (see Remark 4.1). The primary basis of the SCM is the decomposition of the solution into regular and singular parts. Methodologically speaking, the SCM consists in adding some singular test functions to the usual P_1 Lagrange FEM so that one recovers the optimal H^1-convergence rate, even in non-convex domains. In 2D, one may simply add

[1]ENSTA & CNRS UMR 2706, 32, boulevard Victor, 75739 Paris Cedex 15, France. *This author was supported in part by France/Hong Kong Joint Research Scheme.*

[2]Department of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany. *This author was supported by DGA/DSP-ENSTA 00.60.075.00.470.75.01 Research Programme.*

[3]ENSTA & CNRS UMR 2706, 32, boulevard Victor, 75739 Paris Cedex 15, France.

[4]IECN, Université Henri Poincaré Nancy I & INRIA (Projet CALVI), 54506 Vandœuvre-lès-Nancy cedex, France.

[5]Department of Mathematics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong. *This author was fully supported by Hong Kong RGC grants (Project CUHK4048/02P and project 403403).*

one singular test function per reentrant corner.

There exist a couple of numerical methods in the literature for accurately solving 2D Poisson problems in non-convex domains. It was shown in [13] that the SCM can be reformulated so that it coincides with the approach of Moussaoui [27, 1] when L-shaped domains are considered. The SCM differs from the *Dual Singular Function Method* (DSFM) of Blum and Dobrowolski [10] in that it requires no cut-off functions. Actually, when the numerical implementation of the SCM is carried out, the cut-off function is traded for a non-homogeneous boundary condition. Note that Cai and Kim [12] recently proposed a new SFM which involves the evaluations of singular and cut-off functions and the solution of a nonsymmetric elliptic problem. The SCM is clearly different from (anisotropic) mesh refinement techniques [28, 3, 24, 4, 2], and can be applied efficiently to instationary problems (see Remark 4.1), since it does not need mesh refinement and thus larger timesteps may be allowed. However the anisotropic mesh refinement methods have one advantage: they require only a partial knowledge of the most singular part of the solution.

The numerical solution of 3D singular Poisson problems is quite different from the 2D case, and much more difficult. This is a relatively new field of research: most approaches rely on anisotropic mesh refinement, see for instance [3, 5, 24, 4, 25], and [2] and Refs. therein. To our knowledge, this series of papers is the first attempt to generalize the SCM for three-dimensional singular Poisson problems. Specifically, we shall consider the numerical solution of the Poisson problem:
Find $u \in H_0^1(\Omega)$ *such that*

$$-\Delta u = f \quad \text{in} \quad \Omega, \tag{1}$$

where $f \in L^2(\Omega)$, and Ω is a (right) *prismatic domain* described by

$$\Omega = \omega \times Z\,, \tag{2}$$

and ω is a two-dimensional general polygonal domain, Z is an interval varying from 0 to a positive constant L on the x_3-axis. The *bases* of the domain are the subsets of the boundary $\partial\Omega$, which are included in the planes $\{x_3 = 0\}$ and $\{x_3 = L\}$.

The case of an *axisymmetric domain* is considered in the companion paper [14]. When the Poisson problem (1) is solved in this class of domains, two difficulties arise. The first difficulty is that one has to deal with weighted Sobolev spaces, the weights being functions of the distance to the axis. The second one is that there exist two kinds of *geometrical singularities*: reentrant edges like in the prismatic case, and, in addition, *sharp conical vertices*. As for *implementation issues and comparisons* with other methods (such as variants of our method, the FSCM, or mesh refinement techniques [25]), we refer the reader to [15].

The rest of the paper is organized as follows. In the next Section, some theoretical results concerning the regularity of the solution to the Poisson problem in prismatic domains are recalled. *A priori* regularity results of the solution u to (1), and a first splitting of the solution into regular and singular parts, are emphasized. In Section 3, some results about the Fourier expansion along x_3 are recalled and/or proven. This suggests a framework for building the *Fourier Singular Complement Method* (FSCM) for accurately solving the problem (1), using a Fourier expansion in x_3, and an improved variant of the Singular Complement Method [13] in the 2D section ω. In Section 4, we study the variant of the SCM, based on a theoretical splitting of the solution u_μ to 2D problems of the form $-\Delta u_\mu + \mu\, u_\mu = f_\mu$ in ω (with a parameter $\mu \geq 0$ related to the Fourier modes). The main feature of the regular-singular splitting is that it is chosen *independently of* μ; this independence is important, and very helpful, from the computational point of view. Estimates on Sobolev norms of u_μ and its splitting are established. To end this Section, the SCM is considered from a numerical point of view, to approximate u_μ accurately, *via* the discretization of the splitting: the optimal H^1-norm convergence of the order $O(h)$ is recovered. In the last Section, we first prove a refined splitting of the solution u to the 3D Poisson problem under suitable assumptions on the right-hand side f, using the Fourier expansion along x_3. Then, we build the numerical algorithms which define the FSCM, and we show that the FSCM has the optimal convergence of order $O(h + N^{-1})$, where h is the 2D mesh size and N is the number of Fourier modes used.

Throughout this paper, when two quantities a and b are such that $a \leq C\, b$, with a constant $C > 0$ which depends only on the geometry of the domain, we shall use the notation $a \lesssim b$.

2 Poisson problem in prismatic domains

Let us recall that a (right, open) cylinder of $\mathbb{R}^3$, with axis parallel to x_3, is equal to $D \times I$, where D is any connected (open) subset of $\mathbb{R}^2$, and I is any (open) interval of $\mathbb{R}$. Let us proceed then with some remarks on the class of domains Ω, i.e., the prismatic domains. *A priori*, such domains could be considered:

- either as truncated infinite cylinders;
- or as polyhedra.

As it happens, considering Ω as a *polyhedron* is helpful, in a simple manner. Indeed, from [4, 18], we know that, in any polyhedra, the solution u to (1) can be split as

$$\begin{aligned} &u = u_r + u_e + u_v, \text{ with } u_r \in H^2(\Omega), \\ &u_e = \sum_e \mu_e(\rho_e, z_e)\sin(\alpha_e \phi_e), \text{ and } u_v = \sum_v \sum_{-1/2<\lambda_v<1/2} \mu_{v,\lambda_v} \rho_v^{\lambda_v} \Phi_v(\theta_v, \phi_v). \end{aligned} \tag{3}$$

Above, u_r is called the *regular* part, u_e the *edge singularity* part, and u_v the *vertex singularity* part. Note that when $u_e \neq 0$ or $u_v \neq 0$, they do not belong to $H^2(\Omega)$. The summation in u_e is taken over all *reentrant* edges e, (ρ_e, ϕ_e, z_e) denote the local cylindrical coordinates, and π/α_e the dihedral angle (so that $\alpha_e \in]1/2, 1[$). Last, the summation in u_v is taken over all *non-convex* vertices v and over all eigenvalues λ_v of the Laplace-Beltrami operator, which belong to the interval $]-1/2, 1/2[$, and $(\rho_v, \theta_v, \phi_v)$

denote the local spherical coordinates.
In our case, i.e., when Ω is a *prismatic* domain with polygonal bases, it has been shown [29, 2] that the vertex singularity part u_v always vanishes, so (3) reduces to

$$u = u_r + u_s, \text{ with } u_r \in H^2(\Omega) \text{ and } u_s = \sum_e \mu_e(\rho_e, z_e)\sin(\alpha_e\phi_e). \tag{4}$$

Let us describe how one can fall into the other class, that of the *infinite cylinders*. The first step is to introduce a suitable continuation $\tilde{u}$ of the solution u (odd reflection at the bases) along the x_3 direction from Z to $\mathbb{R}$: one builds a problem to be solved in the infinite cylinder $C_\infty = \omega \times \mathbb{R}$. Unfortunately, with this continuation technique, one gets a solution (and data) which is not in $L^2(C_\infty)$. Thus, one introduces in a second step a smooth truncation function η, such that $\eta(x_3)$ is equal to one for $x_3 \in]-L/2, 3L/2[$, and to zero for $|x_3| > 2L$. Then, one multiplies $\tilde{u}$ by η, to obtain a Poisson problem in C_∞ with solution $u^\eta = u\,\eta$. This time, one has $u^\eta \in H^1(C_\infty)$ (and $f^\eta = -\Delta u^\eta \in L^2(C_\infty)$). By construction, the restriction of u^η on Ω coincides with u.

Interestingly, it has been proven in [21, 26], that a splitting similar to (4) holds for u^η. Furthermore, u^η_s can be expressed as

$$u^\eta_s = \gamma^\eta_e(\rho_e, x_3)\rho_e^{\alpha_e}\sin(\alpha_e\phi_e). \tag{5}$$

The function γ^η_e in (5) is often called in mechanics the *stress intensity distribution.* On the one hand, in the original paper [21], γ^η_e is expressed as a convolution product. On the other hand, in [26], it is characterized as the solution to a second order PDE. Finally, the regularity of the singular part u^η_s, can be expressed accurately as follows [26]. Let δ denote the minimal distance between two reentrant edges, and for each reentrant edge e, let $\Omega_e = \{\vec{x} \in \Omega \ : \ d(\vec{x}, e) < \delta/2\}$. Then

$$\begin{cases} u \in H^{1+\alpha-\varepsilon}(\Omega), \ \forall\varepsilon > 0, \ \alpha = \min_e \alpha_e, \\ u \in H^2(\Omega \setminus \cup_e\bar{\Omega}_e), \\ \rho_e^{\beta_e}\partial_i u \in H^1(\Omega_e), \ \forall e, \ \forall\beta_e > 1-\alpha_e, \ i = 1, 2, \\ \partial_3 u \in H^1(\Omega). \end{cases} \tag{6}$$

In Section 6, using a Fourier expansion along the x_3 axis, we recover some properties which are very similar to (4-6).

We end this Section with remarks on other possible boundary conditions.

If the boundary condition for u on the bases of the physical domain Ω are the non-homogeneous Dirichlet boundary condition:

$$u = g \quad \text{at} \quad x_3 = 0 \quad \text{and} \quad x_3 = L,$$

one can set $w = u - \tilde{g}$ with $\tilde{g}$ being a continuation of g into Ω. Then the problem reduces to the case with the solution w satisfying the homogeneous Dirichlet boundary condition, assuming that $\tilde{g} \in H^2(\Omega)$.

If the boundary condition for u is the homogeneous Neumann boundary condition $\partial_n u = 0$ on $\partial\Omega$, then one can replace the $\sin(\alpha_e \phi_e)$ factor in (4) by the expected $\cos(\alpha_e \phi_e)$. Moreover, to obtain an expression like (5), one uses an even reflection of u at the bases of the domain. If we have the non-homogeneous Neumann boundary condition:

$$\partial_n u = g \quad \text{at} \quad x_3 = 0 \quad \text{and} \quad x_3 = L,$$

one may then study the solution

$$w(\vec{x}) = u(\vec{x}) - \int_0^{x_3} \tilde{g}(x_1, x_2, z)\, dz$$

first, which satisfies the homogeneous Neumann boundary conditions at the bases of the domain Ω. Here $\tilde{g}$ is a continuation of g into Ω which is again assumed to belong to $H^2(\Omega)$.

From now on, we assume, for ease of exposition, that the polygon ω has only one reentrant corner C, i.e., with an interior angle larger than π, denoted as π/α, with $1/2 < \alpha < 1$. In particular, the summation which defines the singular part u_s in (4) reduces to exactly one term.

3 Fourier expansion

We devote this Section to some justifications about the Fourier series expansion of the Poisson solution to (1). First, one can show, following for instance Heinrich's proof of Lemma 3.2 in [23], the well-known result

Lemma 3.1 *For any $f \in L^2(\Omega)$, there exist Fourier coefficients defined by*

$$f_k(x_1, x_2) = \frac{2}{L} \int_0^L f(x_1, x_2, x_3) \sin \frac{k\pi}{L} x_3\, dx_3, \quad k = 1, 2, 3, \cdots, \tag{7}$$

such that $f_k \in L^2(\omega)$ and

$$f(x_1, x_2, x_3) = \sum_{k=1}^{\infty} f_k(x_1, x_2) \sin \frac{k\pi}{L} x_3 \qquad a.e.\ in\ \Omega, \tag{8}$$

and

$$\|f\|^2_{L^2(\Omega)} = \frac{L}{2} \sum_{k=1}^{\infty} \|f_k\|^2_{L^2(\omega)} < \infty. \tag{9}$$

If $f \in H^1_0(\Omega)$, then $f_k \in H^1_0(\omega)$ for all k and

$$\|\nabla f\|^2_{L^2(\Omega)} = \frac{L}{2} \sum_{k=1}^{\infty} \left\{ \|\nabla f_k\|^2_{L^2(\omega)} + \left(\frac{k\pi}{L}\right)^2 \|f_k\|^2_{L^2(\omega)} \right\} < \infty. \tag{10}$$

For f in $L^2(\Omega)$, let us introduce the sequence of partial sums $(F_K)_K$ of the Fourier decomposition of f, which converges to f in $L^2(\Omega)$, cf. (8):

$$F_K = \sum_{k=1}^{K} f_k \sin\frac{k\pi}{L}x_3, \text{ for } K > 0. \tag{11}$$

We note that when f is in $H_0^1(\Omega)$, $(F_K)_K$ converges to f in $H_0^1(\Omega)$, according to (10). Also, the sine functions can be replaced by cosine functions with the same argument $\frac{k\pi}{L}x_3$, and (7-10) still holds (for (10), with any f in $H^1(\Omega)$.)

In our subsequent analysis, summations like

$$\sum_{k=1}^{\infty} k^4 \|f_k\|^2_{L^2(\omega)} \tag{12}$$

will appear. The result below provides a characterization of elements f of $L^2(\Omega)$, which are such that (12) is bounded. Let the following Sobolev spaces be introduced:

$$\begin{aligned}
h^1(\Omega) &:= H^1(]0,L[, L^2(\omega)) = \{f \in L^2(\Omega) \ : \ \partial_3 f \in L^2(\Omega)\} \ ; \\
h^1_\diamond(\Omega) &:= H^1_0(]0,L[, L^2(\omega)) = \{f \in h^1(\Omega) \ : \ f_{|\{x_3=0\}} = f_{|\{x_3=L\}} = 0\} \ ; \\
h^2(\Omega) &:= H^2(]0,L[, L^2(\omega)) = \{f \in h^1(\Omega) \ : \ \partial_{33} f \in L^2(\Omega)\}.
\end{aligned}$$

Lemma 3.2 *Given $f \in L^2(\Omega)$, one has the following equivalences*

$$f \in h^1_\diamond(\Omega) \Longleftrightarrow \sum_{k=1}^{\infty} k^2 \|f_k\|^2_{L^2(\omega)} < \infty \ ; \tag{13}$$

$$f \in h^1_\diamond(\Omega) \cap h^2(\Omega) \Longleftrightarrow \sum_{k=1}^{\infty} k^4 \|f_k\|^2_{L^2(\omega)} < \infty. \tag{14}$$

Proof. Let f be in $L^2(\Omega)$.
Assume in addition that $f \in h^1_\diamond(\Omega)$. We note that, by the definition of the Fourier mode f_k and integration by parts (f vanishes at the bases), one has

$$(k\pi) f_k = -2\int_0^L f\left(\cos\frac{k\pi}{L}x_3\right)' dx_3 = 2\int_0^L \partial_3 f \cos\frac{k\pi}{L}x_3 \, dx_3.$$

Since by assumption, $\partial_3 f$ is in $L^2(\Omega)$, one gets the expected $\sum_{k=1}^{\infty} k^2 \|f_k\|^2_{L^2(\omega)} < \infty$.
Let us prove the reciprocal assertion. For f in $L^2(\Omega)$, as the sequence $(F_K)_K$ (see (11)) converges to f in $L^2(\Omega)$, one infers that $(\partial_3 F_K)_K$ converges to $\partial_3 f$ in $H^{-1}(\Omega)$. Now, if the sum is bounded, $(\partial_3 F_K)_K$ is a Cauchy sequence in $L^2(\Omega)$, so it converges in this space, and its limit $\partial_3 f$ is in $L^2(\Omega)$. Since both $(F_K)_K$ and $(\partial_3 F_K)_K$ converge in $L^2(\Omega)$, $(F_K)_K$ converges to f in $h^1(\Omega)$, and, as F_K belongs to $h^1_\diamond(\Omega)$ for all K, f is

also in $h^1_\diamond(\Omega)$, which proves (13).

In order to establish (14), one proceeds similarly, by performing a second integration by parts. Note that for this additional integration by parts, no assumption is required on the trace of f at the bases, since the $(\sin \frac{k\pi}{L} x_3)_k$ vanish there.

$$\frac{k^2\pi^2}{L} f_k = 2\frac{k\pi}{L}\int_0^L \partial_3 f \cos\frac{k\pi}{L}x_3\,dx_3 = -2\int_0^L \partial_{33} f \sin\frac{k\pi}{L}x_3\,dx_3.$$

With this identity, one concludes the proof easily. $\diamond$

Now that the general results have been obtained, we focus on the Poisson problem (1). Consider the weak form of the Poisson problem:

$$a(u,v) = f(v) \quad \forall\, v \in H^1_0(\Omega) \tag{15}$$

where $a(\cdot,\cdot)$ and $f(\cdot)$ are given by

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v\,dx, \quad f(v) = \int_\Omega f\,v\,dx.$$

We expand the solution u in (1) in the Fourier sine series:

$$u(x_1,x_2,x_3) = \sum_{k=1}^{\infty} u_k(x_1,x_2)\sin\frac{k\pi}{L}x_3. \tag{16}$$

Following again Heinrich's proof of Lemma 3.2 in [23], the next two Lemmas hold.

Lemma 3.3 *For any $u, v \in H^1_0(\Omega)$, we have*

$$a(u,v) = \frac{L}{2}\sum_{k=1}^{\infty} a_k(u_k,v_k), \quad f(v) = \frac{L}{2}\sum_{k=1}^{\infty} f_k(v_k),$$

where a_k and f_k are given by

$$a_k(u_k,v_k) = \int_\omega \Big\{\nabla u_k\cdot\nabla v_k + \Big(\frac{k\pi}{L}\Big)^2 u_k v_k\Big\}dx_1dx_2, \quad f_k(v_k) = \int_\omega f_k\,v_k\,dx_1dx_2,$$

and u_k, v_k and f_k are Fourier coefficients of u, $v \in H^1_0(\Omega)$ and $f \in L^2(\Omega)$ respectively.

Lemma 3.4 *For any $f \in L^2(\Omega)$, let $u \in H^1_0(\Omega)$ be the unique weak solution of (15) and u_k and f_k be the Fourier coefficients of u and f. Then $u_k \in H^1_0(\omega)$ is the unique solution of the following 2D weak problem:*
Find $u_k \in H^1_0(\omega)$ such that

$$a_k(u_k,v) = f_k(v) \quad \forall\, v \in H^1_0(\omega). \tag{17}$$

Moreover, u_k satisfies the following a priori *estimates:*

$$\begin{aligned}\int_\omega \Big\{|\nabla u_k|^2 + \Big(\frac{k\pi}{L}\Big)^2 u_k^2\Big\}dx_1dx_2 &\le \Big(\frac{L}{k\pi}\Big)^2 \|f_k\|^2_{L^2(\omega)}, \quad k = 1,2,\cdots, \\ \sum_{k=1}^{\infty} k^2\Big\{\|\nabla u_k\|^2_{L^2(\omega)} + \Big(\frac{k\pi}{L}\Big)^2\|u_k\|^2_{L^2(\omega)}\Big\} &\le \frac{2L}{\pi^2}\|f\|^2_{L^2(\Omega)}.\end{aligned}$$

This means that the k-th Fourier mode of u is characterized as the unique solution to the 2D problem
Find $u_k \in H_0^1(\omega)$ such that

$$-\Delta u_k + \left(\frac{k\pi}{L}\right)^2 u_k = f_k \quad \text{in} \quad \omega; \quad u_k = 0 \quad \text{on} \quad \partial\omega. \tag{18}$$

As Corollaries, one gets a convergence result of the sequence of partial sums $(U_K)_K$ of the Fourier decomposition of u, and also the last result of (6).

Corollary 3.1 *Let $f \in L^2(\Omega)$, and u be the solution to (1). Then $(U_K)_K$ converges to u in $H^1(\Omega)$, and $(\Delta U_K)_K$ converges to $-f$ in $L^2(\Omega)$.*

Proof. The fact that $(U_K)_K$ converges to u in $H^1(\Omega)$ is a consequence of Lemma 3.1. Then, one notes that

$$-\Delta U_K = \sum_{k=1}^{K}(-\Delta u_k + \left(\frac{k\pi}{L}\right)^2 u_k) \sin\frac{k\pi}{L}x_3 \overset{(18)}{=} \sum_{k=1}^{K} f_k \sin\frac{k\pi}{L}x_3 = F_K,$$

which yields the result on the convergence of $(\Delta U_K)_K$. $\diamond$

Corollary 3.2 *Let $f \in L^2(\Omega)$, and u be the solution to (1). Then $\partial_3 u \in H^1(\Omega)$.*

Proof. We prove that, for $i = 1, 2, 3$, $\partial_{i3}u$ belongs to $L^2(\Omega)$.
For $i = 3$, thanks to the last bound of the Lemma 3.4, there holds $\sum_{k=1}^{\infty} k^4 \|u_k\|^2_{L^2(\omega)} < \infty$.
Result (14) yields $u \in h^1_\diamond(\Omega) \cap h^2(\Omega)$, so that $\partial_{33}u$ is in $L^2(\Omega)$.
For $i = 1, 2$, we note that

$$\partial_{i3}U_K = -\frac{\pi}{L}\sum_{k=1}^{K} k\partial_i u_k \cos\frac{k\pi}{L}x_3.$$

According again to the last estimate in Lemma 3.4, $\sum_{k=1}^{\infty} k^2\|\partial_i u_k\|^2_{L^2(\omega)} < \infty$, so $\partial_{i3}u$ is in $L^2(\Omega)$. $\diamond$

To conclude this Section, we note that the Fourier expansion (16) of u together with the series of 2D problems (18) suggest the numerical approximation scheme below, i.e., define the Fourier SCM (FSCM) approximation of the solution u to (15) as follows:

$$U_N^h(x_1, x_2, x_3) = \sum_{k=1}^{N} u_k^h(x_1, x_2) \sin\frac{k\pi}{L}x_3 \tag{19}$$

where N is the total number of Fourier modes used in the approximation, and u_k^h is a suitable approximation of u_k, to be studied in the next two Sections.

4 Regular-singular decomposition in the 2D domain ω: theoretical study

The main interest of this paper is to propose some efficient numerical method for solving the three-dimensional singular Poisson problem (1) in a prismatic domain. Basically, the method reduces the 3D problem into a series of 2D Poisson-like problems, see (18), by the Fourier expansion of the 3D solution along the x_3-direction. This Section will thus focus on the 2D singular Poisson problem:
Find $u_\mu \in H_0^1(\omega)$ such that

$$-\Delta u_\mu \ + \mu \, u_\mu \ = f \quad \text{ in } \ \omega. \tag{20}$$

In the case of the Fourier expansion, one considers $\mu = k^2\pi^2/L^2$ and $f = f_k$ in (20). Due to the presence of the Fourier mode index k, the coefficient μ varies in a large range, from π^2/L^2 to $N^2\pi^2/L^2$, where N is the number of Fourier modes required subsequently in the numerical approximation (cf. Section 6). This brings in one of the main difficulties in the subsequent error estimates, which should hold for all μ's in a large range.

As a preliminary remark, we note that, according to [22], the most singular part of the solution to (20) is of the form $\rho^\alpha \sin(\alpha\theta)$, compared to (4) in 3D.

Let $\gamma_1, \gamma_2, \cdots, \gamma_K$ be the line segments of $\partial\omega$, where γ_1 and γ_2 are two line segments which form the single re-entrant corner of ω. Our numerical method is based on the following important decomposition of the space $L^2(\omega)$ [22]:

$$L^2(\omega) = \Delta[H^2(\omega) \cap H_0^1(\omega)] \overset{\perp}{\oplus} N\,, \tag{21}$$

where N is a space of singular harmonic functions defined by

$$N = \Big\{ p \in L^2(\omega) \ : \ \Delta p = 0, \ p|_{\gamma_k} = 0 \text{ in } (H_{00}^{1/2}(\gamma_k))', \ 1 \le k \le K \Big\}.$$

Above, the space $H_{00}^{1/2}(\gamma_k)$ is made up of elements of $H^{1/2}(\gamma_k)$, such that their continuation to $\partial\omega$ by zero belongs to $H^{1/2}(\partial\omega)$. Its dual space is denoted by $(H_{00}^{1/2}(\gamma_k))'$. To understand that the boundary condition on p holds in this dual space, let us mention that one can prove that, given any $\tilde{\phi}$ in $H^2(\omega) \cap H_0^1(\omega)$, $\partial_n\tilde{\phi}|_{\gamma_k}$ belongs to $H_{00}^{1/2}(\gamma_k)$. Then, the fact that $p|_{\gamma_k} = 0$ simply reflects a surjectivity property, which states that the mapping $\tilde{\phi} \mapsto \partial_n\tilde{\phi}|_{\gamma_k}$ is onto, from $H^2(\omega) \cap H_0^1(\omega)$ to $H_{00}^{1/2}(\gamma_k)$.

As the domain ω has only one re-entrant corner, we know $\dim(N) = 1$, and N=span$\{p_s\}$ for some $p_s \in N \setminus \{0\}$, see Grisvard [22].

Let ϕ_s be an element in $H_0^1(\omega)$, which solves the Poisson problem

$$-\Delta\phi_s = p_s \quad \text{in} \quad \omega\,. \tag{22}$$

Then by the decomposition (21), we can split the solution u_μ to equation (20) as

$$u_\mu = \tilde{u}_\mu + c_\mu \phi_s, \tag{23}$$

where $\tilde{u}_\mu \in H^2(\omega) \cap H_0^1(\omega)$, and is called the regular part of u_μ.

We will devote the rest of this Section to the derivation of some *a priori* estimates for the solution u_μ, its regular part $\tilde{u}_\mu$ and the singularity coefficient c_μ, as well as the solvability of $\tilde{u}_\mu$ and c_μ. Let us first introduce some notation.

Throughout the rest of the paper, α_0 will be a frequently used fixed positive constant lying in the interval $]\frac{1}{2}, \alpha[$, where $\alpha \in]\frac{1}{2}, 1[$ is the singularity exponent. $|\cdot|_s$ is used to denote the semi-norm of the Sobolev space $H^s(\omega)$ for any $s > 0$, $(\cdot,\cdot)$ and $\|\cdot\|_0$ are used to denote the inner product and the norm in the space $L^2(\omega)$. Also, $(\cdot,\cdot)$ will be used for the dual pairing between the space $H_0^1(\omega)$ and $H^{-1}(\omega)$ when necessary.

The following lemma summarizes some a priori estimates on u_μ and c_μ.

Lemma 4.1 *Let u_μ be the solution u_μ to the Poisson problem (20), then we have the following* a priori *estimates:*

$$\mu\,\|u_\mu\|_0 \le \|f\|_0\,, \quad \sqrt{\mu}\,|u_\mu|_1 \;\le\; \frac{1}{\sqrt{2}}\|f\|_0\,, \quad \|\Delta u_\mu\|_0 \le 2\,\|f\|_0\,, \tag{24}$$

$$|c_\mu| \;\lesssim\; \mu^{-\frac{1-\alpha}{2}}\,\|f\|_0 \tag{25}$$

$$|u_\mu|_{1+\alpha_0} \;\lesssim\; \mu^{-\frac{1-\alpha_0}{2}}\,\|f\|_0\,. \tag{26}$$

Proof. Multiplying equation (20) by u_μ and integrating over ω yield

$$|u_\mu|_1^2 + \mu\,\|u_\mu\|_0^2 \le \|f\|_0\,\|u_\mu\|_0\,,$$

which proves the first estimate in (24). Then applying the Cauchy-Schwarz inequality, we further obtain

$$|u_\mu|_1^2 + \mu\,\|u_\mu\|_0^2 \le \frac{1}{2}\mu\,\|u_\mu\|_0^2 + \frac{1}{2\mu}\|f\|_0^2,$$

which leads to the H^1 semi-norm estimate in (24).

The last estimate in (24) follows immediately from $\Delta u_\mu = \mu u_\mu - f$ and the first inequality in (24).

As far as (25) is concerned, it is a simple matter to check that the singularity coefficient c_μ, multiplied by some constant $\beta^\star$, equals the singularity coefficient $c(\mu)$ of [22, pp. 62-69]. Indeed, in Grisvard's papers, u_μ is decomposed into:

$$u_\mu = u_\mu^G + c(\mu)\mathrm{e}^{-\sqrt{\mu}\rho}\xi(\rho)\rho^\alpha \sin(\alpha\phi), \quad u_\mu^G \in H^2(\omega) \cap H_0^1(\omega) \tag{27}$$

where ξ is a smooth cut-off function, equal to one in a neighborhood of 0.

On the other hand one can decompose the singular part in (23) as (cf. [13] or (46) below)

$$c_\mu \phi_s = c_\mu \left(\tilde{\phi} + \beta^\star \rho^\alpha \sin(\alpha\phi)\right), \quad \tilde{\phi} \in H^2(\omega), \quad \beta^\star = \frac{1}{\pi}\|p_s\|_0^2.$$

Using this, (23) and (27), we can write

$$\begin{aligned} & (c_\mu \beta^\star - c(\mu)\xi(\rho))\rho^\alpha \sin(\alpha\phi) \\ = \; & u_\mu - (\tilde{u}_\mu + c_\mu \tilde{\phi}) - c(\mu)\xi(\rho)\rho^\alpha \sin(\alpha\phi) \\ = \; & u_\mu^G + c(\mu)\left(\mathrm{e}^{-\sqrt{\mu}\rho} - 1\right)\xi(\rho)\rho^\alpha \sin(\alpha\phi) - (\tilde{u}_\mu + c_\mu \tilde{\phi}). \end{aligned} \tag{28}$$

Noting that each term on the right-hand side of (28) belongs to $H^2(\omega)$, we must have $c_\mu = c(\mu)/\beta^\star$. But it is shown in [22, ineq. (2.5.5)] that

$$|c(\mu)| \lesssim \mu^{-\frac{1-\alpha}{2}} \|f\|_0, \tag{29}$$

which implies (25).

In order to derive the estimate (26), we shall use (27-30), with the additional norm estimate [22, ineq. (2.5.4)] on the regular part u_μ^G, namely

$$|u_\mu^G|_2 + \sqrt{\mu}|u_\mu^G|_1 + \mu\|u_\mu^G\|_0 \lesssim \|f\|_0. \tag{30}$$

Indeed, from the estimates

$$|u_\mu^G|_1 \lesssim \mu^{-1/2}\|f\|_0, \qquad |u_\mu^G|_2 \lesssim \|f\|_0\,,$$

we have then by standard interpolation theory that

$$|u_\mu^G|_{1+\alpha_0} \lesssim \mu^{-\frac{1-\alpha_0}{2}} \|f\|_0.$$

Next, we use (29) and a direct estimate of the $H^{1+\alpha_0}$ semi-norm to bound the singular part in (27). Actually, there holds

$$|v|_{1+\alpha_0}^2 = \int_{\vec{x}\in\omega}\int_{\vec{x}'\in\omega} \frac{|\nabla v(\vec{x}) - \nabla v(\vec{x}')|^2}{|\vec{x}-\vec{x}'|^{2+2\alpha_0}} d\omega(\vec{x})\, d\omega(\vec{x}'), \quad \forall\, v \in H^{1+\alpha_0}(\omega).$$

Due to the uniform smoothness (in μ) of $\mathrm{e}^{-\sqrt{\mu}\rho}\xi(\rho)\rho^\alpha \sin(\alpha\phi)$ for $\rho \geq \rho_0 > 0$, it is possible to evaluate the integrals only on $\omega_\infty = \{(\rho,\phi) \in]0,\rho_0[\times]0,\pi/\alpha[\}$. Then, one performs the changes of variables $s = \sqrt{\mu}\rho$, $s' = \sqrt{\mu}\rho'$, to find

$$|\mathrm{e}^{-\sqrt{\mu}\rho}\xi(\rho)\rho^\alpha \sin(\alpha\phi)|_{H^{1+\alpha_0}(\omega_\infty)} \leq C(\alpha_0)\mu^{-\frac{\alpha-\alpha_0}{2}}.$$

This with (25) leads to (26). ◇

Now, let us study the solvability of $\tilde{u}_\mu$ and c_μ in decomposition (23). For convenience, we introduce the notation $a_\mu(\cdot,\cdot)$ and the norm $\|\cdot\|_a$:

$$a_\mu(w,v) = (\nabla w, \nabla v) + \mu\,(w,v)\,, \quad \|v\|_a^2 = a_\mu(v,v)\,,$$

and the linear mapping A_μ from $H_0^1(\omega)$ to $H^{-1}(\omega)$, defined by $A_\mu u = -\Delta u + \mu u$, or equivalently by

$${}_{H^{-1}(\omega)}< A_\mu w, v >_{H_0^1(\omega)} = a_\mu(w,v) \quad \forall\, w,v \in H_0^1(\omega).$$

It is not difficult to verify that A_μ is a one-to-one and onto mapping, so it is invertible.

So, we claim that $\tilde{u}_\mu$ and c_μ solve the following coupled system:

$$a_\mu(\tilde{u}_\mu, v) + c_\mu\, a_\mu(\phi_s, v) = (f, v) \quad \forall v \in H_0^1(\omega)\,, \tag{31}$$

$$\left(\|p_s\|_0^2 + \mu|\phi_s|_1^2\right) c_\mu + \mu\,(\tilde{u}_\mu, p_s) = (f, p_s)\,. \tag{32}$$

In fact, by multiplying the equation (20) by p_s and integrating over ω we obtain

$$-(\Delta u_\mu, p_s) + \mu\,(u_\mu, p_s) = (f, p_s)\,,$$

then (32) follows readily from the decomposition (23), the orthogonality between p_s and $\Delta\tilde{u}_\mu$, along with the relation (22) and its following direct consequence

$$|\phi_s|_1^2 = (\phi_s, p_s). \tag{33}$$

On the other hand, the solution u_μ of (20) also satisfies the weak form:

$$(\nabla u_\mu, \nabla v) + \mu\,(u_\mu, v) = (f, v) \quad \forall v \in H_0^1(\omega).$$

This and the decomposition (23) lead to the equation (31).

Below, we show the well-posedness of the system (31)-(32).

Lemma 4.2 *There exists a unique solution $(\tilde{u}_\mu, c_\mu)$ to the coupled system (31)-(32) and the following stability estimates hold:*

$$\begin{aligned} \|\tilde{u}_\mu\|_a &\le \sqrt{2}\left(2\sqrt{\mu}C_P^2 + \frac{1}{\sqrt{\mu}}\right)\|f\|_0\,, \\ |c_\mu| &\le 2\,\frac{\|f\|_0}{\|p_s\|_0}\,, \qquad |\tilde{u}_\mu|_2 \le 4\,\|f\|_0\,, \end{aligned}$$

where C_P is the constant in the Poincaré inequality.

Proof. To see the unique existence, we rewrite (31) as the following operator form:

$$A_\mu\tilde{u}_\mu + c_\mu\, A_\mu\phi_s = f \quad \text{in} \quad H^{-1}(\omega). \tag{34}$$

As the inverse of A_μ exists, we know from (34) that $\tilde{u}_\mu$ can be determined if c_μ is available:

$$\tilde{u}_\mu = A_\mu^{-1}f - c_\mu\,\phi_s\,. \tag{35}$$

This is exactly our original decomposition (23). Substituting this into (32),

$$\left(\|p_s\|_0^2 + \mu|\phi_s|_1^2\right)c_\mu + \mu\,(A_\mu^{-1}f - c_\mu\,\phi_s, p_s) = (f, p_s)\,.$$

With (33), we obtain that

$$c_\mu = \frac{(f - \mu\,A_\mu^{-1}f,\, p_s)}{\|p_s\|_0^2}\,. \tag{36}$$

With c_μ uniquely determined, $\tilde{u}_\mu$ is clearly uniquely determined by (31) or (35).

Next, we derive the stability estimates in Lemma 4.2. We show that these estimates are the consequences of (35-36) and of the following inequality

$$\|A_\mu^{-1}\, g\|_0 \leq \frac{1}{\mu}\, \|g\|_0 \quad \forall\, g \in L^2(\omega). \tag{37}$$

In fact, if (37) is true, then the desired estimate on c_μ follows from (36):

$$|c_\mu| \leq \frac{\|f\| + \mu\, \|A_\mu^{-1} f\|_0}{\|p_s\|_0} \leq 2\frac{\|f\|_0}{\|p_s\|_0}\,.$$

On the other hand, we have from (33) and the Poincaré inequality that

$$\|\phi_s\|_0 \leq C_P\, \|\nabla\phi_s\|_0 \leq C_P^2 \|p_s\|_0\,.$$

Using this and the bound of c_μ, we derive from (31) by taking $v = \tilde{u}_\mu$ that

$$\begin{aligned}\|\nabla\tilde{u}_\mu\|_0^2 + \mu\|\tilde{u}_\mu\|_0^2 &\leq \|f\|_0\|\tilde{u}_\mu\|_0 + |c_\mu|\, (\|\nabla\phi_s\|_0\|\nabla\tilde{u}_\mu\|_0 + \mu\, \|\phi_s\|_0\|\tilde{u}_\mu\|_0) \\ &\leq \|f\|_0\|\tilde{u}_\mu\|_0 + 2C_P\|f\|_0\|\nabla\tilde{u}_\mu\|_0 + 2\,\mu\, C_P^2\|f\|_0\|\tilde{u}_\mu\|_0\,.\end{aligned}$$

Then, an application of the Young inequality yields

$$\|\nabla\tilde{u}_\mu\|_0^2 + \mu\|\tilde{u}_\mu\|_0^2 \leq \frac{1}{2}\mu\, \|\tilde{u}_\mu\|_0^2 + \frac{1}{\mu}\|f\|_0^2 + \frac{1}{2}\|\nabla\tilde{u}_\mu\|_0^2 + 2C_P^2\|f\|_0^2 + 4\mu\, C_P^4\|f\|_0^2\,.$$

This implies

$$\frac{1}{2}\|\tilde{u}_\mu\|_a^2 \leq \left(\frac{1}{\mu} + 2C_P^2 + 4\mu C_P^4\right)\|f\|_0^2 \leq \left(\frac{1}{\sqrt{\mu}} + 2\sqrt{\mu}C_P^2\right)^2 \|f\|_0^2\,,$$

so the desired estimate on $\|\tilde{u}_\mu\|_a$ follows.

We now show the H^2-norm estimate. By the decomposition (35), we have $u_\mu = A_\mu^{-1} f = \tilde{u}_\mu + c_\mu\phi_s$, and

$$-\Delta\tilde{u}_\mu = -\Delta u_\mu + c_\mu\Delta\phi_s = f - \mu\, u_\mu - c_\mu p_s,$$

which gives

$$\|\Delta\tilde{u}_\mu\|_0 \leq \|f\|_0 + \mu\|u_\mu\|_0 + |c_\mu|\, \|p_s\|_0.$$

But we know from Lemma 4.1 that $\mu\|u_\mu\|_0 \leq \|f\|_0$. This, along with the previous bound for c_μ, leads to

$$\|\Delta\tilde{u}_\mu\|_0 \leq 4\|f\|_0.$$

Now, for any $\vec{v} \in H^1(\omega)^2$ such that $\vec{v}\cdot\vec{\tau} = 0$ on $\partial\omega$, with $\vec{\tau}$ the vector tangential to $\partial\omega$, it is well-known (cf. [17]) that (since ω is a polygon)

$$\sum_{1\leq k,l\leq 2} \|\partial_k v_l\|_0^2 = \|\mathrm{curl}\vec{v}\|_0^2 + \|\mathrm{div}\vec{v}\|_0^2.$$

So, by taking $\vec{v} = \nabla \tilde{u}_\mu$, one actually finds

$$|\tilde{u}_\mu|_2 = \|\Delta \tilde{u}_\mu\|_0 \le 4\|f\|_0.$$

Finally, it remains to prove (37). By the definition of $a_\mu(\cdot,\cdot)$, we easily see the following lower bound:

$$_{H^{-1}(\omega)} < A_\mu v, v >_{H_0^1(\omega)} = a_\mu(v,v) \ge \mu \, \|v\|_0^2 \quad \forall\, v \in H_0^1(\omega)\,. \tag{38}$$

Then for any $g \in L^2(\omega) \subset H^{-1}(\omega)$, let $v = A_\mu^{-1}\, g \in H_0^1(\omega)$. One has $A_\mu\, v = g$ in $L^2(\omega)$ and it follows from (38) that

$$\|A_\mu^{-1}\, g\|_0^2 = \|v\|_0^2 \le \frac{1}{\mu}\,(A_\mu\, v, v) = \frac{1}{\mu}\,(g, A_\mu^{-1}\, g) = \frac{1}{\mu}\|g\|_0\, \|A_\mu^{-1}\, g\|_0,$$

which proves (37). ◇

We end this Section with a number of important remarks on the theoretical and practical range of the splitting into regular and singular parts.

Remark 4.1 *Equation (20) is also useful when the* 2D heat equation *is considered in* ω*:*

$$\frac{\partial u}{\partial t} - \Delta u = f \;\; in \;\omega \times]0,T[,$$

with initial condition and (homogeneous) Dirichlet boundary condition. As a matter of fact, assume it is first discretized in time, with a time-step δt, at times $t_m = m\delta t$, $m = 0, 1, \cdots$: let $u^m = u(t_m)$. Then one has to solve in space the implicit problems (with $\theta \in]0,1]$ given)
Find $u^{m+1} \in H_0^1(\omega)$ such that

$$-\Delta u^{m+1} + \frac{1}{\theta\delta t} u^{m+1} = f_{(t_{m+1})} + \frac{1-\theta}{\theta} f_{(t_m)} + \frac{1}{\theta\delta t} u^m + \frac{1-\theta}{\theta}\Delta u^m, \;\; in \;\omega.$$

Above, $\theta = 1$ (resp. $\theta = 1/2$) corresponds to the implicit Euler (resp. Crank-Nicolson) scheme. This is precisely (20) with $\mu = 1/\theta\delta t$.
Clearly, implicit schemes for the 2D wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f \;\; in \;\omega \times]0,T[,$$

lead to other instances of Equation (20).

Remark 4.2 *Both ϕ_s and p_s in (22) are chosen independent of μ, f and u_μ, so their norms will be regarded as some generic constants (i.e., independent of μ, f and u_μ.)*

Remark 4.3 *Instead of the decomposition (23), it seems more natural [21, 22] to take the decomposition $u_\mu = \tilde{u}'_\mu + c_\mu \phi_\mu$, where $\phi_\mu \in H_0^1(\omega)$ depends on the parameter μ, and it is the solution to the problem: $-\Delta\phi_\mu + \mu\, \phi_\mu = p_\mu$ in ω, with $p_\mu \in N_\mu \setminus \{0\}$, where N_μ is given by*

$$N_\mu = \Big\{ p \in L^2(\omega) \;:\; (-\Delta + \mu \mathrm{I})\, p = 0, \;\; p|_{\gamma_k} = 0 \;\; in \; (H_{00}^{1/2}(\gamma_k))', \;\; 1 \le k \le K \Big\}.$$

But the decomposition (23) has an important advantage: the singular part ϕ_s is independent of the parameter μ. As we shall see, this will be much less expensive than using the above more natural decomposition.

5 Discrete formulation in the 2D domain ω: the SCM

In this Section we shall formulate the generalized SCM for solving the coupled system (31)-(32) and derive the error estimates of the approximate solutions. The SCM was first introduced by Assous *et al* [8] for solving the 2D static or unsteady Maxwell equations without charges, and then used in [13] for the 2D Poisson problem. As we will see, the formulation of the SCM for the 2D Poisson-like problem (18) is quite different here due to the involvement of the parameter μ.

Let $\mathcal{T}_h$ be a regular triangulation of the domain ω, with vertices $\{M_j\}_{j=1}^{N_i+N_b}$ and the last N_b vertices lying on the boundary $\partial\omega$. We define V^h to be the continuous piecewise linear finite element space on $\mathcal{T}_h$ with the standard basis functions $\{\psi_j\}_{j=1}^{N_i+N_b}$ (cf. [16]). We further define V_0^h to be the subspace of V^h with all functions vanishing on the boundary of ω. The interpolation associated with the space V^h will be denoted by Π_h.

5.1 Approximation of the singular function p_s

We start with the finite element approximation of the singular function $p_s \in N$ in (22). Recall the splitting (see [13])

$$p_s = \tilde{p} + p_{_P}, \quad \tilde{p} \in H^1(\omega), \quad p_{_P} = \rho^{-\alpha}\sin(\alpha\phi)\,.$$

As p_s is harmonic in ω, one can directly verify that the regular part $\tilde{p}$ in the splitting solves the problem:
Find $\tilde{p} \in H^1(\omega)$ *such that* $\tilde{p} = s$ on $\partial\omega$ and

$$(\nabla\tilde{p}, \nabla v) = 0 \quad \forall\, v \in H_0^1(\omega) \tag{39}$$

where the boundary function s is given by

$$s = 0 \quad \text{on} \quad \gamma_1 \cup \gamma_2; \quad s = -p_{_P} \quad \text{on} \quad \gamma_k \ (3 \le k \le K).$$

For the finite element approximation of the problem (39), we shall use the simple treatment of the boundary condition:

$$\pi_h\, s = \sum_{j=N_i+1}^{N_i+N_b} s(M_j)\psi_j\,. \tag{40}$$

Then we approximate p_s by $p_s^h = \tilde{p}_h + p_{_P}$, where $\tilde{p}_h$ is the piecewise linear finite element solution to the problem (39). Namely, $\tilde{p}_h = \pi_h s + p_h^0$ where $p_h^0 \in V_h^0$ solves

$$(\nabla\tilde{p}_h, \nabla v_h) = 0 \quad \forall\, v_h \in V_0^h\,. \tag{41}$$

The error estimates for the singular function p_s and its finite element approximation p_s^h are summarized in the following lemma.

Lemma 5.1 *We have* [6]

$$|p_s - p_s^h|_1 \lesssim h^{\alpha_0}\,, \quad \|p_s - p_s^h\|_0 \lesssim h^{2\alpha_0}\,.$$

Proof. We introduce a smooth continuation of s into ω:

$$\tilde{s} = -p_{_P}(1-\xi(\rho))\,.$$

Clearly, $\tilde{s} = s$ on $\partial\omega$ and $\tilde{s} \in H^2(\omega)$. Let $p^0 = \tilde{p} - \tilde{s}$. It is known that $\tilde{p} \in H^{1+\alpha_0}(\omega)$, so we have $p^0 \in H^{1+\alpha_0}(\omega) \cap H_0^1(\omega)$. It follows from (39) that

$$(\nabla p^0, \nabla v) = -(\nabla \tilde{s}, \nabla v) \quad \forall\, v \in H_0^1(\omega)\,. \tag{42}$$

Recall Π_h is the interpolant associated with V^h, thus we can rewrite the finite element solution $\tilde{p}_h$ to the system (41) as $\tilde{p}_h = \Pi_h \tilde{s} + p_h^0$ with $p_h^0 \in V_0^h$ now solving

$$(\nabla p_h^0, \nabla v_h) = -(\nabla \Pi_h \tilde{s}, \nabla v_h) \quad \forall\, v_h \in V_0^h\,, \tag{43}$$

by noting $\Pi_h \tilde{s} = \pi_h\, s$ on $\partial\omega$.

Now we are ready to derive the error estimates. It is clear from (42) and (43) that

$$(\nabla(p^0 - p_h^0), \nabla v_h) = (\nabla(\Pi_h \tilde{s} - \tilde{s}), \nabla v_h) \quad \forall\, v_h \in V_0^h\,. \tag{44}$$

Using this, we obtain for any $q_h \in V_0^h$ that

$$\|\nabla(p^0 - q_h)\|^2 \geq \|\nabla(p^0 - p_h^0)\|^2 + 2(\nabla(\Pi_h \tilde{s} - \tilde{s}), \nabla(p_h^0 - q_h)),$$

taking $q_h = \Pi_h p^0$ above and using the Young inequality leads to

$$\begin{aligned} |p^0 - p_h^0|_1^2 &\leq |p^0 - \Pi_h p^0|_1^2 + 2\,|\Pi_h \tilde{s} - \tilde{s}|_1\,(|p_h^0 - p^0|_1 + |p^0 - \Pi_h p^0|_1) \\ &\leq 2\,|p^0 - \Pi_h p^0|_1^2 + \frac{1}{2}|p_h^0 - p^0|_1^2 + 3\,|\Pi_h \tilde{s} - \tilde{s}|_1^2\,. \end{aligned}$$

Then by the standard interpolation results we obtain

$$|p^0 - p_h^0|_1^2 \leq 4\,|p^0 - \Pi_h p^0|_1^2 + 6\,|\Pi_h \tilde{s} - \tilde{s}|_1^2 \lesssim h^{2\alpha_0}|p^0|_{1+\alpha_0}^2 + h^2\,|\tilde{s}|_2^2.$$

This leads to the desired H^1-norm error estimate:

$$\begin{aligned} |p_s - p_s^h|_1 &= |\tilde{p} - \tilde{p}_h|_1 = |p^0 + \tilde{s} - p_h^0 - \Pi_h \tilde{s}|_1 \\ &\leq |p^0 - p_h^0|_1 + |\tilde{s} - \Pi_h \tilde{s}|_1 \lesssim h^{\alpha_0} + h|\tilde{s}|_2 \lesssim h^{\alpha_0}. \end{aligned}$$

Finally, we apply the Nitsche trick to derive the L^2-norm error estimate. Let $w \in H_0^1(\omega)$ be the solution to the variational problem

$$(\nabla w, \nabla v) = (p^0 - p_h^0, v) \quad \forall\, v \in H_0^1(\omega). \tag{45}$$

[6] By construction, neither p_s nor p_s^h belong to $H^1(\omega)$, due to the presence of $p_{_P}$, but the following holds:

$$p_s - p_s^h = \tilde{p} - \tilde{p}_h \in H^1(\omega).$$

By the elliptic theory, we know $w \in H^{1+\alpha_0}(\omega)$ and

$$|w|_{1+\alpha_0} \lesssim \|p^0 - p_h^0\|_0.$$

Let w_h be the finite element approximation of w: $w_h \in V_0^h$ solves

$$(\nabla w_h, \nabla v_h) = (p^0 - p_h^0, v_h) \quad \forall\, v_h \in V_0^h.$$

Taking $v_h = w_h$ above and using the Poincaré inequality, we know

$$|w_h|_1 \lesssim \|p^0 - p_h^0\|_0.$$

Also, by the standard error estimate, we have

$$|w - w_h|_1 \lesssim h^{\alpha_0} |w|_{1+\alpha_0} \lesssim h^{\alpha_0} \|p^0 - p_h^0\|_0.$$

Now, taking $v = p^0 - p_h^0$ in (45) and using (44) and the duality argument, we obtain

$$\begin{aligned}
\|p^0 - p_h^0\|_0^2 &= (\nabla w, \nabla (p^0 - p_h^0)) \\
&= (\nabla (w - w_h), \nabla (p^0 - p_h^0)) + (\nabla w_h, \nabla (p^0 - p_h^0)) \\
&= (\nabla (w - w_h), \nabla (p^0 - p_h^0)) + (\nabla (\Pi_h \tilde{s} - \tilde{s}), \nabla (w_h - w)) \\
&\quad + (\nabla (\Pi_h \tilde{s} - \tilde{s}), \nabla w) \\
&\leq |w - w_h|_1 \, |p^0 - p_h^0|_1 + |\Pi_h \tilde{s} - \tilde{s}|_1 \, |w_h - w|_1 + |\Pi_h \tilde{s} - \tilde{s}|_{1-\alpha_0} |w|_{1+\alpha_0} \\
&\lesssim h^{2\alpha_0} \|p^0 - p_h^0\|_0 + h^{1+\alpha_0} |\tilde{s}|_2 \|p^0 - p_h^0\|_0 \,,
\end{aligned}$$

which leads to the desired L^2-norm error estimate:

$$\|p_s - p_s^h\|_0 \leq \|p^0 - p_h^0\|_0 + \|\tilde{s} - \Pi_h \tilde{s}\|_0 \lesssim h^{2\alpha_0} + h^2 |\tilde{s}|_2 \lesssim h^{2\alpha_0}. \qquad \diamond$$

Remark 5.1 *Following the proof given in [1], one can improve the results of the previous Lemma. Indeed, one can derive the estimates* $|p_s - p_s^h|_1 \lesssim h^{\alpha}$ *and* $\|p_s - p_s^h\|_0 \lesssim h^{2\alpha}$*, with slightly more restrictive assumptions on the mesh.*

5.2 Approximation of the singular part ϕ_s

In order to approximate the singular part ϕ_s in the decomposition $u_\mu = \tilde{u}_\mu + c_\mu \, \phi_s$, we recall (cf. [13]) that $\phi_s \in H_0^1(\omega)$ solves the elliptic problem (22) and has the following decomposition:

$$\phi_s = \tilde{\phi} + \beta^\star \phi_P, \quad \tilde{\phi} \in H^2(\omega), \quad \beta^\star = \frac{1}{\pi} \|p_s\|_0^2, \quad \phi_P = \rho^\alpha \sin(\alpha \phi)\,. \tag{46}$$

Using (22), we see that $\tilde{\phi}$, satisfying $\tilde{\phi} = -\beta^\star \phi_P$ on $\partial\omega$, solves the variational problem:

$$(\nabla \tilde{\phi}, \nabla v) = (p_s, v) \quad \forall\, v \in H_0^1(\omega). \tag{47}$$

The next step is to consider the finite element approximation of $\tilde{\phi}$ in V^h:

$$\tilde{\phi}_h = -\beta_h^\star \pi_h \phi_P + \phi_h^0,$$

where π_h is defined as in (40), $\beta_h^\star$ is computed using $\beta_h^\star = \dfrac{1}{\pi}\displaystyle\int_\omega (p_s^h)^2 d\omega$, and $\phi_h^0 \in V_0^h$ is the solution to the problem:

$$(\nabla\tilde{\phi}_h, \nabla v_h) = (p_s^h, v_h) \quad \forall\, v_h \in V_0^h. \tag{48}$$

Then we propose to compute the finite element approximation of ϕ_s by

$$\phi_s^h = \tilde{\phi}_h + \beta_h^\star \phi_P\,.$$

Below, we derive the error estimates for this approximation.

Lemma 5.2 *The following error estimates hold*

$$|\phi_s - \phi_s^h|_1 \lesssim h\,, \quad \|\phi_s - \phi_s^h\|_a \lesssim \sqrt{\mu}\, h\,.$$

Proof. We first estimate the error $\tilde{\phi} - \tilde{\phi}_h$. Subtracting (48) from (47) yields

$$(\nabla(\tilde{\phi} - \tilde{\phi}_h), \nabla v_h) = (p_s - p_s^h, v_h) \quad \forall\, v_h \in V_0^h\,,$$

thus we obtain for any $w_h \in V^h$ satisfying $w_h - \tilde{\phi}_h \in V_0^h$,

$$|\tilde{\phi} - w_h|_1^2 = |\tilde{\phi} - \tilde{\phi}_h|_1^2 + |\tilde{\phi}_h - w_h|_1^2 + 2(p_s - p_s^h, \tilde{\phi}_h - w_h),$$

which with the Young inequality and the Poincaré inequality gives

$$\begin{aligned}
|\tilde{\phi} - \tilde{\phi}_h|_1^2 &\le |\tilde{\phi} - w_h|_1^2 + 2C_P\|p_s - p_s^h\|_0(|\tilde{\phi} - \tilde{\phi}_h|_1 + |\tilde{\phi} - w_h|_1) \\
&\lesssim 2\,|\tilde{\phi} - w_h|_1^2 + \frac{1}{2}|\tilde{\phi} - \tilde{\phi}_h|_1^2 + \|p_s - p_s^h\|_0^2.
\end{aligned} \tag{49}$$

Noting that $\tilde{\phi} = -\beta^\star \phi_P$ on $\partial\omega$, so $\beta_h^\star \Pi_h \tilde{\phi} = \beta^\star \tilde{\phi}_h$ on $\partial\omega$. Let $w_h = \beta_h^\star \Pi_h \tilde{\phi}/\beta^\star$, then $w_h - \tilde{\phi}_h \in V_0^h$. With this w_h, we derive from (49) and Lemma 5.1 that

$$\begin{aligned}
|\tilde{\phi} - \tilde{\phi}_h|_1^2 &\lesssim h^2 + (\beta^\star)^{-2}|\beta^\star\tilde{\phi} - \beta_h^\star \Pi_h \tilde{\phi}|_1^2 \\
&\lesssim h^2 + |\beta^\star - \beta_h^\star|^2\, |\tilde{\phi}|_1^2 + |\beta_h^\star|^2\, |\tilde{\phi} - \Pi_h\tilde{\phi}|_1^2.
\end{aligned} \tag{50}$$

But using the definitions of $\beta^\star$ and $\beta_h^\star$, we have

$$|\beta^\star - \beta_h^\star| = \frac{1}{\pi}\Big|\|p_s\|_0^2 - \|p_s^h\|_0^2\Big| \lesssim \|p_s - p_s^h\|_0 \lesssim h^{2\alpha_0}\,. \tag{51}$$

It follows from (50) and the property $\tilde{\phi} \in H^2(\omega)$ that

$$|\tilde{\phi} - \tilde{\phi}_h|_1 \lesssim h\,.$$

This with (51) and the decompositions of ϕ_s and ϕ_s^h gives the desired H^1-norm estimate:

$$|\phi_s - \phi_s^h|_1 \;\le\; |\tilde{\phi} - \tilde{\phi}_h|_1 + |\beta^\star - \beta_h^\star|\,|\phi_P|_1 \lesssim h\,.$$

Finally, by noting that both ϕ_s and ϕ_s^h vanish on γ_1 and γ_2, we can apply the Poincaré inequality to the function $\phi_s - \phi_s^h$ to get

$$\|\phi_s - \phi_s^h\|_0 \le C_P'|\phi_s - \phi_s^h|_1\,.$$

Then the desired estimate on $\|\phi_s - \phi_s^h\|_a$ follows from

$$\|\phi_s - \phi_s^h\|_a^2 = |\phi_s - \phi_s^h|_1^2 + \mu\,\|\phi_s - \phi_s^h\|_0^2 \lesssim h^2 + \mu\, h^2. \qquad \diamond$$

5.3 Approximation of $\tilde{u}_\mu$ and c_μ in decomposition (23)

Noting that $\tilde{u}_\mu$ and c_μ solve the coupled system (31) and (32), it is natural to formulate their finite element approximations as follows:
Find $\tilde{u}_\mu^h \in V_0^h$ *and* $c_\mu^h \in \mathbb{R}^1$ *such that*

$$a_\mu(\tilde{u}_\mu^h, v) + c_\mu^h\, a_\mu(\phi_s^h, v) = (f, v) \quad \forall v \in V_0^h, \tag{52}$$

$$\Big(\|p_s^h\|_0^2 + \mu|\phi_s^h|_1^2\Big) c_\mu^h + \mu\,(\tilde{u}_\mu^h, p_s^h) = (f, p_s^h)\,, \tag{53}$$

where ϕ_s^h and p_s^h are the finite element approximations of ϕ_s and p_s, see Subsect. 5.1-5.2.

However, this formulation requires solving a coupled system, and it poses some difficulty in getting the error estimates as it does not fall into any existing saddle-point-like framework. Instead, we are going to propose a more efficient approximation which enables us to find $\tilde{u}_\mu^h \in V_0^h$ and c_μ^h separately. In fact, we can use the formula (36) to first find c_μ^h, and then use (52) to find $\tilde{u}_\mu^h \in V_0^h$. This leads to the following algorithm to find $\tilde{u}_\mu^h \in V_0^h$ and c_μ^h. Let $C^\star > 0$ be a fixed constant.

SCM Algorithm for finding $\tilde{u}_\mu^h \in V_0^h$ and $c_\mu^h \in \mathbb{R}^1$.

Step 1. *Find* $z_\mu^h \in V_0^h$ *such that*

$$a_\mu(z_\mu^h, v) = (f, v) \quad \forall\, v \in V_0^h\,. \tag{54}$$

Compute c_μ^h as follows:

$$c_\mu^h = \frac{(f - \mu\, z_\mu^h,\, p_s^h)}{\|p_s^h\|_0^2} \quad \text{if} \quad \sqrt{\mu} < C^\star\, h^{-\frac{1}{2-\alpha_0}}\,; \tag{55}$$

and

$$c_\mu^h = 0 \quad \text{if} \quad \sqrt{\mu} \geq C^\star\, h^{-\frac{1}{2-\alpha_0}}\,. \tag{56}$$

Step 2. *Find* $\tilde{u}_\mu^h \in V_0^h$ *such that*

$$a_\mu(\tilde{u}_\mu^h, v) + c_\mu^h\, a_\mu(\phi_s^h, v) = (f, v) \quad \forall v \in V_0^h\,. \tag{57}$$

Remark 5.2 *In practice (see [15]), the conditions (55-56) mean that only a few coefficients* $(c_\mu^h)_\mu$ *need to be computed, with respect to the total number of Fourier modes.*

Below, we shall derive the error estimates on $(c_\mu - c_\mu^h)$ and $(\tilde{u}_\mu - \tilde{u}_\mu^h)$. Recall the formula (36) for c_μ:

$$c_\mu = \frac{(f - \mu\, z_\mu,\, p_s)}{\|p_s\|_0^2}\,, \tag{58}$$

where $z_\mu = A_\mu^{-1} f \in H_0^1(\omega)$ solves

$$a_\mu(z_\mu, v) = (f, v) \quad \forall\, v \in H_0^1(\omega)\,. \tag{59}$$

Clearly $z_\mu = u_\mu$, the solution to the equation (20). But a different notation z_μ is used here for convenience, since the numerical approximation z_μ^h is derived with the *standard* piecewise linear FEM.

Lemma 5.3 *For the solution z_μ to the problem (59) and its piecewise linear finite element approximation z_μ^h in (54), we have the following error estimates*

$$\|z_\mu - z_\mu^h\|_0 \;\leq\; \mu^{-1}\|f\|_0\,, \tag{60}$$
$$\|z_\mu - z_\mu^h\|_0 \;\lesssim\; h^{2\alpha_0}\mu^{\alpha_0-1}(1+\sqrt{\mu}h)^2\|f\|_0\,, \tag{61}$$

while for the coefficient c_μ in (58) and its approximation c_μ^h in (55), we have

$$|c_\mu - c_\mu^h| \lesssim (h^{2\alpha_0}\mu^{\alpha_0}(1+\sqrt{\mu}h)^2 + h)\,\|f\|_0\,. \tag{62}$$

Proof. It follows from (59) and (54) that

$$a_\mu(z_\mu - z_\mu^h, z_\mu - z_\mu^h) = a_\mu(z_\mu, z_\mu - z_\mu^h) = (f, z_\mu - z_\mu^h).$$

This implies

$$|z_\mu - z_\mu^h|_1^2 + \mu\,\|z_\mu - z_\mu^h\|_0^2 \leq \|f\|_0\,\|z_\mu - z_\mu^h\|_0,$$

thus (60) follows by the Young inequality.

We next show (61). Again it follows from (54) and (59) that

$$\|z_\mu - z_\mu^h\|_a \leq \|z_\mu - v_h\|_a \quad \forall\, v_h \in V_0^h.$$

But, by standard interpolation theory, we know that

$$|z_\mu - \Pi_h z_\mu|_1 \lesssim h^{\alpha_0}|z_\mu|_{1+\alpha_0}, \text{ and } \|z_\mu - \Pi_h z_\mu\|_0 \lesssim h^{1+\alpha_0}|z_\mu|_{1+\alpha_0}.$$

Therefore, we reach

$$\|z_\mu - z_\mu^h\|_a \lesssim (1+\sqrt{\mu}\,h)h^{\alpha_0}|z_\mu|_{1+\alpha_0}\,. \tag{63}$$

On the other hand, for any $g \in L^2(\omega)$, define $w \in H_0^1(\omega)$ such that

$$a_\mu(w, v) = (g, v) \quad \forall\, v \in H_0^1(\omega). \tag{64}$$

Using the duality and (64), we have

$$\begin{aligned}
\|z_\mu - z_\mu^h\|_0 &= \sup_{g\in L^2(\omega)} \frac{(z_\mu - z_\mu^h, g)}{\|g\|_0} = \sup_{g\in L^2(\omega)} \frac{a_\mu(w, z_\mu - z_\mu^h)}{\|g\|_0} \\
&= \sup_{g\in L^2(\omega)} \frac{a_\mu(w - \Pi_h w, z_\mu - z_\mu^h)}{\|g\|_0} \leq \sup_{g\in L^2(\omega)} \frac{\|w - \Pi_h w\|_a\,\|z_\mu - z_\mu^h\|_a}{\|g\|_0}.
\end{aligned}$$

Using the interpolation result and the same derivation as in (63) and the *a priori* estimate (26) (with u and f replaced by w and g), we obtain

$$\begin{aligned}
\|z_\mu - z_\mu^h\|_0 &\lesssim \sup_{g\in L^2(\omega)} \frac{h^{2\alpha_0}(1+\sqrt{\mu}h)^2|w|_{1+\alpha_0}|z_\mu|_{1+\alpha_0}}{\|g\|_0} \\
&\lesssim h^{2\alpha_0}\mu^{\alpha_0-1}(1+\sqrt{\mu}h)^2\|f\|_0\,,
\end{aligned}$$

which proves (61).

It remains to prove (62). We have from (55) and (58) that

$$
\begin{aligned}
c_\mu - c_\mu^h &= \frac{(f-\mu\, z_\mu,\, p_s)}{\|p_s\|_0^2} - \frac{(f-\mu\, z_\mu^h,\, p_s^h)}{\|p_s^h\|_0^2} \\
&= \Big\{\frac{(f,\, p_s)}{\|p_s\|_0^2} - \frac{(f,\, p_s^h)}{\|p_s^h\|_0^2}\Big\} + \mu\Big\{\frac{(z_\mu^h,\, p_s^h)}{\|p_s^h\|_0^2} - \frac{(z_\mu,\, p_s)}{\|p_s\|_0^2}\Big\} := I_1 + I_2\,.
\end{aligned}
$$

For I_1, we have from Lemma 5.1 that

$$|I_1| \lesssim h\,\|f\|_0\,.$$

For I_2, we further write it as follows

$$I_2 = \mu\,\frac{(z_\mu^h - z_\mu,\, p_s^h)}{\|p_s^h\|_0^2} + \mu\,\frac{(z_\mu,\, p_s^h - p_s)}{\|p_s^h\|_0^2} + \mu\,(z_\mu,\, p_s)\Big\{\frac{1}{\|p_s^h\|_0^2} - \frac{1}{\|p_s\|_0^2}\Big\}\,.$$

Then using estimate (61) and Lemma 5.1, we can derive

$$|I_2| \lesssim \mu\,\|z_\mu - z_\mu^h\|_0 + h\,\|f\|_0 \lesssim (h^{2\alpha_0}\mu^{\alpha_0}(1+\sqrt{\mu}h)^2 + h)\,\|f\|_0\,.$$

This with the estimate of I_1 gives (62). $\diamond$

In the rest of this Section, we shall estimate the error between the solution u_μ to the elliptic problem (20) and its SCM approximation u_μ^h. We note that the decomposition of u_μ is equal to:

$$u_\mu = \tilde{u}_\mu + c_\mu\,\phi_s = \tilde{u}_\mu + c_\mu\,(\tilde{\phi} + \beta^\star\phi_P)\,. \tag{65}$$

So, we propose its SCM approximation u_μ^h of the form:

$$u_\mu^h = \tilde{u}_\mu^h + c_\mu^h\,\phi_s^h = \tilde{u}_\mu^h + c_\mu^h\,(\tilde{\phi}_h + \beta_h^\star\phi_P). \tag{66}$$

We shall derive the error estimate on $u_\mu - u_\mu^h$. Let us start with the estimate of $(\tilde{u}_\mu - \tilde{u}_\mu^h)$. We have

Lemma 5.4 *The following error estimate holds*

$$\|\tilde{u}_\mu - \tilde{u}_\mu^h\|_a^2 \lesssim \sqrt{\mu}\,(h^2\,\|f\|_0^2 + |c_\mu - c_\mu^h|^2)\,.$$

Proof. Subtracting (52) from (31) we have

$$a_\mu(\tilde{u}_\mu - \tilde{u}_\mu^h, v_h) + c_\mu a_\mu(\phi_s, v_h) - c_\mu^h a_\mu(\phi_s^h, v_h) = 0 \quad \forall\, v_h \in V_0^h.$$

Using this we obtain for any $w_h \in V_0^h$,

$$\|\tilde{u}_\mu - w_h\|_a^2 = \|\tilde{u}_\mu - \tilde{u}_\mu^h\|_a^2 + \|\tilde{u}_\mu^h - w_h\|_a^2 + 2c_\mu^h a_\mu(\phi_s^h, \tilde{u}_\mu^h - w_h) - 2c_\mu a_\mu(\phi_s, \tilde{u}_\mu^h - w_h),$$

which implies

$$
\begin{aligned}
&\|\tilde{u}_\mu - \tilde{u}_\mu^h\|_a^2 \\
\le\;& \|\tilde{u}_\mu - w_h\|_a^2 + 2c_\mu\,a_\mu(\phi_s - \phi_s^h, \tilde{u}_\mu^h - w_h) + 2(c_\mu - c_\mu^h)a_\mu(\phi_s^h, \tilde{u}_\mu^h - w_h) \qquad (67)\\
\le\;& \|\tilde{u}_\mu - w_h\|_a^2 + 2\,|c_\mu|\,\|\phi_s - \phi_s^h\|_a\,\|\tilde{u}_\mu^h - w_h\|_a + 2\,|c_\mu - c_\mu^h|\,\|\phi_s^h\|_a\,\|\tilde{u}_\mu^h - w_h\|_a\,.
\end{aligned}
$$

Now, there holds $\|\phi_s\|_a - \|\phi_s - \phi_s^h\|_a \le \|\phi_s^h\|_a \le \|\phi_s\|_a + \|\phi_s - \phi_s^h\|_a$. Using Lemma 5.2 and $\|\phi_s\|_a^2 = |\phi_s|_1^2 + \mu\,\|\phi_s\|_0^2$, we find $\|\phi_s^h\|_a \approx \sqrt{\mu}\,\|\phi_s\|_0$. Using the interpolation results, we obtain

$$\|\tilde{u}_\mu - \Pi_h \tilde{u}_\mu\|_a^2 \le |\tilde{u}_\mu - \Pi_h \tilde{u}_\mu|_1^2 + \mu\,\|\tilde{u}_\mu - \Pi_h \tilde{u}_\mu\|_0^2 \lesssim h^2\,|\tilde{u}_\mu|_2^2,$$

thus letting $w_h = \Pi_h \tilde{u}_\mu$ in (67) and using Lemma 4.2, we derive

$$\begin{aligned}\|\tilde{u}_\mu - \tilde{u}_\mu^h\|_a^2 &\lesssim h^2\,|\tilde{u}_\mu|_2^2 + \sqrt{\mu} h^2 \|f\|_0\,|\tilde{u}_\mu|_2 + \sqrt{\mu}\,h\,|c_\mu - c_\mu^h|\,|\tilde{u}_\mu|_2 \\ &\lesssim \sqrt{\mu}\,(h^2\,\|f\|_0^2 + |c_\mu - c_\mu^h|^2)\,. \qquad \diamond\end{aligned}$$

Theorem 5.1 *Let u_μ be the solution to the equation (20) and u_μ^h be its finite element approximation given in (66). Then the following error estimate holds:*

$$\exists C > 0 \text{ such that } \forall \mu,\ \|u_\mu - u_\mu^h\|_a \le C\,\mu\,h\,\|f\|_0.$$

Proof. It follows from (65) and (66) that

$$u_\mu - u_\mu^h = (\tilde{u}_\mu - \tilde{u}_\mu^h) + c_\mu(\phi_s - \phi_s^h) + \phi_s^h(c_\mu - c_\mu^h).$$

Then we obtain, using Lemmas 5.4, 5.2 and 4.2, that

$$\begin{aligned}\|u_\mu - u_\mu^h\|_a^2 &\le 3\Big\{\|\tilde{u}_\mu - \tilde{u}_\mu^h\|_a^2 + |c_\mu|^2\,\|\phi_s - \phi_s^h\|_a^2 + \|\phi_s^h\|_a^2 |c_\mu - c_\mu^h|^2\Big\} \\ &\lesssim \mu\,h^2\,\|f\|_0^2 + \mu\,|c_\mu - c_\mu^h|^2\,.\end{aligned}$$

To prove the desired estimate, we need simply

$$|c_\mu - c_\mu^h|^2 \lesssim \mu\,h^2\,\|f\|_0^2\,. \tag{68}$$

First consider the case (56), i.e., $\sqrt{\mu} \ge C^\star\,h^{-\frac{1}{2-\alpha_0}}$. This condition is equivalent to

$$h^{-2}\mu^{\alpha_0 - 2} \lesssim 1.$$

Then (68) comes directly from this condition, $c_\mu^h = 0$ and (25) as follows:

$$|c_\mu - c_\mu^h|^2 = c_\mu^2 \lesssim \mu^{\alpha_0 - 1}\|f\|_0^2 \lesssim \mu\,h^2\,(h^{-2}\mu^{\alpha_0-2})\|f\|_0^2 \lesssim \mu\,h^2\,\|f\|_0^2\,.$$

For the remaining case (55), we have $\sqrt{\mu} < C^\star\,h^{-\frac{1}{2-\alpha_0}}$, or $h^2 \lesssim \mu^{-(2-\alpha_0)}$. On the one hand, since $\alpha_0 < 1$, $\sqrt{\mu} h \lesssim h^{\frac{1-\alpha_0}{2-\alpha_0}} \lesssim 1$. On the other hand, since $2\alpha_0 - 1 > 0$, $h^{4\alpha_0 - 2} \lesssim \mu^{-(2\alpha_0-1)(2-\alpha_0)}$. But one infers from (62) and these inequalities that

$$|c_\mu - c_\mu^h|^2 \lesssim (h^{4\alpha_0}\mu^{2\alpha_0} + h^2)\,\|f\|_0^2 \lesssim h^2\,(\mu^{2\alpha_0 - (2\alpha_0-1)(2-\alpha_0)} + 1)\,\|f\|_0^2\,.$$

To conclude, (68) follows from this and the fact that, as $\alpha_0 \in]\frac{1}{2}, 1[$, the exponent of μ is bounded by

$$2\alpha_0 - (2\alpha_0 - 1)(2 - \alpha_0) = 2\alpha_0^2 - 3\alpha_0 + 2 = 1 + (2\alpha_0 - 1)(\alpha_0 - 1) < 1\,. \qquad \diamond$$

6 Fourier Singular Complement Methods

In order to define the numerical part of the Fourier Singular Complement Method, let us prove a result which can be viewed as the mathematical foundation of the FSCM, from the Fourier point of view. It allows to recover (4-6), for sufficiently smooth right-hand sides.

Let u be the solution to the Poisson problem (1) and u_k be its Fourier coefficients in (16). By Lemma 3.4, we know that $u_k(x_1, x_2)$ solves the 2D problem (17-18). And using (23) we can decompose u_k as follows:

$$u_k = \tilde{u}_k + c_k\,\phi_s \tag{69}$$

where $\tilde{u}_k \in H^2(\omega) \cap H^1_0(\omega)$ and $\phi_s \in H^1_0(\omega)$ solves (22).

Lemma 6.1 *Let $f \in h^2(\Omega) \cap h^1_\diamond(\Omega)$, and $u \in H^1_0(\Omega)$ be the solution to (1). Then*

$$u = \tilde{u} + \gamma(x_3)\phi_s,\ \ \textit{with}\ \tilde{u} \in H^2(\Omega) \cap H^1_0(\Omega),\ \ \gamma \in H^2(]0, L[) \cap H^1_0(]0, L[). \tag{70}$$

Proof. Let $(U_K)_K$ be the Fourier sequence of u. Recall that $(U_K)_K$ converges to u in $H^1_0(\Omega)$, and $(\Delta U_K)_K$ converges to $-f$ in $L^2(\Omega)$. From (69), let us split the Fourier sequence into regular and singular parts, as

$$U_K = \widetilde{U}_K + \gamma_K(x_3)\,\phi_s,\ \text{with}\ \widetilde{U}_K = \sum_{k=1}^{K} \tilde{u}_k \sin\frac{k\pi}{L}x_3,\ \gamma_K(x_3) = \sum_{k=1}^{K} c_k \sin\frac{k\pi}{L}x_3.$$

We shall prove below that $(\gamma_K)_K$ converges in $H^2(]0, L[) \cap H^1_0(]0, L[)$, and $(\widetilde{U}_K)_K$ converges in $H^2(\Omega) \cap H^1_0(\Omega)$.

As far as the singular part is concerned, from (14) and the bound on $|c_k|$ in Lemma 4.2, we obtain that $\sum_{k=1}^{\infty} k^4|c_k|^2 < \infty$. Since we are dealing with the 1D Fourier sequence $(\gamma_K)_K$ (with sine functions), it is well-known that it converges to a limit, subsequently called γ, in $H^2(]0, L[) \cap H^1_0(]0, L[)$. Then, one finds that $(\gamma_K\,\phi_s)_K$ converges to $\gamma\,\phi_s$ in $H^1_0(\Omega)$, and that $(\Delta(\gamma_K\,\phi_s))_K$ converges in $L^2(\Omega)$, to $\gamma''\,\phi_s - \gamma p_s$.

For the regular part, we note that since there holds $\widetilde{U}_K = U_K - \gamma_K\,\phi_s$, $(\widetilde{U}_K)_K$ converges in $H^1_0(\Omega)$, to a limit called $\tilde{u}$, which is equal to

$$\tilde{u} = u - \gamma\,\phi_s.$$

Moreover, $(\Delta\widetilde{U}_K)_K$ converges in $L^2(\Omega)$, to $\Delta\tilde{u}$.

To conclude the proof, one has to establish that $\tilde{u}$ is an element of $H^2(\Omega)$. From Corollary 3.2, we know already that $\partial_3\tilde{u}$ is in $H^1(\Omega)$. So one has to check that $\partial_{ij}\tilde{u}$ is in $L^2(\Omega)$, for $i,j \in \{1,2\}$. But this follows from the estimate on $|\tilde{u}_k|_2$ in Lemma 4.2, and on the expression of the second order partial derivatives of $\widetilde{U}_K$, that is

$$\partial_{ij}\widetilde{U}_K = \sum_{k=1}^{K} \partial_{ij}\tilde{u}_k \sin\frac{k\pi}{L}x_3. \qquad \diamond$$

Remark 6.1 *In the more general case, i.e., $f \in L^2(\Omega)$, one gets only a convergence of $(\gamma_K)_K$ in $H^{1-\alpha}(]0,L[)$, see [11]. This precludes a convergence of the singular part in the desired Sobolev spaces, i.e., $H^1(\Omega)$ with $L^2(\Omega)$ Laplacian.*

In order to build the numerical schemes which completely define the FSCM, we introduce $u_k^h(x_1,x_2)$ the SCM approximation to $u_k(x_1,x_2)$. It is the same as u_μ^h in (66), but with μ replaced by $k^2\pi^2/L^2$, that is,

$$u_k^h = \tilde{u}_k^h + c_k^h\,\phi_s^h.$$

We then rephrase the 2D SCM Algorithm (54-57). This gives

<u>**Step 1**</u>. *Find $z_k^h \in V_0^h$ such that*

$$a_k(z_k^h, v) = (f, v) \quad \forall\, v \in V_0^h\,. \tag{71}$$

Compute c_k^h as follows:

$$c_k^h = \frac{(f - \dfrac{k^2\pi^2}{L^2}\, z_k^h,\, p_s^h)}{\|p_s^h\|_0^2} \quad \text{if} \quad k < C^\star\frac{L}{\pi}\,h^{-\frac{1}{2-\alpha_0}}\,; \tag{72}$$

and

$$c_k^h = 0 \quad \text{if} \quad k \geq C^\star\frac{L}{\pi}\,h^{-\frac{1}{2-\alpha_0}}\,. \tag{73}$$

<u>**Step 2**</u>. *Find $\tilde{u}_k^h \in V_0^h$ such that*

$$a_k(\tilde{u}_k^h, v) + c_k^h\, a_k(\phi_s^h, v) = (f, v) \quad \forall v \in V_0^h\,. \tag{74}$$

As mentioned already, only a few coefficients $(c_k^h)_k$ are actually computed.

Following (19), we finally define the FSCM approximation to the solution u to (1) as follows:

$$U_N^h(x_1,x_2,x_3) = \sum_{k=1}^{N} u_k^h(x_1,x_2)\sin\frac{k\pi}{L}x_3\,.$$

Then we have the final error estimate below

Theorem 6.1 *Assume that $f \in h_\diamond^1(\Omega) \cap h^2(\Omega)$.*
The following error estimate holds:

$$\|\nabla(u - U_N^h)\|_{L^2(\Omega)} \lesssim (h + N^{-1})\Big\{\|f\|_{L^2(\Omega)} + \|\partial_{33} f\|_{L^2(\Omega)}\Big\}\,.$$

Proof. Using the Fourier expansion of u and the definition of U_N^h, we have, cf. (10),

$$\begin{aligned}\|\nabla(u-U_N^h)\|_{L^2(\Omega)}^2 &= \frac{L}{2}\sum_{k=1}^{N}\Big(\|\nabla(u_k-u_k^h)\|_0^2+(\frac{k\pi}{L})^2\|u_k-u_k^h\|_0^2\Big)\\ &\quad+\frac{L}{2}\sum_{k>N}\Big(\|\nabla u_k\|_0^2+(\frac{k\pi}{L})^2\|u_k\|_0^2\Big)\\ &=: \mathrm{I}_1+\mathrm{I}_2.\end{aligned}$$

According to Lemma 3.4, we derive

$$\begin{aligned}\mathrm{I}_2 &= \frac{L}{2}\sum_{k>N}\Big(\|\nabla u_k\|_0^2+(\frac{k\pi}{L})^2\|u_k\|_0^2\Big)\\ &\le \frac{L}{2}N^{-2}\sum_{k>N}k^2\Big(\|\nabla u_k\|_0^2+(\frac{k\pi}{L})^2\|u_k\|_0^2\Big)\\ &\le \Big(\frac{L}{\pi}\Big)^2 N^{-2}\,\|f\|_{L^2(\Omega)}^2\,.\end{aligned}$$

For I_1, we have

$$\mathrm{I}_1=\frac{L}{2}\sum_{k=1}^{N}\|u_k-u_k^h\|_a^2\,.$$

According to Theorem 5.1 we have

$$\|u_k-u_k^h\|_a^2\lesssim k^4\,h^2\,\|f_k\|_0^2\,.$$

Using this and (14), we obtain the estimate of I_1:

$$\mathrm{I}_1\lesssim h^2\sum_{k=1}^{N}k^4\,\|f_k\|_0^2\lesssim h^2\,\|\partial_{33}f\|_{L^2(\Omega)}^2,$$

which, together with the previous estimate of I_2, leads to the desired error estimate. $\diamond$

7 Conclusion

The optimal convergence rate of the FSCM in prismatic domains, has been proven for the Poisson problem with homogeneous Dirichlet boundary conditions. Assuming that the right-hand side f is slightly more regular than $f\in L^2(\Omega)$, i.e., that f belongs to $h^2(\Omega)\cap h^1_\diamond(\Omega)$, the convergence rate of the FSCM in H^1-norm is like

$$\|u-U_N^h\|_1\le C_f(h+N^{-1}),$$

where h is the 2D mesh size, and N is the number of Fourier modes used.

The same result also holds for the discretization of the Poisson problem with a homogeneous Neumann boundary condition, or for the Poisson problem with non-homogeneous boundary conditions, provided there exist sufficiently smooth liftings.

Further, it is no difficulty to consider the case of a prismatic domain Ω with several reentrant edges, i.e., ω with several reentrant corners.

As far as the assumptions on the right-hand side f are concerned, a few remarks can be made. It seems that, in a prismatic domain Ω such as the one we considered here, the boundary condition on the bases was omitted in [2]. Nevertheless, this condition does not exist in the case of an axisymmetric domain, see [14], nor in the case of an infinite cylinder. In other words, $f \in h^2(\Omega)$ is enough in those types of domains. In the case of a Poisson problem with Neumann boundary conditions, one has to replace the vanishing trace conditions at the bases by the familiar $\partial_3 f = 0$ at the same bases.

As mentioned already, this paper is the first part of a three-part article [14, 15]. In the companion paper [14], the FSCM is analysed theoretically and its numerical approximation is built, in *axisymmetric domains with conical vertices and reentrant edges*. There are two difficulties which are inherent in this class of domains. The first one is the weights, which have to be introduced in the 2D sections. The second one is the addition of *sharp vertex* singularities, which have to be taken into account separately. In [15], the FSCM is analyzed from a numerical point of view (complexity, implementation issues, numerical experiments, etc.), and it is compared to other methods, such as mesh refinement techniques, or variants of the FSCM (2D SCM with the λ-approach [13]; 3D discretization of the regular part, etc.) in prismatic or axisymmetric domains. In particular, the use of the FFT to aproximate the sine functions in x_3 is motivated and justified there.

As noted in Remark 4.1, one can apply the same theoretical and numerical techniques to the 2D heat or wave equations, with any L^2-smooth (in space) right-hand side. For these PDEs, the singular functions p_s and ϕ_s do not depend on the time-step.
Finally, the results, can also be viewed as the first effort towards the discretization of electromagnetic fields in prismatic domains, with *continuous numerical approximations*, the importance of which is well-known, cf. [9]. As a matter of fact, the SCM developed in [8, 7, 19] for 2D electromagnetic computations can be generalized, based on the results obtained here.

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