HAL open science

On the Cauchy problem in Sobolev spaces for nonlinear Schrodinger equations with potential

Rémi Carles

To cite this version:

Rémi Carles. On the Cauchy problem in Sobolev spaces for nonlinear Schrodinger equations with potential. Portugal. Math. (N.S.), 2008, 65 (2), pp.191-209. hal-00093953v2

HAL Id: hal-00093953

https://hal.science/hal-00093953v2

Submitted on 23 Jan 2007

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

hal-00093953, version 2 - 23 Jan 2007

ON THE CAUCHY PROBLEM IN SOBOLEV SPACES FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH POTENTIAL

RÉMI CARLES

Abstract. We consider the Cauchy problem for nonlinear Schrödinger equations in the presence of a smooth, possibly unbounded, potential. No assumption is made on the sign of the potential. If the potential grows at most linearly at infinity, we construct solutions in Sobolev spaces (without weight), locally in time. Under some natural assumptions, we prove that the H^1-solutions are global in time. On the other hand, if the potential has a super-linear growth, then the Sobolev regularity of positive order is lost instantly, not matter how large it is, unless the initial datum decays sufficiently fast at infinity.

1. Introduction

We consider the Cauchy problem for the (nonlinear) Schrödinger equation

$$\begin{aligned} i\partial_t u + \frac{1}{2}\Delta u &= V(x)u + f\left(|u|^2\right)u, \quad (t,x)\in \mathbb{R}_+\times\mathbb{R}^d,\\ u_{|t=0} &= a_0 \in H^s\left(\mathbb{R}^d\right), \quad s\geqslant 0, \end{aligned} \tag{1.1}$$

where the potential V is smooth and sub-quadratic (see below), the nonlinearity f is sufficiently smooth, and the initial data a_0 may or may not belong to weighted L^2 spaces $\mathcal{F}(H^k)$ (sometimes denoted L^2_k), where $\mathcal{F}$ stands for the Fourier transform. Note that we consider only propagation in the future; this choice is made only to simplify some statements. We show that if the potential V is sub-linear, then (1.1) is locally well-posed in $H^1(\mathbb{R}^d)$, upon suitable assumptions on f. On the other hand, if V is super-linear (e.g. harmonic potential), then (1.1) is ill-posed in all Sobolev spaces of positive order; this is not a nonlinear result, since it holds even when $f\equiv 0$. This is heuristically reasonable, at least in the case of the harmonic oscillator: the potential rotates the phase space, so the natural space for the initial data is of the form $H^s\cap\mathcal{F}(H^s)$. If $a_0\in H^s\setminus\mathcal{F}(H^k)$, for $s\geqslant k>0$, then $u(t,\cdot)\notin H^k(\mathbb{R}^d)$ for arbitrarily small $t>0$. For the linear equation, this can be seen via the Fourier integral representation (Mehler's formula in the case of the harmonic potential). The proof we present treats both linear and nonlinear cases.

Before going further into details, we clarify our assumptions. We define the Fourier transform as

$$\mathcal{F}\varphi(\xi) = \widehat{\varphi}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi}\varphi(x)dx.$$

Denote $\langle x\rangle = (1+|x|^2)^{1/2}$. For $s\geqslant 0$, we define

$$H^s\left(\mathbb{R}^d\right) = \{\varphi\in L^2(\mathbb{R}^d)\ ;\ \xi\mapsto \langle\xi\rangle^s\,\widehat{\varphi}(\xi)\in L^2(\mathbb{R}^d)\} \quad ; \quad H^\infty\left(\mathbb{R}^d\right) = \cap_{s\geqslant 0}H^s\left(\mathbb{R}^d\right).$$

In particular, $\mathcal{F}(H^s)$ is just the weighted L^2 space:

$$\mathcal{F}(H^s) = \{\varphi\in L^2(\mathbb{R}^d)\ ;\ x\mapsto \langle x\rangle^s\,\varphi(x)\in L^2(\mathbb{R}^d)\}.$$

2000 *Mathematics Subject Classification.* 35A05; 35A07; 35B33; 35B65; 35Q55.

Support by the French ANR project SCASEN is acknowledged.

Assumption. *We assume that the potential is smooth, real-valued and sub-quadratic:* $V \in C^\infty(\mathbb{R}^d;\mathbb{R})$*, and* $\partial^\alpha V \in L^\infty(\mathbb{R}^d)$ *for all* $|\alpha| \geqslant 2$*.*

Definition. We say that V is *sub-linear* if $\partial^\alpha V \in L^\infty(\mathbb{R}^d)$ as soon as $|\alpha| \geqslant 1$. We say that V is *super-linear* if $\nabla_x V$ is unbounded.

Remark. For super-quadratic potentials, the theory must be modified. First, if V is super-quadratic and negative, then H is not essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$ ([11, 16]). If V is super-quadratic and positive, then even the local existence results are different. We refer to [20, 21] for very interesting results in this direction.

The construction of the parametrix for the propagator of $H = -\frac{1}{2}\Delta + V$ provided by D. Fujiwara [12, 13] shows that $U(t) = e^{-itH}$, which is L^2-unitary, satisfies the local dispersion estimate: there exists $\delta > 0$ such that

$$\|U(t)\|_{L^1 \to L^\infty} \lesssim \frac{1}{|t|^{d/2}}, \text{ for } |t| \leqslant \delta. \tag{1.2}$$

One can infer local and global existence results for (1.1) if $a_0 \in D(\sqrt{H})$ when $V \geqslant 0$, under suitable assumptions on the nonlinearity f, as proved initially by Oh [15]. The assumption $V \geqslant 0$ is actually not necessary, and one can prove the local existence results of Oh in weighted Sobolev spaces of the form $H^s \cap \mathcal{F}(H^s)$ thanks to Strichartz estimates (see e.g. [5], and [2] where global existence results are recalled for potentials V which are not necessarily non-negative). In all this paper, u is assumed to be a mild solution to (1.1), that is, to solve

$$u(t) = U(t)a_0 - i\int_0^t U(t-s)\left(f\left(|u(s)|^2\right)u(s)\right)ds.$$

In Proposition 1.2 though, we construct a classical solution for (1.1).

When $V \geqslant 0$ and $f(|u|^2) = \mu|u|^{2\sigma}$, one can prove global existence in $D(\sqrt{H})$ for the solution u of (1.1) under suitable assumptions on μ and σ, thanks to the following conservations:

$$\text{Mass: } \frac{d}{dt}\left(\|u(t)\|_{L^2}^2\right) = 0.$$

$$\text{Energy: } \frac{d}{dt}\left(\frac{1}{2}\|\nabla u(t)\|_{L^2}^2 + \frac{\mu}{\sigma+1}\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2} + \int_{\mathbb{R}^d} V(x)|u(t,x)|^2 dx\right) = 0.$$

The question we ask is: What remains when we do not assume $V|a_0|^2 \in L^1(\mathbb{R}^d)$? Roughly speaking, the local existence results remain when V is sub-linear, but fail when V is super-linear (we prove the latter under slightly more restrictive assumptions on V, see Th. 1.5). Note that in the above example, if we assume $0 < \sigma < 2/d$, then one can prove the existence of a global solution, with an L^2 regularity, as in [19]. Our goal is to understand better the relevance of Sobolev spaces with positive index, when no extra decay of the initial datum is assumed.

We recall a particular case of [4, Lemma 1]:

Lemma 1.1. *There exist* $T > 0$ *and a unique solution* $\phi_{\rm eik} \in C^\infty([0,T]\times\mathbb{R}^n)$ *to:*

$$\partial_t \phi_{\rm eik} + \frac{1}{2}|\nabla_x \phi_{\rm eik}|^2 + V = 0 \quad ; \quad \phi_{{\rm eik}|t=0} = 0\,. \tag{1.3}$$

This solution is sub-quadratic: $\partial_x^\alpha \phi_{\rm eik} \in L^\infty([0,T]\times\mathbb{R}^n)$ *as soon as* $|\alpha| \geqslant 2$*.*

Example. If $V(x) = \frac{1}{2}\sum_{j=1}^d \omega_j^2 x_j^2$ with $\omega_j \geqslant 0$, then

$$\phi_{\rm eik}(t,x) = -\sum_{j=1}^d \frac{\omega_j}{2} x_j^2 \tan(\omega_j t)\,.$$

This shows that in general, the above result is really local in time, due to the formation of caustics.

Example. If $V(x) = \langle x\rangle^a$, with $0 < a \leqslant 2$, then we can see that caustics appear in finite time even if the potential V is sub-linear.

Proposition 1.2. *Let $d \geqslant 1$.*
(1) *If $f \equiv 0$ (linear equation), assume that $a_0 \in H^s(\mathbb{R}^d)$ for some $s \geqslant 0$. Then* (1.1) *has a unique solution u such that $u \cdot e^{-i\phi_{\rm eik}} \in C([0,T]; H^s)$, where $\phi_{\rm eik}$ and T are given by Lemma 1.1.*
(2) *For the nonlinear equation, assume that f is smooth, $f \in C^\infty(\mathbb{R}_+;\mathbb{C})$, and that $a_0 \in H^s(\mathbb{R}^d)$ for some $s > d/2$. Then* (1.1) *has a unique solution u such that $u \cdot e^{-i\phi_{\rm eik}} \in C([0,T]; H^s)$, where $\phi_{\rm eik}$ and T are given by Lemma 1.1.*

Proposition 1.3. *Let $d \geqslant 1$, $a_0 \in H^1(\mathbb{R}^d)$, and assume that V is sub-linear and that the nonlinearity f is of the form*

$$f(|u|^2) = \mu|u|^{2\sigma}, \quad \text{with } \mu \in \mathbb{R},\ \sigma > 0, \text{ and } \sigma < \frac{2}{d-2} \text{ if } d \geqslant 3.$$

Then there exists $\tau = \tau(d, \|a_0\|_{H^1}, \mu, \sigma) > 0$ such that (1.1) *has a unique solution $u \in C([0,\tau]; H^1) \cap L^{\frac{4\sigma+4}{d\sigma}}\left([0,\tau]; W^{1,2\sigma+2}\right)$.*
If moreover $\sigma < 2/d$ or $\mu \geqslant 0$, then this solution is global in time:

$$u \in C(\mathbb{R}_+; H^1) \cap L^{\frac{4\sigma+4}{d\sigma}}_{\rm loc}\left(\mathbb{R}_+; W^{1,2\sigma+2}\right).$$

Remark. Even the local result is not a consequence of Proposition 1.2: the regularity required on the initial data is not the same. The reason is that Proposition 1.2 is established without dispersive or Strichartz estimates, while the local existence result in Proposition 1.3 is proven thanks to (local in time) Strichartz estimates.

We also discuss the local Cauchy problem in $H^s(\mathbb{R}^d)$, $s > 0$, in Section 4. The main point consists in showing that in the presence of a sub-linear potential, local Strichartz estimates are available in Sobolev and Besov spaces. We prove:

Proposition 1.4. *Let V be sub-linear, $0 < s < d/2$ and $0 < \sigma \leqslant \frac{2}{d-2s}$. If σ is not an integer, assume that $[s] < 2\sigma$. Then there exist $T > 0$ and a unique solution*

$$u \in C([0,T]; H^s) \cap L^\gamma([0,T]; B^s_{\rho,2})$$

to (3.1)*, where*

$$\rho = \frac{2\sigma+2}{1+\frac{2\sigma s}{d}} \quad ; \quad \gamma = \frac{4\sigma+4}{\sigma(d-2s)}.$$

We now come to the non-existence result:

Theorem 1.5. *Let $d \geqslant 1$, and f be smooth, $f \in C^\infty(\mathbb{R}_+;\mathbb{R})$. Assume that V is super-linear, and that there exist $0 < k(\leqslant 1)$ and $C > 0$ such that*

$$|\nabla V(x)| \leqslant C\langle x\rangle^k, \quad \forall x \in \mathbb{R}^d,$$

and $\omega, \omega' \in \mathbb{S}^{d-1}$ such that

$$|\omega \cdot \nabla V(x)| \geqslant c|\omega' \cdot x|^k \text{ as } |x| \to \infty, \text{ for some } c > 0. \tag{1.4}$$

Then there exists $a_0 \in H^\infty(\mathbb{R}^d)$ such that for arbitrarily small $t > 0$ and all $s > 0$, the solution $u(t,\cdot)$ to (1.1) *provided by Proposition 1.2 fails to be in $H^s(\mathbb{R}^d)$.*

Example. As a potential V, we may consider any non-trivial quadratic form, or $V(x) = \pm\langle x'\rangle^a$, with $1 < a \leqslant 2$, for some decomposition $x = (x', x'')$.

Remark. Note that no assumption is made on the growth of the nonlinearity at infinity: the above result reveals a *geometric phenomenon*, and not an ill-posedness result like for super-critical nonlinearities without a potential ([1, 3, 8, 14]).

In Section 2, we outline the proof of Proposition 1.2, which is a particular case of [4, Proposition 3]. We establish Proposition 1.3 in Section 3. We extend the local theory to all the spaces $H^s(\mathbb{R}^d)$ for $s > 0$ in Section 4, where we prove Proposition 1.4. Finally, Theorem 1.5 is proved in Section 5.

2. Preliminary remarks

In this section, we outline the proof of Proposition 1.2, which is a straightforward consequence of the analysis in [4], with the choice $\varepsilon = 1$. This will also guide us for the proof of Theorem 1.5.

First, Lemma 1.1 is a straightforward consequence of the local Hamilton-Jacobi theory, Gronwall lemma, and a global inversion theorem, which can be found for instance in [17, Th. 1.22] or [10, Prop. A.7.1]. To prepare the proof of Theorem 1.5, we recall some details. Let $x(t,y)$ and $\xi(t,y)$ solve

$$\begin{cases} \partial_t x(t,y) = \xi\left(t,y\right) \quad ; \quad x(0,y) = y, \\ \partial_t \xi(t,y) = -\nabla_x V\left(x(t,y)\right) \quad ; \quad \xi(0,y) = 0. \end{cases} \tag{2.1}$$

The local Hamilton-Jacobi theory provides a solution to (1.3) in the neighborhood of every point where $y \mapsto x(t,y)$ is invertible. The theory is global in space (not in time, in general) thanks to the global inversion theorem mentioned above, and to Gronwall lemma. The gradient of $\phi_{\rm eik}$ is given by

$$\nabla_x \phi_{\rm eik}(t,x) = \xi(t, y(t,x)), \tag{2.2}$$

where $y(t,x)$ is the inverse mapping of $y \mapsto x(t,y)$. Introduce the Jacobi determinant

$$J_t(y) = \det \nabla_y x(t,y). \tag{2.3}$$

The global inversion theorem can be applied since there exists $C > 0$ such that

$$C^{-1} \leqslant J_t(y) \leqslant C, \quad \forall (t,y) \in [0,T] \times \mathbb{R}^d. \tag{2.4}$$

The change of unknown function $u(t,x) = a(t,x)e^{i\phi_{\rm eik}(t,x)}$ turns (1.1) into the equivalent Cauchy problem:

$$\partial_t a + \nabla \phi_{\rm eik} \cdot \nabla a + \frac{1}{2} a \Delta \phi_{\rm eik} = \frac{i}{2}\Delta a - i f\left(|a|^2\right) a \quad ; \quad a_{|t=0} = a_0. \tag{2.5}$$

The major difference with (1.1) is that the potential V is no longer present in the equation. The idea is to view the left hand side as a transport operator with velocity $\nabla\phi_{\rm eik}$ and a renormalization factor along the characteristics, $\frac{1}{2}a\Delta\phi_{\rm eik}$. We can then reduce the problem of existence of solutions of (2.5), to the existence of *a priori* estimates, thanks to a mollification procedure. Since we seek $a \in C([0,T];H^s)$, we note that the term $i\Delta$ on the right hand side is skew-symmetric, and has no contribution in the energy estimates. To take advantage of this property, we do not rewrite (2.5) along the characteristics, but notice that from Lemma 1.1, $\|a\Delta\phi_{\rm eik}\|_{L^\infty_t H^s} \lesssim \|a\|_{L^\infty_t H^s}$. For the convective term, we use Lemma 1.1, and an integration by parts: if $\alpha \in \mathbb{N}^d$ is such that $|\alpha| \leqslant s$, we write

$$\begin{aligned} \mathrm{Re}\int \partial_x^\alpha \bar a \partial_x^\alpha \left(\nabla\phi_{\rm eik}\cdot\nabla a\right) dx =& \mathrm{Re}\int \partial_x^\alpha \bar a \left(\nabla\phi_{\rm eik}\cdot\nabla \partial_x^\alpha a\right) dx \\ &+ \sum_{|\beta|\geqslant 1} c_{\alpha,\beta}\, \mathrm{Re}\int \partial_x^\alpha \bar a \nabla \partial_x^\beta \phi_{\rm eik}\cdot \nabla \partial_x^{\alpha-\beta} a dx \\ =& \frac{1}{2}\int \nabla\phi_{\rm eik}\cdot\nabla|\partial_x^\alpha a|^2 dx + \mathcal{O}\left(\|a\|^2_{L^\infty_t H^s}\right) \\ =& \mathcal{O}\left(\|a\|^2_{L^\infty_t H^s}\right). \end{aligned}$$

If s is not an integer, we can use interpolation. Proposition 1.2 follows easily, since $s > d/2$ ensures that $H^s(\mathbb{R}^d)$ is an algebra.

Remark 2.1. Let $I \subset [0,T]$ be a compact time interval. The approach of [4] recalled above shows that the map $F \mapsto \mathtt{a}$, where

$$\partial_t \mathtt{a} + \nabla\phi_{\rm eik}\cdot\nabla\mathtt{a} + \frac{1}{2}\mathtt{a}\Delta\phi_{\rm eik} = \frac{i}{2}\Delta\mathtt{a} + F \quad ; \quad \mathtt{a}_{|t=0} = 0,$$

sends $L^1(I;L^2)$ to $C\cap L^\infty(I;L^2)$ continuously:

$$\|\mathtt{a}\|_{L^\infty(I;L^2)} \leqslant C\|F\|_{L^1(I;L^2)},$$

where C depends only on d and $\|\nabla^2\phi_{\rm eik}\|_{L^\infty(I;L^2)}$.

3. Sub-linear potentials

3.1. **Local H^1 theory.** To prove the first part of Proposition 1.3, the idea is to keep the same proof as without potential. The gradient does not commute with H, but we have:

$$\left(i\partial_t + \frac{1}{2}\Delta\right)\nabla u = V(x)\nabla u + u\nabla V(x) + \mu\nabla\left(|u|^{2\sigma}u\right).$$

The new term is $u\nabla V(x)$, that is, u multiplied by a bounded term. Recall that $U(t) = e^{-itH}$. We show that for $\tau > 0$ sufficiently small, there exists u such that:

$$u(t) = U(t)a_0 - i\mu\int_0^t U(t-s)\left(|u|^{2\sigma}u\right)(s)ds =: \Phi(u)(t). \tag{3.1}$$

We see that

$$\begin{aligned}\nabla\Phi(u)(t) &= U(t)\nabla a_0 - i\mu\int_0^t U(t-s)\nabla\left(|u|^{2\sigma}u\right)(s)ds\\ &\quad - i\int_0^t U(t-s)\left(\Phi(u)(s)\nabla V\right)ds.\end{aligned} \tag{3.2}$$

Recall that (q,r) is Schrödinger-admissible in $\mathbb{R}^d$ if

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad 2 \leqslant r \leqslant \frac{2d}{d-2}, \ (q,r) \neq (2,\infty).$$

It follows from [13] that Strichartz estimates are available for $U(t)$ (see e.g. [5]): for all admissible pairs (q,r), (q_1,r_1) and (q_2,r_2), there exist C_r and C_{r_1,r_2} such that for any compact interval I and any $\varphi \in L^2(\mathbb{R}^d)$, $F \in L^{q_2}(I;L^{r_2}(\mathbb{R}^d))$,

$$\begin{aligned}\|U(\cdot)\varphi\|_{L^q(I;L^r)} &\leqslant C_r(1+|I|)^{\frac{1}{q}}\|\varphi\|_{L^2},\\ \left\|\int_0^t U(t-s)F(s)ds\right\|_{L^{q_1}(I;L^{p_1})} &\leqslant C_{r_1,r_2}(1+|I|)^{\frac{1}{q_1}}\|F\|_{L^{q_2'}(I;L^{p_2'})},\end{aligned} \tag{3.3}$$

where r' stands for the Hölder conjugate exponent of r. Note that the powers of $|I|$ on the right hand sides are sharp in general, for H may have eigenvalues. For (q,r) an admissible pair, define

$$Y_{r,\rm loc}(I) := \left\{u \in C(I;H^1);\ Au \in L^q_{\rm loc}(I;L^r)\cap L^\infty_{loc}(I;L^2)\ \forall A \in \{\mathrm{Id},\nabla\}\right\}.$$

Introduce the following Lebesgue exponents:

$$r = 2\sigma+2 \quad ; \quad q = \frac{4\sigma+4}{d\sigma} \quad ; \quad k = \frac{2\sigma(2\sigma+2)}{2-(d-2)\sigma}\cdot \tag{3.4}$$

Then (q,r) is the (admissible) pair of the proposition, and

$$\frac{1}{r'} = \frac{2\sigma}{r} + \frac{1}{r} \quad ; \quad \frac{1}{q'} = \frac{2\sigma}{k} + \frac{1}{q}\cdot$$

For $\tau > 0$ and any pair (a,b), we use the notation

$$\|f\|_{L^a_\tau L^b} = \|f\|_{L^a([0,\tau];L^b)}.$$

We first prove that there exists $\tau > 0$ such that the set

$$X_\tau := \big\{u \in Y_{r,\mathrm{loc}}([0,\tau]);\ \|u\|_{L^\infty_\tau L^2} \leqslant 2\|a_0\|_{L^2},\ \|u\|_{L^q_\tau L^r} \leqslant 2C_r\|a_0\|_{L^2}\ ,\\ \|\nabla u\|_{L^\infty_\tau L^2} \leqslant 2\|\nabla a_0\|_{L^2}\ ,\quad \|\nabla u\|_{L^q_\tau L^r} \leqslant 2C_r\|\nabla a_0\|_{L^2}\big\}$$

is stable under the map Φ, where C_r is the constant of the homogeneous Strichartz inequality (3.3). Then choosing τ even smaller, Φ is a contraction on $L^q([0,\tau];L^r)$.

Let $u \in X_\tau$. For $\tau \leqslant 1$, (3.3) yields:

$$\begin{aligned}\|\Phi(u)\|_{L^\infty_\tau L^2} &\leqslant \|U(t)a_0\|_{L^2} + C_{2,r}|\mu|\left\||u|^{2\sigma}u\right\|_{L^{q'}_\tau L^{r'}}\\ &\leqslant \|a_0\|_{L^2} + C\,\|u\|^{2\sigma}_{L^k_\tau L^r}\,\|u\|_{L^q_\tau L^r}\ .\end{aligned}$$

Sobolev embedding yields:

$$\|u\|_{L^k_\tau L^r} \leqslant C\tau^{\frac{1}{k}}\|u\|_{L^\infty_\tau H^1}.$$

It follows that

$$\|\Phi(u)\|_{L^\infty_\tau L^2} \leqslant \|a_0\|_{L^2} + C\tau^{\frac{2\sigma}{k}}\|a_0\|^{2\sigma+1}_{H^1}.$$

The same computations yield:

$$\|\Phi(u)\|_{L^q_\tau L^r} \leqslant C_r\|a_0\|_{L^2} + \widetilde{C}\tau^{\frac{2\sigma}{k}}\|a_0\|^{2\sigma+1}_{H^1}.$$

Similarly,

$$\begin{aligned}\|\nabla\Phi(u)\|_{L^\infty_\tau L^2} &\leqslant \|\nabla a_0\|_{L^2} + C_{2,r}|\mu|\left\|\nabla\left(|u|^{2\sigma}u\right)\right\|_{L^{q'}_\tau L^{r'}} + C_{2,2}\|\nabla V\|_{L^\infty}\|\Phi(u)\|_{L^1_\tau L^2}\\ &\leqslant \|\nabla a_0\|_{L^2} + C\,\|u\|^{2\sigma}_{L^k_\tau L^r}\,\|\nabla u\|_{L^q_\tau L^r} +\\ &\quad + \tau C_{2,2}\|\nabla V\|_{L^\infty}\left(\|a_0\|_{L^2} + C\tau^{\frac{2\sigma}{k}}\|a_0\|^{2\sigma+1}_{H^1}\right),\end{aligned}$$

and

$$\begin{aligned}\|\nabla\Phi(u)\|_{L^q_\tau L^r} \leqslant{}& C_r\|\nabla a_0\|_{L^2} + \widetilde{C}\,\|u\|^{2\sigma}_{L^k_\tau L^r}\,\|\nabla u\|_{L^q_\tau L^r} +\\ &+ \tau C_{r,2}\|\nabla V\|_{L^\infty}\left(\|a_0\|_{L^2} + C\tau^{\frac{2\sigma}{k}}\|a_0\|^{2\sigma+1}_{H^1}\right).\end{aligned}$$

Therefore Φ leaves X_τ stable for

$$\tau\|\nabla V\|_{L^\infty}\|a_0\|_{L^2} + \tau^{\frac{1}{k}}\|u\|_{L^\infty_\tau H^1} \ll 1. \tag{3.5}$$

To complete the proof of the first part of the proposition, it is enough to prove contraction for small τ in the weaker metric $L^q([0,\tau];L^r)$. We have:

$$\begin{aligned}\left\|\Phi(u_2) - \Phi(u_1)\right\|_{L^q_\tau L^r} &\leqslant C\left\|\left(|u_2|^{2\sigma}u_2 - |u_1|^{2\sigma}u_1\right)\right\|_{L^{q'}_\tau L^{r'}}\\ &\leqslant C\left(\|u_1\|^{2\sigma}_{L^k_\tau L^r} + \|u_2\|^{2\sigma}_{L^k_\tau L^r}\right)\|u_2 - u_1\|_{L^q_\tau L^r}\,.\end{aligned}$$

As above, we have the estimate

$$\|u_j\|^{2\sigma}_{L^k_\tau L^r} \leqslant C\tau^{\frac{1}{k}}\|u_j\|_{L^\infty_\tau H^1}.$$

Therefore, contraction follows for τ sufficiently small, according to (3.5).

3.2. **Global existence in H^1.** If V is sub-linear and unbounded, then the energy

$$E = \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 + \frac{\mu}{\sigma+1}\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2} + \int_{\mathbb{R}^d} V(x)|u(t,x)|^2 dx$$

may not be defined initially, if we simply require $a_0 \in H^1(\mathbb{R}^d)$. To complete the proof of Proposition 1.3, the idea is to notice that the time derivative of the "bad" term in the energy is controlled by the H^1 norm of the solution. We present the computations at a formal level only, and refer to [5] for a justification method which uses the multiplication by Gaussians. We have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} V(x)|u(t,x)|^2 dx &= 2\operatorname{Re}\int_{\mathbb{R}^d} V(x)\overline{u}\partial_t u dx = 2\operatorname{Im}\int_{\mathbb{R}^d} V(x)\overline{u}\left(i\partial_t u\right) dx \\ &= -\operatorname{Im}\int_{\mathbb{R}^d} V(x)\overline{u}\Delta u dx = \operatorname{Im}\int_{\mathbb{R}^d} \overline{u}\nabla V(x)\cdot\nabla u dx.\end{aligned}$$

We infer, thanks to the conservation of mass:

$$\begin{aligned}\frac{1}{2}\|\nabla u(t)\|_{L^2}^2 + \frac{\mu}{\sigma+1}\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant &\frac{1}{2}\|\nabla a_0\|_{L^2}^2 + \frac{\mu}{\sigma+1}\|a_0\|_{L^{2\sigma+2}}^{2\sigma+2} \\ &+ \|\nabla V\|_{L^\infty}\|a_0\|_{L^2}\int_0^t \|\nabla u(s)\|_{L^2} ds.\end{aligned}$$

When $\mu \geqslant 0$, this yields the estimate

$$\|\nabla u(t)\|_{L^2}^2 \lesssim 1 + \int_0^t \|\nabla u(s)\|_{L^2} ds,$$

hence $\|\nabla u(t)\|_{L^2}$ grows at most exponentially.

If $\sigma < 2/d$ and $\mu < 0$, Gagliardo–Nirenberg inequality and the conservation of mass yield:

$$\|\nabla u(t)\|_{L^2}^2 \lesssim 1 + \|\nabla u(t)\|_{L^2}^{d\sigma} + \int_0^t \|\nabla u(s)\|_{L^2} ds.$$

Using Young inequality

$$\|\nabla u(t)\|_{L^2}^{d\sigma} \leqslant C_\epsilon + \epsilon\|\nabla u(t)\|_{L^2}^2,$$

and choosing $\epsilon > 0$ sufficiently small, we conclude as before. This completes the proof of Proposition 1.3.

4. On the local Cauchy problem in H^s: proof of Proposition 1.4

When $a_0 \in H^s(\mathbb{R}^d)$ with $s > 0$ not necessarily equal to one, and V is sub-linear, it is still possible to establish a local in time theory. Without potential, $V \equiv 0$, Proposition 1.4 was proved by T. Cazenave and F. Weissler [6, Theorem 1.1, (i)–(ii)]. As in this paper, we shall not define Besov spaces by using a dyadic decomposition, but rather use their characterization in terms of interpolation between Sobolev spaces. We first recall the argument when $V \equiv 0$, and then show how it can be adapted to infer Proposition 1.4.

4.1. **Proof when $V \equiv 0$.** The idea is to apply a fixed point argument, as in Section 3.1. However, when $s < d/2$ is not an integer, it becomes delicate to estimate the H^s norm of the nonlinearity. This is why in [6], the authors work in Besov spaces. When s is an integer, the above result can be refined. We shall not recall this aspect more precisely, and simply refer to [6]. The proof proceeds in three steps. The authors first establish Strichartz estimates for the free group $e^{i\frac{t}{2}\Delta}$ in (homogeneous) Besov spaces [6, Th. 2.2]. Next, they prove estimates for the nonlinear term, in homogeneous Besov spaces as well [6, Th. 3.1]. Finally,

these tools, along with Strichartz estimates, make it possible to apply a fixed point argument, and prove Proposition 1.4 when $V \equiv 0$.

Denote

$$\mathcal{I}(t)F := \int_0^t U(t-s)F(s)ds.$$

The first step yields, for $s > 0$, and (q,r), (q_j, r_j) admissible pairs:

$$\begin{aligned} &\|U(\cdot)\varphi\|_{L^q(\mathbb{R}_+;\dot B^s_{r,2})} \leqslant C_r \|\varphi\|_{\dot H^s}, \\ &\|\mathcal{I}(\cdot)F\|_{L^{q_1}(I;\dot B^s_{r_1,2})} \leqslant C_{r_1,r_2} \|F\|_{L^{q_2'}(I;\dot B^s_{r_2,2})}, \end{aligned} \tag{4.1}$$

where C_{r_1,r_2} does not depend on the time interval I. Next, under the assumptions of Proposition 1.4, we have

$$\left\| |u|^{2\sigma} u \right\|_{\dot B^s_{\rho',2}} \lesssim \|u\|^{2\sigma+1}_{\dot B^s_{\rho,2}}. \tag{4.2}$$

Proposition 1.4 follows from (4.1), (4.2), Hölder's inequality and a fixed point argument.

Remark 4.1. Note that (4.1) and (4.2) still hold if we replace homogeneous Besov spaces with inhomogeneous ones. This remark simplifies the generalization to the case when V is sub-linear.

4.2. **Strichartz estimates in Besov spaces with a sub-linear potential.** We show that when V is sub-linear, (4.1) still holds, up to two modifications:

- The Strichartz inequalities hold on *finite* time intervals only.
- We replace the homogeneous Besov spaces with inhomogeneous ones.

The first point is unavoidable, as recalled in Section 3.1. Since we shall prove a local in time result, in the rest of this section we consider time intervals of length at most one. The second point is here to consider pseudo-differential operators with smooth symbols which do not contain x-variable.

If $P = P(D)$ is a pseudo-differential operator with smooth symbol, we have:

$$\begin{aligned} [P, U(t)]\varphi &= -i \int_0^t U(t-s)[P,V]U(s)\varphi ds = -i\mathcal{I}(t)\left([P,V]U(\cdot)\varphi\right), \\ [P, \mathcal{I}(t)]F &= -i \int_0^t U(t-s)[P,V]\mathcal{I}(s)F ds = -i\mathcal{I}(t)\left([P,V]\mathcal{I}(\cdot)\varphi\right). \end{aligned}$$

First, assume $0 < s < 1$. For I a time interval with $|I| \leqslant 1$, (3.3) yields:

$$\begin{aligned} \|PU(t)\varphi\|_{L^q(I;L^r)} &\leqslant \|U(t)P\varphi\|_{L^q(I;L^r)} + \|\mathcal{I}(t)\left([P,V]U(\cdot)\varphi\right)\|_{L^q(I;L^r)} \\ &\lesssim \|P\varphi\|_{L^2} + \|[P,V]U(\cdot)\varphi\|_{L^1(I;L^2)} \\ &\lesssim \|P\varphi\|_{L^2} + \|[P,V]U(\cdot)\varphi\|_{L^\infty(I;L^2)}. \end{aligned}$$

Similarly,

$$\|P\mathcal{I}(t)F\|_{L^{q_1}(I;L^{r_1})} \lesssim \|PF\|_{L^{q_2'}(I;L^{r_2'})} + \|[P,V]\mathcal{I}(\cdot)F\|_{L^\infty(I;L^2)}.$$

For $s > 0$, let $P_s = (I-\Delta)^{s/2}$. By [9, Th. 2] (see also [18, § 3.6]), we know that if in addition $s \leqslant 1$, then $[P_s, V]$ is bounded from L^2 to L^2, with norm controlled by $C\|\nabla V\|_{L^\infty}$ for some universal constant C. We infer, when $s \leqslant 1$,

$$\begin{aligned} \|P_s U(t)\varphi\|_{L^q(I;L^r)} &\lesssim \|P_s\varphi\|_{L^2} + \|U(\cdot)\varphi\|_{L^\infty(I;L^2)} \\ &\lesssim \|P_s\varphi\|_{L^2} + \|\varphi\|_{L^2} \lesssim \|P_s\varphi\|_{L^2}, \end{aligned}$$

where we have used Strichartz estimates (3.3). This means:

$$\|U(\cdot)\varphi\|_{L^q(I;W^{s,r})} \lesssim \|\varphi\|_{H^s} \tag{4.3}$$

Similarly, when $s \leqslant 1$,

$$\|\mathcal{I}(\cdot)F\|_{L^{q_1}(I;W^{s,r_1})} \lesssim \|F\|_{L^{q_2'}(I;W^{s,r_2'})}. \tag{4.4}$$

For $s > 1$, replace P_s with the family $(P_{s-m} \circ \partial^\alpha)_{|\alpha|\leqslant m}$, where $m = [s]$. Reasoning as above, we see that since $\partial^\alpha V \in L^\infty(\mathbb{R}^d)$ for all $|\alpha| \geqslant 1$, (4.3) and (4.4) hold for all $s > 0$.

Interpolating (as in [6], up to replacing homogeneous spaces by their inhomogeneous counterparts), we conclude:

$$\begin{aligned} \|U(\cdot)\varphi\|_{L^q(I;B^s_{r,2})} &\leqslant C_r\|\varphi\|_{H^s}, \\ \|\mathcal{I}(\cdot)F\|_{L^{q_1}(I;B^s_{r_1,2})} &\leqslant C_{r_1,r_2}\|F\|_{L^{q_2'}(I;B^s_{r_2,2})}, \end{aligned} \tag{4.5}$$

where the constants C_r and C_{r_1,r_2} do not depend on I, provided that $|I| \leqslant 1$.

Conclusion. Since (4.2) holds with homogeneous Besov spaces replaced by their inhomogeneous counterparts, the fixed point argument used in [6] can be transported here. This completes the proof of Proposition 1.4.

5. Loss of Sobolev regularity: proof of Theorem 1.5

5.1. **A decomposition suggested by geometric optics.** The idea is to resume the approach of weakly nonlinear geometric optics recalled in Section 2. We consider an intermediary function defined by leaving out the term $i\Delta a$ in (2.5): without this term, (2.5) is an ordinary differential equation along the characteristics of the transport operator with velocity $\nabla\phi_{\rm eik}$ (*i.e.* the bicharacteristics associated to H).

Recall that a solves (2.5), and define b as the solution on $[0,T]$ to:

$$\partial_t b + \nabla\phi_{\rm eik}\cdot\nabla b + \frac{1}{2}b\Delta\phi_{\rm eik} = -if\left(|b|^2\right)b \quad ; \quad b_{|t=0} = a_0. \tag{5.1}$$

To see that b solves an ordinary differential equation along the rays of geometric optics (the projections of the Hamilton flow (2.1) on the physical space), introduce

$$\beta(t,y) = b\left(t, x(t,y)\right)\sqrt{J_t(y)},$$

where $x(t,y)$ is given by (2.1) and the Jacobi determinant is defined by (2.3). This change of unknown function makes sense for $t \in [0,T]$, where $y \mapsto x(t,y)$ is a global diffeomorphism. Then (5.1) is equivalent to

$$\partial_t\beta(t,y) = -if\left(J_t(y)^{-1}|\beta(t,y)|^2\right)\beta(t,y) \quad ; \quad \beta(0,y) = a_0(y). \tag{5.2}$$

Since in Theorem 1.5, we assume that f is real-valued, we note that

$$\partial_t|\beta|^2 = 0,$$

so that (5.2) is just a *linear* ordinary differential equation:

$$\beta(t,y) = a_0(y)\exp\left(-i\int_0^t f\left(J_s(y)^{-1}|a_0(y)|^2\right)ds\right).$$

We infer

$$b(t,x) = \frac{1}{\sqrt{J_t(y(t,x))}}a_0\left(y(t,x)\right)\exp\left(-i\int_0^t f\left(J_s\left(y(t,x)\right)^{-1}|a_0\left(y(t,x)\right)|^2\right)ds\right).$$

The main observation is that (2.4) implies that $b \in C([0,T];H^s(\mathbb{R}^d))$ for all $s \geqslant 0$. Let $r = a - b$: for every $t \in [0,T]$, $r(t,\cdot) \in H^\infty(\mathbb{R}^d)$. For $1 \leqslant j \leqslant d$, $x_j r$ solves:

$$\begin{aligned} \partial_t(x_j r) + \nabla\phi_{\rm eik}\cdot\nabla(x_j r) + \frac{1}{2}x_j r\Delta\phi_{\rm eik} &= \frac{i}{2}\Delta(x_j r) + r\partial_j\phi_{\rm eik} - i\partial_j r + \frac{i}{2}x_j\Delta b \\ &\quad - ix_j\left(f\left(|b+r|^2\right)(b+r) - f\left(|b|^2\right)b\right), \\ x_j r_{|t=0} &= 0. \end{aligned}$$

Notice that the fundamental theorem of calculus yields:

$$\begin{aligned} x_j\left(f\left(|a|^2\right)a - f\left(|b|^2\right)b\right) &= x_j\left(f\left(|b+r|^2\right)(b+r) - f\left(|b|^2\right)b\right) \\ &= x_j r\int_0^1 \partial_z F\left(b+sr\right)ds + x_j\overline{r}\int_0^1 \partial_{\overline{z}} F\left(b+sr\right)ds, \end{aligned}$$

where $F(z) = f(|z|^2)z$. In particular, we know that

$$\int_0^1 \partial_z F\left(b+sr\right)ds, \int_0^1 \partial_{\overline{z}} F\left(b+sr\right)ds \in C\cap L^\infty(I\times\mathbb{R}^d).$$

Reasoning as in Remark 2.1, we see that:

$$\|xr\|_{L^\infty([0,t];L^2)} \leqslant C\left(1+\|x\Delta b\|_{L^1([0,t];L^2)}\right).$$

We must make sure that the last term is, or can be chosen, finite. We shall demand $x\Delta b \in L^\infty([0,T];L^2)$. In view of (2.4), this requirement is met as soon as $a_0 \in H^\infty(\mathbb{R}^d)$ is such that $x\Delta a_0, xa_0|\nabla a_0|^2 \in L^2(\mathbb{R}^d)$. We then have:

$$\begin{aligned} &\text{If } a_0 \in H^\infty(\mathbb{R}^d) \text{ is such that } x\Delta a_0, xa_0|\nabla a_0|^2 \in L^2(\mathbb{R}^d), \text{ then:} \\ &a = b+r, \text{ with } b,r \in C([0,T];H^s)\ \forall s\geqslant 0, \text{ and } xr\in C([0,T];L^2). \end{aligned} \tag{5.3}$$

5.2. **Small time approximation of $\nabla\phi_{\mathrm{eik}}$.** We now prove that for small times, $\nabla\phi_{\mathrm{eik}}(t,x)$ can be approximated by $-t\nabla V(x)$.

Lemma 5.1. *Assume that there exist $0\leqslant k\leqslant 1$ and $C>0$ such that*

$$|\nabla V(x)| \leqslant C\left\langle x\right\rangle^k, \quad \forall x\in\mathbb{R}^d.$$

Then there exist $T_0, C_0 > 0$ such that

$$|\nabla\phi_{\mathrm{eik}}(t,x) + t\nabla V(x)| \leqslant C_0 t^2\left\langle x\right\rangle^k, \quad \forall t\in[0,T_0].$$

Proof of Lemma 5.1. We infer from (1.3) and Lemma 1.1 that

$$|\partial_t\nabla\phi_{\mathrm{eik}}(t,x) + \nabla V(x)| \leqslant \|\nabla^2\phi_{\mathrm{eik}}(t)\|_{L^\infty}|\nabla\phi_{\mathrm{eik}}(t,x)| \lesssim |\nabla\phi_{\mathrm{eik}}(t,x)|. \tag{5.4}$$

From (2.1) and (2.2), we also have

$$\begin{aligned} |\nabla\phi_{\mathrm{eik}}(t,x)| = |\xi\left(t,y(t,x)\right)| &= \left|\int_0^t \nabla V\left(x(s,y(t,x))\right)ds\right| \\ &\lesssim \int_0^t |\nabla V\left(y(t,x)\right)|\,ds + \int_0^t |x(s,y(t,x)) - y(t,x)|\,ds. \end{aligned}$$

We claim that

$$|x(t,y)-y| \lesssim t^2\left\langle y\right\rangle^k. \tag{5.5}$$

Indeed, we have from (2.1),

$$\begin{aligned} |x(t,y)-y| &= \left|\int_0^t \partial_t x(s,y)ds\right| = \left|\int_0^t\int_0^s \nabla V\left(x(s',y)\right)ds'ds\right| \\ &= \left|\int_0^t (t-s')\nabla V\left(x(s',y)\right)ds'\right| \\ &= \left|\int_0^t (t-s)\nabla V\left(y\right)ds + \int_0^t (t-s)\left(\nabla V\left(x(s,y)\right) - \nabla V\left(y\right)\right)ds\right| \\ &\lesssim t^2\left\langle y\right\rangle^k + \int_0^t (t-s)\,|x(s,y)-y|\,ds, \end{aligned}$$

and (5.5) follows from Gronwall lemma. We infer that for $t>0$ sufficiently small,

$$|y(t,x)-x| \lesssim t^2\left\langle x\right\rangle^k,$$

and therefore,

$$\begin{aligned}|\nabla\phi_{\rm eik}(t,x)| &\lesssim \int_0^t |\nabla V\left(y(t,x)\right)|\,ds + \int_0^t |x(s,y(t,x)) - y(t,x)|\,ds\\ &\lesssim \int_0^t |\nabla V\left(x\right)|\,ds + \int_0^t |x - y(t,x)|\,ds + \int_0^t |x(s,y(t,x)) - y(t,x)|\,ds\\ &\lesssim t\left\langle x\right\rangle^k + t^3\left\langle x\right\rangle^k + \int_0^t s^2\left\langle y(t,x)\right\rangle^k ds\\ &\lesssim t\left\langle x\right\rangle^k + t^3\left\langle x\right\rangle^k + t^3\left(\left\langle x\right\rangle^k + t^{2k}\left\langle x\right\rangle^{2k}\right).\end{aligned}$$

Then (5.4) yields

$$|\partial_t\nabla\phi_{\rm eik}(t,x) + \nabla V(x)| \lesssim t\left\langle x\right\rangle^k,$$

Lemma 5.1 follows by integration in time. □

We infer that for $t > 0$ small enough,

$$|\omega\cdot\nabla\phi_{\rm eik}(t,x)| \gtrsim t|\omega\cdot\nabla V(x)|. \tag{5.6}$$

5.3. **Conclusion.** Consider

$$a_0(x) = \frac{1}{\left\langle x\right\rangle^{d/2}\log\left(2+|x|^2\right)}. \tag{5.7}$$

As is easily checked, a_0 meets the requirements of the first line of (5.3). Denote

$$v = be^{i\phi_{\rm eik}} \quad ; \quad w = re^{i\phi_{\rm eik}}.$$

Obviously, $u = v + w$. We see from (5.3) and (5.6) that $v(t,\cdot)\in L^2(\mathbb{R}^d)\setminus H^1(\mathbb{R}^d)$ for $t > 0$ sufficiently small, under the assumptions of Theorem 1.5. On the other hand, $w(t,\cdot)\in H^1(\mathbb{R}^d)$ for all $t\in[0,T]$, hence $u(t,\cdot)\in L^2(\mathbb{R}^d)\setminus H^1(\mathbb{R}^d)$ for $0 < t \ll 1$.

To complete the proof of Theorem 1.5, we now just have to see that the same holds if we replace $H^1(\mathbb{R}^d)$ with $H^s(\mathbb{R}^d)$ for $0 < s < 1$. We use the following characterization of $H^s(\mathbb{R}^d)$ (see e.g. [7]): for $\varphi\in L^2(\mathbb{R}^d)$ and $0 < s < 1$,

$$\varphi\in H^s(\mathbb{R}^d) \iff \iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|\varphi(x+y)-\varphi(x)|^2}{|y|^{d+2s}}dxdy < \infty.$$

Since $w(t,\cdot)\in H^1$ for all $t\in[0,T]$, we shall prove that $v(t,\cdot)\in L^2\setminus H^s$ for t sufficiently small. Let $0 < s < 1$. We prove that for $0 < t \ll 1$,

$$I := \int_{|y|\leqslant 1}\int_{x\in\mathbb{R}^d}\frac{|v(t,x+y)-v(t,x)|^2}{|y|^{d+2s}}dxdy = \infty.$$

To apply a fractional Leibnitz rule, write

$$\begin{aligned}v(t,x+y) - v(t,x) &= \left(b(t,x+y) - b(t,x)\right)e^{i\phi_{\rm eik}(t,x+y)}\\ &\quad + \left(e^{i\phi_{\rm eik}(t,x+y)} - e^{i\phi_{\rm eik}(t,x)}\right)b(t,x).\end{aligned}$$

In view of the inequality $|\alpha-\beta|^2 \geqslant \alpha^2/2 - \beta^2$, we have:

$$\begin{aligned}|v(t,x+y)-v(t,x)|^2 &\geqslant \frac{1}{2}\left|\left(e^{i\phi_{\rm eik}(t,x+y)} - e^{i\phi_{\rm eik}(t,x)}\right)b(t,x)\right|^2\\ &\quad - |b(t,x+y)-b(t,x)|^2.\end{aligned}$$

We can leave out the last term, since $b(t,\cdot)\in H^\infty$ for $t\in[0,T]$:

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|b(t,x+y)-b(t,x)|^2}{|y|^{d+2s}}dxdy < \infty, \quad \forall t\in[0,T].$$

We now want to prove

$$\int_{|y|\leqslant 1}\int_{x\in\mathbb{R}^d}|b(t,x)|^2\frac{\left|\sin\left(\frac{\phi_{\rm eik}(t,x+y)-\phi_{\rm eik}(t,x)}{2}\right)\right|^2}{|y|^{d+2s}}dxdy=\infty.$$

Lemma 1.1 yields:

$$\left(\partial_t+\nabla\phi_{\rm eik}\cdot\nabla\right)\nabla^2\phi_{\rm eik}\in L^\infty\left([0,T]\times\mathbb{R}^d\right)^{d^2}\quad;\quad \nabla^2\phi_{{\rm eik}|t=0}=0.$$

Therefore,

$$\left\|\nabla^2\phi_{\rm eik}(t,\cdot)\right\|_{L^\infty(\mathbb{R}^d)^{d^2}}=\mathcal{O}(t)\quad\text{as } t\to 0.$$

We infer:

$$\phi_{\rm eik}(t,x+y)-\phi_{\rm eik}(t,x)=y\cdot\nabla\phi_{\rm eik}(t,x)+\mathcal{O}(t|y|^2),\quad\text{uniformly for } x\in\mathbb{R}^d,$$

and

$$\sin\left(\frac{\phi_{\rm eik}(t,x+y)-\phi_{\rm eik}(t,x)}{2}\right)=\sin\left(\frac{y\cdot\nabla\phi_{\rm eik}(t,x)}{2}\right)\cos\left(\mathcal{O}(t|y|^2)\right)$$
$$+\cos\left(\frac{y\cdot\nabla\phi_{\rm eik}(t,x)}{2}\right)\sin\left(\mathcal{O}(t|y|^2)\right).$$

The second term is $\mathcal{O}(t|y|^2)$. Using the estimate $|\alpha-\beta|^2\geqslant\alpha^2/2-\beta^2$ again, we see that the integral corresponding to the second term is finite, and can be left out. To prove that

$$I'=\int_{|y|\leqslant 1}\int_{x\in\mathbb{R}^d}|b(t,x)|^2\frac{\left|\sin\left(\frac{y\cdot\nabla\phi_{\rm eik}(t,x)}{2}\right)\right|^2}{|y|^{d+2s}}dxdy=\infty\quad\text{for } 0<t\ll 1,$$

we can localize y in a small conic neighborhood of $\omega\mathbb{R}\cap\{|y|\leqslant 1\}$:

$$\mathcal{V}_\epsilon=\{|y|\leqslant 1\ ;\ |y-(y\cdot\omega)\omega|\leqslant\epsilon|y|\},\quad 0<\epsilon\ll 1.$$

For $0<\epsilon,t\ll 1$, (5.6) yields:

$$\left|\sin\left(\frac{y\cdot\nabla\phi_{\rm eik}(t,x)}{2}\right)\right|\gtrsim t|y\cdot\omega|\times|\omega\cdot\nabla V(x)|,\quad y\in\mathcal{V}_\epsilon.$$

Introduce a conic localization for x close to ω', excluding the origin:

$$\mathcal{U}_\epsilon=\{|x|\geqslant 1\ ;\ |x-(x\cdot\omega')\omega|\leqslant\epsilon|x|\}.$$

Change the variable in the y-integral: for t and ϵ sufficiently small, and $x\in\mathcal{U}_\epsilon$, set

$$y'=\omega\cdot\nabla\phi_{\rm eik}(t,x)y.$$

This change of variable is admissible, from (1.4) and (5.6). For $0<\epsilon,t\ll 1$, we have:

$$\begin{aligned}I'&\geqslant\int_{y\in\mathcal{V}_\epsilon}\int_{x\in\mathbb{R}^d}|b(t,x)|^2\frac{\left|\sin\left(\frac{y\cdot\nabla\phi_{\rm eik}(t,x)}{2}\right)\right|^2}{|y|^{d+2s}}dxdy\\&\gtrsim\int_{x\in\mathcal{U}_\epsilon}|b(t,x)|^2|\omega\cdot\nabla\phi_{\rm eik}(t,x)|^{2s}\left(\int_{y\in|\omega\cdot\nabla\phi_{\rm eik}(t,x)|\mathcal{V}_\epsilon}\frac{dy}{|y|^{d+2s-2}}\right)dx\\&\gtrsim\int_{x\in\mathcal{U}_\epsilon}|b(t,x)|^2|\omega\cdot\nabla\phi_{\rm eik}(t,x)|^{2s}\left(\int_{y\in ct\mathcal{V}_\epsilon}\frac{dy}{|y|^{d+2s-2}}\right)dx.\end{aligned}$$

The assumption (1.4), the expression of b and the choice (5.7) for a_0 then show that for $0<t\ll 1$, $I=\infty$. This completes the proof of Theorem 1.5.

References

[1] N. Burq, P. Gérard, and N. Tzvetkov, *Multilinear eigenfunction estimates and global existence for the three dimensional nonlinear Schrödinger equations*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 2, 255–301.

[2] R. Carles, *Linear vs. nonlinear effects for nonlinear Schrödinger equations with potential*, Commun. Contemp. Math. **7** (2005), no. 4, 483–508.

[3] ———, *Geometric optics and instability for semi-classical Schrödinger equations*, Arch. Ration. Mech. Anal. (2007), to appear (`doi:10.1007/s00205-006-0017-5`).

[4] ———, *WKB analysis for nonlinear Schrödinger equations with potential*, Comm. Math. Phys. **269** (2007), no. 1, 195–221.

[5] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, vol. 10, New York University Courant Institute of Mathematical Sciences, New York, 2003.

[6] T. Cazenave and F. Weissler, *The Cauchy problem for the critical nonlinear Schrödinger equation in H^s*, Nonlinear Anal. TMA **14** (1990), 807–836.

[7] J.-Y. Chemin, *Perfect incompressible fluids*, Oxford Lecture Series in Mathematics and its Applications, vol. 14, The Clarendon Press Oxford University Press, New York, 1998, Translated from the 1995 French original by I. Gallagher and D. Iftimie.

[8] M. Christ, J. Colliander, and T. Tao, *Ill-posedness for nonlinear Schrödinger and wave equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, to appear. See also `arXiv:math.AP/0311048`.

[9] R. Coifman and Y. Meyer, *Commutateurs d'intégrales singulières et opérateurs multilinéaires*, Ann. Inst. Fourier (Grenoble) **28** (1978), no. 3, xi, 177–202.

[10] J. Dereziński and C. Gérard, *Scattering theory of quantum and classical N-particle systems*, Texts and Monographs in Physics, Springer Verlag, Berlin Heidelberg, 1997.

[11] N. Dunford and J. T. Schwartz, *Linear operators. Part II: Spectral theory. Self adjoint operators in Hilbert space*, With the assistance of William G. Bade and Robert G. Bartle, Interscience Publishers John Wiley & Sons New York-London, 1963.

[12] D. Fujiwara, *A construction of the fundamental solution for the Schrödinger equation*, J. Analyse Math. **35** (1979), 41–96.

[13] ———, *Remarks on the convergence of the Feynman path integrals*, Duke Math. J. **47** (1980), no. 3, 559–600.

[14] G. Lebeau, *Non linear optic and supercritical wave equation*, Bull. Soc. Roy. Sci. Liège **70** (2001), no. 4-6, 267–306 (2002), Hommage à Pascal Laubin.

[15] Y.-G. Oh, *Cauchy problem and Ehrenfest's law of nonlinear Schrödinger equations with potentials*, J. Diff. Eq. **81** (1989), no. 2, 255–274.

[16] M. Reed and B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975.

[17] J. T. Schwartz, *Nonlinear functional analysis*, Gordon and Breach Science Publishers, New York, 1969, Notes by H. Fattorini, R. Nirenberg and H. Porta, with an additional chapter by Hermann Karcher, Notes on Mathematics and its Applications.

[18] M. Taylor, *Pseudodifferential operators and nonlinear PDE*, Progress in Mathematics, vol. 100, Birkhäuser Boston Inc., Boston, MA, 1991.

[19] Y. Tsutsumi, *L^2–solutions for nonlinear Schrödinger equations and nonlinear groups*, Funkcial. Ekvac. **30** (1987), no. 1, 115–125.

[20] K. Yajima and G. Zhang, *Smoothing property for Schrödinger equations with potential superquadratic at infinity*, Comm. Math. Phys. **221** (2001), no. 3, 573–590.

[21] ———, *Local smoothing property and Strichartz inequality for Schrödinger equations with potentials superquadratic at infinity*, J. Differential Equations **202** (2004), no. 1, 81–110.

Wolfgang Pauli Institute, c/o Inst. f. Math., Universität Wien, Nordbergstr. 15, A-1090 Wien, Austria[1]

E-mail address: `Remi.Carles@math.cnrs.fr`

[1]Future affiliation: Univ. Montpellier, UMR CNRS 5149, 34095 Montpellier cedex 5, France.