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Hyperbolic representation of light propagation in a multilayer medium

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By analogy with the representation of the polarization of light on the Poincaré sphere, we describe the propagation and the reflection/transmission of light in a multilayer on a hyperbolic surface. We show that the propagation of light corresponds to a classical rotation on this surface and that its reflection/transmission corresponds to a hyperbolic rotation.

1. INTRODUCTION

It is common to describe the evolution of the polarization of light interacting with birefringent and/or rotator systems on the Poincaré sphere. This representation permits a simple and elegant geometrical interpretation of the geometric phase arising when the polarization state of light evolves by passing through birefringent (or rotator) devices. In this physical problem, the polarization state of light is classically described by a complex vector (two components), whereas its evolution is given by the well-known 2×2 Jones matrix of the system.¹⁻³

In the description of the reflection/transmission and of the propagation of light in a multilayer, the fields are represented by their forward and backward components (see Fig. 1), and their evolutions are obtained from the Abeles⁴ reflection/transmission and propagation 2×2 matrices. Thus there is a similarity in the mathematical treatment between the polarization in birefringent/rotator systems and the reflection/transmission in a multilayer device. So it would be of interest to find a geometrical interpretation of the evolution of the electric field as well as of the geometric phase⁵ that appears in a multilayer system.

To obtain such a geometrical representation, we have to find an invariant quantity for the electric field in the multilayer system. This task is presented in Section 2 by considering propagating and conservative waves only, i.e., when there are neither absorption nor evanescent waves in the layers. In Section 3, we study the properties of the propagation matrix and of the reflection/transmission matrix, and we show that these matrices conserve our invariant quantity. In Section 4, we introduce a three-dimensional hyperboloid that permits a geometrical representation of the evolution of the electric field. Then we show that the propagation matrix acts as classical rotations and that the reflection/transmission matrices act as hyperbolic rotations. In Sections 5 and 6, we discuss

all these results, and we finally look at the special case of evanescent waves. We show that, even in that case, a geometrical representation can be made on the hyperboloid surface by making some assumptions.

2. LOOKING FOR AN INVARIANT: THE POYNTING VECTOR

To find a geometrical representation of the electric field in a multilayer system, we will make use of the spinor theory of polarization.

Classically, the polarization state of a monochromatic plane wave traveling in the z direction is described by a two-component complex vector $|\psi\rangle = (E_x, E_y)$, which forms a unitary spinor (i.e., $|E_x|^2 + |E_y|^2 = 1$). In fact, it is more practical^{6,7} to use the following unitary spinor:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \frac{E_x + iE_y}{\sqrt{2}} \\ \frac{E_x - iE_y}{\sqrt{2}} \end{pmatrix}. \quad (1)$$

This polarization state can also be represented in the three-dimensional real space by a point A with three components (x, y, z) given by the following relations:

$$x = \langle \Psi | \sigma_x | \Psi \rangle = \langle \Psi | \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} | \Psi \rangle, \quad (2)$$

$$y = \langle \Psi | \sigma_y | \Psi \rangle = \langle \Psi | \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} | \Psi \rangle, \quad (3)$$

$$z = \langle \Psi | \sigma_z | \Psi \rangle = \langle \Psi | \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} | \Psi \rangle. \quad (4)$$

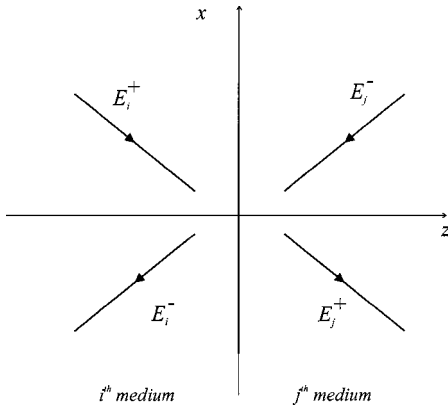


Fig. 1. Decomposition of the electric field in forward and backward components at an interface.

Here, σ_x , σ_y , and σ_z are the well-known Pauli matrices. It is straightforward to demonstrate (when there is no polarizer or dissipative element) that the components x , y , and z satisfy

$$x^2 + y^2 + z^2 = 1. \quad (5)$$

It follows from Eq. (5) that the point A lies on a sphere, which is called the Poincaré sphere. This result shows why it is better to use the spinor $|\Psi\rangle$ than $|\psi\rangle$: With the former, a circular polarization corresponds to a pole of the Poincaré sphere.

As long as there is no dissipative element, the quantity $I_+ = |E_x|^2 + |E_y|^2 = \langle \Psi | \sigma_0 | \Psi \rangle$ is invariant (here σ_0 is the 2×2 unity matrix). The subscript $+$ in I_+ is used to point out that this invariant is the sum of the squared modulus of each spinor component.

After that brief recall of the geometrical representation of a polarization state, let us look at a possible description of the electric field in a multilayer in terms of spinors. We consider monochromatic plane waves, and we first restrict our study to propagating waves only (that is, we do not consider the case of evanescent waves or of absorbing media). The electric (and magnetic) field inside a layer is described as the superposition of two fields: a forward field $E_+(z)$ traveling in the positive z direction and a backward field $E_-(z)$ traveling in the negative z direction (see Fig. 1). The amplitudes of both fields form a complex vector:

$$|\varphi(z)\rangle = \begin{pmatrix} E^+(z) \\ E^-(z) \end{pmatrix}. \quad (6)$$

These fields can propagate between two interfaces. They can also be reflected and transmitted at each interface. These phenomena are described with 2×2 matrices. The propagation matrix gives the link between the vector $|\varphi(z_i)\rangle$ describing the field at the coordinate z_i and the vector $|\varphi(z_j)\rangle$ describing it at the coordinate $z_j = z_i + d_{ji}$. It can be written as

$$|\varphi(z_j)\rangle = [\beta_{ji}] |\varphi(z_i)\rangle = \begin{bmatrix} \exp(i\beta_{ji}) & 0 \\ 0 & \exp(-i\beta_{ji}) \end{bmatrix} |\varphi(z_i)\rangle, \quad (7)$$

with

$$\beta_{ji} = k_i^z(z_j - z_i) = k_i^z d_{ji}, \quad (8)$$

where k_i^z is the component of the wave vector of light in the positive z direction in the i th considered medium and d_{ji} is the distance of propagation in the z direction. Hereafter, all matrices are denoted with a $[\]$ symbol.

The reflection/transmission of light at an interface between two media identified with subscripts i and $j = i + 1$ is also described by a 2×2 matrix. It is well-known that such a matrix is polarization dependent. For simplicity, we consider the case of a TE polarization state only, but there would be no difficulty to obtain the equivalent expression for a TM polarization state. So, for a TE polarization, we obtain

$$\begin{aligned} |\varphi_j\rangle &= \frac{1}{2k_j^z} \begin{bmatrix} k_j^z + k_i^z & k_j^z - k_i^z \\ k_j^z - k_i^z & k_j^z + k_i^z \end{bmatrix} |\varphi_i\rangle \\ &= \frac{1}{t_{ji}} \begin{bmatrix} 1 & r_{ji} \\ t_{ji} & 1 \end{bmatrix} |\varphi_i\rangle = [R_{ji}] |\varphi_i\rangle, \end{aligned} \quad (9)$$

where $|\varphi_i\rangle$ and $|\varphi_j\rangle$ are the electric complex vectors before and behind the interface. We denote this matrix $[R_{ji}]$. In Eq. (9), r_{ji} and t_{ji} are the Fresnel reflection and transmission coefficients at the interface between the j th medium and the i th = (j th - 1) medium.

As for the geometrization of a polarization state, we need to find an invariant equivalent to that of Eq. (5). For this, we will write the Poynting vector in two successive media. The notation is given in Fig. 2. When two fields \mathbf{E}_A and \mathbf{E}_B (with real wave vectors \mathbf{k}_A and \mathbf{k}_B) interfere, the Poynting vector can be shown to be

$$\begin{aligned} \mathbf{P} &= \frac{1}{2\mu_0\omega} [|\mathbf{E}_A|^2 \mathbf{k}_A + |\mathbf{E}_B|^2 \mathbf{k}_B] \\ &+ \frac{1}{4\mu_0\omega} [\mathbf{E}_A \cdot \mathbf{E}_B^* + \mathbf{E}_A^* \cdot \mathbf{E}_B] (\mathbf{k}_A + \mathbf{k}_B), \end{aligned} \quad (10)$$

where $*$ means complex conjugation. In such an equation, we assume that both fields have the same polarization (TE or TM) and that the wave vectors are real. In the case of Fig. 2, the fields \mathbf{E}_A and \mathbf{E}_B correspond to the couples $\{\mathbf{E}_1^+, \mathbf{E}_1^-\}$ or $\{\mathbf{E}_2^+, \mathbf{E}_2^-\}$ with associated wave-vector components $\{(k_1^x, 0, k_1^z), (k_1^x, 0, -k_1^z)\}$ and $\{(k_2^x, 0, k_2^z), (k_2^x, 0, -k_2^z)\}$, respectively (and we have $k_1^x = k_2^x$). By using Eq. (10) with $\mathbf{E}_A = \mathbf{E}_1^+$ and $\mathbf{E}_B = \mathbf{E}_1^-$, one can easily find the z component of the Poynting vector at a point 1 in the first medium just before an interface. We find

$$P_1^z = \frac{k_1^z}{2\mu_0\omega} [|\mathbf{E}_1^+|^2 - |\mathbf{E}_1^-|^2]. \quad (11)$$

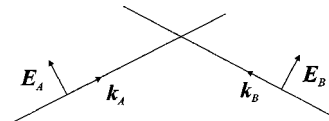


Fig. 2. Notation for the calculation of the Poynting vector of two interfering fields defined by the wave vectors \mathbf{k}_A and \mathbf{k}_B and the electric fields \mathbf{E}_A and \mathbf{E}_B .

Similarly, we obtain the following at a point 2 in the second medium just behind the interface:

$$P_2^z = \frac{k_2^z}{2\mu_0\omega} [|E_2^+|^2 - |E_2^-|^2]. \quad (12)$$

Equations (11) and (12) define the invariant quantity for the multilayer.

To show this, we consider the case of light propagating from a point 1 with z_1 coordinate to another point 2 with z_2 coordinate in the same medium. In that case, using Eq. (7), we find that the couples $\{E_1^+, E_1^-\}$ and $\{E_2^+, E_2^-\}$ are simply linked by the two relations

$$E_2^+ = \exp(i\beta)E_1^+, \quad (13)$$

$$E_2^- = \exp(-i\beta)E_1^-. \quad (14)$$

Inserting Eqs. (13) and (14) into Eqs. (11) and (12) shows that $P_1^z = P_2^z$ when light propagates inside the same medium.

Let us now consider the case of light transmitted from a point 1 just before an interface to a point 2 just behind it. Using in that case Eq. (9), we find that the couples $\{E_1^+, E_1^-\}$ and $\{E_2^+, E_2^-\}$ are now linked by the relations

$$E_2^+ = \frac{1}{t_{21}}E_1^+ + \frac{r_{21}}{t_{21}}E_1^-, \quad (15)$$

$$E_2^- = \frac{r_{21}}{t_{21}}E_1^+ + \frac{1}{t_{21}}E_1^-, \quad (16)$$

where r_{21} and t_{21} are the Fresnel reflection and transmission coefficients at the interface between the second medium and the first medium.

With some easy calculus, it is then simple to see that the quantities P_1^z and P_2^z are also equal whatever may be the polarization (TE or TM). So the z component of the Poynting vector is an invariant quantity for a multilayer made of nonabsorbing media when there are no evanescent waves.

The invariant quantities (11) and (12) depend on the variables E_1^+, E_1^-, E_2^+ , and E_2^- but also on the two components \mathbf{k}_1^z and \mathbf{k}_2^z . To obtain a simpler invariant relation, we introduce, as in Refs. 8 and 9, the variable

$$e_i^\pm = \sqrt{k_i^z}E_i^\pm \quad (17)$$

for the forward and backward fields in the i th medium. Using Eqs. (11), (12), and (17), we can now introduce the invariant quantity I_- . When written in the i th medium, its expression is

$$I_- = |e_i^+|^2 - |e_i^-|^2, \quad (18)$$

which is the same expression as that in Ref. 9. Using the new ‘‘spinor’’ $|\Phi_i\rangle$, which is expressed as

$$|\Phi_i\rangle = \begin{pmatrix} e_i^+ \\ e_i^- \end{pmatrix}, \quad (19)$$

we can write that invariant quantity as

$$I_- = \langle \Phi_i | \sigma_z | \Phi_i \rangle. \quad (20)$$

In terms of the e_i^\pm fields only, the new expression of the $[R_{ji}]$ [Eq. (9)] matrix (which we denote $[r_{ji}]$) is

$$[r_{ji}] = \frac{1}{2\sqrt{k_i^z k_j^z}} \begin{bmatrix} k_j^z + k_i^z & k_j^z - k_i^z \\ k_j^z - k_i^z & k_j^z + k_i^z \end{bmatrix}. \quad (21)$$

Note that, in terms of the e_i^\pm fields, both matrices $[r_{ji}]$ and $[\beta]$ have a unit determinant.

3. GEOMETRICAL REPRESENTATION OF THE ELECTRIC FIELD

We have shown that in a multilayer where there are neither absorption nor evanescent waves, I_- is an invariant quantity. As for the polarization, our aim is now to obtain a three-dimensional geometrical representation of the state of the field e_i^\pm in the i th medium. The idea is to define three coordinates x_i , y_i , and z_i that satisfy an expression equivalent to that of Eq. (5). Let us define these coordinates as

$$x_i = \langle \Phi_i | \sigma_x | \Phi_i \rangle = [e_i^{+*}e_i^- + e_i^+e_i^{-*}], \quad (22)$$

$$y_i = -\langle \Phi_i | \sigma_y | \Phi_i \rangle = i[e_i^{+*}e_i^- - e_i^+e_i^{-*}], \quad (23)$$

$$z_i = \langle \Phi_i | \sigma_z | \Phi_i \rangle = |e_i^+|^2 - |e_i^-|^2. \quad (24)$$

It is easy to verify that these three coordinates satisfy the following equation:

$$z_i^2 - x_i^2 - y_i^2 = [|e_i^+|^2 - |e_i^-|^2]^2 = I_-^2, \quad (25)$$

which is the equation of a hyperboloid made of two sheets. Such a surface is represented in Fig. 3. As in the case of the geometrical representation of all the different states of polarization on the Poincaré sphere, we can normalize the invariant I_- without loss of generality and write

$$x_i = \frac{\langle \Phi_i | \sigma_x | \Phi_i \rangle}{\langle \Phi_i | \sigma_z | \Phi_i \rangle}, \quad (26)$$

$$y_i = \frac{\langle \Phi_i | \sigma_y | \Phi_i \rangle}{\langle \Phi_i | \sigma_z | \Phi_i \rangle}, \quad (27)$$

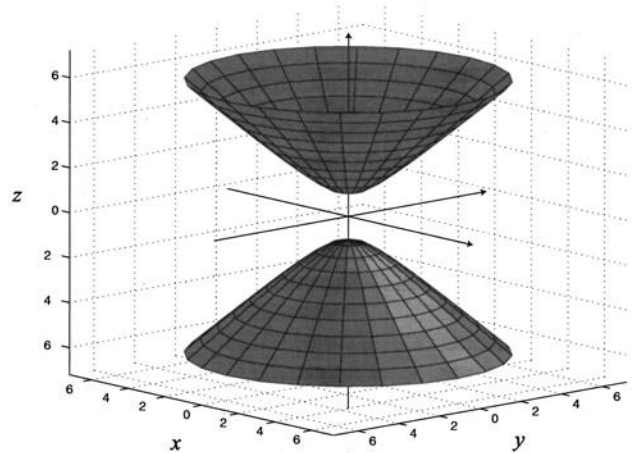


Fig. 3. Hyperboloid surface of two sheets described by the equation $z^2 - x^2 - y^2 = 1$. The geometrical representation of the fields e_i^\pm needs only the positive sheet ($z > 0$) when all the sources are positioned at the left of the multilayer.

$$z_i = \frac{\langle \Phi_i | \sigma_0 | \Phi_i \rangle}{\langle \Phi_i | \sigma_z | \Phi_i \rangle}, \quad (28)$$

$$z_i^2 = 1 + x_i^2 + y_i^2. \quad (29)$$

With such definitions of the three coordinates, the positive sheet of the hyperboloid corresponds to the case where the Poynting vector is oriented from the left to the right, whereas the negative sheet corresponds to the case where the Poynting vector is oriented from the right to the left. Hereafter, we suppose that the three components x_i , y_i , and z_i are normalized by I_- .

4. PROPERTIES OF THE $[\beta]$ AND $[r]$ MATRICES

What is the geometric effect associated with the propagation matrix $[\beta]$? After a propagation inside the same medium, the spinor is obtained from

$$|\Phi_j\rangle = [\beta_{ji}]|\Phi_i\rangle. \quad (30)$$

Then the point A_j , which represents the new field state \mathbf{e}_j on the hyperboloid, is given by the three following coordinates:

$$x_j = \langle \Phi_j | \sigma_x | \Phi_j \rangle / I_- = \langle \Phi_i | [\beta_{ji}]^\dagger \sigma_x [\beta_{ji}] | \Phi_i \rangle / I_-, \quad (31)$$

$$y_j = -\langle \Phi_j | \sigma_y | \Phi_j \rangle / I_- = -\langle \Phi_i | [\beta_{ji}]^\dagger \sigma_y [\beta_{ji}] | \Phi_i \rangle / I_-, \quad (32)$$

$$z_j = \langle \Phi_j | \sigma_0 | \Phi_j \rangle / I_- = \langle \Phi_i | [\beta_{ji}]^\dagger \sigma_0 [\beta_{ji}] | \Phi_i \rangle / I_-, \quad (33)$$

where the dagger means the conjugate transpose of the matrix. After some calculations, we find that (we note that $\beta_{ji} = \beta$)

$$x_j = x_i \cos 2\beta - y_i \sin 2\beta, \quad (34)$$

$$y_j = x_i \sin 2\beta + y_i \cos 2\beta, \quad (35)$$

$$z_j = z_i. \quad (36)$$

It follows from Eqs. (34)–(36) that the propagation of the field inside the same medium is equivalent to a rotation by the angle 2β about the z axis of the hyperboloid.

Let us now study the geometric effect associated with the transmission of light at an interface. The same equations as Eqs. (31)–(33), but with the reflection/transmission matrix $[r_{ji}]$ instead of $[\beta_{ji}]$, give the following relations:

$$x_j = \frac{(k_j^2 + k_i^2)x_i + (k_j^2 - k_i^2)z_i}{2k_i k_j}, \quad (37)$$

$$y_j = y_i, \quad (38)$$

$$z_j = \frac{(k_j^2 - k_i^2)x_i + (k_j^2 + k_i^2)z_i}{2k_i k_j}, \quad (39)$$

where we have dropped the z superscript from the wave vectors k_i and k_j to lighten the notation.

Writing

$$\cosh 2\theta_{ji} = \frac{k_j^2 + k_i^2}{2k_i k_j}, \quad (40)$$

$$\sinh 2\theta_{ji} = \frac{k_j^2 - k_i^2}{2k_i k_j}, \quad (41)$$

we can restate Eqs. (37)–(39) (noting that $\theta_{ji} = \theta$) as

$$x_j = (\cosh 2\theta)x_i + (\sinh 2\theta)z_i, \quad (42)$$

$$y_j = y_i, \quad (43)$$

$$z_j = (\sinh 2\theta)x_i + (\cosh 2\theta)z_i. \quad (44)$$

Equations (37)–(39) can be shown to represent a hyperbolic rotation around the y axis.

With this notation, we find that the Fresnel reflection coefficient can be written in the TE polarization case as

$$r_{ji} = \frac{k_j - k_i}{k_j + k_i} = \tanh \theta_{ji}. \quad (45)$$

In a previous paper,¹⁰ we introduced a hyperbolic formulation of the reflection, transmission, and propagation effects in multilayer systems. In that paper, we used a hyperbolic angle B defined by [see Eq. (15) of Ref. 10]

$$\sinh B = \frac{2r}{1 - r^2}. \quad (46)$$

With our present notation, this angle B is simply equal to the angle 2θ of the hyperbolic rotation.

Thus, to every possible state of a plane monochromatic wave of a given invariant (i.e., $I_- = \text{constant}$), there corresponds one point on the sheet of the hyperboloid [cf. Eq. (29)] and vice versa. Since $2\theta_{ji}$ is positive or negative according to whether the wave propagates to a second medium optically denser or less dense than the first [see Eq. (45)], it follows from Eqs. (42)–(44) that x increases (decreases) when light propagates to a denser (less dense) medium. Figure 4 shows an example of the field state in a multilayer. The rotations around the z axis correspond to the propagation, whereas the hyperbolic rotations correspond to the reflection/transmission between two media.

5. DISCUSSION

Equations (34)–(36), (37)–(39), and (42)–(44) show that different states of the field inside a multilayer can have a

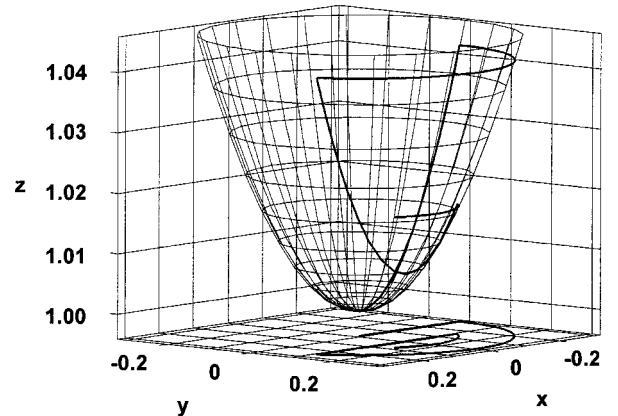


Fig. 4. Example of a path on the hyperboloid surface. The multilayer consists of three media ($n_1 = 1$, $n_2 = 1.5$, $n_3 = 2$ and $d_1 = 100$ nm, $d_2 = 301$ nm). The wavelength is $\lambda = 0.9$ μm .

geometrical representation on a hyperboloid. To be clear, we discuss the physical meaning of the various points on the surface (this is analogous to the interpretation of the Poincaré sphere where the poles represent circularly polarized light, the equator represents linearly polarized light, and positions around the sphere represent the angle of polarization). On the hyperboloid, similar considerations can be made:

- Equations (26)–(29) show that the vertex ($x = y = 0, z = +1$) of the upper sheet represents light propagating in the forward direction only ($e_i^+ \neq 0, e_i^- = 0$, and $I_- = +1$), whereas the vertex ($x = y = 0, z = -1$) of the lower sheet represents light propagating in the backward direction only ($e_i^+ = 0, e_i^- \neq 0$, and $I_- = -1$). To be clear in what follows, we describe only what happens on the positive sheet of the hyperboloid.

- When we move up from that vertex ($x = y = 0, z = +1$) on the hyperboloid, the ratio $|e_i^+/e_i^-|$ decreases progressively. The reflected and forward waves have the same amplitude as z tends to infinity.

- When the amplitudes $|e_i^+|$ and $|e_i^-|$ are fixed (that is to say, when $z = \text{constant}$), the phase between the forward (e_i^+) and reverse (e_i^-) waves varies as we go around the hyperbola. A complete rotation around the z axis corresponds to a layer that has an optical thickness of one half-wave. In particular, in the xz plane ($y = 0$), the forward and reverse waves are in phase for positive values of x and out of phase for negative values of x . Similarly, in the yz plane ($x = 0$), $|e_i^+|$ is exactly $\pi/2$ out of phase with $|e_i^-|$ when y is positive and is $-\pi/2$ out of phase with $|e_i^-|$ when y is negative.

- With the above knowledge, one can interpret the meaning of rotations and hyperbolic rotations: That is, rotations about the z axis correspond to the phase changing between the forward and reverse waves but not to a change in their relative amplitude (they thus correspond to the propagation of light inside the same layer). The hyperbolic rotations correspond to a change in the amplitude of the forward and reverse waves, as happens, for example, when light passes from one side to the other of a boundary between two media of different refractive indices.

We now illustrate such considerations with the simple example of a device made of two plane-parallel interfaces separating three media (numbered 1, 2, and 3) with optical indices of n_1, n_2 , and n_3 , respectively. We suppose that $n_3 > n_2 > n_1$. Let us also consider that the incident wave in the first medium n_1 propagates in the positive z direction (i.e., the path stays on the positive sheet).

- Everywhere in the last medium (number 3) $e_3^- = 0$, so that the point D of the hyperboloid that corresponds to the field is located at $D \equiv (x_D = 0, y_D = 0, z_D = 1)$ [use Eqs. (22)–(24) and note that for a given multilayer, when there is only one source of the field (at $z = -\infty$), the final state always corresponds to the point $D \equiv (x_D = 0, y_D = 0, z_D = 1)$].

- Let us calculate the coordinates of the point C corresponding to the field inside medium 2 just before the last interface. By inverting Eqs. (42)–(44), we obtain the coordinates of the point C from those of the point D :

$$x_C = -(\sinh \theta_{32})z_D, \quad (47)$$

$$y_C = y_D, \quad (48)$$

$$z_C = (\cosh \theta_{32})z_D. \quad (49)$$

The first of these three equations shows that, as expected, x_C is negative (positive) when $n_2 < n_3$ ($n_2 > n_3$).

- We now calculate the coordinates of the point B corresponding to the field inside medium 2 just behind the first interface. By inverting Eqs. (34)–(36), we obtain the coordinates (x_B, y_B, z_B) from the knowledge of (x_C, y_C, z_C) :

$$x_B = (\cos 2\beta)x_C + (\sin 2\beta)y_C, \quad (50)$$

$$y_B = -(\sin 2\beta)x_C + (\cos 2\beta)y_C, \quad (51)$$

$$z_B = z_C. \quad (52)$$

This point B is thus obtained from a rotation of C by the angle -2β around the z axis (the rotation angle is positive when going from B to C).

- The point A in the front of the first interface is obtained from B by using equations similar to Eqs. (47)–(49).

- Another example is the case when the final point D corresponding to the field in the last medium is the same as the initial point A . This can, for example, be the case when an electromagnetic wave propagates through a homogeneous nonabsorbing film. Figure 5, which represents only the projection of the path of light on the Oxy plane, illustrates the problem. The refraction of light at an interface being always described by a rotation around the Oy axis [Eqs. (42)–(44)], such a case can occur only when the point C (corresponding to the field just before the second and last interface) is on the Ox axis (see Fig. 5).

- Two cases can then be considered:

- The first case corresponds to Fig. 5(a). This figure shows that two conditions have to be satisfied. The first is $2\beta = 2\pi m$, where m is an integer. With the use of Eq. (8), this corresponds to $n \cos \theta d = m\lambda/2$. The other condition to get the final point $D \equiv A$ is that the hyperbolic rotation describing the last refraction be the opposite of the first; that is,

$$\sinh 2\theta_{32} = -\sinh 2\theta_{21}. \quad (53)$$

Using Eq. (41), we obtain (in the case of normal incidence) the condition

$$(n_1 - n_3)(n_1 n_3 + n_2^2) = 0. \quad (54)$$

The case of Fig. 5(a) can then be obtained only when the first and last refractive indices are equal ($n_1 = n_3$).

The second possible case is described in Fig. 5(b). In that case, similar considerations lead to the two conditions

$$n \cos \theta d = m\lambda/4, \quad (55)$$

$$\sinh 2\theta_{21} = \sinh 2\theta_{32}. \quad (56)$$

The latter condition gives

$$(n_1 + n_3)(n_1 n_3 - n_2^2) = 0. \quad (57)$$

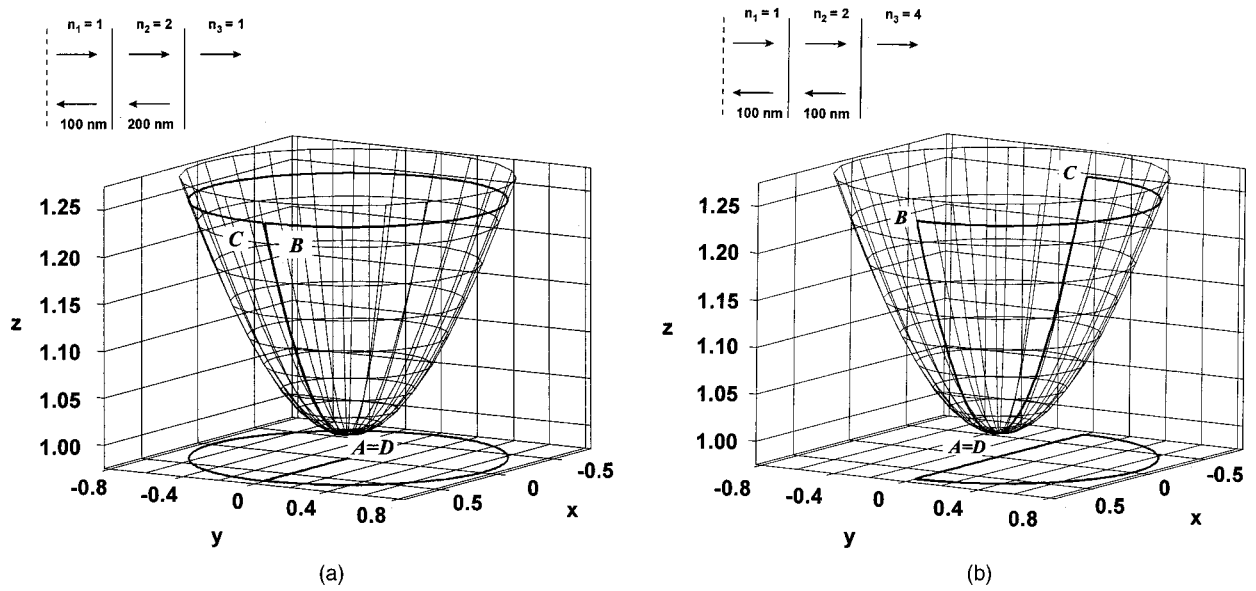


Fig. 5. Projection on the Oxy plane of the path of the state of light on the hyperboloid of Fig. 3 in the two cases where the electromagnetic field is the same before and behind a plane-parallel plate. (a) Three media with $\{n_1, n_2, n_3\} = \{1, 2, 1\}$ refractive indices. The optical path of the light is equal to $\lambda/2$. (b) Three media with $\{n_1, n_2, n_3\} = \{1, 2, 4\}$ refractive indices ($n_1 n_3 = n_2^2$). The optical path of the light is equal to $\lambda/2$.

In this discussion, it is also of interest to know the expression of the most general matrix $[M]$ for which the quantity I_- is an invariant. Let the spinors $|\Phi_2\rangle$ and $|\Phi_1\rangle$ represent the \mathbf{e} fields behind and before a multilayer. The overall system is then defined by the 2×2 matrix $[M]$, and we have

$$|\Phi_2\rangle = [M]|\Phi_1\rangle. \quad (58)$$

With the use of Eq. (20), the condition for I_- to be invariant is

$$\langle \Phi_2 | \sigma_z | \Phi_2 \rangle = \langle \Phi_1 | \sigma_z | \Phi_1 \rangle, \quad (59)$$

so that, with the use of Eq. (58),

$$[M]^\dagger \sigma_z [M] = \sigma_z. \quad (60)$$

The general expression of the matrix $[M]$ can be easily found. It is

$$[M] = \begin{bmatrix} \rho \exp(i\alpha) & \sqrt{\rho^2 - 1} \exp[i(\gamma - \delta + \alpha)] \\ \sqrt{\rho^2 - 1} \exp(i\delta) & \rho \exp(i\gamma) \end{bmatrix}, \quad (61)$$

where ρ , α , γ , and δ are the four parameters of the matrix. If, moreover, we require that $\det([M]) = 1$, as for the $[r_{ji}]$ and $[\beta]$ matrices, then the above expression is reduced to

$$[M] = \begin{bmatrix} \rho \exp(i\alpha) & \sqrt{\rho^2 - 1} \exp(-i\delta) \\ \sqrt{\rho^2 - 1} \exp(i\delta) & \rho \exp(-i\alpha) \end{bmatrix} \\ = \begin{bmatrix} a & b \\ b^* & a^* \end{bmatrix}, \quad (62)$$

where a and b are two complex numbers with $|a|^2 - |b|^2 = 1$. Such a matrix is known as unimodular and “second-order quasi-unitary”¹¹ and belongs to the $SU(1, 1)$ group.

For propagating waves, the $[r_{ji}]$ and $[\beta]$ matrices obviously satisfy Eq. (62). In that case, the $[M]$ matrix in fact reduces to the propagation matrix given by Eq. (7) by substituting $\rho=1$ and $\alpha = \beta_{ji}$ into Eq. (62). It reduces similarly to the reflection matrix given by Eq. (9) by substituting $\rho = 1/t_{ji}$ and $\alpha = \delta = 0$ into Eq. (62). This is no longer the case when total reflection occurs on one interface. In that other case, $\exp(\pm i\beta_{ji})$ in Eq. (7) is to be replaced by $\exp(\pm \beta_{ji})$, and the Fresnel coefficients r_{ji} and t_{ji} are complex. Thus the $[\beta]$ matrix still belongs to the $SU(1, 1)$ group, but the $[r]$ matrix does not.

6. CONCLUSION

We have thus shown that, in the case where there are neither absorption nor evanescent waves, the propagation of light through a multilayer can be illustrated by a path drawn on a hyperboloid. The case of light propagating in a multilayer where there are evanescent waves in one or more layers is more complicated. In fact, in that case neither the propagation matrix $[\beta]$ nor the reflection/transmission matrix $[r_{ji}]$ has the form of Eq. (62). Nevertheless, it is easy to verify that any part of the multilayer where evanescent waves appear can be globally described by a matrix of the form (62) provided that fields are propagative before and after that part. This means that the state of the field, which is characterized by a point A in the $\{x, y, z\}$ space, does not remain on the hyperboloid when there are evanescent waves but that it comes back on it when the waves become propagative again. Of course, it is possible to consider the projection of the complete path of A on the hyperboloid (as we do for the description on the Poincaré sphere when using a po-

larizer that does not conserve the modulus of the electric field). We may expect that such a path could lead to the phase discussed in Ref. 5.

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