A nonlinearly ill-posed problem of reconstructing the temperature from interior data

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A nonlinearly ill-posed problem of reconstructing the temperature from interior data *

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Abstract. We consider the problem of reconstructing, from the interior data $u(x,1)$, a function u satisfying a nonlinear elliptic equation

$$\Delta u = f(x,y,u(x,y)), \qquad x \in \mathbb{R}, y > 0.$$

The problem is ill-posed. Using the method of Green function, the method of Fourier transforms and the method of truncated high frequencies, we shall regularize the problem. Error estimate is given.

Keywords Fourier transform, Contraction, ill-posedness.
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1 Introduction

In this paper, we consider the problem of reconstructing the temperature of a body from interior measurements. In fact, in many engineering contexts (see, e.g., [BBC]), we cannot attach a temperature sensor at the surface of a body (e.g., the skin of a missile). Hence, to get the temperature distribution on the surface, we have to use the temperature measured inside the body.

Precisely, we consider a two-dimensional body represented by the half-plane $\mathbb{R} \times \mathbb{R}^+$. Letting $u(x,y)$ be the temperature of the body at $(x,y) \in \mathbb{R} \times \mathbb{R}^+$ and letting $f \equiv f(x,y,u)$ be a (nonlinear) heat source, we have the following nonlinearly nonhomogeneous equation

$$\Delta u = f, \qquad x \in \mathbb{R}, y > 0 \tag{1}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. We assume that the temperature on the line $y = 1$ is known, i. e.,

$$u(x,1) = \varphi(x), \tag{2}$$

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and that

$$u(x,y) \to 0 \quad \text{as} \quad |x|, y \to \infty. \tag{3}$$

The problem can be referred as a *sideways elliptic problem* and the interior measurement $\varphi(x)$ is also called (in geology) the borehole measurement. The problem can be splitted into two problems:

Problem 1: finding the function u satisfying

$$\Delta u = f, \qquad x \in \mathbb{R}, y > 1 \tag{4}$$

subject to the conditions (2), (3). Generally, the problem is well-posed. Using the solution of the problem we can calculate the quantity $u_y(x,1) = \phi(x)$.

Problem 2: finding a function $u(x,y) \quad x \in \mathbb{R}, 0 < y < 1$
satisfying

$$\Delta u = f, \qquad x \in \mathbb{R}, 0 < y < 1 \tag{5}$$

subject to the conditions $u(x,1) = \varphi(x), u_y(x,1) = \phi(x)$.

The latter problem is a Cauchy elliptic problem and, as known, it is severely ill-posed. Hence, a regularization is in order.

The homogeneous problem ($f \equiv 0$) was studied, by various methods in many papers. Using the mollification method, the homogeneous sideways parabolic problems were considered in [HRS, HR1, AS, L, LV, M] and the references therein. Similarly, the number of papers devoted to the Cauchy problem for linear homogeneous elliptic equation are very rich (see, e.g., [HR, T, B, CHWY, KS]).

Although there are many papers on homogeneous cases, we only find a few papers on nonhomogeneous sideways problems (for both parabolic and elliptic equations). Especially, the papers on the nonlinear case are very rare. In [QT], we use the method of integral equations to consider a sideways elliptic equation with a nonlinear heat source. However, we cannot get an effective method to regularize the problem. In the present paper, we shall consider the problem with a nonlinear heat source $f \equiv f(x,y,u(x,y))$. The remainder of our paper is divided in three sections. In Section 2, using Green functions, we shall transform Problems 1 and 2 into integral equations. In Section 3, we shall prove that Problem 1 has a unique solution. Moreover, we shall give an effective way to approximate the quantity $u_y(x,1)$. In Section 4, using $\varphi(x), u_y(x,1)$, we shall regularize Problem 2. The main result of the section and of the paper is Theorem 3 (in Subsection 4.2). The method of truncated high frequencies (of Fourier images) will be used and the regularized solution can be found as a fixed point of a contraction. In our knowledge, the latter method is new. Error estimates are given.

2 Integral equations of Problems 1 and 2

2.1 Problem 1

Put

$$N(x,y,\xi,\eta) = -\frac{1}{4\pi}\ln\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta-2)^2}. \tag{6}$$

For $y>1, x\in\mathbb{R}$, integrating the identity

$$\frac{\partial}{\partial\xi}(uN_\xi - Nu_\xi) + \frac{\partial}{\partial\eta}(uN_\eta - Nu_\eta) = -Nf \tag{7}$$

over the domain $(-m,m)\times(1,n)\backslash B((x,y),\varepsilon)$, where $B((x,y),\varepsilon)$ is the ball with center at (x,y) and radius $\varepsilon>0$, and letting $n\to\infty, m\to\infty, \varepsilon\to 0$, we get, after some rearrangements,

$$u(x,y) = Au(x,y), \tag{8}$$

where

$$Au(x,y) = h(x,y,\varphi) - \int_{-\infty}^{+\infty}\int_{1}^{+\infty} N(x,y;\xi,\eta)f(\xi,\eta,u(\xi,\eta))d\xi d\eta \tag{9}$$

and

$$h(x,y,\varphi) = \int_{-\infty}^{+\infty} N_\eta(x,y,\xi,1)\varphi(\xi)d\xi.$$

2.2 Problem 2

We repeat $\psi(x) = u_y(x,1)$ with u being an exact solution of Problem 1. We note that we only consider φ as an exact data. The function ψ is a processed data dependent on φ. In fact, in Lemma 1 (Section 3), we shall construct a function ψ_ε which is an approximation of ψ.

Consider

$$\Delta u = f(x,y,u(x,y)), \quad x\in\mathbb{R},\ y\in(0,1)$$

subject to the boundary conditions below

$$u(x,1) = \varphi(x), \quad x\in\mathbb{R}, \tag{10}$$

$$\frac{\partial u}{\partial y}(x,1) = \psi(x), \quad x\in\mathbb{R}.$$

We assume in addition that the exact solution u satisfying $\widehat{\varphi}(\zeta)e^{|\zeta|}, \widehat{\psi}(\zeta)e^{|\zeta|}\in L^2(\mathbb{R})$. We divide this problem into two problems.

Problem 2.1 Consider the problem

$$\Delta v = 0, \quad x\in\mathbb{R}, \quad y\in(0,1), \tag{11}$$

$$v(x,1) = \varphi(x), x \in \mathbb{R}, \tag{12}$$

$$\frac{\partial v}{\partial y}(x,1) = \psi(x), x \in \mathbb{R}. \tag{13}$$

We shall prove that, under appropriate conditions, the problem has a unique solution v_0 approximated by a regularized solution v_ε.

Problem 2.2 Let u_0 be an exact solution of (10). If we put $w_0 = u_0 - v_0$ then w_0 is the solution of the problem

$$\Delta w = g(x,y,w), \;\; x \in \mathbb{R}, \;\; y \in (0,1) \tag{14}$$

$$w(x,1) = 0, \;\; x \in \mathbb{R}, \tag{15}$$

$$\frac{\partial w}{\partial y}(x,1) = 0, \;\; x \in \mathbb{R} \tag{16}$$

where $g(x,y,w) = f(x,y,w+v_0)$. We shall find a w_ε which is an approximation of w_0 and estimate $\|w_\varepsilon - w_0\|_2$ where $\|.\|_2$ is the norm in $L^2(\mathbb{R} \times (0,1))$. Let $u_\varepsilon = v_\varepsilon + w_\varepsilon$, we shall estimate $\|u_\varepsilon - u_0\|_2$.

2.2.1 An integral equation of Problem 2.1

Let

$$\Gamma(x,y,\xi,\eta) = -\frac{1}{4\pi} \ln\left[(x-\xi)^2 + (y-\eta)^2\right]$$

and

$$G(x,y,\xi,\eta) = \Gamma(x,y,\xi,\eta) - \Gamma(x,-y,\xi,\eta). \tag{17}$$

For $x \in \mathbb{R}, 0 < y < 1$, integrating the identity

$$\frac{\partial}{\partial \xi}\left(-vG_\xi + Gv_\xi\right) + \frac{\partial}{\partial \eta}\left(-vG_\eta + Gv_\eta\right) = 0$$

over the domain $(-n,n) \times (0,1) \backslash B((x,y),\varepsilon)$ and letting $n \to \infty, \varepsilon \to 0$, we get, after some rearrangements

$$\begin{aligned} v(x,y) \;\; = \;\; & -\int_{-\infty}^{+\infty} \left[\varphi(\xi) G_\eta(x,y,\xi,1) - G(x,y,\xi,1)\psi(\xi)\right] d\xi + \\ & + \int_{-\infty}^{+\infty} G_\eta(x,y,\xi,0) v(\xi,0) d\xi. \end{aligned} \tag{18}$$

Letting $y \to 1$ in (18), we have

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{1}{(x-\xi)^2+1}v(\xi,0)d\xi+\int_{-\infty}^{+\infty}\left[-\varphi(\xi)G_\eta(x,1,\xi,1)+G(x,1,\xi,1)\psi(\xi)\right]d\xi=\varphi(x).$$

It can be rewritten as

$$F_{(1)}*v_{(0)}(x)=\pi K_{(1)}(x)+\frac{\sqrt{\pi}}{\sqrt{2}}\varphi(x) \tag{19}$$

where

$$\begin{aligned} K_{(y)}(x) &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left[-\varphi(\xi)G_\eta(x,y,\xi,1)+G(x,y,\xi,1)\psi(\xi)\right]d\xi, \\ F_{(y)}(x) &\equiv \frac{y}{x^2+y^2}, v_{(y)}(x)=v(x,y). \end{aligned} \tag{20}$$

Letting

$$M_{(y,1)}(x)=\frac{1-y}{x^2+(y-1)^2}-\frac{1+y}{x^2+(y+1)^2}$$

and

$$L_{(\eta,y)}(x)\equiv\ln\frac{x^2+(y-\eta)^2}{x^2+(y+\eta)^2}\quad(0<y,\eta<1,\ x\in\mathbb{R}),$$

we have the Fourier transform of M, L, F as followed

$$\begin{aligned} \hat{F}_{(y)}(\zeta) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}F_{(y)}(x)e^{-ix\zeta}dx=\frac{\sqrt{\pi}}{\sqrt{2}}e^{-y|\zeta|}, \\ \hat{L}_{(\eta,y)}(\zeta) &= \sqrt{2\pi}\frac{1}{|\zeta|}\left[e^{-(y+\eta)|\zeta|}-e^{-|y-\eta||\zeta|}\right], \\ \hat{M}_{(y,1)}(\zeta) &= \frac{\sqrt{\pi}}{\sqrt{2}}\left[e^{(y-1)|\zeta|}-e^{-(y+1)|\zeta|}\right]. \end{aligned} \tag{21}$$

From (20), (21), we get

$$K_{(y)}(x)=-\frac{1}{4\pi}\left[2\varphi*M_{(y,1)}(x)-\psi*L_{(1,y)}(x)\right]. \tag{22}$$

From (19), (20), we have

$$\hat{v}_{(0)}(\zeta)=e^{|\zeta|}\left(\sqrt{2\pi}\hat{K}_{(1)}(\zeta)+\hat{\varphi}(\zeta)\right). \tag{23}$$

Taking the Fourier transform of (18), we get

$$\begin{aligned}\hat{v}_{(y)}(\zeta) &= e^{|\zeta|}\hat{F}_{(y)}(\zeta)\left(2\hat{K}_{(1)}(\zeta)+\frac{\sqrt{2}}{\sqrt{\pi}}\hat{\varphi}(\zeta)\right)-\sqrt{2\pi}\hat{K}_{(y)}(\zeta)\\ &= \frac{1}{2}\hat{\varphi}(\zeta)\left[e^{(1-y)|\zeta|}+e^{(y-1)|\zeta|}\right]-\frac{1}{2\,|\zeta|}\widehat{\psi}(\zeta)\left[e^{(1-y)|\zeta|}-e^{(y-1)|\zeta|}\right]\equiv\aleph(\zeta,y). \quad (24)\end{aligned}$$

2.2.2 An integral equation of Problem 2.2

We recall that w_0 an exact solution of Problem $(14)-(16)$.

Let $v_0\in L^2(\mathbb{R}\times(0,1))$ be the exact solution of $(11)-(13)$ and $v_\varepsilon\in L^2(\mathbb{R}\times(0,1))$ be a regularized solution.

We write $w_{(y)}(x)=w(x,y)$ and $f_{(\eta,w_0,v_0)}(\xi)=f(\xi,\eta,v_0(\xi,\eta)+w_0(\xi,\eta))$.

For $x\in\mathbb{R}, 0<y<1$, let G be defined in (17), integrating the identity

$$\frac{\partial}{\partial\xi}(-w_0G_\xi+Gw_{0\xi})+\frac{\partial}{\partial\eta}(-w_0G_\eta+Gw_{0\eta})=Gf_{(\eta,w_0,v_0)}(\xi) \quad (25)$$

over the domain $(-n,n)\times(0,1)\backslash B((x,y),\varepsilon)$ and letting $n\to\infty,\varepsilon\to 0$, we get, after some rearrangements,

$$w_0(x,y)=\int_{-\infty}^{+\infty}w_0(\xi,0)G_\eta(x,y,\xi,0)d\xi-\int_{-\infty}^{+\infty}\int_0^1 G(x,y,\xi,\eta)f_{(\eta,w_0,v_0)}(\xi)d\xi d\eta. \quad (26)$$

From (26), we have

$$\begin{aligned}w_0(x,y) &= \frac{1}{\pi}\int_{-\infty}^{+\infty}w_0(\xi,0)\frac{y}{(x-\xi)^2+y^2}d\xi\\ &\quad+\frac{1}{4\pi}\int_0^1\int_{-\infty}^{+\infty}\ln\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2}f_{(\eta,w_0,v_0)}(\xi)d\xi d\eta. \quad (27)\end{aligned}$$

Letting $y\to 1$, we get

$$\frac{1}{4}\int_0^1\int_{-\infty}^{+\infty}\ln\frac{(x-\xi)^2+(1-\eta)^2}{(x-\xi)^2+(1+\eta)^2}f_{(\eta,w_0,v_0)}(\xi)d\xi d\eta+\int_{-\infty}^{+\infty}w_0(\xi,0)\frac{1}{(x-\xi)^2+1}d\xi=0. \quad (28)$$

From (21), we get

$$\begin{aligned}&F_{(y)}(x)\equiv\frac{y}{x^2+y^2}, L_{(\eta,y)}(x)\equiv\ln\frac{x^2+(y-\eta)^2}{x^2+(y+\eta)^2}\quad(0<y,\eta<1,\ x\in\mathbb{R}),\\ &\hat{F}_{(y)}(\zeta)=\frac{\sqrt{\pi}}{\sqrt{2}}e^{-y|\zeta|}, \hat{L}_{(\eta,y)}(\zeta)=\sqrt{2\pi}\frac{1}{|\zeta|}\left[e^{-(y+\eta)|\zeta|}-e^{-|y-\eta||\zeta|}\right]. \quad (29)\end{aligned}$$

We write

$$w_{0(y)}(x) \equiv w_0(x,y).$$

From (28), (29) can be rewritten as

$$w_{0(0)} * F_{(1)}(x) + \frac{1}{4}\int_0^1 L_{(\eta,1)} * f_{(\eta,w_0,v_0)}(x)d\eta = 0.$$

Taking the Fourier transform, we have

$$\widehat{w_{0(0)}}(\zeta).\hat{F}_{(1)}(\zeta) + \frac{1}{4}\int_0^1 \hat{L}_{(\eta,1)}(\zeta)\hat{f}_{(\eta,w_0,v_0)}(\zeta)d\eta = 0.$$

From (29), we have

$$\widehat{w_{0(0)}}(\zeta) = -\frac{1}{2}\int_0^1 \frac{1}{|\zeta|}\left[e^{-\eta|\zeta|} - e^{\eta|\zeta|}\right]\hat{f}_{(\eta,w_0,v_0)}(\zeta)d\eta. \tag{30}$$

From (26), we have

$$w_{0(y)}(x) = \frac{\sqrt{2}}{\sqrt{\pi}}w_{0(0)} * F_{(y)}(x) + \frac{1}{2\sqrt{2\pi}}\int_0^1 L_{(\eta,y)} * f_{(\eta,w_0,v_0)}(x)d\eta. \tag{31}$$

Taking the Fourier transform of (31), we get

$$\widehat{w_{0(y)}}(\zeta) = \frac{\sqrt{2}}{\sqrt{\pi}}\widehat{w_0}_{(0)}(\zeta).\hat{F}_{(y)}(\zeta) + \frac{1}{2\sqrt{2\pi}}\int_0^1 \hat{L}_{(\eta,y)}(\zeta).\hat{f}_{(\eta,w_0,v_0)}(\zeta)d\eta. \tag{32}$$

From (29) and (30), Eq. (32) takes the form

$$\widehat{w_0}_{(y)}(\zeta) = \frac{1}{2}\int_0^1 \frac{1}{|\zeta|}\left[e^{(\eta-y)|\zeta|} - e^{-|y-\eta||\zeta|}\right]\hat{f}_{(\eta,w_0,v_0)}(\zeta)d\eta \tag{33}$$

for ζ. We have

$$\widehat{w_0}_{(y)}(\zeta) = \frac{1}{2\sqrt{2\pi}}\int_0^1\int_{-\infty}^{+\infty} \frac{1}{|\zeta|}\left[e^{(\eta-y)|\zeta|} - e^{-|y-\eta||\zeta|}\right] f_{(\eta,w_0,v_0)}(\xi)e^{-i\xi\zeta}d\xi d\eta. \tag{34}$$

3 Finding the solution of Problem 1

From (8)-(9), we readily get the following result

Theorem 1. *(see [QT]) Suppose that for all* $(\xi,\eta,\zeta)\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}$

$$\left|f'_\zeta(\xi,\eta,\zeta)\right|\le p(\xi,\eta), \tag{35}$$

where $p(\xi,\eta)\in L^1(\mathbb{R}\times(1,+\infty)), p\ge 0$ *satisfies*

$$K\equiv\sup_{(x,y)\in\mathbb{R}\times(1,+\infty)}\left|\int\limits_{-\infty}^{+\infty}\int\limits_{1}^{+\infty}N(x,y;\xi,\eta)p(\xi,\eta)d\xi d\eta\right|<1. \tag{36}$$

Put

$$J=\left\{u\in C(\mathbb{R}\times(1,+\infty))\,\middle|\,\lim_{\sqrt{x^2+y^2}\to+\infty}u(x,y)=0\right\}. \tag{37}$$

Then $A: J\to J$ *is a contraction and hence* u *is uniquely determined and can be found by successive approximation.*

Let $\psi(x)=u_y(x,1)$ with u be an exact solution in the half plane $x\in\mathbb{R}, y>1$. From a measured data φ_ε of $\varphi(x)=u(x,1)$, we construct ψ_ε being an aproximation of ψ and estimate $\|\psi_\varepsilon-\psi\|_{L^2(\mathbb{R})}$ by the following lemma

Lemma 1

Let u *be a solution of (8) as in Theorem 1 and let* $\varphi\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ *satisfy* $\hat\varphi(\zeta)e^{|\zeta|}\in L^2(\mathbb{R})$.

Suppose that f *satisfies the conditions*

$$f(\zeta,\eta,0)=0$$

and

$$|f(\xi,\eta,\zeta_1)-f(\xi,\eta,\zeta_2)|\le|p(\xi,\eta)|\,|\zeta_1-\zeta_2|\quad for\ \ all\ \ (\xi,\eta)\in\mathbb{R}\times\mathbb{R}^+,\zeta_1,\zeta_2\in\mathbb{R}$$

where $p\in L^1(\mathbb{R}\times(1,+\infty))$,

$$K\equiv\sup_{(x,y)\in\mathbb{R}\times(1,+\infty)}\int\limits_{-\infty}^{+\infty}\int\limits_{1}^{+\infty}N(x,y;\xi,\eta)\,|p(\xi,\eta)|\,d\xi d\eta<1$$

and

$$L=\sqrt{\int\limits_{-\infty}^{+\infty}\left[\int\limits_{1}^{+\infty}\int\limits_{-\infty}^{+\infty}\frac{\eta-1}{(x-\xi)^2+(1-\eta)^2}\,|p(\xi,\eta)|\,d\xi d\eta\right]^2dx}<+\infty.$$

For every $0 < \varepsilon < 1$, we call $\varphi_\varepsilon \in L^2(\mathbb{R})$ a measured data such that

$$\|\varphi_\varepsilon - \varphi\|_{L^2(\mathbb{R})} < \varepsilon.$$

From φ_ε, we can construct a function $\psi_\varepsilon \in L^2(\mathbb{R})$ such that

$$\|\psi_\varepsilon - \psi\|_{L^2(\mathbb{R})} < C\varepsilon^{1/2}$$

where C is independent on ε.

Proof
Let

$$k(x,y,u) = -\int_{-\infty}^{+\infty}\int_{1}^{+\infty} N(x,y;\xi,\eta) f(\xi,\eta,u(\xi,\eta)) d\xi d\eta$$

and

$$h(x,y,\varphi) = \int_{-\infty}^{+\infty} N_\eta(x,y,\xi,1)\varphi(\xi)d\xi.$$

We have $u(x,y) = h(x,y,\varphi) + k(x,y,u)$.
Put

$$\widetilde{\varphi_\varepsilon}(x) = \frac{1}{\sqrt{2\pi}} \int_{|\zeta|<\frac{1}{\varepsilon^{1/2}}} \widehat{\varphi_\varepsilon}(\zeta) e^{i\zeta x} d\zeta.$$

We have

$$\widehat{\widetilde{\varphi_\varepsilon}}(\zeta) = \begin{cases} \widehat{\varphi_\varepsilon}(\zeta) & |\zeta| < \frac{1}{\varepsilon^{1/2}} \\ 0 & |\zeta| \geq \frac{1}{\varepsilon^{1/2}} \end{cases} \tag{38}$$

Let u_ε be a solution of (8) with φ replaced by $\widetilde{\varphi_\varepsilon}$, i.e.

$$u_\varepsilon = h(x,y,\widetilde{\varphi_\varepsilon}) + k(x,y,u_\varepsilon). \tag{39}$$

We denote

$$\begin{aligned} h(x,y,\varphi) &= h(x,y), \\ h(x,y,\widetilde{\varphi_\varepsilon}) &= h_\varepsilon(x,y), \\ k(x,y,u) &= k(x,y), \\ k(x,y,u_\varepsilon) &= k_\varepsilon(x,y). \end{aligned} \tag{40}$$

Let

$$\psi_\varepsilon(x) = h_{\varepsilon y}(x,1) + k_{\varepsilon y}(x,1). \tag{41}$$

We have

$$\begin{aligned} h(x,y) &= \int_{-\infty}^{+\infty} N_\eta(x,y;\xi,1)\varphi(\xi)d\xi \\ &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{y-1}{(x-\xi)^2+(y-1)^2}\varphi(\xi)d\xi. \end{aligned} \tag{42}$$

If we put $F_{(y)}(x) \equiv \frac{y}{x^2+y^2}, y>0$, then

$$\hat{F}_{(y)}(\zeta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F_{(y)}(x)e^{-ix\zeta}dx = \frac{\sqrt{\pi}}{\sqrt{2}}e^{-y|\zeta|}.$$

Taking the Fourier transform of (42), we get

$$\widehat{h}(\zeta,y) = \frac{\sqrt{2}}{\sqrt{\pi}}\widehat{F_{(y-1)} * \varphi}(\zeta) = \widehat{\varphi}(\zeta)e^{-(y-1)|\zeta|}$$

and

$$\widehat{h}_y(\zeta,y) = -|\zeta|\,\widehat{\varphi}(\zeta)e^{-(y-1)|\zeta|}.$$

Similarly, we have

$$\widehat{h_\varepsilon}(\zeta,y) = \widehat{\widetilde{\varphi}}(\zeta)e^{-(y-1)|\zeta|}.$$

We shall find an estimation of $\|h_{\varepsilon y}(.,1) - h_y(.,1)\|_{L^2(\mathbb{R})}$.
Using the inequality $u^4 < e^{2u} \quad \forall u > 1$, we have

$$\begin{aligned} \|h_{\varepsilon y}(.,1) - h_y(.,1)\|^2_{L^2(\mathbb{R})} &= \int_{-\infty}^{+\infty} |\zeta|^2 \left|\widehat{\varphi}(\zeta) - \widehat{\widetilde{\varphi}_\varepsilon}(\zeta)\right|^2 d\zeta \\ &\le \int_{|\zeta|<\frac{1}{\varepsilon^{1/2}}} |\zeta|^2 \left|\widehat{\varphi}(\zeta) - \widehat{\varphi_\varepsilon}(\zeta)\right|^2 d\zeta + \int_{|\zeta|>\frac{1}{\varepsilon^{1/2}}} \frac{e^{2|\zeta|}}{|\zeta|^2}\left|\widehat{\varphi}(\zeta)\right|^2 d\zeta. \end{aligned}$$

Therefore

$$\|h_{\varepsilon y}(.,1) - h_y(.,1)\|^2_{L^2(\mathbb{R})} < \varepsilon + \varepsilon \left\|e^{|\zeta|}\widehat{\varphi}(\zeta)\right\|^2_{L^2(\mathbb{R})} = C_1^2\varepsilon$$

where $C_1 = \sqrt{1 + \|e^{|\zeta|}\widehat{\varphi}(\zeta)\|^2_{L^2(\mathbb{R})}}$.
Hence

$$\|h_{\varepsilon y}(.,1) - h_y(.,1)\|_{L^2(\mathbb{R})} < C_1\sqrt{\varepsilon}. \tag{43}$$

We have

$$k(x,y) = -\int_{1}^{+\infty}\int_{-\infty}^{+\infty} N(x,y;\xi,\eta) f(\xi,\eta,u(\xi,\eta)) d\xi d\eta.$$

It follows that

$$k_y(x,1) = \frac{1}{\pi}\int_{1}^{+\infty}\int_{-\infty}^{+\infty} \frac{1-\eta}{(x-\xi)^2+(1-\eta)^2} f(\xi,\eta,u(\xi,\eta)) d\xi d\eta.$$

Similarly, we get

$$k_{\varepsilon y}(x,1) = \frac{1}{\pi}\int_{1}^{+\infty}\int_{-\infty}^{+\infty} \frac{1-\eta}{(x-\xi)^2+(1-\eta)^2} f(\xi,\eta,u_\varepsilon(\xi,\eta)) d\xi d\eta.$$

We have

$$\begin{aligned}
&\|k_{\varepsilon y}(.,1) - k_y(.,1)\|_{L^2(\mathbb{R})} = \\
&= \frac{1}{\pi}\sqrt{\int_{-\infty}^{+\infty}\left(\int_{1}^{+\infty}\int_{-\infty}^{+\infty} \frac{1-\eta}{(x-\xi)^2+(1-\eta)^2}(f(\xi,\eta,u(\xi,\eta)) - f(\xi,\eta,u_\varepsilon(\xi,\eta)) d\xi d\eta\right)^2 dx} \\
&\leq \frac{1}{\pi}\sqrt{\int_{-\infty}^{+\infty}\left[\int_{1}^{+\infty}\int_{-\infty}^{+\infty} \frac{\eta-1}{(x-\xi)^2+(1-\eta)^2}\,|p(\xi,\eta)|\,|u-u_\varepsilon|\,d\xi d\eta\right]^2 dx} \\
&\leq \frac{1}{\pi}\,\|u-u_\varepsilon\|_\infty\, L \qquad (44)
\end{aligned}$$

where

$$L = \sqrt{\int_{-\infty}^{+\infty}\left[\int_{1}^{+\infty}\int_{-\infty}^{+\infty} \frac{\eta-1}{(x-\xi)^2+(1-\eta)^2}\,|p(\xi,\eta)|\,d\xi d\eta\right]^2 dx} < +\infty.$$

Moreover, we have

$$\begin{aligned}
\|u-u_\varepsilon\|_\infty \;\leq\; \sup_{(x,y)\in\mathbb{R}\times(1,+\infty)} \Bigg| &\int_{-\infty}^{+\infty} N_\eta(x,y,\xi,1)\,[\varphi(\xi) - \widetilde{\varphi_\varepsilon}(\xi)]\,d\xi \\
&- \int_{-\infty}^{+\infty}\int_{1}^{+\infty} N(x,y,\xi,\eta)\,(f(\xi,\eta,u(\xi,\eta)) - f(\xi,\eta,u_\varepsilon(\xi,\eta)))\,d\xi d\eta\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{(x,y)\in\mathbb{R}\times(1+\infty)} \left[\int_{-\infty}^{+\infty} \frac{1}{\pi} \frac{y-1}{(x-\xi)^2+(y-1)^2} \left|\varphi(\xi)-\widetilde{\varphi_\varepsilon}(\xi)\right| d\xi \right. \\
&\quad \left. + \int_{-\infty}^{+\infty}\int_{1}^{+\infty} N(x,y,\xi,\eta) \left|p(\xi,\eta)\right| \left|u(\xi,\eta)-u_\varepsilon(\xi,\eta)\right| d\xi d\eta \right] \\
&\leq \left\|\varphi-\widetilde{\varphi_\varepsilon}\right\|_{L^\infty(\mathbb{R})} + K \left\|u-u_\varepsilon\right\|_\infty
\end{aligned}
$$

where

$$
K = \sup_{(x,y)\in\mathbb{R}\times(1+\infty)} \int_{-\infty}^{+\infty}\int_{1}^{+\infty} N(x,y,\xi,\eta) \left|p(\xi,\eta)\right| d\xi d\eta \in (0,1). \tag{45}
$$

Hence

$$
\left\|u-u_\varepsilon\right\|_\infty \leq \frac{1}{1-K} \left\|\varphi-\widetilde{\varphi_\varepsilon}\right\|_{L^\infty(\mathbb{R})}. \tag{46}
$$

We get

$$
\begin{aligned}
\left\|\varphi-\widetilde{\varphi_\varepsilon}\right\|_{L^\infty(\mathbb{R})} &= \sup_{x\in\mathbb{R}} \left|\widetilde{\varphi_\varepsilon}(x)-\varphi(x)\right| \\
&= \sup_{x\in\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left| \int_{|\zeta|<\frac{1}{\varepsilon^{1/2}}} \widehat{\varphi_\varepsilon}(\zeta) e^{i\zeta x} d\zeta - \int_{-\infty}^{+\infty} \widehat{\varphi}(\zeta) e^{i\zeta x} d\zeta \right| \\
&\leq \frac{1}{\sqrt{2\pi}} \left[\int_{|\zeta|<\frac{1}{\varepsilon^{1/2}}} \left|\widehat{\varphi_\varepsilon}(\zeta)-\widehat{\varphi}(\zeta)\right| d\zeta + \int_{|\zeta|>\frac{1}{\varepsilon^{1/2}}} \left|\widehat{\varphi}(\zeta)\right| d\zeta \right] \\
&\leq \frac{1}{\sqrt{2\pi}} \left[\sqrt{\int_{-\frac{1}{\varepsilon^{1/2}}}^{\frac{1}{\varepsilon^{1/2}}} d\zeta \int_{-\frac{1}{\varepsilon^{1/2}}}^{\frac{1}{\varepsilon^{1/2}}} \left|\widehat{\varphi_\varepsilon}(\zeta)-\widehat{\varphi}(\zeta)\right|^2 d\zeta} + \right. \\
&\quad \left. + \sqrt{\int_{|\zeta|>\frac{1}{\varepsilon^{1/2}}} e^{2|\zeta|} \left|\widehat{\varphi}(\zeta)\right|^2 d\zeta . \int_{|\zeta|>\frac{1}{\varepsilon^{1/2}}} e^{-2|\zeta|} d\zeta} \right] \\
&\leq \frac{1}{\sqrt{2\pi}} \left[\frac{\sqrt{2}}{\varepsilon^{1/4}} \left\|\varphi-\varphi_\varepsilon\right\|_{L^2(\mathbb{R})} + \sqrt{\frac{2}{3}} \left\|e^{|\zeta|}\widehat{\varphi}(\zeta)\right\|_{L^2(\mathbb{R})} \varepsilon^{3/4} \right] \\
&< C_2 \varepsilon^{3/4}
\end{aligned} \tag{47}
$$

where

$$C_2 = \frac{1}{\sqrt{2\pi}} \left[\sqrt{2} + \sqrt{\frac{2}{3}} \left\| e^{|\zeta|} \widehat{\varphi}(\zeta) \right\|_{L^2(\mathbb{R})} \right].$$

Inequalities $(44) - (47)$ give

$$\|k_{\varepsilon y}(.,1) - k_y(.,1)\|_{L^2(\mathbb{R})} \leq \frac{L}{\pi(1-K)} \|\varphi - \widetilde{\varphi}_\varepsilon\|_\infty < \frac{LC_2}{\pi(1-K)} \varepsilon^{3/4}. \tag{48}$$

In view of $(38) - (41), (43)$ and (48), we have

$$\begin{aligned} \|\psi_\varepsilon - \psi\|_{L^2(\mathbb{R})} &= \|u_{\varepsilon y}(.,1) - u_y(.,1)\|_{L^2(\mathbb{R})} \\ &= \|h_{\varepsilon y}(.,1) + k_{\varepsilon y}(.,1) - h_y(.,1) - k_y(.,1)\|_{L^2(\mathbb{R})} \\ &\leq \|h_{\varepsilon y}(.,1) - h_y(.,1)\|_{L^2(\mathbb{R})} + \|k_{\varepsilon y}(.,1) - k_y(.,1)\|_{L^2(\mathbb{R})} \\ &< \frac{LC_2}{\pi(1-K)} \varepsilon^{3/4} + C_1 \varepsilon^{1/2} \leq \left(\frac{LC_2}{\pi(1-K)} + C_1 \right) \varepsilon^{1/2}. \end{aligned}$$

This complete the proof of Lemma 1.

4 Regularization of problem

We shall first regularize the Problem 2.1 (Subsection 4.1). Using the approximated solution v_ε of Problem 2.1, we shall regularize the Problem 2.2 (Subsection 4.2).

4.1 Problem 2.1 (Problem (24))

We have the following result

Theorem 1

Let $\varphi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Suppose $\widehat{\varphi}(\zeta) e^{|\zeta|} \in L^2(\mathbb{R}), \widehat{\psi}(\zeta) e^{|\zeta|} \in L^2(\mathbb{R})$. Then Problem $(11) - (13)$ has a unique solution $v_0 \in L^2(\mathbb{R} \times (0,1))$.

Proof

From (24) and the inequality $\frac{e^{|x|}-1}{|x|} \leq e^{|x|}$ we have

$$|\aleph(\zeta, y)| \leq e^{|\zeta|} |\widehat{\varphi}(\zeta)| + e^{|\zeta|} \left| \widehat{\psi}(\zeta) \right| \in L^2(\mathbb{R}), \text{ for all } 0 \leq y \leq 1$$

where $\aleph$ is as in (24).

Hence Problem $(11) - (13)$ has a (unique) solution $v_0 \in L^2(\mathbb{R} \times (0,1))$.

The proof is completed.

Theorem 2

Let assumptions in Theorem 1 hold.

For every $0 < \varepsilon < e^{-3}$, let $\varphi_\varepsilon \in L^2(\mathbb{R})$ be measured data such that $\|\varphi_\varepsilon - \varphi\|_{L^2(\mathbb{R})} < \varepsilon$.

If, in addition, the assumptions of Lemma 1 hold and we have $|\zeta| \widehat{\varphi}(\zeta) e^{|\zeta|} \in L^2(\mathbb{R})$, $|\zeta| \widehat{\psi}(\zeta) e^{|\zeta|} \in L^2(\mathbb{R})$.

Then, from φ_ε, we can construct a regularized solution v_ε such that $\|v_0 - v_\varepsilon\|_2 < D\left(\ln\frac{1}{\varepsilon}\right)^{-1}$, where $\|.\|_2$ is the norm in $L^2(\mathbb{R}\times(0,1))$ and D is a positive constant independent of ε.

Proof

We recall that $\widetilde{\varphi_\varepsilon}$ is defined in Lemma 1, we get

$$\begin{aligned}
\|\widetilde{\varphi_\varepsilon} - \varphi\|^2_{L^2(\mathbb{R})} &= \int\limits_{|\zeta|<\varepsilon^{-1/2}} |\widehat{\widetilde{\varphi_\varepsilon}}(\zeta) - \widehat{\varphi}(\zeta)|^2 \, d\zeta + \int\limits_{|\zeta|>\varepsilon^{-1/2}} |\widehat{\varphi}(\zeta)|^2 \, d\zeta \\
&< \varepsilon^2 + \int\limits_{|\zeta|>\varepsilon^{-1/2}} \frac{|\widehat{\varphi}(\zeta)|^2 \, e^{2|\zeta|}}{|\zeta|^4} d\zeta \\
&< \varepsilon^2 + \varepsilon^2 \left\|\widehat{\varphi}(\zeta) e^{|\zeta|}\right\|^2_{L^2(\mathbb{R})} = \varepsilon^2 C_3^2
\end{aligned}$$

where $C_3 = \sqrt{1 + \|\widehat{\varphi}(\zeta)e^{|\zeta|}\|^2_{L^2(\mathbb{R})}}$.

Therefore $\|\widetilde{\varphi_\varepsilon} - \varphi\|_{L^2(\mathbb{R})} < C_3\varepsilon$.

Put

$$\phi_\varepsilon(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\frac{1}{6}\ln\frac{1}{\varepsilon}}^{\frac{1}{6}\ln\frac{1}{\varepsilon}} \widehat{\widetilde{\varphi}}(\zeta) e^{i\zeta x} d\zeta.$$

We have

$$\widehat{\phi_\varepsilon}(\zeta) = \begin{cases} \widehat{\widetilde{\varphi_\varepsilon}}(\zeta) & |\zeta| < \frac{1}{6}\ln\frac{1}{\varepsilon} \\ 0 & |\zeta| \geq \frac{1}{6}\ln\frac{1}{\varepsilon} \end{cases} .$$

It follows that

$$\begin{aligned}
\left\| \left(\widehat{\phi_\varepsilon}(\zeta) - \widehat{\varphi}(\zeta)\right) e^{|\zeta|} \right\|^2_{L^2(\mathbb{R})} &= \int\limits_{-\frac{1}{6}\ln\frac{1}{\varepsilon}}^{\frac{1}{6}\ln\frac{1}{\varepsilon}} \left| \widehat{\widetilde{\varphi_\varepsilon}}(\zeta) - \widehat{\varphi}(\zeta) \right|^2 e^{2|\zeta|} d\zeta + \int\limits_{|\zeta|>\frac{1}{6}\ln\frac{1}{\varepsilon}} |\widehat{\varphi}(\zeta)|^2 \, e^{2|\zeta|} d\zeta \\
&\leq e^{\frac{1}{3}\ln\frac{1}{\varepsilon}} \|\widetilde{\varphi_\varepsilon} - \varphi\|^2_{L^2(\mathbb{R})} + \frac{1}{\left(\frac{1}{6}\ln\frac{1}{\varepsilon}\right)^2} \left\|\zeta\widehat{\varphi}(\zeta) e^{|\zeta|}\right\|^2_{L^2(\mathbb{R})} \\
&< C_3^2 e^{\frac{1}{3}\ln\frac{1}{\varepsilon}} \varepsilon^2 + \frac{1}{\left(\frac{1}{6}\ln\frac{1}{\varepsilon}\right)^2} \left\|\zeta\widehat{\varphi}(\zeta) e^{|\zeta|}\right\|^2_{L^2(\mathbb{R})} \\
&< C_3^2 \varepsilon^{5/3} + \frac{36}{\ln^2\frac{1}{\varepsilon}} \left\|\zeta\widehat{\varphi}(\zeta) e^{|\zeta|}\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Using the inequality $\varepsilon^{5/3} < \frac{1}{\ln^2\frac{1}{\varepsilon}}$ as $\varepsilon < e^{-3}$, we get

$$\left\| \left(\widehat{\phi_\varepsilon}(\zeta) - \widehat{\varphi}(\zeta)\right) e^{|\zeta|} \right\|^2_{L^2(\mathbb{R})} < C_4^2 \frac{1}{\ln^2\frac{1}{\varepsilon}},$$

where

$$C_4 = \sqrt{C_3^2 + 36 \left\| \zeta \widehat{\varphi}(\zeta) e^{|\zeta|} \right\|^2_{L^2(\mathbb{R})}}. \tag{49}$$

Using lemma 1, there exists $\psi_\varepsilon \in L^2(\mathbb{R})$ such that

$$\|\psi_\varepsilon - \psi\|_{L^2(\mathbb{R})} < C\varepsilon^{1/2}.$$

Put

$$\Psi_\varepsilon(x) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{1}{6}\ln\frac{1}{\varepsilon}}^{\frac{1}{6}\ln\frac{1}{\varepsilon}} \widehat{\Psi_\varepsilon}(\zeta) e^{i\zeta x} d\zeta.$$

We have

$$\widehat{\Psi_\varepsilon}(\zeta) = \begin{cases} \widehat{\psi_\varepsilon}(\zeta) & |\zeta| < \frac{1}{6}\ln\frac{1}{\varepsilon} \\ 0 & |\zeta| \geq \frac{1}{6}\ln\frac{1}{\varepsilon} \end{cases}.$$

Similarly, using the inequality $\varepsilon^{2/3} < \frac{1}{\ln^2 \frac{1}{\varepsilon}}$ as $\varepsilon < e^{-3}$, we get

$$\left\| \left(\widehat{\Psi_\varepsilon}(\zeta) - \widehat{\psi}(\zeta) \right) e^{|\zeta|} \right\|^2_{L^2(\mathbb{R})} < C_5^2 \frac{1}{\ln^2 \frac{1}{\varepsilon}},$$

where

$$C_5 = \sqrt{C^2 + 36 \left\| \zeta \widehat{\psi}(\zeta) e^{|\zeta|} \right\|^2_{L^2(\mathbb{R})}}. \tag{50}$$

We put

$$\aleph_\varepsilon(\zeta, y) = \frac{1}{2} \widehat{\phi}_\varepsilon(\zeta) \left[e^{(1-y)|\zeta|} + e^{(y-1)|\zeta|} \right] - \frac{1}{2|\zeta|} \widehat{\Psi_\varepsilon}(\zeta) \left[e^{(1-y)|\zeta|} - e^{(y-1)|\zeta|} \right]$$

and

$$v_\varepsilon(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \aleph_\varepsilon(\zeta, y) e^{i\zeta x} d\zeta.$$

From (24), (50), we get

$$\begin{aligned} &\|v_\varepsilon - v_0\|_2 = \|\aleph_\varepsilon - \aleph\|_2 \\ &\leq \frac{1}{2} \left(\left\| \left(\widehat{\varphi}(\zeta) - \widehat{\phi}_\varepsilon(\zeta) \right) \left[e^{(1-y)|\zeta|} + e^{(y-1)|\zeta|} \right] \right\|_2 + \left\| \frac{\left(\widehat{\psi}(\zeta) - \widehat{\Psi_\varepsilon}(\zeta) \right)}{|\zeta|} \left[e^{(1-y)|\zeta|} - e^{(y-1)|\zeta|} \right] \right\|_2 \right) \\ &\leq \left\| (\widehat{\varphi}(\zeta) - \widehat{\phi}_\varepsilon(\zeta)) e^{|\zeta|} \right\|_2 + \left\| (\widehat{\psi}(\zeta) - \widehat{\Psi_\varepsilon}(\zeta)) e^{|\zeta|} \right\|_2 \\ &< (C_4 + C_5) \left(\ln \frac{1}{\varepsilon} \right)^{-1}. \end{aligned}$$

Hence $\|v_0 - v_\varepsilon\|_2 < D\left(\ln \frac{1}{\varepsilon}\right)^{-1}$.

The proof is completed.

4.2 Problem 2.2 (Problem (34))

Theorem 3

Let assumptions in Theorem 1 and Theorem 2 hold.

Let $v_0 \in L^2(\mathbb{R}\times(0,1))$ be an exact solution of $(11)-(13)$ and $v_\varepsilon \in L^2(\mathbb{R}\times(0,1))$ be the regularized solution of $(11)-(13)$.

Suppose that f satisfies the conditions (27) and $|p(\xi,\eta)| \le k\ \forall(\xi,\eta)\in\mathbb{R}\times(0,1)$.

Assume in addition that the exact solution $u_0 = v_0 + w_0 \in L^2(\mathbb{R}\times(0,1))$ of (10) satisfying

$$e^{3|\zeta|}\left|\widehat{f}_{(\eta,v_0,w_0)}(\zeta)\right| \in L^2(\mathbb{R}\times(0,1)).$$

Then there exists a regularized solutions w_ε and $u_\varepsilon = v_\varepsilon + w_\varepsilon$ of (10) such that

$$\|w_\varepsilon - w_0\|_2 < C\left(\ln\frac{1}{\varepsilon}\right)^{-1/2}$$

and

$$\|u_\varepsilon - u_0\|_2 < E\left(\ln\frac{1}{\varepsilon}\right)^{-1/2}$$

where C and E independent of ε.

Proof

From (33), we have

$$\begin{aligned}\widehat{w_0}_{(y)}(\zeta) &= \frac{1}{2}\int_0^1 \frac{1}{|\zeta|}\left[e^{(\eta-y)|\zeta|} - e^{-|y-\eta||\zeta|}\right]\widehat{f}_{(\eta,w_0,v_0)}(\zeta)d\eta \\ &= \frac{1}{2}\int_y^1 \frac{1}{|\zeta|}\left[e^{(\eta-y)|\zeta|} - e^{(y-\eta)|\zeta|}\right]\widehat{f}_{(\eta,w_0,v_0)}(\zeta)d\eta.\end{aligned}$$

We put

$$T\left(w^{(\alpha)}(x,y)\right) = \frac{1}{2\sqrt{2\pi}}\int_y^1\int_{-\alpha}^{\alpha}\frac{1}{|\zeta|}\left[e^{(\eta-y)|\zeta|} - e^{(y-\eta)|\zeta|}\right]\widehat{f}_{(\eta,w^{(\alpha)},v_\varepsilon)}(\zeta)e^{i\zeta x}d\zeta d\eta$$

We shall prove

$$\left\|T^m(w^{(\alpha)}(.,y)) - T^m(w^{1(\alpha)}(.,y))\right\|^2_{L^2(\mathbb{R})} \le \left(k^2e^{2\alpha}\right)^m\frac{(1-y)^m}{m!}\left|\left|\left|w^{(\alpha)} - w^{1(\alpha)}\right|\right|\right|^2 \qquad (51)$$

for every $\alpha > 0, y \in (0,1), m \ge 1$ and $w^{(\alpha)}, w^{1(\alpha)} \in C([0,1];L^2(\mathbb{R}))$, where $|||.|||$ denotes sup-norm in $C([0,1];L^2(\mathbb{R}))$.

We shall prove (51) by induction.
We define

$$\chi_{[-\alpha,\alpha]}(\zeta) = \begin{cases} 1 & |\zeta| \le \alpha \\ 0 & |\zeta| > \alpha \end{cases} .$$

For $m = 1$, noting that

$$\widehat{Tw^{(\alpha)}}(\zeta, y) = \frac{1}{2}\chi_{[-\alpha,\alpha]}(\zeta) \int_y^1 \frac{1}{|\zeta|} \left[e^{(\eta-y)|\zeta|} - e^{(y-\eta)|\zeta|}\right] \widehat{f}_{(\eta, w^{(\alpha)}, v_\varepsilon)}(\zeta) d\eta,$$

we have

$$\begin{aligned}
&\left\| Tw^{(\alpha)}(.,y) - Tw^{1(\alpha)}(.,y) \right\|^2_{L^2(\mathbb{R})} = \\
&= \left\| \widehat{Tw^{(\alpha)}}(.,y) - \widehat{Tw^{1(\alpha)}}(.,y) \right\|^2_{L^2(\mathbb{R})} \\
&= \frac{1}{4} \left\| \int_y^1 \chi_{[-\alpha,\alpha]}(\zeta) \frac{e^{(\eta-y)|\zeta|} - e^{(y-\eta)|\zeta|}}{|\zeta|} \left(\widehat{f}_{(\eta, w^{(\alpha)}, v_\varepsilon)}(\zeta) - \widehat{f}_{(\eta, w^{1(\alpha)}, v_\varepsilon)}(\zeta) \right) d\eta \right\|^2_{L^2(\mathbb{R})} \\
&\le \int_{-\infty}^{+\infty} \left| \int_y^1 \chi_{[-\alpha,\alpha]}(\zeta) e^{(\eta-y)|\zeta|} \left| \widehat{f}_{(\eta, w^{(\alpha)}, v_\varepsilon)}(\zeta) - \widehat{f}_{(\eta, w^{1(\alpha)}, v_\varepsilon)}(\zeta) \right| d\eta \right|^2 d\zeta \\
&\le \int_{-\infty}^{+\infty} \left| \int_y^1 e^{(\eta-y)\alpha} \left| \widehat{f}_{(\eta, w^{(\alpha)}, v_\varepsilon)}(\zeta) - \widehat{f}_{(\eta, w^{1(\alpha)}, v_\varepsilon)}(\zeta) \right| d\eta \right|^2 d\zeta \\
&\le e^{2\alpha}(1-y) \int_{-\infty}^{+\infty} \int_y^1 \left| \widehat{f}_{(\eta, w^{(\alpha)}, v_\varepsilon)}(\zeta) - \widehat{f}_{(\eta, w^{1(\alpha)}, v_\varepsilon)}(\zeta) \right|^2 d\eta d\zeta \\
&\le k^2 e^{2\alpha}(1-y) \int_y^1 \int_{-\infty}^{+\infty} \left| w^{(\alpha)}(x,\eta) - w^{1(\alpha)}(x,\eta) \right|^2 dx d\eta \\
&\le e^{2\alpha}(1-y)k^2 \int_y^1 \left\| w^{(\alpha)}(.,\eta) - w^{1(\alpha)}(.,\eta) \right\|^2_{L^2(\mathbb{R})} d\eta \\
&\le e^{2\alpha}(1-y)k^2 \left| \left\| w^{(\alpha)} - w^{1(\alpha)} \right\| \right|^2 .
\end{aligned}$$

Therefore (51) holds.

Suppose that (51) holds for $m = j$. We shall prove that (51) holds for $m = j+1$. We have

$$\begin{aligned}
&\left\|T^{j+1}\left(w^{(\alpha)}(.,y)\right) - T^{j+1}\left(w^{1(\alpha)}(.,y)\right)\right\|^2_{L^2(\mathbb{R})} = \\
&= \left\|\widehat{T}\left(T^j\left(w^{(\alpha)}(.,y)\right)\right) - \widehat{T}\left(T^j\left(w^{1(\alpha)}(.,y)\right)\right)\right\|^2_{L^2(\mathbb{R})} \\
&\leq k^2 e^{2\alpha}(1-y)\int_y^1 \left\|T^j\left(w^{(\alpha)}(.,\eta)\right) - T^j\left(w^{1(\alpha)}(.,\eta)\right)\right\|^2_{L^2(\mathbb{R})} d\eta \\
&\leq k^2 e^{2\alpha}(1-y)\int_y^1 \left(k^2 e^{2\alpha}\right)^j \frac{(1-\eta)^j}{j!} \left|\left\|w^{(\alpha)} - w^{1(\alpha)}\right\|\right|^2 d\eta \\
&\leq \left(k^2 e^{2\alpha}\right)^{j+1} \frac{(1-y)^{j+1}}{(j+1)!} \left|\left\|w^{(\alpha)} - w^{1(\alpha)}\right\|\right|^2.
\end{aligned}$$

Therefore, by the induction principle, (51) holds for every m. From (51), we get

$$\left|\left\|T^m\left(w^{(\alpha)}\right) - T^m\left(w^{1(\alpha)}\right)\right\|\right|^2 \leq \left(k^2 e^{2\alpha}\right)^m \frac{1}{m!} \left|\left\|w^{(\alpha)} - w^{1(\alpha)}\right\|\right|^2.$$

Since $\lim_{m\to\infty} \left(k^2 e^{2\alpha}\right)^m \frac{1}{m!} = 0$, there exists a positive integer number m_0 such that

$$\left(k^2 e^{2\alpha}\right)^{m_0} \frac{1}{m_0!} < 1.$$

Hence T^{m_0} is a contraction in $C\left([0,1]; L^2(\mathbb{R})\right)$. It follows that the equation $T^{m_0}(w^{(\alpha)}) = w^{(\alpha)}$ has a unique solution $w^{(\alpha)} \in C\left([0,1]; L^2(\mathbb{R})\right)$.

We claim that $T(w^{(\alpha)}) = w^{(\alpha)}$. In fact, one has $T(T^{m_0}(w^{(\alpha)})) = T(w^{(\alpha)})$. Hence $T^{m_0}(T(w^{(\alpha)})) = T(w^{(\alpha)})$. By the uniqueness of the fixed point of T^{m_0}, one has $T(w^{(\alpha)}) = w^{(\alpha)}$, i.e., the equation $T(w^{(\alpha)}) = w^{(\alpha)}$ has a unique solution $w^{(\alpha)} \in C\left([0,1]; L^2(\mathbb{R})\right)$.

We have

$$\begin{aligned}
&4\left\|w_{0(y)} - w^{(\alpha)}(.,y)\right\|^2_{L^2(\mathbb{R})} \\
&= \int_{-\alpha}^{\alpha} \left| \int_y^1 \frac{\left[e^{(\eta-y)|\zeta|} - e^{(y-\eta)|\zeta|}\right]}{|\zeta|} \left[\widehat{f}_{(\eta,w_0,v_0)}(\zeta) - \widehat{f}_{(\eta,w^{(\alpha)},v_\varepsilon)}(\zeta)\right] d\eta \right|^2 d\zeta \\
&+ \int_{|\zeta|>\alpha} \left| \int_y^1 \frac{\left[e^{(\eta-y)|\zeta|} - e^{(y-\eta)|\zeta|}\right]}{|\zeta|} \widehat{f}_{(\eta,w_0,v_0)}(\zeta) d\eta \right|^2 d\zeta
\end{aligned}$$

$$\leq 4 \int_{-\alpha}^{\alpha} \left| \int_{y}^{1} e^{(\eta-y)|\zeta|} \left| \widehat{f}_{(\eta,w_0,v_0)}(\zeta) - \widehat{f}_{(\eta,w^{(\alpha)},v_\varepsilon)}(\zeta) \right| d\eta \right|^2 d\zeta$$

$$+ 4 \int_{|\zeta|>\alpha} \left| \int_{y}^{1} e^{(\eta-y)|\zeta|} \widehat{f}_{(\eta,w_0,v_0)}(\zeta) d\eta \right|^2 d\zeta$$

$$\leq 4 \int_{-\alpha}^{\alpha} \left| \int_{y}^{1} e^{(\eta-y)\alpha} \left| \widehat{f}_{(\eta,w_0,v_0)}(\zeta) - \widehat{f}_{(\eta,w^{(\alpha)},v_\varepsilon)}(\zeta) \right| d\eta \right|^2 d\zeta$$

$$+ 4e^{-2y\alpha} \int_{|\zeta|>\alpha} \left| \int_{y}^{1} e^{(\eta-y)|\zeta|+y\alpha} \widehat{f}_{(\eta,w_0,v_0)}(\zeta) d\eta \right|^2 d\zeta$$

$$\leq 4e^{-2y\alpha} \left\{ \int_{-\alpha}^{\alpha} \int_{y}^{1} e^{2\eta\alpha} \left| \widehat{f}_{(\eta,w_0,v_0)}(\zeta) - \widehat{f}_{(\eta,w^{(\alpha)},v_\varepsilon)}(\zeta) \right|^2 d\eta d\zeta \right.$$

$$\left. + \int_{|\zeta|>\alpha} \left| \int_{y}^{1} e^{(\eta-y)|\zeta|+y\alpha} \widehat{f}_{(\eta,w_0,v_0)}(\zeta) d\eta \right|^2 d\zeta \right\}$$

$$\leq 4e^{-2y\alpha} \left\{ \int_{y}^{1} e^{2\eta\alpha} \left(\int_{-\infty}^{+\infty} \left| f_{(\eta,w_0,v_0)}(\xi) - f_{(\eta,w^{(\alpha)},v_\varepsilon)}(\xi) \right|^2 d\xi \right) d\eta \right.$$

$$\left. + \int_{|\zeta|>\alpha} \left| \int_{y}^{1} e^{|\zeta|+\alpha} \left| \widehat{f}_{(\eta,w_0,v_0)}(\zeta) \right| d\eta \right|^2 d\zeta \right\}$$

$$\leq 4e^{-2y\alpha} \left\{ \int_{y}^{1} e^{2\eta\alpha} \int_{-\infty}^{+\infty} k^2 \left| v_0(\xi,\eta) - v_\varepsilon(\xi,\eta) + w_0(\xi,\eta) - w^{(\alpha)}(\xi,\eta) \right|^2 d\xi d\eta \right.$$

$$\left. + \int_{|\zeta|>\alpha} e^{2(|\zeta|+\alpha)} \int_{y}^{1} \left| \widehat{f}_{(\eta,w_0,v_0)}(\zeta) \right|^2 d\eta d\zeta \right\}$$

$$\leq 4e^{-2y\alpha} \left\{ 2k^2 \int_{y}^{1} e^{2\eta\alpha} \left\| w_0(.,\eta) - w^{(\alpha)}(.,\eta) \right\|^2_{L^2(\mathbb{R})} d\eta \right.$$

$$\left. + 2 \int_{0}^{1} \int_{-\infty}^{+\infty} e^{2\alpha} k^2 \left| v_0(\xi,\eta) - v_\varepsilon(\xi,\eta) \right|^2 d\xi d\eta \right.$$

$$
\begin{aligned}
&+ \int\limits_{|\zeta|>\alpha} e^{2(|\zeta|+\alpha)} \int\limits_{y}^{1} \left| \widehat{f}_{(\eta,w_0,v_0)}(\zeta) \right|^2 d\eta d\zeta \Bigg\} \\
&\leq 4e^{-2y\alpha} \Bigg\{ 2k^2 \int\limits_{y}^{1} e^{2\eta\alpha} \left\| w_0(.,\eta) - w^{(\alpha)}(.,\eta) \right\|^2_{L^2(\mathbb{R})} d\eta \\
&+ 2k^2 e^{2\alpha} \left\| v_0 - v_\varepsilon \right\|_2^2 \\
&+ \int\limits_{|\zeta|>\alpha} e^{2(|\zeta|+\alpha)} \int\limits_{y}^{1} \left| \widehat{f}_{(\eta,w_0,v_0)}(\zeta) \right|^2 d\eta d\zeta \Bigg\}.
\end{aligned}
$$

We put

$$M_1 = 2k^2 e^{2\alpha} \left\| v_0 - v_\varepsilon \right\|_2^2$$

and

$$M_2 = \int\limits_{|\zeta|>\alpha} e^{2(|\zeta|+\alpha)} \int\limits_{y}^{1} \left| \widehat{f}_{(\eta,w_0,v_0)}(\zeta) \right|^2 d\eta d\zeta.$$

Therefore

$$e^{2y\alpha} \left\| w_{o(y)} - w^{(\alpha)}(.,y) \right\|^2_{L^2(\mathbb{R})} \leq 2k^2 \int\limits_{y}^{1} e^{2\eta\alpha} \left\| w_o(.,\eta) - w^{(\alpha)}(.,\eta) \right\|^2_{L^2(\mathbb{R})} d\eta + M_1 + M_2.$$

Using Gronwall's inequality, we have

$$e^{2y\alpha} \left\| w_o(.,y) - w^{(\alpha)}(.,y) \right\|^2_{L^2(\mathbb{R})} \leq (M_1 + M_2)\, e^{2k^2(1-y)},$$

hence

$$\left\| w_o(.,y) - w^{(\alpha)}(.,y) \right\|^2_{L^2(\mathbb{R})} \leq e^{-2\alpha y}(M_1 + M_2) e^{2k^2(1-y)}. \tag{52}$$

From Theorem 2, we get

$$
\begin{aligned}
e^{-2\alpha y} M_1 &= 2e^{2(1-y)\alpha} k^2 \left\| v_0 - v_\varepsilon \right\|_2^2 \\
&< 2D^2 e^{2(1-y)\alpha} k^2 \left(\ln \frac{1}{\varepsilon} \right)^{-2}.
\end{aligned} \tag{53}
$$

We have

$$e^{-2\alpha y} M_2 = e^{-2\alpha y} \int\limits_{|\zeta|>\alpha} e^{2(|\zeta|+\alpha)} \int\limits_{y}^{1} \left| \widehat{f}_{(\eta,v_0,w_0)}(\zeta) \right|^2 d\eta d\zeta$$

$$\leq e^{-2\alpha y} \int\limits_{|\zeta|>\alpha} \int\limits_{0}^{1} \frac{e^{6|\zeta|}}{e^{2\alpha}} \left| \widehat{f}_{(\eta,v_0,w_0)}(\zeta) \right|^2 d\eta d\zeta$$

$$\leq e^{-2\alpha(y+1)} \left\| e^{3|\zeta|} \left| \widehat{f}_{(\eta,v_0,w_0)}(\zeta) \right| \right\|_2^2. \tag{54}$$

From (52) − (54) and choosing $\alpha = \frac{1}{2(1-y)} \ln\left(\ln \frac{1}{\varepsilon}\right)$, we get

$$\begin{aligned} \left\| w_o(.,y) - w^{(\alpha)}(.,y) \right\|_{L^2(\mathbb{R})}^2 &< e^{2k^2} \left[2k^2 D^2 \left(\ln \frac{1}{\varepsilon} \right)^{-1} + \left\| e^{3|\zeta|} \left| \widehat{f}_{(\eta,v_0,w_0)}(\zeta) \right| \right\|_2^2 \left(\ln \frac{1}{\varepsilon} \right)^{-1} \right] \\ &< C^2 \left(\ln \frac{1}{\varepsilon} \right)^{-1}. \end{aligned}$$

Therefore

$$\left\| w_o - w^{(\alpha)} \right\|_2 < C \left(\ln \frac{1}{\varepsilon} \right)^{-1/2}.$$

Denoting $w_\varepsilon = w^{(\alpha)}$ and $u_\varepsilon = v_\varepsilon + w_\varepsilon$, we get

$$\begin{aligned} \|u_0 - u_\varepsilon\|_2 &= \|w_0 + v_0 - v_\varepsilon - w_\varepsilon\|_2 \\ &\leq \left\| w_0 - w^{(\alpha)} \right\|_2 + \|v_0 - v_\varepsilon\|_2 \\ &\leq C \left(\ln \frac{1}{\varepsilon} \right)^{-1/2} + D \left(\ln \frac{1}{\varepsilon} \right)^{-1} \\ &\leq (C + D) \left(\ln \frac{1}{\varepsilon} \right)^{-1/2}. \end{aligned}$$

This completes the proof of Theorem 3.

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