



**HAL**  
open science

# A regularized Galerkin symmetric BIE formulation for mixed elastic boundary-value problems

Marc Bonnet

## ► To cite this version:

Marc Bonnet. A regularized Galerkin symmetric BIE formulation for mixed elastic boundary-value problems. Boundary Elements Abstracts and Newsletters, 1993, 4, pp.109-113. <hal-00092364>

**HAL Id: hal-00092364**

**<https://hal.science/hal-00092364v1>**

Submitted on 10 Mar 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

# A REGULARIZED GALERKIN SYMMETRIC BIE FORMULATION FOR MIXED ELASTIC BOUNDARY VALUE PROBLEMS

Marc Bonnet

CNRS, Laboratoire de Mécanique des Solides, Ecole Polytechnique, 91128 Palaiseau Cedex, France

## 1. Introduction and motivation

Although variational Galerkin-type boundary integral formulations have received a lot of attention, mostly in the applied mathematics field (see e.g. Refs 2, 8, 13, 19) they have led to relatively few developments and applications in the field of solid mechanics (see Refs 1, 14, 15, 18), compared to the very popular collocation boundary element methods (CBEM). Usually variational methods involve double surface integrals (see however Ref.7, where a symmetric BIE formulation with a *single* surface integration is discussed). Variational approaches have nevertheless attractive conceptual features over usual collocation methods: known convergence results are more extensive<sup>3</sup> and actually better than those of CBEM, and they lead to symmetric stiffness-like BEM matrices. Moreover, as it will be seen below, the variational traction boundary integral equation (TBIE) requires  $C^{0,\alpha}$  interpolation of the densities instead of  $C^{1,\alpha}$  at the collocation point for the collocation TBIE. The usual  $C^0$  conformal shape functions may then be used. On the other hand, the implementation of this approach for general engineering problems is somewhat more involved than using collocation, which explains perhaps its relative lack of popularity.

In the present paper, the author investigates the statement of a Galerkin weighted residual symmetric BIE formulation for the mixed boundary value problem of linear elasticity. More precisely, the objective is to present an application of previous works on indirect regularization of displacement CBIEs<sup>4,5,6</sup> and of traction BIEs<sup>4</sup> to the derivation of a Galerkin formulation for mixed elastic boundary value problems in which either the inner or the outer integration is at most weakly singular while the other integration is regular. As will be seen below, this may improve existing Galerkin formulations for elastic problems found in the recent literature.

We consider the static response of a 3D isotropic, either bounded or unbounded, elastic body  $\Omega$  (Lame constants  $\lambda, \mu$ ), subjected to given tractions  $\mathbf{t}^d$  on a portion  $S_r$  of its boundary  $\partial\Omega$  while a displacement  $\mathbf{u}^d$  is prescribed over the complementary part  $S_u = \partial\Omega - S_r$  of the boundary (body forces are not considered here). The displacement field  $\mathbf{u}(\mathbf{y})$  ( $\mathbf{y} \in \Omega$ ) is then governed by the homogeneous Navier equation together with Hooke's law:

$$C_{isab}u_{a,b}(\mathbf{y}) = 0 \quad (1)$$

with  $C_{ispq} = \lambda\delta_{is}\delta_{pq} + \mu(\delta_{ip}\delta_{sq} + \delta_{iq}\delta_{ps})$

and the above-indicated boundary conditions and, if  $\Omega$  is unbounded, suitable decay conditions at infinity. The comma, in eqn(1) and throughout the paper, indicates partial differentiation with respect to the components of  $\mathbf{y}$ , and Einstein summation convention is used. Assume that each point of  $\partial\Omega$  has a unique tangent plane, the unit normal vector  $\mathbf{n}$  being directed towards the exterior of the material body.

Let  $\mathbf{x} \in \mathbb{R}^3$ , and  $U_i^k(\mathbf{x}, \mathbf{y})$  be the  $i$ -th component of the (singular) displacement field at  $\mathbf{y}$  due to the unit point force acting on  $\mathbf{x}$  along  $\mathbf{e}_k$  direction in an infinite elastic medium (Kelvin fundamental displacement), and  $\Sigma_{is}^k(\mathbf{x}, \mathbf{y})$  the  $is$ -component of the Kelvin fundamental stress tensor, given by:

$$\Sigma_{is}^k(\mathbf{x}, \mathbf{y}) = C_{ispq}U_{p,q}^k(\mathbf{x}, \mathbf{y}) \quad (2)$$

In eqn(2) and throughout the present paper, the comma used with two point kernels denotes differentiation with respect to the second argument  $\mathbf{y}$ . The Kelvin solution possesses the following well-known symmetry properties:

$$U_i^k(\mathbf{x}, \mathbf{y}) = U_i^k(\mathbf{y}, \mathbf{x}) = U_i^k(\mathbf{x}, \mathbf{y}) \quad (3)$$

$$\frac{\partial}{\partial x_r} U_i^k(\mathbf{x}, \mathbf{y}) = -\frac{\partial}{\partial y_r} U_i^k(\mathbf{x}, \mathbf{y}) = -U_{i,r}^k(\mathbf{x}, \mathbf{y})$$

and singularity properties ( $r = \|\mathbf{x} - \mathbf{y}\|$  denoting the euclidean distance between  $\mathbf{x}$  and  $\mathbf{y}$ ):

$$U_i^k(\mathbf{x}, \mathbf{y}) = O(1/r) \quad U_{i,r}^k(\mathbf{x}, \mathbf{y}) = O(1/r^2) \quad (4)$$

$$\Sigma_{is}^k(\mathbf{x}, \mathbf{y}) = O(1/r^2) \quad \Sigma_{is,r}^k(\mathbf{x}, \mathbf{y}) = O(1/r^3)$$

Let  $\mathbf{z}$  be a fixed point, either interior or exterior to  $\Omega$ , (ie. not on the boundary  $\partial\Omega$ ). The integral boundary representation formulas for displacement  $\mathbf{u}$  and stress  $\boldsymbol{\sigma}$  read:

$$\kappa u_k(\mathbf{z}) + \int_{\partial\Omega} u_i(\mathbf{y})n_s(\mathbf{y})\Sigma_{is}^k(\mathbf{z}, \mathbf{y}) dS\mathbf{y} \quad (5)$$

$$- \int_{\partial\Omega} t_i(\mathbf{y})U_i^k(\mathbf{z}, \mathbf{y}) dS\mathbf{y} = 0$$

$$\kappa\sigma_{ab}(\mathbf{z}) - C_{abkr} \int_{\partial\Omega} u_i(\mathbf{y})n_s(\mathbf{y})\Sigma_{is,r}^k(\mathbf{z}, \mathbf{y}) dS\mathbf{y} \quad (6)$$

$$+ \int_{\partial\Omega} t_i(\mathbf{y})\Sigma_{ab}^i(\mathbf{z}, \mathbf{y}) dS\mathbf{y} = 0$$

where  $\kappa = 1$  ( $\mathbf{z}$  interior to  $\Omega$ ) or  $\kappa = 0$  ( $\mathbf{z}$  exterior to  $\Omega$ ). In eqn(6), use has been made of eqn(3), which allows the exchange of differentiations with respect to the source point  $\mathbf{z}$  and the integration point  $\mathbf{y}$ .

Let  $\mathbf{x} \in \partial\Omega$  be a fixed boundary point with a unique outward unit normal  $\mathbf{n}(\mathbf{x})$ . The limiting case  $\mathbf{z} \rightarrow \mathbf{x}$  in eqn(5) and in the dot-product of eqn(6) with  $\mathbf{n}(\mathbf{x})$  yield respectively the so-called displacement and traction collocation boundary integral equations, which are the basis of various collocation BEM formulations. As a matter of fact, due to the well-known singularity properties of the Kelvin fundamental solutions

and their derivatives, eqn(4), these boundary integral equations contain integrals which are classically defined as the result of specific limiting processes, namely Cauchy principal value for the  $r^{-2}$  kernels and Hadamard finite part for the  $r^{-3}$  kernels. More recently, it has been shown (see Ref.6), that these singular integrals, suitably rewritten using so-called 'indirect regularization' methods, are actually weakly singular integrals provided certain regularity requirements for  $\partial\Omega$ ,  $\mathbf{u}$  and  $\mathbf{t}$  are met, (see also Ref.11).

An alternative approach is to consider eqns(5) and (6) in a weighted-residual sense, taking the work-like inner products of eqn(5) with  $\tilde{t}_k(\mathbf{z})$  and eqn(6) with  $\tilde{u}_j(\mathbf{z})n_j(\mathbf{z})$ , where  $\tilde{t}_k(\mathbf{z})$  and  $\tilde{u}_j(\mathbf{z})$  denote fictitious traction and displacement fields on  $\partial\Omega$ , and then integrate the result over  $\partial\Omega$  with respect to  $\mathbf{z}$ . This is the basic well-known idea for obtaining Galerkin formulations, which are the subject of the present paper. By performing this manipulation, one can in principle arrive to a symmetric weighted-residuals formulation if additional requirements on  $\tilde{u}_j(\mathbf{z})$ ,  $\tilde{t}_k(\mathbf{z})$ .

However, difficulties arise from the strongly singular and hypersingular nature of the Kelvin kernels, so that Galerkin formulations cannot be obtained in a straightforward way using the procedure outlined above. In Ref.18, the authors consider an auxiliary closed surface  $\tilde{S}$  such that  $\partial\Omega$  lies in the interior of  $\tilde{S}$ , and define the fictitious fields  $\tilde{t}_k(\mathbf{z})$  and  $\tilde{u}_j(\mathbf{z})$  for  $\mathbf{z} \in \tilde{S}$ , thus obtaining a formulation in which the inner integral is taken on  $\partial\Omega$  and the outer integral on  $\tilde{S}$ . At this stage, both surfaces and all surface fields are discretized in a usual boundary element fashion. Finally, in order to tackle the singularity problem, they take the limit  $\tilde{S} \rightarrow \partial\Omega$  at the element integral level. Their numerical implementation of the method hence relies upon an analytical treatment of the limiting process in the singular double element integrals, which they actually perform for 2D situations, straight constant or linear elements, using complex variable techniques. While fully valid in a fundamental viewpoint, this treatment may induce severe practical limitations regarding the choice of elements and interpolation functions.

The purpose of the present paper is the statement of a regularized symmetric Galerkin BIE formulation, identical to the one of Sirtori *et al.* in its essence but applicable to any choice of boundary element interpolation. Our formulation relies upon suitable treatments, including regularization, of the displacement and traction boundary integral equations, which are made *before* any discretization. This avoids the subsequent cumbersome limiting process at the element integral level.

## 2.Regularized displacement BIE

This step relies upon the *generalized rigid-body identity*<sup>6</sup> which reads:

$$\forall \mathbf{z} \in \mathbb{R}^3 - \partial\Omega$$

$$(\kappa - \gamma)\delta_{ki} + \int_{\partial\Omega} n_s(\mathbf{y}) \begin{cases} \gamma = 1 & \Omega \text{ unbounded} \\ \gamma = 0 & \Omega \text{ bounded} \end{cases} \quad (7)$$

$$\Sigma_{i,s}^k(\mathbf{z}, \mathbf{y}) dS\mathbf{y} = 0$$

Now eqn(7) is multiplied by  $u_k(\mathbf{z})$  and subtracted from eqn(5), yielding:

$$\gamma u_k(\mathbf{z}) + \int_{\partial\Omega} (u_i(\mathbf{y}) - u_i(\mathbf{z})) \Sigma_{i,s}^k(\mathbf{z}, \mathbf{y}) dS\mathbf{y}$$

$$- \int_{\partial\Omega} t_i(\mathbf{y}) U_i^k(\mathbf{z}, \mathbf{y}) dS\mathbf{y} = 0 \quad (8)$$

where the coefficient  $\gamma$  assumes the same meaning as in eqn(7). Finally, let  $\mathbf{x} \in \partial\Omega$  and assume  $u_i \in C^{0,\alpha}$  at  $\mathbf{x}$ , where  $C^{m,\alpha}$  denotes the set of functions  $m$  times continuously differentiable such that  $\exists(\alpha, C) > 0$ ,  $|u_{i,m}(\mathbf{x}) - u_{i,m}(\mathbf{y})| \leq C \|\mathbf{x} - \mathbf{y}\|^\alpha$ . Then all integrals in eqn(8) are weakly singular in the limit  $\mathbf{z} \rightarrow \mathbf{x}$ . As a consequence, eqn(8) taken for  $\mathbf{x} \in \partial\Omega$  is the regularized DBIE.<sup>4</sup>

Now, one can take the inner product of eqn(8) written for  $\mathbf{z} = \mathbf{x} \in \partial\Omega$  with a fictitious surface traction field  $\tilde{t}_k(\mathbf{x})$  and integrate the result over  $\partial\Omega$  again. This leads to:

$$\gamma \int_{\partial\Omega} \tilde{t}_k(\mathbf{x}) \left\{ u_k(\mathbf{x}) + \int_{\partial\Omega} (u_i(\mathbf{y}) \right.$$

$$\left. - u_i(\mathbf{x}) \right\} \Sigma_{i,s}^k(\mathbf{x}, \mathbf{y}) dS\mathbf{y} \Big\} dS\mathbf{x} \quad (9)$$

$$- \int_{\partial\Omega} \int_{\partial\Omega} t_i(\mathbf{y}) \tilde{t}_k(\mathbf{x}) U_i^k(\mathbf{x}, \mathbf{y}) dS\mathbf{y} dS\mathbf{x} = 0$$

It is worth noting that, for the result of eqn(9) to be valid, the surface fields  $\mathbf{t}$  and  $\mathbf{u}$  must respectively be piecewise continuous and  $C^{0,\alpha}$  (the latter condition ensures that the factor  $\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})$  actually weakens the singularity of the inner integral). These are indeed the very conditions under which the integrals in the collocation DBIE, eqn (5), are convergent, see Ref.12.

## 3.Regularized traction BIE

Here, following the approach of Sirtori *et al.*,<sup>18</sup> an auxiliary regular surface  $\tilde{S}$ , which completely surrounds  $\partial\Omega$ , is introduced, such that there exists a one-to-one mapping  $\mathbf{x} \in \partial\Omega \rightarrow \mathbf{z} \in \tilde{S}$ . Let us introduce a fictitious displacement field  $\tilde{u}_j(\mathbf{z})$  defined for  $\mathbf{z} \in \tilde{S}$ . Now the inner product of eqn(6) with  $\tilde{u}_j(\mathbf{z}) \otimes n_j(\mathbf{z})$  is taken and integrated over  $\tilde{S}$ , which gives:

$$\kappa \int_{\tilde{S}} \tilde{u}_a(\mathbf{z}) \left\{ \sigma_{ab}(\mathbf{z}) n_b(\mathbf{z}) + \int_{\partial\Omega} t_i(\mathbf{y}) n_b(\mathbf{z}) \right.$$

$$\left. \Sigma_{ab}^i(\mathbf{z}, \mathbf{y}) dS\mathbf{y} \right\} dS\mathbf{z} - C_{abkr} \int_{\tilde{S}} n_b(\mathbf{z}) \tilde{u}_a(\mathbf{z}) \quad (10)$$

$$\int_{\partial\Omega} u_i(\mathbf{y}) n_s(\mathbf{y}) \Sigma_{i,s,r}^k(\mathbf{z}, \mathbf{y}) dS\mathbf{y} dS\mathbf{z} = 0$$

Let us investigate the two double surface integrals in eqn(10). One has:

$$\int_{\tilde{S}} \int_{\partial\Omega} t_i(\mathbf{y}) \tilde{u}_a(\mathbf{z}) n_b(\mathbf{z}) \Sigma_{ab}^i(\mathbf{z}, \mathbf{y}) dS\mathbf{y} dS\mathbf{z}$$

$$= \int_{\partial\Omega} t_i(\mathbf{y}) \left\{ \int_{\tilde{S}} (\tilde{u}_a(\mathbf{z}) - \tilde{u}_a(\mathbf{y})) n_b(\mathbf{z}) \Sigma_{ab}^i(\mathbf{z}, \mathbf{y}) dS\mathbf{z} \right.$$

$$\left. + \tilde{u}_a(\mathbf{y}) \left[ \int_{\tilde{S}} n_b(\mathbf{z}) \Sigma_{ab}^i(\mathbf{z}, \mathbf{y}) dS\mathbf{z} \right] \right\} dS\mathbf{y} \quad (11)$$

$$= \int_{\partial\Omega} t_i(\mathbf{y}) \left\{ \tilde{u}_i(\mathbf{y})(\gamma - \kappa) + \int_{\tilde{S}} (\tilde{u}_a(\mathbf{z}) \right.$$

$$\left. - \tilde{u}_a(\mathbf{y})) n_b(\mathbf{z}) \Sigma_{ab}^i(\mathbf{z}, \mathbf{y}) dS\mathbf{z} \right\} dS\mathbf{y}$$

where identity, eqn(7) has been used and integrations over  $\tilde{S}$  and  $\partial\Omega$  exchanged. At this point, one can consider the limiting case  $\tilde{S} \rightarrow \partial\Omega$  in eqn(11). Since the inner integral over  $\tilde{S}$  becomes only weakly singular, while the subsequent integral over  $\partial\Omega$  is regular, the limiting case of eqn(11) is simply the same formula where  $\partial\Omega$  is substituted to  $\tilde{S}$  and  $\mathbf{x} \in \partial\Omega$  to  $\mathbf{z} \in \tilde{S}$ :

$$\lim_{\mathbf{z} \in \tilde{S} \rightarrow \mathbf{x} \in \partial\Omega} \int_{\tilde{S}} \int_{\partial\Omega} t_i(\mathbf{y}) \tilde{u}_a(\mathbf{x}) n_b(\mathbf{x}) \Sigma_{ab}^i(\mathbf{x}, \mathbf{y}) dS\mathbf{y} dS\mathbf{z}$$

$$= \int_{\partial\Omega} t_i(\mathbf{y}) \left\{ \tilde{u}_i(\mathbf{y})(\gamma - \kappa) + \int_{\partial\Omega} (\tilde{u}_a(\mathbf{x}) \right.$$

$$\left. - \tilde{u}_a(\mathbf{y})) n_b(\mathbf{x}) \Sigma_{ab}^i(\mathbf{x}, \mathbf{y}) dS\mathbf{x} \right\} dS\mathbf{y} \quad (12)$$

In eqn(12), the outer integral is weakly singular integrals while the inner is regular (they have to be considered in this order, because the inner integral, considered alone, is of the strongly singular type).

Next, let us consider the integral in eqn(10) containing the hypersingular kernel  $C_{abkr} \Sigma_{i,r}^k$ . In this case, we will use two successive integrations by parts. Following Becache<sup>1</sup> and Nishimura & Kobayashi,<sup>14</sup> there exists a fourth-order tensor  $\mathbf{B}(\mathbf{z}, \mathbf{y})$ , associated to the Kelvin stress tensor, such as, for any  $\mathbf{z} \neq \mathbf{y}$ :

$$C_{ijab}\Sigma_{kl,b}^a(\mathbf{z}, \mathbf{y}) = e_{jfq}e_{lhs}B_{iqks, fh}(\mathbf{z}, \mathbf{y}) \quad (13)$$

$$B_{iqks}(\mathbf{z}, \mathbf{y}) = -\frac{\mu e_{iep}e_{kgr}}{8\pi(\lambda + 2\mu)} \frac{(\delta_{eg} - r_{,e}r_{,g})}{r} [2\lambda\delta_{pq}\delta_{rs} + (\lambda + 2\mu)(\delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr})] \quad (14)$$

where each are the components of the permutation tensor. Moreover, the Stokes formula for a closed surface  $\Gamma$  reads:

$$\int_{\Gamma} e_{abc}n_a(\mathbf{y})f_{,b}(\mathbf{y})dS_{\mathbf{y}} = 0 \quad (15)$$

for any continuous and piecewise continuously differentiable function  $f(\mathbf{y})$  (if  $f(\mathbf{y})$  is only piecewise continuous, identity (15) would also present contour integrals over lines across which  $f$  has a jump). It is worth emphasizing that  $n_{\alpha}(\mathbf{y})f_{,b}(\mathbf{y})$  can be expressed in terms of *tangential* derivatives of  $f$ . Using eqn(13) and applying eqn(15) twice (once for each of the two integration variables  $\mathbf{y} \in \partial\Omega$  and  $\mathbf{z} \in S$ ), one has:

$$\begin{aligned} & \int_{\tilde{S}} \tilde{u}_i(\mathbf{y})n_j(\mathbf{z})C_{ijab} \int_{\partial\Omega} \Sigma_{kl,b}^a(\mathbf{z}, \mathbf{y})n_l(\mathbf{y})u_k(\mathbf{y})dS_{\mathbf{y}}dS_{\mathbf{z}} \\ &= - \int_{\tilde{S}} \tilde{u}_i(\mathbf{y})e_{jfq}n_j(\mathbf{z}) \int_{\partial\Omega} B_{iqks, f}(\mathbf{z}, \mathbf{y})e_{lhs}n_l(\mathbf{y}) \\ & u_{k, h}(\mathbf{y})dS_{\mathbf{y}} = \int_{\tilde{S}} e_{jfq}n_j(\mathbf{z})\tilde{u}_{i, f}(\mathbf{y}) \\ & \int_{\partial\Omega} B_{iqks}(\mathbf{z}, \mathbf{y})e_{lhs}n_l(\mathbf{y})u_{k, h}(\mathbf{y})dS_{\mathbf{y}} \end{aligned} \quad (16)$$

Equation (14) shows that the kernel  $B_{iqks}(\mathbf{x}, \mathbf{y})$  has a  $\|\mathbf{x}-\mathbf{y}\|^{-1}$  singularity. Hence, the first (either inner or outer) surface integration in eqn(16) is of a weakly singular nature, while the other is regular. As a consequence, the limiting case  $\tilde{S} \rightarrow \partial\Omega$  of eqn(16), here again, is simply the same formula where  $\partial\Omega$  is substituted to  $S$  and  $\mathbf{x} \in \partial\Omega$  to  $\mathbf{x} \in S$ :

$$\begin{aligned} & \lim_{\mathbf{z} \in \tilde{S} \rightarrow \mathbf{x} \in \partial\Omega} \int_{\tilde{S}} \tilde{u}_i(\mathbf{y})n_j(\mathbf{z})C_{ijab} \int_{\partial\Omega} \Sigma_{kl,b}^a(\mathbf{z}, \mathbf{y})n_l(\mathbf{y}) \\ & u_k(\mathbf{y})dS_{\mathbf{y}}dS_{\mathbf{z}} = \int_{\partial\Omega} e_{jfq}n_j(\mathbf{x})\tilde{u}_{i, f}(\mathbf{x}) \\ & \int_{\partial\Omega} B_{iqks}(\mathbf{x}, \mathbf{y})e_{lhs}n_l(\mathbf{y})u_{k, h}(\mathbf{y})dS_{\mathbf{y}}dS_{\mathbf{x}} \end{aligned} \quad (17)$$

Finally, collecting eqns(12), (17), the limiting form of (10) when  $\tilde{S} \rightarrow \partial\Omega$  is given by:

$$\begin{aligned} & \gamma \int_{\partial\Omega} t_i(\mathbf{x}) \left\{ \tilde{u}_i(\mathbf{x}) + \int_{\partial\Omega} (\tilde{u}_a(\mathbf{x}) - \tilde{u}_a(\mathbf{y}))n_b(\mathbf{x}) \right. \\ & \left. \Sigma_{ab}^i(\mathbf{x}, \mathbf{y})dS_{\mathbf{y}} \right\} dS_{\mathbf{x}} - \int_{\partial\Omega} e_{jfq}n_j(\mathbf{x})\tilde{u}_{i, f}(\mathbf{x}) \\ & \int_{\partial\Omega} B_{iqks}(\mathbf{x}, \mathbf{y})e_{lhs}n_l(\mathbf{y})u_{k, h}(\mathbf{y})dS_{\mathbf{y}}dS_{\mathbf{x}} = 0 \end{aligned} \quad (18)$$

where the coefficient  $\alpha$  assumes the same meaning as in eqn(7).

One can note that, for the result (18) to be valid, the surface fields  $\mathbf{t}$  and  $\mathbf{u}$  must respectively be piecewise continuous and  $C^{0,\alpha}$ . These regularity requirements are identical to those for eqn(9). Likewise, the surface fictitious displacement  $\tilde{\mathbf{u}}$  must also be  $C^{0,\alpha}$ . In contrast, one can note that the collocation traction BIE eqn(6) requires that  $\mathbf{u}$  be  $C^{1,\alpha}$ , see Refs 6, 9 and 11. The latter fact is recognized as serious as far as numerical implementation is considered because the development of general  $C^1$  boundary elements for 3D problems is problematic.

#### 4. Galerkin boundary integral formulation for the mixed boundary-value problem

Now let us apply the previous results to the statement of a variational BIE formulation for the mixed elastic boundary-value problem. More precisely, eqns(9) and (18) are considered with  $\tilde{\mathbf{u}} \in \mathcal{V}_u$ ,  $\tilde{\mathbf{t}} \in \mathcal{V}_T$

$$\mathcal{V}_u = \{\tilde{\mathbf{u}} \mid \tilde{\mathbf{u}} \in C^{0,\alpha}(\partial\Omega) \text{ and } \tilde{\mathbf{u}}(\mathbf{x}) = 0 \forall \mathbf{x} \in S_u\} \quad (19)$$

$$\mathcal{V}_T = \{\tilde{\mathbf{t}} \mid \tilde{\mathbf{t}} \text{ piecewise continuous on } \partial\Omega \text{ and } \tilde{\mathbf{t}}(\mathbf{x}) = 0 \forall \mathbf{x} \in S_T\} \quad (20)$$

Splitting  $\partial\Omega$  into its components  $S_u$  and  $S_T$ , eqns(9), and (18) then lead to the following variational equations:

$$\begin{cases} \mathcal{B}_{uu}(\mathbf{u}, \tilde{\mathbf{u}}) + \mathcal{B}_{tu}(\mathbf{t}, \tilde{\mathbf{u}}) = \mathcal{L}_{uu}(\tilde{\mathbf{u}}) + \mathcal{L}_{tu}(\tilde{\mathbf{u}}) \\ \mathcal{B}_{ut}(\mathbf{u}, \tilde{\mathbf{t}}) + \mathcal{B}_{tt}(\mathbf{t}, \tilde{\mathbf{t}}) = \mathcal{L}_{ut}(\tilde{\mathbf{t}}) + \mathcal{L}_{tt}(\tilde{\mathbf{t}}) \end{cases} \quad (21)$$

$$\forall \tilde{\mathbf{u}} \in \mathcal{V}_u, \forall \tilde{\mathbf{t}} \in \mathcal{V}_T$$

where

$$\mathcal{B}_{ut}(\mathbf{u}, \tilde{\mathbf{t}}) = \int_{S_u} \int_{S_T} (u_i(\mathbf{y}) - u_i(\mathbf{x}))\tilde{t}_k(\mathbf{x})\Sigma_{is}^k(\mathbf{x}, \mathbf{y})dS_{\mathbf{y}}dS_{\mathbf{x}}$$

$$\begin{aligned} \mathcal{B}_{tt}(\mathbf{u}, \tilde{\mathbf{t}}) &= - \int_{S_u} \int_{S_u} t_i(\mathbf{y})\tilde{t}_k \\ & (\mathbf{x})U_i^k(\mathbf{x}, \mathbf{y})dS_{\mathbf{y}}dS_{\mathbf{x}} \end{aligned} \quad (23)$$

$$\begin{aligned} \mathcal{L}_t(\tilde{\mathbf{t}}) &= - \int_{S_u} \tilde{t}_k(\mathbf{x}) \left\{ \gamma u_k^D(\mathbf{x}) + \int_{S_u} (u_i^D(\mathbf{y}) - u_i^D(\mathbf{x}))\tilde{t}_k(\mathbf{x})\Sigma_{is}^k(\mathbf{x}, \mathbf{y})dS_{\mathbf{y}} \right\} dS_{\mathbf{x}} \\ & + \int_{S_u} \int_{S_T} t_i^D(\mathbf{y})\tilde{t}_k(\mathbf{x})U_i^k(\mathbf{x}, \mathbf{y})dS_{\mathbf{y}}dS_{\mathbf{x}} \end{aligned} \quad (24)$$

$$\begin{aligned} \mathcal{B}_{tu}(\mathbf{t}, \tilde{\mathbf{u}}) &= \int_{S_T} \int_{S_u} t_i(\mathbf{y})(\tilde{u}_a(\mathbf{x}) - \tilde{u}_a(\mathbf{y}))n_b(\mathbf{x})\Sigma_{ab}^i(\mathbf{x}, \mathbf{y})dS_{\mathbf{y}}dS_{\mathbf{x}} \end{aligned} \quad (25)$$

$$\mathcal{B}_{uu}(\mathbf{u}, \tilde{\mathbf{u}}) = - \int_{S_T} e_{jfq}n_j(\mathbf{x})\tilde{u}_{i, f}(\mathbf{x}) \int_{S_T} B_{iqks}(\mathbf{x}, \mathbf{y})e_{lhs}n_l(\mathbf{y})u_{k, h}(\mathbf{y})dS_{\mathbf{y}}dS_{\mathbf{x}}$$

$$\begin{aligned} \mathcal{L}_u(\tilde{\mathbf{u}}) &= - \int_{S_T} t_i^D(\mathbf{x}) \left\{ \gamma \tilde{u}_i(\mathbf{x}) + \int_{S_T} (\tilde{u}_a(\mathbf{x}) - \tilde{u}_a(\mathbf{y}))n_b(\mathbf{x})\Sigma_{ab}^i(\mathbf{x}, \mathbf{y})dS_{\mathbf{y}} \right\} dS_{\mathbf{x}} \\ & + \int_{S_T} e_{jfq}n_j(\mathbf{x})\tilde{u}_{i, f}(\mathbf{y}) \int_{S_u} B_{iqks}(\mathbf{x}, \mathbf{y})e_{lhs}n_l(\mathbf{y})u_{k, h}(\mathbf{y})dS_{\mathbf{y}}dS_{\mathbf{x}} \end{aligned} \quad (27)$$

The given displacements  $\mathbf{u}^D$  and tractions  $\mathbf{t}^D$  are incorporated in the linear forms  $\mathcal{L}_u$  and  $\mathcal{L}_t$ . From eqns(21) to (27) and taking the symmetry properties of eqn(3) into account, the following symmetry properties,

which are expected as classical features of Galerkin BIE methods like e.g. in Ref. 18, are apparent:

$$\begin{aligned} \forall \mathbf{u}, \tilde{\mathbf{u}} \in \mathcal{V}_u, \forall \mathbf{t}, \tilde{\mathbf{t}} \in \mathcal{V}_T \\ \mathcal{B}_{uu}(\mathbf{u}, \tilde{\mathbf{u}}) = \mathcal{B}_{uu}(\tilde{\mathbf{u}}, \mathbf{u}) \quad \mathcal{B}_{tt}(\mathbf{t}, \tilde{\mathbf{t}}) = \mathcal{B}_{tt}(\tilde{\mathbf{t}}, \mathbf{t}) \quad (28) \\ \mathcal{B}_{ut}(\mathbf{u}, \tilde{\mathbf{t}}) = \mathcal{B}_{tu}(\tilde{\mathbf{t}}, \mathbf{u}) \end{aligned}$$

The variational BIE formulation (21) is the main result of the present paper. Indeed it must be understood as a reformulation of the variational symmetric BIE formulation developed in Ref. 18. The difference between the two formulations, which constitutes the main motivation of this paper, is that our result (21) has undergone a preliminary regularization treatment, using singularity exclusion methods developed in our previous works (e.g. in Ref. 6) together with a stress function and integration by parts method which has been previously used, e.g. in Refs 1 and 14 for the BIE modelling of crack problems. As a result, our variational formulation can be expected to be more flexible in a BEM implementation point of view, since boundary element geometrical and field interpolation of any degree can be incorporated into eqn(21) without difficulty, as will be briefly shown in the next section.

## 5. Numerical implementation of the regularized variational BIE

The symmetry properties (28) allows a reduction of computational effort for both the matrix building and the linear system solution steps, which have been extensively discussed elsewhere. We rather wish to focus the discussion on the treatment of the singular surface integrals which appear in eqn(21), in the simplest case of isoparametric conformal BEM interpolation. The surface  $\partial\Omega$  is divided into boundary elements, which are mapped onto a *reference element*  $E_p$ , which is generally the square  $\xi = (\xi_1, \xi_2) \in [-1, 1]^2$  or the triangle  $0 \leq \xi_1 + \xi_2 \leq 1$ . Let the discussion be restricted to the consideration of a single element  $E$ . The location of a point  $\mathbf{y}$  on  $E$  is expressed in terms of  $n$  shape functions  $N^k$  and  $n$  geometrical nodes  $\mathbf{A}^k$  ( $k = 1, \dots, n$ ) located on the boundary of  $E$ :

$$\mathbf{Oy} = N^k(\xi) \mathbf{OA}^k \quad (\xi \in E_0) \quad (29)$$

Then, for  $(\alpha, \beta) = 1, 2$ , the natural basis  $(\mathbf{a}_\alpha)$ , metric tensor  $(g_{\alpha\beta})$  and unit normal  $\mathbf{n}$  on  $E$  are given by:

$$\begin{aligned} \mathbf{a}_\alpha(\xi) = N^k_{,\alpha}(\xi) \mathbf{OA}^k \quad g_{\alpha\beta}(\xi) = \mathbf{a}_\alpha(\xi) \cdot \mathbf{a}_\beta(\xi) \\ \sqrt{\hat{g}(\xi)} \mathbf{n}(\xi) = \mathbf{a}_1 \wedge \mathbf{a}_2 \quad \hat{g}(\xi) = (g_{11}g_{22} - g_{12}^2)(\xi) \quad (30) \\ (\xi \in E_0) \end{aligned}$$

The shape functions  $N^q$  are also used for the interpolation of the fields  $\mathbf{u}, \mathbf{t}, \tilde{\mathbf{u}}, \tilde{\mathbf{t}}$ . Moreover, it can be shown that:

$$\begin{aligned} \forall \mathbf{y} \in E \quad \sqrt{\hat{g}(\xi)} \mathbf{n}(\xi) e_{abc} n_b(\mathbf{y}(\xi)) f_{,c}(\mathbf{y}(\xi)) \\ = e_{\alpha\beta\gamma} \frac{\partial f}{\partial \xi_\alpha} \mathbf{a}_\beta \quad (31) \end{aligned}$$

Let  $\eta = (\eta_1, \eta_2)$  denote the antecedent of  $\mathbf{x}$  on  $E_0$ . Following a common practice in BEM (see e.g. Ref. 17), set  $\xi_1 = \eta_1 + \rho \cos \alpha$ ,  $\xi_2 = \eta_2 + \rho \sin \alpha$ . Then:

$$dS_{\mathbf{y}} = \sqrt{\hat{g}(\xi)} d\xi_1 d\xi_2 = \sqrt{\hat{g}(\xi)} \rho d\rho d\alpha = J(\xi) \rho d\rho d\alpha \quad (32)$$

and the regular quantities  $\hat{N}^q(\rho, \alpha; \eta)$ ,  $\hat{r}(\rho, \alpha; \eta)$ ,  $\hat{U}_i^k(\rho, \alpha; \eta)$ ,  $\hat{\Sigma}_{is}^k(\rho, \alpha; \eta)$ ,  $\hat{A}_{iqks, eg}(\rho, \alpha; \eta)$  are defined by:

$$N^q(\xi) - N^q(\eta) = \rho \hat{N}^q(\rho, \alpha; \eta)$$

$$\begin{aligned} r(\mathbf{x}, \mathbf{y}) = || N^k(\xi) \mathbf{OA}^k || = \rho \hat{r}(\rho, \alpha; \eta) \\ U_i^k(\mathbf{x}, \mathbf{y}) = \frac{1}{\rho} \hat{U}_i^k(\rho, \alpha; \eta) \quad (33) \\ \Sigma_{is}^k(\mathbf{x}, \mathbf{y}) = \frac{1}{\rho^2} \hat{\Sigma}_{is}^k(\rho, \alpha; \eta) \\ B_{iqks}(\mathbf{x}, \mathbf{y}) = \frac{1}{\rho} \hat{B}_{iqks}(\rho, \alpha; \eta) \end{aligned}$$

where  $\hat{r}(\rho, \alpha; \eta) \neq 0$  and  $\hat{U}(\rho, \alpha; \eta)$ ,  $\hat{\Sigma}(\rho, \alpha; \eta)$  and  $\hat{A}_{iqks, eg}(\rho, \alpha; \eta)$  are regular at  $\rho = 0$ . Hence the singular surface integrals which occur in eqn(21) can be written:

$$\begin{aligned} \int_E n_s(\mathbf{y}) \Sigma_{is}^k(\mathbf{x}, \mathbf{y}) (u_i(\mathbf{y}) - u_i(\mathbf{x})) dS_{\mathbf{y}} \\ = u_i^q \int_E n_s(\mathbf{y}) \hat{\Sigma}_{is}^k(\rho, \alpha; \eta) \hat{M}^q(\rho, \alpha; \eta) J(\xi) d\rho d\alpha \quad (34) \end{aligned}$$

$$\begin{aligned} \int_E B_{iqks}(\mathbf{x}, \mathbf{y}) e_{lhs} n_l(\mathbf{y}) u_{k,h}(\mathbf{y}) dS_{\mathbf{y}} \\ = u_i^q \int_E \hat{B}_{iqks}(\rho, \alpha; \eta) e_{\alpha\beta\gamma} \frac{\partial M^q}{\partial \xi_\alpha} (\mathbf{a}_\beta \cdot \mathbf{e}_l) d\rho d\alpha \quad (35) \end{aligned}$$

$$\begin{aligned} \int_E U_i^k(\mathbf{x}, \mathbf{y}) t_i(\mathbf{y}) dS_{\mathbf{y}} \\ = t_i^q \int_E \hat{U}_i^k(\rho, \alpha; \eta) M^q(\xi) J(\xi) d\rho d\alpha \quad (36) \end{aligned}$$

Equations (34), (35) and (36) take full advantage of the regularization. Their numerical evaluation can be performed with standard product Gaussian quadrature formulas, provided a further coordinate change  $(\rho, \alpha) \rightarrow (v_1, v_2)$  is made in eqn(34) in order to recover an integral over the square  $[-1, 1]^2$ .

The variational formulation (21) involves *double* surface integrals, but at most one of the two surface integration is singular, the other being always regular. In such case, the singular integration should be performed first.

## 6. Concluding comments

In this paper, a symmetric Galerkin regularized BIE formulation is established. Its main feature is that, before any discretization, one of the successive surface integrations is at most weakly singular while the other is regular. Thus, the present formulation, which may be understood as an improvement of the one developed in Ref. 18, provides a basis for the numerical treatment of general 3D situations. Conventional boundary element interpolations of any degree for the geometry as well as the unknowns can be implemented in a straightforward manner, using conventional techniques for numerical integration together with a now well-established singularity cancellation method at the shape function level.

The present approach can be readily applied to other similar situations, such as potential problems or elastodynamic problems. Moreover, similar formulations are already known in the literature, notably about variational formulations for crack problems (see references quoted herein).

## References

1. Becache, E. Résolution par une méthode d'équations intégrales d'un problème de diffraction d'ondes élastiques transitoires par une fissure, Thèse de doctorat, Université Paris 6, 1991.

2. Bendali, A. Problème aux limites extérieur et intérieur pour le système de Maxwell en régime harmonique, Internal report no. 59, Centre de Mathématiques Appliquées, Ecole Polytechnique, Palaiseau, France, 1980.
3. Beskos, D.E. (ed), *Boundary Element Methods in Mechanics*, Computational Methods in Mechanics, Vol. 3, North Holland, 1987.
4. Bonnet, M. Méthode des équations intégrales régularisées en élastodynamique tridimensionnelle, Thèse de doctorat, publiée dans le Bulletin EDF/DER série C, nr. 1/2, 1987.
5. Bonnet, M. Regularized boundary integral equations for three dimensional bounded or unbounded elastic bodies containing curved cracks of arbitrary shape under dynamic loading, *Boundary Element Techniques: Applications in Engineering*, ed. C.A. Brebbia & N.G. Zamani, Computational Mechanics Publications, Southampton, 1989.
6. Bonnet, M. & Bui, H.D. Regularization of the displacement and traction BIE for 3D elastodynamics using indirect methods, *Advances in Boundary Element Techniques*, ed. J.H. Kane, G. Maier, N. Tosaka & S.N. Atluri, Springer-Verlag, 1992, (to appear).
7. Bui, H.D. On the variational boundary integral equations in elastodynamics with the use of conjugate functions, *J. Elast.*, 1992, **28**, 247-256.
8. Cortey-Dumont, Ph. Simulation numérique de problèmes de diffraction d'ondes par une fissure, Thèse d'Etat, Université Paris VI, 1985.
9. Guiggiani, M., Krishnasamy, G., Rudolphi, T.J. & Rizzo, F.J. A general algorithm for the numerical solution of hypersingular boundary integral equations, *ASME J. Appl. Mech.*, 1992, (to appear).
10. Hamdi, M.A. Formulation variationnelle par équations intégrales pour le calcul de champs acoustiques linéaires proches et lointains, Thèse d'Etat, UTC Compiègne, 1982.
11. Krishnasamy, G., Rizzo, F.J. & Rudolphi, T.J. Hypersingular boundary integral equations: their occurrence, interpretation, regularization and computation, *Developments in Boundary Element Methods*, Vol. 7: Advanced Dynamic Analysis, ed. P.K. Banerjee & S. Kobayashi, Elsevier, 1992, Chap.7.
12. Kupradze, V.D. (ed), *Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*, North Holland, 1979.
13. Nedelec, J.C. Integral equations with non integrable kernels, *Integral Equations and Operator Theory*, Vol. 5, 1982, pp. 562-572.
14. Nishimura, N. & Kobayashi, S. A boundary integral equation method for an inverse problem related to crack detection, *Int. J. Num. Meth. Eng.*, 1991, **32**, 1371-1387.
15. Parreira, P. & Guiggiani, M. On the implementation of the Galerkin approach in the boundary element method, *Computers & Structures*, 1989, **33**, 269-279.
16. Polizzotto, C. An energy approach to the boundary element method. Part I: elastic solids, *Comp. Meth. Appl. Mech. Eng.*, 1988, **69**, 167-184.
17. Rizzo, F.J., Shippy, D.J. & Rezayat, M. A boundary integral equation method for radiation and scattering of elastic waves in three-dimensions, *Int. J. Num. Meth. in Eng.*, 1985, **21**, 115-129.
18. Sirtori, S., Maier, G., Novati, G. & Miccoli, S. A Galerkin symmetric boundary element method in elasticity: formulation and implementation, *Int. J. Num. Meth. Engng*, 1992, (to appear).
19. Wendland, W.L. Mathematical properties and asymptotic error estimates for elliptic boundary element methods, *Advanced Boundary Element Methods*, ed. T.A. Cruse, Springer-Verlag, 1988, pp. 475-489.