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► **To cite this version:**

Rachel-Marie Pradeilles-Duval, Claude Stolz. Mechanical transformations and discontinuities along a moving surface. *Journal of the Mechanics and Physics of Solids*, 1995, 43 (1), pp.91-121. <10.1016/0022-5096(94)00061-9>. <hal-00092042>

HAL Id: hal-00092042

<https://hal.science/hal-00092042v1>

Submitted on 4 Apr 2024

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MECHANICAL TRANSFORMATIONS AND DISCONTINUITIES ALONG A MOVING SURFACE

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Partial damage is considered in this paper. It is defined as a transformation of mechanical properties between two materials along a surface, in elasticity or anelasticity. The transformation is controlled by a generalized energy criterion of the Griffith type. The equilibrium state of the structure, the total dissipation with increasing loading and the associated rate boundary value problem are studied within the framework of generalized standard materials. A global formulation of the rate boundary value problem allows us to characterize the actual equilibrium state assumed to be known. Finally, the condition of stability and the possibility of bifurcation are given, as are applications to simple composite structures.

1. INTRODUCTION

In the recent past, the propagation of damage has been studied in connection with fracture mechanics. Different approaches based on macroscopic or microscopic description of mechanical degradation properties have been proposed.

Because of the loading, damage in continuum mechanics can be induced by the initiation and growth of micro-cracks and micro-cavities. According to empirical descriptions based on the concept of effective stress in the sense of Kachanov, many works describe damage with macroscopic parameters, in the context of a formalism similar to that used in plasticity (Dragon and Mroz, 1979; Lemaitre and Chaboche, 1985).

Several papers have also dealt with the relation between microscopic and macroscopic behavior. These descriptions, which are based on the evolution of microscopic properties, propose to take the growth of pores or micro-cracks into account, through the idea that when some threshold value of stress, strain or embedded energy is reached, the material can not support further tensile loading. Then, a connection with fracture mechanics can be made in an asymptotic sense (Bui and Ehrlacher, 1980). Variational formulations were performed to describe the evolution of the surface between the sound and damaged materials (Bui *et al.*, 1981). Whatever the chosen

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criterion may be, among other problems, the evolution of the structure is studied. Some applications have been attempted for particular geometries of the initial defect and loading, including the study of stability of the response (Dems and Mroz, 1985).

In previous studies, two mechanisms were considered. The first involves a volume description of damage, the second is characterized by the evolution of a moving interface along which mechanical transformation occurs. This paper is concerned mostly with the latter description, but connections with the first approach will be explored. It turns out that the relationships between the discontinuities of mechanical fields give us the whole knowledge of the interface behavior.

In the following development, partial localized damage will be considered, after the discussion of the general features of the model. At each time, two materials coexist in the structure, and the structure is heterogeneous. This study is concerned with the motion of the existing interface which separates the two phases. It is a perfect one. The emergence of the damaged material when the entire body is undamaged is not considered here. The aim of this paper is slightly different from the study of the nucleation process proposed recently by Lusk (1994).

The equilibrium state is described using the stationarity of potential energy. When a threshold is reached, one material changes into another at the interface, which consequently becomes a moving surface along which the mechanical fields exhibit discontinuities.

The conservation law of the total energy gives us the expression of the global dissipation appearing in the structure when the loading increases. It is related to the propagation of the interface and it yields the definition of the energy release rate associated with the transformation. Assuming that the phase transformation is governed by a generalized Griffith's energy criterion, the rate boundary value problem must be solved, taking into account the coupling between the mechanical behavior and the moving interface. The local equations are written first, then a global formulation is obtained, which is related to a variational inequality.

Even if the local behavior is stable, the study of the existence and the uniqueness of the response to a prescribed loading path must be performed, in a global manner, with the help of the variational formulation. Those conclusions provide the characteristics of the actual equilibrium state. The stability and the bifurcation analysis along a path of equilibrium are discussed via an appropriate extension of the analysis of Nguyen (1984), as proposed in Pradeilles and Stolz (1991), and more recently in Pradeilles-Duval (1992).

Some applications are then discussed, in which the influence of the interface geometry is emphasized. It is to be noticed that a density of interfacial energy has an influence on the stability (Nguyen *et al.*, 1989).

The case of a composite spheres assemblage where such an irreversible process occurs between the core and the shell is then analyzed. This description shows that it would be necessary to study the stability of the overall behavior when a homogenization procedure is performed in nonlinear mechanics, even if the relation between local and overall quantities can be performed in a general manner, thanks to some localization process (Bui *et al.*, 1982; Pradeilles and Stolz, 1992; Stolz, 1983; Suquet *et al.*, 1983).

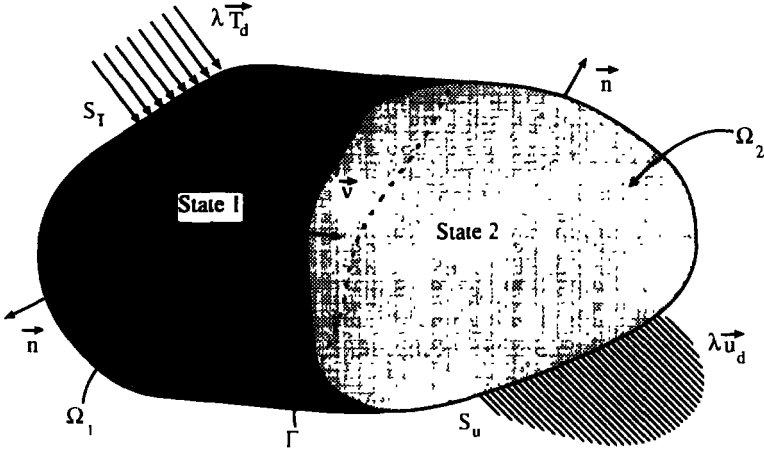


Fig. 1. General features of the problem.

2. SOME GENERAL FEATURES

Let Ω_i denote the domain, composed of two distinct volumes Ω_1 and Ω_2 which are occupied by two materials with different mechanical characteristics. The perfect interface between them is assumed to be a regular surface and is denoted by Γ_i . Material 1 changes into material 2, along Γ_i , by an irreversible process. Hence, Γ_i moves with a normal velocity ϕ , positive along Γ_i . The external surface $\partial\Omega_i$ is composed of two parts $\partial\Omega_{i,u}$ and $\partial\Omega_{i,T}$ on which the displacements $\lambda \mathbf{u}^d$ and the loading $\lambda \mathbf{T}^d$ are prescribed, respectively, in terms of a loading parameter λ (see Fig. 1).

The subscript i is used to denote material i . The actual state is characterized by the displacement field \mathbf{u} . It is associated to the strain field $\varepsilon = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^t)$; the internal variables α (which could represent plastic deformation, a volume damage parameter etc.) and the position of the interface Γ_i . The total potential energy of the structure has the following form :

$$E^{\text{pot}}(\mathbf{u}, \alpha, \Gamma_i, \lambda) = \int_{\Omega_i} w d\omega - \int_{\partial\Omega_{i,T}} \lambda \mathbf{T}^d \mathbf{u} da = \int_{\Omega_1} w_1 d\omega + \int_{\Omega_2} w_2 d\omega - \int_{\partial\Omega_{i,T}} \lambda \mathbf{T}^d \mathbf{u} da. \quad (1)$$

where w_i denotes the density of the free energy in the domain Ω_i and is a function of α and of ε . σ denotes the stress tensor field that is thermodynamically associated with $\varepsilon \sigma = \partial w_i / \partial \varepsilon$, while \mathbf{A} is the thermodynamic force associated with α : $\mathbf{A} = -\partial w_i / \partial \alpha$.

It is important to point out that the potential energy represents the global free energy in a thermodynamic description ; then the position of the interface Γ_i becomes an internal variable.

In what follows, we adopt the following notations :

- \mathbf{n} is the vector normal to $\partial\Omega_i$, exterior to Ω_i ;
- \mathbf{v} is the vector normal to Γ_i , external to Ω_1 ; the pair $(\boldsymbol{\tau}, \boldsymbol{\eta})$ denotes the orthonormal right-handed basis of Γ_i such that $\boldsymbol{\tau} \wedge \boldsymbol{\eta} = \mathbf{v}$;
- $[[f]]_\Gamma$ is the jump of f across Γ_i , $[[f]]_\Gamma = f_1 - f_2$.

2.1. Characterization of the equilibrium state

Assuming that, at time t , the spatial distribution of the two phases is known, the displacement field \mathbf{u} is the field that solves the problem defined on a heterogeneous elastic structure by the stationarity of the potential energy :

$$E^{\text{pot}},_{\mathbf{u}} \delta \mathbf{u} = 0, \forall \delta \mathbf{u} \text{ kinematically admissible with } 0 \text{ on } \partial\Omega_u.$$

This means :

$$0 = \sum_{i=1}^2 \int_{\Omega_i} \boldsymbol{\sigma} : \nabla \delta \mathbf{u} \, d\omega - \int_{\partial\Omega_T} \lambda \mathbf{T}^d \cdot \delta \mathbf{u} \, da, \quad (2)$$

$\forall \delta \mathbf{u}$ differentiable in Ω_1 and Ω_2 , and $\delta \mathbf{u}$ is continuous across Γ_i and $\delta \mathbf{u} = \mathbf{0}$ on $\partial\Omega_u$.

This variational problem can also be described by the following set of local equations :

- local constitutive relation

$$\boldsymbol{\sigma} = \frac{\partial w}{\partial \boldsymbol{\varepsilon}}, \quad (3)$$

- equilibrium equations

$$\begin{aligned} \operatorname{div} \boldsymbol{\sigma} &= \mathbf{0} \text{ over } \Omega_i, \\ [[\boldsymbol{\sigma} \cdot \mathbf{v}]]_\Gamma &= \mathbf{0} \text{ on } \Gamma_i, \\ \boldsymbol{\sigma} \cdot \mathbf{n} &= \lambda \mathbf{T}^d \text{ on } \partial\Omega_T, \end{aligned} \quad (4)$$

- kinematic relations on admissible displacements

$$\begin{aligned} [[\mathbf{u}]]_\Gamma &= \mathbf{0} \text{ on } \Gamma_i, \\ \mathbf{u} &= \lambda \mathbf{u}^d \text{ on } \partial\Omega_u, \end{aligned} \quad (5)$$

- compatibility relation

$$2\boldsymbol{\varepsilon} = \nabla \mathbf{u}' + \nabla \mathbf{u}. \quad (6)$$

The condition of a perfect interface yields the continuity of the displacement [equation (5)] and of the stress vector [equation (4)] on Γ_i . Under the assumption of elastic stability, the previous problem has a unique solution that determines the actual equilibrium state.

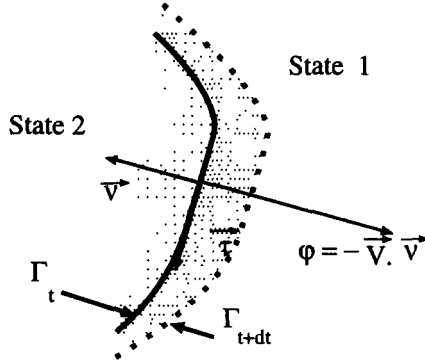


Fig. 2. Propagation of the interface Γ_t .

3. QUASI-STATIC EVOLUTION

Let us consider any positive loading rate $\dot{\lambda}$. The inertia effects are neglected. Some evolutions are induced in the mechanical quantities and the possible propagation of the interface must be studied according to a given evolution law, chosen in a manner that is appropriate to describe the irreversibility of the transformation.

First, the conservation law of the total energy leads to the definition of the total dissipation. That shows the important role of the energy release rate linked with the propagation of the interface Γ_t . The associated evolution law is given next. Finally, in order to characterize this evolution in terms of stability of the actual equilibrium state and bifurcations, the rate boundary value problem is formulated through a global formulation.

3.1. Evolution of the interface

Any variation of λ can create a propagation of Γ_t , which is characterized by the normal velocity ϕ , assumed to be positive (Fig. 2). During the propagation of the interface, perfect contact between the two materials is assumed to be maintained. Thus, using the definition given in Appendix A, the compatibility equation of Hadamard takes the following form :

$$D_\phi([\mathbf{u}]_\Gamma) = [[\mathbf{v}]]_\Gamma - \phi[[\nabla\mathbf{u} \cdot \mathbf{v}]]_\Gamma = \mathbf{0}. \quad (7)$$

In the above formula, \mathbf{v} denotes the velocity field $\mathbf{v} = (\partial\mathbf{u}/\partial t)$. More generally, the rate of any quantity f is defined by $\dot{f} = (\partial f/\partial t)$.

Because the evolution is assumed to proceed without a shock, every state during the quasi-static evolution is an equilibrium state. Thus, the continuity of the stress vector across Γ_t is maintained when the front moves. Thus, the second Hadamard's relation is obtained. In the three-dimensional case, it is written :

$$D_\phi([\sigma \cdot \mathbf{v}]_\Gamma) = [[\dot{\sigma} \cdot \mathbf{v}]]_\Gamma + \text{div}_{\Gamma_t}([\sigma \phi]_\Gamma) = \mathbf{0}, \quad (8)$$

where $\text{div}_{\Gamma_t}(f) = \text{div}(f) - \mathbf{v} \cdot \nabla f \cdot \mathbf{v} = \boldsymbol{\tau} \cdot \nabla f \cdot \boldsymbol{\tau} + \boldsymbol{\eta} \cdot \nabla f \cdot \boldsymbol{\eta}$.

In the two-dimensional case, one obtains :

$$D_\phi([\sigma \cdot \mathbf{v}]_\Gamma) = \llbracket \dot{\sigma} \cdot \mathbf{v} \rrbracket_\Gamma + \frac{\partial \llbracket \sigma \cdot \boldsymbol{\tau} \phi \rrbracket_\Gamma}{\partial s} = \mathbf{0}. \quad (9)$$

3.2. Analysis of the total dissipation

Taking into account the conservation law for the total energy during the evolution of the system, the total dissipation associated with the loading rate $\dot{\lambda}$ is :

$$\begin{aligned} D &= \int_{\partial\Omega_T} \dot{\lambda} \mathbf{T}^d \cdot \mathbf{v} \, da + \int_{\partial\Omega_u} \dot{\lambda} \mathbf{u}^d \cdot \boldsymbol{\sigma} \cdot \mathbf{n} \, da - \frac{d}{dt} \left(\sum_{i=1}^2 \left[\int_{\Omega_i} w_i(\boldsymbol{\varepsilon}) \, d\omega \right] \right) \\ &= -\frac{dP}{dt} - \int_{\partial\Omega_T} \dot{\lambda} \mathbf{T}^d \cdot \mathbf{u} \, da + \int_{\partial\Omega_u} \dot{\lambda} \mathbf{u}^d \cdot \boldsymbol{\sigma} \cdot \mathbf{n} \, da, \end{aligned} \quad (10)$$

where P denotes E^{pot} at the equilibrium state. In this case, using the properties of the equilibrium solution and Hadamard's relations, the total dissipation may be divided into two terms such that

$$\begin{aligned} D &= \sum_{i=1}^2 \int_{\Omega_i} \mathbf{A} \dot{\boldsymbol{\alpha}} \, d\omega + \int_{\Gamma} G \phi \, da \\ &= \sum_{i=1}^2 \int_{\Omega_i} \mathbf{A} \dot{\boldsymbol{\alpha}} \, d\omega + \int_{\Gamma} (\llbracket [w] \rrbracket_\Gamma - (\mathbf{v} \cdot \boldsymbol{\sigma}) \cdot \llbracket [\nabla \mathbf{u} \cdot \mathbf{v}] \rrbracket_\Gamma) \phi \, da \\ &= D_P + D_{\text{Rupt}} \geq 0. \end{aligned}$$

The first term is the volume dissipation due to the evolution of internal variables (it could be plastic dissipation, etc.). The second is due to the transformation between the two materials ; it is a surface density defined along Γ , and linearly dependent on the normal velocity ϕ . Two quantities appear here : the energy release rate G which corresponds to the thermodynamic force associated with the normal velocity of the interface and the thermodynamic force \mathbf{A} associated with the internal variables. The energy release rate of the propagation of the interface is defined by :

$$G = \llbracket [w] \rrbracket_\Gamma - (\mathbf{v} \cdot \boldsymbol{\sigma}) \cdot \llbracket [\nabla \mathbf{u} \cdot \mathbf{v}] \rrbracket_\Gamma.$$

This quantity has an analogous form to the driving traction acting on a surface of strain discontinuity proposed by Abeyaratne and Knowles (1990). The criteria which guide the evolution of the internal variables and of the interface may be written in terms of these quantities.

3.3. Criterion and evolution law

To study the evolution of the system, complementary relations must be considered to describe irreversibility. In this paper, an energy criterion is chosen as a generalized form of the well-known theory of Griffith. We assume :

- if $g(G(s)) < 0$, then no propagation can occur ; thus $\phi(s) = 0$;
- if $g(G(s)) = 0$, then propagation can occur and $\phi(s) \geq 0$.

Here s denotes a point on Γ_r . This is a local energy criterion. The function $g(G)$ is assumed to be convex and associated with the set $C(G)$, defined by :

$$C(G) = \{G \text{ such that } g(G) \leq 0\}.$$

Thus, at each equilibrium state, the interface Γ_r can be divided into two subsets where the propagation is either possible or not. They are denoted by Γ_{rupt} and $\Gamma_r/\Gamma_{\text{rupt}}$, respectively, and are defined by :

$$\Gamma_{\text{rupt}} = \{s \in \Gamma_r \text{ such that } g(G(s)) = 0\} \text{ and } \Gamma_r/\Gamma_{\text{rupt}} = \{s \in \Gamma_r \text{ such that } g(G(s)) < 0\}.$$

In the same spirit, the evolution criterion of the internal variable is written as follows.

- If $f_i(\mathbf{A}(x)) < 0$, then there is no evolution of α at the point x , so $\dot{\alpha}(x) = 0$.
- If $f_i(\mathbf{A}(x)) = 0$, then the evolution of α is possible at x , so $\dot{\alpha}(x) \neq 0$.

It is highlighted that the criteria in the different materials can be different when they are associated with different convex sets :

$$C_i(\mathbf{A}) = \{\mathbf{A}, \text{ such that } f_i(\mathbf{A}) \leq 0\}.$$

An example of useful sets is :

$$\Omega_p = \{x \in \Omega_r, \text{ such that } f_i(\mathbf{A}) = 0\}.$$

3.4. Normality law

The previous assumptions (Section 3.3) ensure that the framework of generalized standard materials (Nguyen, 1984) can be used. Then the rate of the internal variable and the normal velocity of the interface are prescribed by the normality flow rules associated with the criteria given previously. Thus, one is lead to the definition of Lagrangian multipliers c and μ which verify the following relations :

$$\left. \begin{array}{l} \text{on } \Gamma_r, \quad \phi(s) = c(s) \frac{\partial g}{\partial G} \text{ with } g(G(s))c(s) = 0 \text{ and } c(s) \geq 0; \\ \text{on } \Omega_r, \quad \dot{\alpha}(x) = \mu(x) \frac{\partial f_i}{\partial \mathbf{A}} \text{ under the assumption } f_i(\mathbf{A}(x))\mu(x) = 0 \text{ and } \mu \geq 0. \end{array} \right\} \quad (11)$$

During the transformation, characterized by the normal velocity ϕ of the interface, the criterion must be satisfied. It follows that the consistency equation

$$D_\phi(g(G(s))) = \frac{\partial g}{\partial G} D_\phi(G(s)) = 0$$

must be satisfied. It is necessary to study the evolution of the energy release rate according to ϕ . By using Hadamard's relations and other properties of the mechanical fields along Γ_r , we can derive the following formulas :

$$D_\phi(G) = [[\boldsymbol{\tau} \cdot \boldsymbol{\sigma}]_\Gamma \cdot (\nabla \mathbf{v}_1 \cdot \boldsymbol{\tau}) - (\mathbf{v} \cdot \dot{\boldsymbol{\sigma}}_2) \cdot [[\nabla \mathbf{u} \cdot \mathbf{v}]_\Gamma - [[\mathbf{A}\dot{\boldsymbol{\alpha}}]_\Gamma + \phi[[\mathbf{A}(\nabla \boldsymbol{\alpha} \cdot \mathbf{v})]_\Gamma - \phi \left\{ \left(\boldsymbol{\tau} \cdot \frac{\partial \sigma_2}{\partial s} \right) \cdot [[\nabla \mathbf{u} \cdot \mathbf{v}]_\Gamma - (\boldsymbol{\tau} \cdot \boldsymbol{\sigma}_2) \cdot \frac{\partial [[\nabla \mathbf{u} \cdot \mathbf{v}]_\Gamma}{\partial s} + [[\boldsymbol{\sigma} : (\nabla \boldsymbol{\varepsilon} \cdot \mathbf{v}) - (\mathbf{v} \cdot \boldsymbol{\sigma}) \cdot (\nabla \nabla \mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{v}]]_\Gamma \right\} \right],$$

in the 2D case, and

$$D_\phi(G) = [[\boldsymbol{\tau} \cdot \boldsymbol{\sigma}]_\Gamma \cdot (\nabla \mathbf{v}_1 \cdot \boldsymbol{\tau}) + [[\boldsymbol{\eta} \cdot \boldsymbol{\sigma}]_\Gamma \cdot (\nabla \mathbf{v}_1 \cdot \boldsymbol{\eta}) - (\mathbf{v} \cdot \dot{\boldsymbol{\sigma}}_2) \cdot [[\nabla \mathbf{u} \cdot \mathbf{v}]_\Gamma - [[\mathbf{A}\dot{\boldsymbol{\alpha}}]_\Gamma - \phi \{ \text{div}_{\Gamma} \sigma_2 \cdot [[\nabla \mathbf{u} \cdot \mathbf{v}]_\Gamma - (\boldsymbol{\tau} \cdot \boldsymbol{\sigma}_2) \cdot \nabla [[\nabla \mathbf{u} \cdot \mathbf{v}]_\Gamma \cdot \boldsymbol{\tau} - (\boldsymbol{\eta} \cdot \boldsymbol{\sigma}_2) \cdot \nabla [[\nabla \mathbf{u} \cdot \mathbf{v}]_\Gamma \cdot \boldsymbol{\eta} \} - \phi \{ [[\boldsymbol{\sigma} : (\nabla \boldsymbol{\varepsilon} \cdot \mathbf{v}) - (\mathbf{v} \cdot \boldsymbol{\sigma}) \cdot (\nabla \nabla \mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{v}]]_\Gamma - [[\mathbf{A}(\nabla \boldsymbol{\alpha} \cdot \mathbf{v})]_\Gamma \} \},$$

in the 3D case.

3.5. Formulation of the rate boundary value problem during a quasi-static evolution

During the quasi-static evolution of the system, the loading is increased step by step so that each state of the body is in equilibrium. At a given equilibrium state, we study the evolution of the system associated with some prescribed loading rate. The velocity, \mathbf{v} , the Lagrangian multipliers associated with the rate of internal variables, μ , and the normal velocity of the interface, c , need to be determined. The velocity \mathbf{v} is a solution of the variational problem :

$$\left. \begin{aligned} &\text{Find } \mathbf{v} \text{ continuous and differentiable in } \Omega_1 \text{ and } \Omega_2, \\ &\mathbf{v} = \dot{\lambda} \mathbf{u}^d \text{ on } \partial \Omega_u, \\ &[[\mathbf{v}]]_\Gamma = \phi [[\nabla \mathbf{u} \cdot \mathbf{v}]]_\Gamma \text{ on } \Gamma, \\ &\text{such that } \frac{d}{dt} (E^{\text{pot}}_{,u} \delta \mathbf{u}) = 0, \forall \delta \mathbf{u} \text{ kinematically admissible with } 0 \text{ on } \partial \Omega_u. \end{aligned} \right\} \quad (12)$$

The latter relation means that :

$$\sum_{i=1}^2 \int_{\Omega_i} -\text{div } \dot{\boldsymbol{\sigma}} \cdot \delta \mathbf{u} \, d\omega + \int_{\Gamma} [[\mathbf{v} \cdot \dot{\boldsymbol{\sigma}} \cdot \delta \mathbf{u}]]_\Gamma - [[\boldsymbol{\sigma} : \nabla \delta \mathbf{u}]]_\Gamma \phi \, da + \int_{\partial \Omega_T} (\mathbf{n} \cdot \dot{\boldsymbol{\sigma}} - \dot{\lambda} \mathbf{T}^d) \cdot \delta \mathbf{u} \, da = 0,$$

for every $\delta \mathbf{u}$ kinematically admissible with 0 on $\partial \Omega_u$.

In order to determine μ and c , (12) must be completed by the evolution laws due to the normality flow rules associated with the criteria of anelasticity and of propagation of Γ_t .

3.5.1. Local formulation. The rate boundary value problem may also be written in terms of a system of local equations defined on the actual geometry of the system. Find (\mathbf{v}, c, μ) such that :

- equilibrium equations :

$$\text{div } \dot{\boldsymbol{\sigma}} = \mathbf{0} \text{ in } \Omega_t, \quad (13)$$

- constitutive law :

$$\begin{cases} \dot{\sigma} = \frac{\partial^2 w_i}{\partial \varepsilon \partial \varepsilon} \dot{\varepsilon} + \frac{\partial^2 w_i}{\partial \varepsilon \partial \alpha} \mu \frac{\partial f_i}{\partial \mathbf{A}}, \\ -\dot{\mathbf{A}} = \frac{\partial^2 w_i}{\partial \alpha \partial \varepsilon} \dot{\varepsilon} + \frac{\partial^2 w_i}{\partial \alpha \partial \alpha} \mu \frac{\partial f_i}{\partial \mathbf{A}}, \end{cases} \quad (14)$$

or

$$[\dot{\sigma}, -\dot{\mathbf{A}}] = w_i' \left[\dot{\varepsilon}, \mu \frac{\partial f_i}{\partial \mathbf{A}} \right] \text{ in } \Omega_i,$$

- compatibility relation :

$$\dot{\varepsilon} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}') \text{ in } \Omega_i, \quad (15)$$

- boundary conditions :

$$\begin{aligned} & \text{on } \partial \Omega_u, \quad \mathbf{v} = \lambda \mathbf{u}^d, \\ & \text{on } \partial \Omega_T, \quad \dot{\sigma} \cdot \mathbf{n} = \dot{\lambda} \mathbf{T}^d, \\ & \text{on } \Gamma_i, \quad \llbracket \mathbf{v} \rrbracket_{\Gamma} - c \frac{\partial g}{\partial G} \llbracket \nabla \mathbf{u} \cdot \mathbf{v} \rrbracket_{\Gamma} = \mathbf{0}, \end{aligned} \quad (16)$$

$$\llbracket \dot{\sigma} \cdot \mathbf{v} \rrbracket_{\Gamma} + \text{div}_{\Gamma} \left(\llbracket \sigma \rrbracket_{\Gamma} c \frac{\partial g}{\partial G} \right) = \mathbf{0},$$

- local evolution law on the interface :

$$D_{\phi}(G)(c'(s) - c(s)) \frac{\partial g}{\partial G} \leq 0, \quad (17)$$

where $c(s) \geq 0$ and $\forall c'(s) \geq 0$ on Γ_{rupt} and $c(s) = c'(s) = 0$ if $s \in \Gamma_i / \Gamma_{\text{rupt}}$,

$$\dot{\mathbf{A}}(\mu'(x) - \mu(x)) \frac{\partial f_i}{\partial \mathbf{A}} \leq 0, \quad (18)$$

where $\mu(x) \geq 0$ and $\forall \mu'(x) \geq 0$ in Ω_p and $\mu = \mu' = 0$ in $\Omega_i / \{\Omega_{1p} \cup \Omega_{2p}\}$.

The previous set of equations defines a problem of heterogeneous elasticity with respect to \mathbf{v} with internal constraints due to the anelasticity and propagation of Γ_i given by (μ, c) .

It is highlighted that the non-homogeneous boundary conditions on Γ_i are linearly dependent on c . The structure studied here is the same used in the solution of the equilibrium state in Section 2.1.

3.5.2 Global formulation. In order to perform the solution of the rate boundary value problem from the point of view of existence and uniqueness of the solution which means stability of the actual equilibrium state and non-bifurcation in the evolution, a global formulation is developed next. It is shown that the solution of the previous problem is the same as that of the following one :

Find $(\mathbf{v}, \mu, c) \in \mathbf{K}$ such that :

$$\frac{\partial F}{\partial \mathbf{v}} (\mathbf{v}' - \mathbf{v}) + \frac{\partial F}{\partial \mu} (\mu' - \mu) + \frac{\partial F}{\partial c} (c' - c) \geq 0, \forall (\mathbf{v}', \mu', c) \in \mathbf{K}, \quad (19)$$

where K and F are defined as follows. K is the set of fields (\mathbf{v}, μ, c) satisfying:

- \mathbf{v} is differentiable in Ω_1 and Ω_2 , satisfies boundary conditions on $\partial\Omega_u$ and the first Hadamard's relation on Γ ,

$$\|[\mathbf{v}]_{\Gamma} = \frac{\partial g}{\partial G} c \|[\nabla \mathbf{u} \cdot \mathbf{v}]_{\Gamma};$$

- μ is admissible, i.e. $\mu \geq 0$ in Ω_p and $\mu = 0$ in $\Omega_i / \{\Omega_{1p} \cup \Omega_{2p}\}$;
- c is admissible with generalized Griffith's criterion, i.e. $c \geq 0$ on Γ_{rupt} and $c = 0$ on $\Gamma_i / \Gamma_{\text{rupt}}$.
- The function F is given in the 2D case by:

$$\begin{aligned} F(\mathbf{v}, \mu, c, \lambda) &= \sum_{i=1}^2 \int_{\Omega_i} \frac{1}{2} \left[\dot{\varepsilon}, \mu \frac{\partial f_i}{\partial \mathbf{A}} \right] w_i'' \left[\dot{\varepsilon}, \mu \frac{\partial f_i}{\partial \mathbf{A}} \right] d\omega - \int_{\Omega_T} \lambda \cdot \mathbf{T}^d \cdot \mathbf{v} da \\ &\quad - \int_{\Gamma} \frac{\partial g}{\partial G} c \left\{ \|[\boldsymbol{\tau} \cdot \boldsymbol{\sigma}]_{\Gamma} \cdot (\nabla \mathbf{v}_1 \cdot \boldsymbol{\tau}) - \left[\mathbf{A} \mu \frac{\partial f}{\partial \mathbf{A}} \right]_{\Gamma} \right\} da \\ &\quad + \int_{\Gamma} \frac{c^2}{2} \frac{\partial g}{\partial G} \left\{ \|[\boldsymbol{\sigma} : (\nabla \boldsymbol{\varepsilon} \cdot \mathbf{v}) - (\mathbf{v} \cdot \boldsymbol{\sigma}) \cdot (\nabla \nabla \mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{v}] - \mathbf{A}(\nabla \boldsymbol{\alpha} \cdot \mathbf{v})]_{\Gamma} \right. \\ &\quad \left. + \left(\boldsymbol{\tau} \cdot \frac{\partial \boldsymbol{\sigma}_2}{\partial s} \right) \|[\nabla \mathbf{u} \cdot \mathbf{v}]_{\Gamma} - (\boldsymbol{\tau} \cdot \boldsymbol{\sigma}_2) \cdot \frac{\partial \|[\nabla \mathbf{u} \cdot \mathbf{v}]_{\Gamma}\|}{\partial s} \right\} \frac{\partial g}{\partial G} da. \end{aligned} \quad (20)$$

- In the general 3D case, it takes the form:

$$\begin{aligned} F(\mathbf{v}, \mu, c, \lambda) &= \sum_{i=1}^2 \int_{\Omega_i} \frac{1}{2} \left[\dot{\varepsilon}, \mu \frac{\partial f_i}{\partial \mathbf{A}} \right] w_i'' \left[\dot{\varepsilon}, \mu \frac{\partial f_i}{\partial \mathbf{A}} \right] d\omega \\ &\quad - \int_{\Gamma_{\text{rupt}}} \lambda \mathbf{T}^d \cdot \mathbf{v} da - \int_{\Gamma} \frac{\partial g}{\partial G} c \left\{ \|[\boldsymbol{\tau} \cdot \boldsymbol{\sigma}]_{\Gamma} \cdot (\nabla \mathbf{v}_1 \cdot \boldsymbol{\tau}) \right. \\ &\quad \left. + \|[\boldsymbol{\eta} \cdot \boldsymbol{\sigma}]_{\Gamma} \cdot (\nabla \mathbf{v}_1 \cdot \boldsymbol{\eta}) - \left[\mathbf{A} \mu \frac{\partial f}{\partial \mathbf{A}} \right]_{\Gamma} \right\} da \\ &\quad + \int_{\Gamma} \frac{c^2}{2} \frac{\partial g}{\partial G} \left\{ \text{div}_{\Gamma} \boldsymbol{\sigma}_2 \cdot \|[\nabla \mathbf{u} \cdot \mathbf{v}]_{\Gamma} \right. \\ &\quad - (\boldsymbol{\tau} \cdot \boldsymbol{\sigma}_2) \cdot \nabla \|[\nabla \mathbf{u} \cdot \mathbf{v}]_{\Gamma} \cdot \boldsymbol{\tau} - (\boldsymbol{\eta} \cdot \boldsymbol{\sigma}_2) \cdot \nabla \|[\nabla \mathbf{u} \cdot \mathbf{v}]_{\Gamma} \cdot \boldsymbol{\eta} \\ &\quad \left. + \|[\boldsymbol{\sigma} : (\nabla \boldsymbol{\varepsilon} \cdot \mathbf{v}) - (\mathbf{v} \cdot \boldsymbol{\sigma}) \cdot (\nabla \nabla \mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{v}]_{\Gamma} - \|[\mathbf{A}(\nabla \boldsymbol{\alpha} \cdot \mathbf{v})]_{\Gamma} \right\} \frac{\partial g}{\partial G} da. \end{aligned} \quad (21)$$

The proof in the plane case is given in Appendix B. It is worth noticing that, by using the global formulation, we get an additional inequality on the interface Γ_r . It shows that propagation of the interface and evolution of the internal variable in any area in the neighborhood of the interface cannot occur simultaneously.

When material 2 is totally damaged ($w_2 = 0$), the previous analysis was given in

Bui and Ehrlicher (1980). The global formulation of the rate boundary value problem [equation (19)] when the damage is complete leads to the following quadratic functional, similar to the result given in Bui *et al.* (1981) and Stolz (1983).

$$F(\mathbf{v}, c, \dot{\lambda}) = \int_{\Omega_1} \frac{1}{2} \dot{\varepsilon}' w_1'' \dot{\varepsilon} \, d\omega - \int_{\Gamma_{\Omega_T}} \dot{\lambda} \mathbf{T}^d \cdot \mathbf{v} \, da \\ - \int_{\Gamma} \frac{\partial g}{\partial G} c \{ \sigma_1 : \nabla \mathbf{v}_1 \} \, da + \int_{\Gamma} \frac{c^2}{2} \frac{\partial g}{\partial G} \{ \sigma_1 : (\nabla \varepsilon_1 \cdot \mathbf{v}) \} \frac{\partial g}{\partial G} \, da.$$

3.5.3 Dual formulation. In what follows, a dual formulation is given for the general case. The plane strain case can easily be deduced from the discussion that follows.

Given the equilibrium state in terms of stress tensor σ , internal variable α and position of the interface Γ , the complementary potential energy is called w_i^* and we have the following relations:

$$\varepsilon = \frac{\partial w_i^*}{\partial \sigma} \text{ and } -\mathbf{A} = \frac{\partial w_i^*}{\partial \alpha}.$$

The rate boundary value problem to be considered is the determination of $(\dot{\sigma}, \mu, c)$ in the set of admissible fields L , defined by:

- $\dot{\sigma}$ is such that $\text{div } \dot{\sigma} = \mathbf{0}$ in $\Omega_1 \cup \Omega_2$. It is admissible with boundary conditions on $\partial\Omega_T$: $\dot{\sigma} \cdot \mathbf{n} = \dot{\lambda} \mathbf{T}^d$ and it satisfies the second Hadamard's relation on Γ : $[[\dot{\sigma} \cdot \mathbf{v}]]_{\Gamma} + \text{div}_{\Gamma} [[\sigma (\partial g / \partial G) c]]_{\Gamma} = \mathbf{0}$;
- c and μ are admissible with the normality laws previously introduced, i.e.: $c \geq 0$ on Γ_{rupt} ; $c = 0$ on $\Gamma_i / \Gamma_{\text{rupt}}$; $\mu \geq 0$ in $\Omega_{1p} \cup \Omega_{2p}$; $\mu = 0$ in $\Omega_i / \{\Omega_{1p} \cup \Omega_{2p}\}$.

Let H denote the following quadratic function of elements of L ,

$$H(\dot{\sigma}, \mu, c, \dot{\lambda}) = - \sum_{i=1}^2 \int_{\Omega_i} \frac{1}{2} \left[\dot{\sigma}, \mu \frac{\partial f_i}{\partial \mathbf{A}} \dot{\mathbf{A}} \right]' w_i^{*''} \left[\dot{\sigma}, \mu \frac{\partial f_i}{\partial \mathbf{A}} \right] \, d\omega \\ - \int_{\Gamma_{\Omega_u}} \dot{\lambda} \mathbf{u}^d \cdot (\dot{\sigma} \cdot \mathbf{n}) \, da \\ - \int_{\Gamma} \frac{\partial g}{\partial G} c \left\{ [[\mathbf{v} \cdot \nabla \mathbf{u}]]_{\Gamma} \cdot (\dot{\sigma}_2 \cdot \mathbf{v}) + \left[\left[\mathbf{A} \mu \frac{\partial f}{\partial \mathbf{A}} \right] \right]_{\Gamma} \right\} \, da \\ - \int_{\Gamma} \frac{c^2}{2} \frac{\partial g}{\partial G} \{ \text{div}_{\Gamma} (\sigma_2) \cdot [[\nabla \cdot \mathbf{u}]]_{\Gamma} \\ - (\boldsymbol{\tau} \cdot \sigma_2) \cdot \nabla [[\nabla \mathbf{u} \cdot \mathbf{v}]]_{\Gamma} \cdot \boldsymbol{\tau} - (\boldsymbol{\eta} \cdot \sigma_2) \cdot \nabla [[\nabla \mathbf{u} \cdot \mathbf{v}]]_{\Gamma} \cdot \boldsymbol{\eta} \\ + [[\sigma \cdot (\nabla \varepsilon \cdot \mathbf{v}) - (\mathbf{v} \cdot \sigma) \cdot (\nabla \nabla \mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{v}]]_{\Gamma} - [[\mathbf{A} \nabla \alpha \cdot \mathbf{v}]]_{\Gamma} \} \frac{\partial g}{\partial G} \, da. \quad (22)$$

Then the variational inequality becomes : find $(\dot{\sigma}, \mu, c) \in L$ such that

$$\frac{\partial H}{\partial \dot{\sigma}}(\dot{\sigma}' - \dot{\sigma}) + \frac{\partial H}{\partial \mu}(\mu' - \mu) + \frac{\partial H}{\partial c}(c' - c) \leq 0, \forall (\dot{\sigma}', \mu', c') \in L, \quad (23)$$

which is equivalent to

$$\begin{aligned} 0 \geq & \sum_{i=1}^2 \int_{\Omega} \dot{\mathbf{A}} \frac{\partial f_i}{\partial \mathbf{A}}(\mu' - \mu) d\omega - \int_{\partial\Omega_u} (\lambda \mathbf{u}^d - \mathbf{v}) \cdot (\dot{\sigma}' - \dot{\sigma}) \cdot \mathbf{n} da \\ & + \int_{\Gamma} \{ |[\mathbf{v}]|_{\Gamma} - \phi |[\mathbf{v} \cdot \nabla \mathbf{u}]|_{\Gamma} \} \cdot (\dot{\sigma}'_2 - \dot{\sigma}_2) \cdot \mathbf{v} da - \int_{\Gamma} \phi \left| \left[\mathbf{A} \frac{\partial f}{\partial \mathbf{A}}(\mu' - \mu) \right] \right|_{\Gamma} da \\ & + \int_{\Gamma} \frac{\partial g}{\partial G}(c' - c) D_{\phi}(G) da. \end{aligned}$$

In the derivation of the above relations, the properties of $\dot{\sigma}$, c and μ have been used. Finally, the solution of (23) gives the local boundary conditions, and the evolution relations on Γ , in the *plastic* domain Ω_p . The additional relation on Γ , which deals with the possibility of simultaneous propagation and evolution of the internal variable is also obtained.

3.5.4 Remarks. It is noted that the values of F and H for the solutions to the rate boundary value problem are opposite, because if (\mathbf{v}, μ, c) and $(\dot{\sigma}, \mu, c)$ are the solutions of (19) and (23), respectively, then $\dot{\sigma}$ and $\dot{\epsilon}$ are related by the constitutive laws. So, it follows that :

$$\begin{aligned} F(\mathbf{v}, \mu, c, \dot{\lambda}) + H(\dot{\sigma}, \mu, c, \dot{\lambda}) &= \sum_{i=1}^2 \int_{\Omega} \dot{\sigma} : \varepsilon(\mathbf{v}) d\omega \\ &+ \int_{\partial\Omega_u} \lambda \mathbf{u}^d \cdot \dot{\sigma} \cdot \mathbf{n} da - \int_{\partial\Omega_T} \dot{\lambda} \mathbf{T}^d \cdot \mathbf{v} da \\ &- \int_{\Gamma} \phi \{ |[\mathbf{v} \cdot \nabla \mathbf{u}]|_{\Gamma} \cdot (\dot{\sigma}_2 \cdot \mathbf{v}) + |[\boldsymbol{\tau} \cdot \boldsymbol{\sigma}]|_{\Gamma} \cdot (\nabla \mathbf{v}_1 \cdot \boldsymbol{\tau}) \\ &+ |[\boldsymbol{\eta} \cdot \boldsymbol{\sigma}]|_{\Gamma} \cdot (\nabla \mathbf{v}_1 \cdot \boldsymbol{\eta}) \} da \\ &= 0. \end{aligned}$$

This property leads to a mixed formulation via the potential I :

$$I(\mathbf{v}, \dot{\sigma}, \mu, c, \dot{\lambda}) = F(\mathbf{v}, \mu, c, \dot{\lambda}) + H(\dot{\sigma}, \mu, c, \dot{\lambda}).$$

The solution of the rate boundary value problem is a stationary point of I .

4. CHARACTERIZATION OF THE ACTUAL STATE

As it was underlined before, at any given loading rate, any propagation normal velocity and any rate of internal variables (here defined by $\dot{\lambda}$, c and μ), there exists a displacement velocity, denoted by $\mathbf{v}(c, \mu, \dot{\lambda})$, which is the solution to an elasticity

problem with given initial stresses. Thus, denoting the value of F when $\mathbf{v} = \mathbf{v}(c, \mu, \lambda)$ by $W(c, \mu, \lambda)$, the above inequality can be written in the following form.

Find (μ, c) in \tilde{K} such that

$$\frac{\partial W}{\partial \mu}(\mu' - \mu) + \frac{\partial W}{\partial c}(c' - c) \geq 0, \forall (\mu', c') \in K. \quad (24)$$

K represents the set of admissible fields (μ, c) which allow an evolution of the internal variable and a propagation of the interface when the associated criterion is satisfied. Thus these fields are such that

$$\begin{aligned} \mu &\geq 0 \text{ in } \Omega_p, \\ \mu &= 0 \text{ in } \Omega_t / \{\Omega_{1p} \cup \Omega_{2p}\}, \\ c &\geq 0 \text{ on } \Gamma_{\text{rupt}}, \\ c &= 0 \text{ on } \Gamma_t / \Gamma_{\text{rupt}}. \end{aligned}$$

The existence and the uniqueness of the solution of the rate boundary value problem are dependent on the existence and the uniqueness of the solution of this latter inequality.

4.1 Criterion of existence—stability of the actual state

As in Nguyen and Stolz (1986), there exists a solution to the rate boundary value problem from the actual equilibrium state, whatever may be the increasing of the loading parameter, which means that the actual equilibrium state is stable if

$$\delta c \frac{\partial^2 W}{\partial c^2} \delta c + \delta \mu \frac{\partial^2 W}{\partial \mu^2} \delta \mu > 0, \forall (\delta \mu, \delta c) \in \tilde{V}^*,$$

where \tilde{V}^* is the previous set K without the trivial element defined by c and μ equal to 0 on Γ , and Ω , respectively.

4.2 Uniqueness and non-bifurcation criterion

The solution to the rate problem is unique, i.e. the actual equilibrium state is not a bifurcation point if

$$\delta c \frac{\partial^2 W}{\partial c^2} \delta c + \delta \mu \frac{\partial^2 W}{\partial \mu^2} \delta \mu > 0, \forall (\delta \mu, \delta c) \in \tilde{V}^{0*},$$

where \tilde{V}^{0*} is the vector space generated by K without the trivial element defined above.

5. APPLICATIONS

In order to predict the existence and the uniqueness of a solution to the rate boundary value problem from the actual known equilibrium state, some simple

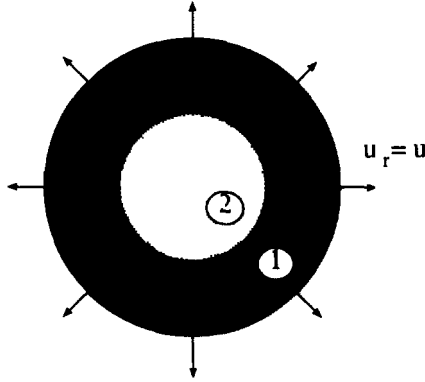


Fig. 3. Damaged composite sphere.

structures are now analyzed. Composite spheres are studied. They are constituted by a core with the damaged material (material 2) and a shell with the sound material (material 1). The loading is given through a prescribed displacement on the external surface, or a prescribed force. Both materials are assumed to be isotropic and linearly elastic. Their elastic characteristics are denoted by (E_i, ν_i) with $i \in \{1, 2\}$. Material 1 transforms irreversibly into material 2 along Γ_i , with the generalized Griffith's criterion (see Section 3.3):

- if $G(s) < G_C$ then there is no propagation of Γ_i at the point s ; $\phi(s) = 0$;
- if $G(s) = G_C$ then the propagation may occur at the point s and $\phi(s) \geq 0$.

5.1. Elastic composite spheres under radial extension

The system is made of a sphere whose external radius is denoted by R . The core is constituted by material 2, and the shell between the interface and the external volume is made of the sound material. The interface radius is denoted by a and the volume concentration of the damaged material is $c = (a^3/R^3)$. Let c_{mit} represent the initial volume concentration of material 2. The loading applied on the external surface is a displacement along the radial vector (see Fig. 3). Only spherical propagations are considered here.

The materials in the composite sphere are assumed to be isotropic and linearly elastic. Their mechanical properties are the shear moduli, denoted by μ_1 and μ_2 , and the bulk moduli, denoted by k_1 and k_2 , where k_1 is assumed to be greater than k_2 .

The displacement vectors at the equilibrium state are given by Appendix C. The energy release rate is uniform on the interface Γ_i , and is given by:

$$G = \frac{9u^2}{R^2 D(c)^2} (k_1 - k_2)(3k_2 + 4\mu_1)(3k_1 + 4\mu_1).$$

As the loading increases, three stages can be observed.

- First, when the loading is smaller than the critical value $u_C(c_{\text{mit}})$, i.e.

$$u_c(c_{\text{int}}) = \sqrt{\frac{G_c R^2 D (c_{\text{int}})^2}{9(k_1 - k_2)(3k_2 + 4\mu_1)(3k_1 + 4\mu_1)}}$$

the threshold is not reached and the interface can not propagate. The external behavior is given by :

$$p = \sigma_{,r}(r = R) = \frac{3u k_1(3k_2 + 4\mu_1) - 4\mu_1(k_1 - k_2)c_{\text{int}}}{R(3k_2 + 4\mu_1 + 3c_{\text{int}}(k_1 - k_2))} = 3k_{\text{int}} \frac{u}{R}.$$

- When the loading reaches the critical value $u_c(c_{\text{int}})$, the energy release rate reaches the threshold value G_c . The interface can propagate under the condition that the criterion remains verified. So, it gives us the link between the velocity of the interface \dot{a} and the loading rate \dot{u} . The quadratic function that enables us to characterize the evolution of the structure is given by :

$$\begin{aligned} \frac{1}{2} \ddot{a} W''_{,uu} \ddot{a} &= \frac{18\pi \dot{a}^2 u^2 \sqrt{c}(k_1 - k_2)}{RD^3} \\ &\times [36(k_1 - k_2)\mu_1(3k_2 + 4\mu_1)(1 - c) + 54k_2(k_1 - k_2)(3k_1 + 4\mu_1)c \\ &+ 6k_2(3k_1 + 4\mu_1)D + 72\mu_1(k_1 - k_2)(3k_2 + 4\mu_1)(2 + c)] \\ &\geq 0, \forall \dot{a} \text{ an admissible velocity of the interface.} \end{aligned}$$

Consequently, the actual state of equilibrium is stable. This interface propagates until the whole sphere transforms into material 2. During this evolution,

$$\frac{\delta p}{\delta u} = -4 \frac{\mu_1}{R}.$$

When the interface reaches the external surface, the corresponding loading is

$$u_{\text{Final}} = u_c(c = 1) = R \sqrt{\frac{G_c(3k_1 + 4\mu_1)}{9(k_1 - k_2)(3k_2 + 4\mu_1)}}.$$

- When the loading increases from u_{Final} to infinity, the structure has the behavior of a sphere of material 2 :

$$p = 3k_2 \frac{u}{R}.$$

The global behavior of the composite sphere is illustrated in Fig. 4.

5.2. Composite sphere with two elastoplastic phases

The two materials are now assumed to be isotropic and elastoplastic. Initially, there is no plastic strain in the structure. Their elastic mechanical characteristics are the shear moduli, denoted by μ_1 and μ_2 , and the bulk moduli, denoted by k_1 and k_2 ; k_1 is assumed to be greater than k_2 . The loading applied on the external surface is a displacement along the radial vector (see Fig. 3). Only spherical propagations are considered here.

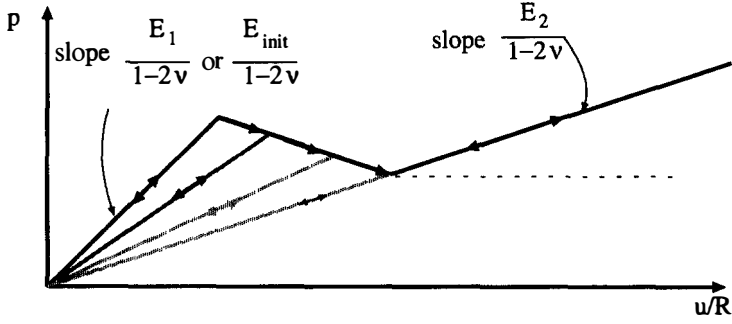


Fig. 4. Global behavior under spheric loading of an elastic composite sphere.

The state variables are the displacements \mathbf{u} , the plastic strain tensor ε_p and the hardening variables α . Their volume density of free energy is written as:

$$w = w_{el}(\varepsilon - \varepsilon_p) + \frac{1}{2} C \alpha : \alpha.$$

This kind of model includes kinematic hardening. The thermodynamic forces associated with the state variables are described as follows:

- the stress tensor σ with the strain tensor, $\sigma = (\partial w_{el} / \partial \varepsilon) = k \text{tr}(\varepsilon) \text{Id} + 2\mu(\varepsilon - \text{tr}(\varepsilon) \text{Id}) - 2\mu \varepsilon_p$,
- $-\sigma$ with the plastic strain tensor,
- a tensor denoted by X with the hardening variables.

Here Id denotes the identity tensor. The criterion is given by a convex function :

$$f(\sigma) = |\sigma_{rr} - \sigma_{\theta\theta} - X_{rr} + X_{\theta\theta}| - \sigma_Y \leq 0.$$

The constitutive equations imply the following relations:

$$\begin{aligned} \dot{\varepsilon}_p &= 0, \text{ if } f < 0 \text{ or } (f = 0 \text{ and } \dot{f} < 0), \\ \dot{\varepsilon}_p &= \frac{3}{2C} (\dot{\sigma}_{rr} - \dot{\sigma}_{\theta\theta}) \{ \mathbf{e}_r \otimes \mathbf{e}_r - \frac{1}{2} \mathbf{e}_\theta \otimes \mathbf{e}_\theta - \frac{1}{2} \mathbf{e}_\phi \otimes \mathbf{e}_\phi \}, \text{ if } f = 0 \text{ and } \dot{f} = 0, \\ \dot{X} &= \frac{2C}{3} \dot{\varepsilon}_p, \\ \dot{\alpha} &= \dot{\varepsilon}_p, \end{aligned}$$

where $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ is the spherical orthonormal right-handed basis.

At the equilibrium without plastic zone, the displacement vectors are given in the previous section. The energy release rate is uniform on the interface Γ_r and is given by:

$$G = \frac{9u^2}{R^2 D(c)^2} (k_1 - k_2)(3k_2 + 4\mu_1)(3k_1 + 4\mu_1).$$

First, when the loading is smaller than the critical value $u_c(c_{init})$, defined above, we

obtain an elastic response. Because the stress in the core has a uniform pressure, no plastic zone develops in material 2. However, it may exist in material 1, and then the plastic zone becomes a shell around the core. The external radius of the plastic zone is denoted by r_p . The behavior of the structure and the evolution of the internal state are characterized by four stages.

- (1) At the beginning, the threshold value of both the damage and plasticity criterion are not reached. Thus there is no plastic zone and the interface does not move when $u < u_c(c_{int})$ and $u < u_p(c_{int})$ where

$$u_p(c) = R \frac{\sigma_Y [3k_2 + 4\mu_1 + 3c(k_1 - k_2)]}{18\mu_1(k_1 - k_2)}.$$

The global behavior is obtained by $\sigma_{rr}(r = R) = 3k_{int}(u/R)$.

- (2) As the loading increases, two situations may appear :

- If $u_c(c_{int}) < u_p(c_{int})$ then $u_c(c) < u_p(c)$, whatever the value of c . So, there is never a plastic zone in material 1. The evolution of the structure is analyzed as in the previous section when both materials are elastic.
- If $u_c(c_{int}) \geq u_p(c_{int})$, then, when the loading reaches $u_p(c_{int})$ a plastic zone appears. The structure is then composed by three different regions : the first is made of material 2, the second of material 1 in its plastic domain, and the third of material 1 without any plastic deformation. The displacements in each zone are given in Appendix C. The global behavior is such that

$$\frac{u}{R} = \frac{2\mu_1\sigma_Y}{F_1} \left\{ \ln \frac{c_P}{c} + 1 \right\} + \frac{\sigma_Y(3k_1 + 4\mu_1)F_2c_P}{18\mu_1F_1(k_1 - k_2)c} + \frac{\sigma_Yc_P}{6\mu_1}.$$

The energy release rate on the interface between materials 2 and 1 is :

$$G(c, c_P) = \frac{\sigma_Y^2(3k_1 + 4\mu_1)}{2F_1} + \frac{\sigma_Y^2(3k_1 + 4\mu_1)^2F_2c_P^2}{72\mu_1^2(k_1 - k_2)F_1c^2}.$$

Because of the relation between u and r_p , a critical value of the displacement, denoted by $\tilde{u}_c(c)$, may be extracted such that $G(c, c_P) = G_C$ when $u = \tilde{u}_c(c)$. While $u \in [u_p(c_{int}), \tilde{u}_c(c_{int})]$, the plastic zone size in material 1 increases but the damage interface does not propagate.

- (3) When the loading increases, there are two possibilities :

- If the plastic radius reaches the external surface ($r_p = R$), before the damage criterion is reached, then the structure is divided into two regions, one made of material 2, the other of material 1 with plastic strain. The displacement and stress fields are given in Appendix C, and the energy release rate is written :

$$G = \frac{\sigma_Y^2(3k_1 + 4\mu_1)}{2F_1} + \frac{9F_1F_2(k_1 - k_2)}{2} \frac{\left\{ \frac{u}{R} + \frac{2\mu_1\sigma_Y}{F_1} \ln c \right\}^2}{\{F_2 + 3(3\mu_1 + C)(k_1 - k_2)c\}^2}.$$

In this case, there is propagation of the interface only if

$$\frac{\dot{u}}{R} = \frac{3\dot{a}c}{a} \left[-\frac{2\mu_1\sigma_Y}{F_1c} + \frac{3(k_1-k_2)(3\mu_1+C)\left(\frac{u}{R} + \frac{2\mu_1\sigma_Y}{F_1}\ln c\right)}{F_2+3(3\mu_1+C)(k_1-k_2)c} \right].$$

- If the damage criterion is reached before the whole elastic area of material 1 transforms into the plastic zone, then during the propagation of the interface,

$$\frac{\dot{a}}{a} = \frac{\dot{r}_P}{r_P}, \quad \frac{\dot{u}}{R} = \frac{\sigma_Y \dot{r}_P r_P^2}{2\mu_1 R^3},$$

$$\dot{\sigma}_{rr}(r=R) = -4\mu_1 \frac{\dot{u}}{R},$$

and the global behavior of the structure is given by

$$\frac{u}{R} = \frac{u_{PC_{\text{mit}}}}{R} + \frac{\sigma_Y(r_P^3 - r_{PC_{\text{mit}}}^3)}{2\mu_1 R^3},$$

$$\sigma_{rr}(r=R) = \frac{6\sigma_Y k_1 \mu_1}{F_1} \left\{ \ln \frac{c_P}{c_{\text{mit}}} + 1 + \frac{(k_1 + 4\mu_1) F_2 c_P}{36\mu_1^2 (k_1 - k_2) c_{\text{mit}}} \right\} - \frac{2\sigma_Y c_P}{3}.$$

In previous relations, $u_{PC_{\text{mit}}}$ is the critical value of the loading when the damage criterion is reached in the presence of plastic area. When $r_P = R$, then the results of the previous case may be used.

- (4) Finally, the damaged area reaches the external surface and the plastic zone completely disappears in the structure. The global behavior of the composite sphere is now that of material 2.

The evolution is summarized in Figs 5, 6 and 7.

5.3. Composite spheres assemblage

In this section, the composite spheres assemblage of Hashin (1983) is analyzed. The system is composed of the compact assemblage of spheres whose external radius is variable in order to fill the whole domain as shown in Fig. 8. The microscopic structure is constituted by small composite spheres with a core of material 2 and a shell of material 1 ; both materials are linearly elastic and homogeneous.

As in the general case, material 2 transforms into material 1 ; the transformation is irreversible and the generalized Griffith's criterion based on the energy release rate of the transformation is used. The volume concentration of material 2 is denoted by c . Applying the same method as that of Bretheau *et al.* (1992) and Herve and Zaoui (1992), the assemblage is considered as well-disordered. Using the particular "three phase model" introduced by Christensen and Lo (1979) the homogeneous equivalent medium denoted by material 0 has an unknown behavior (see Fig. 9). In phase i , the local characteristics are the bulk modulus and the shear modulus denoted respectively by

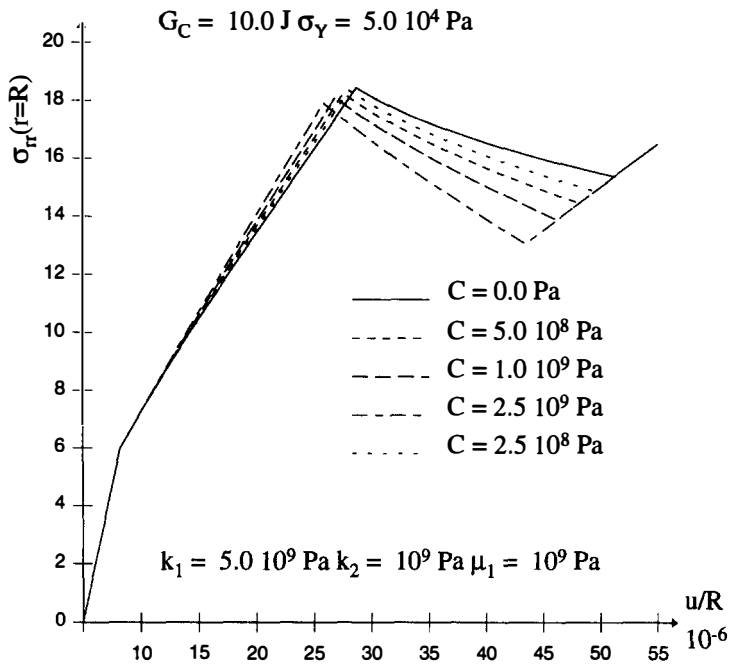


Fig. 5. Global behavior of an elastoplastic composite sphere . influence of C

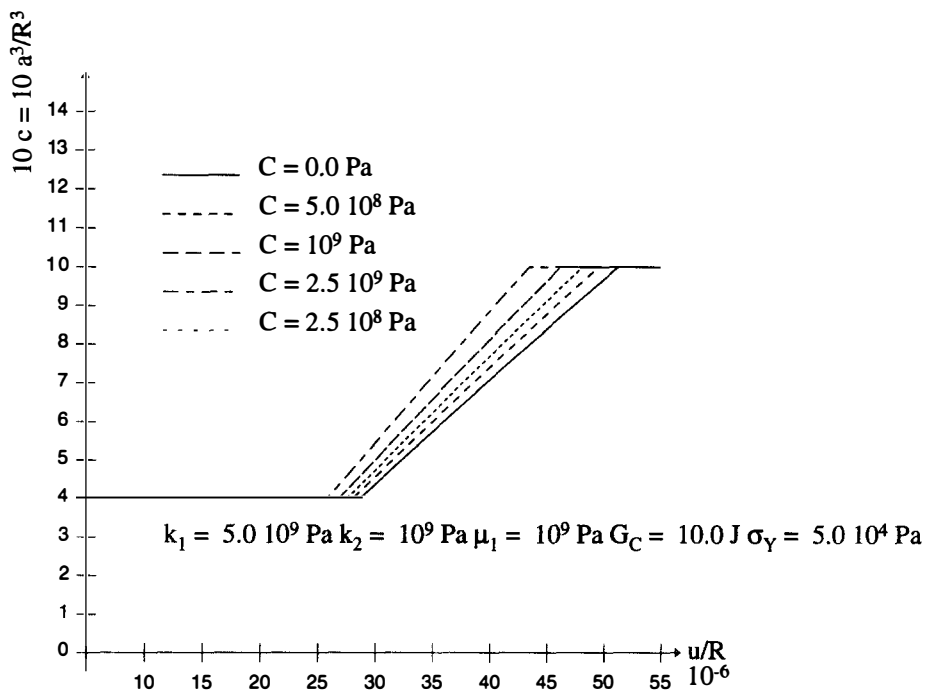


Fig. 6. Evolution of $c = (a^3/R^3)$: influence of C .

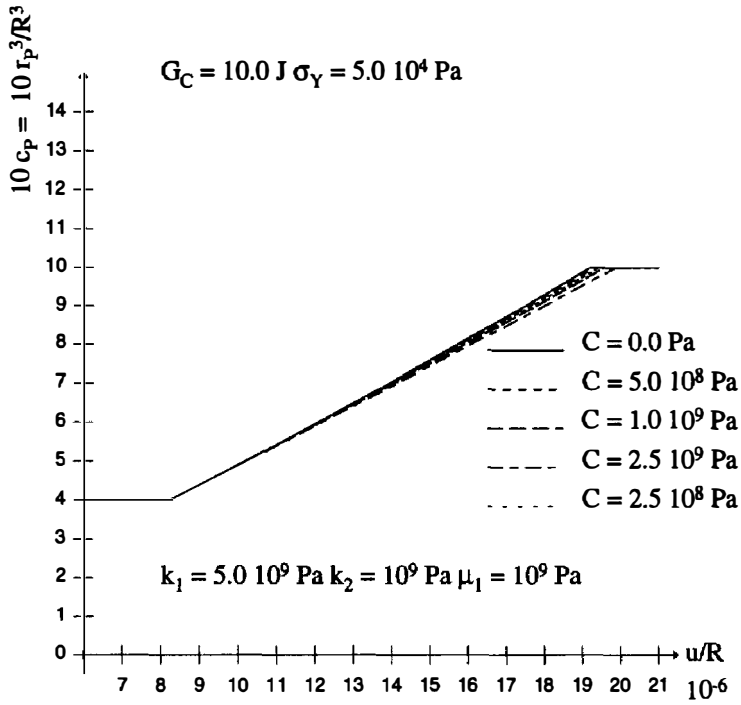


Fig. 7. Evolution of (r_p^3/R^3) : influence of C .

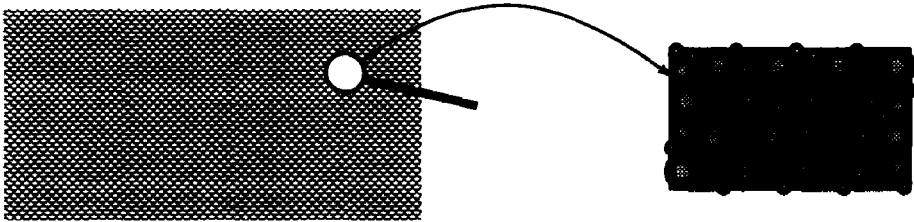


Fig. 8. Assemblage of composite spheres.

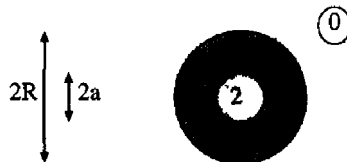


Fig. 9. Three-phase model.

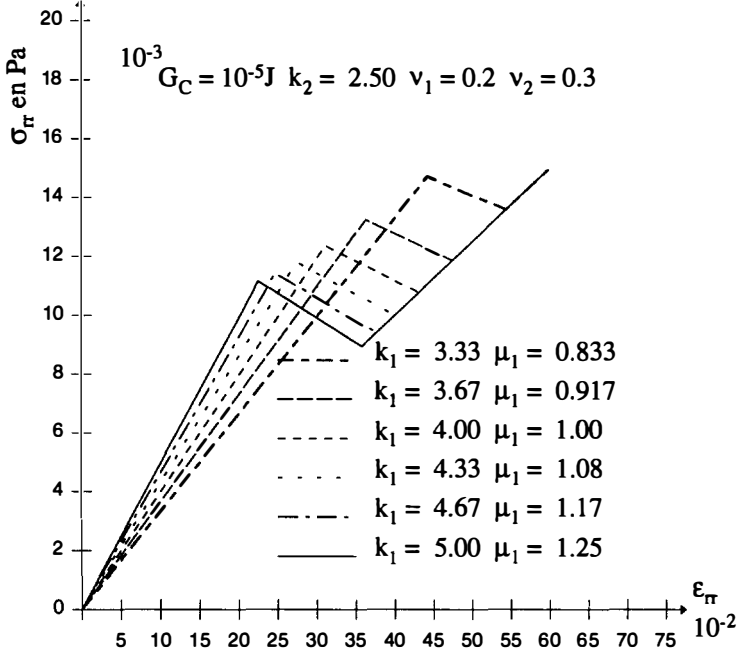


Fig. 10. $\nu_1 = 0.2$ and $\nu_2 = 0.3$.

$$k_i = \frac{E_i}{3(1-2\nu_i)} \quad \text{and} \quad \mu_i = \frac{E_i}{2(1+\nu_i)},$$

where E_i is the Young modulus and ν_i is the Poisson ratio.

In what follows, k_1 is assumed to be greater than k_2 .

5.3.1. *Macroscopic behavior under isotropic extension with one family of composite spheres.* There exists only one family of composite spheres in the structure ; that means that the ratio a/R is given and equals $\sqrt[3]{c}$. Using analytical results obtained in Bretheau *et al.* (1992) and Christensen and Lo (1979), one gets the effective bulk modulus of the material 0, denoted by k_0 :

$$k_0 = k_1 + c \frac{(k_2 - k_1)}{1 + \frac{3(k_2 - k_1)(1 - c)}{(4\mu_1 + 3k_1)}}. \quad (25)$$

On the interface, the energy release rate is

$$G = \frac{\theta_0^2 (4\mu_1 + 3k_1)(4\mu_1 + 3k_2)(k_1 - k_2)}{2[4\mu_1 + 3k_2 + 3c(k_1 - k_2)]^2}. \quad (26)$$

where θ_0 represents the uniform strain given at infinity. When a generalized Griffith's criterion is taken into account for the damage transformation, as G reaches the critical threshold G_C , the ratio a/R will increase such that G remains equal to G_C . That is why the macroscopic behavior is given by Figs 10 and 11.

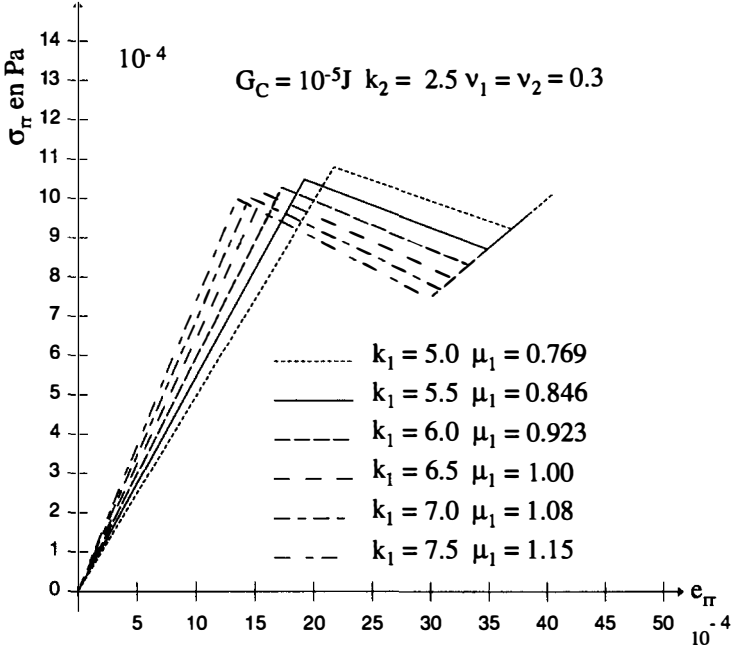


Fig. 11 $\nu_1 = \nu_2 = 0.3$. Macroscopic behavior of composite spheres assemblage: influence of (E_1/E_2) .

5.3.2. *Macroscopic behavior under isotropic extension with two families of composite spheres.* In what follows, the macroscopic evolution of a composite spheres assemblage is analyzed when two different families coexist in the structure. They are supposed to be perfectly disordered. c_1 (respectively, c_2) denotes the volume fraction of material 2 in the first family (respectively, in the second). Thus, $c = fc_1 + (1-f)c_2$.

The macroscopic bulk modulus is

$$k_0 = k_1 - \frac{(4\mu_1 + 3k_1)(k_1 - k_2)\{fc_1D_1 + (1-f)c_2D_2\}}{(4\mu_1 + 3k_2)\{fD_2 + (1-f)D_1\} + 3(k_1 - k_2)\{fc_1D_2 + (1-f)c_2D_1\}}, \quad (27)$$

where $D_i = 2[4\mu_1 + 3k_2 + 3c_i(k_1 - k_2)]^2$. If G_i denotes the energy release rate in the family i ,

$$(G_1 - G_2)(c_1 - c_2)(\mu_1 - \mu_2) > 0. \quad (28)$$

So, the global behavior of the system will be as follows:

- At the beginning, the macroscopic behavior is elastic, with k_0 calculated with the initial concentrations $c_{1\text{mit}}$ and $c_{2\text{mit}}$.
- At higher loading parameter θ_0 , the energy criterion may be reached.
 - If $\mu_1 > \mu_2$, the difference between the two concentrations will increase until the larger reaches the value 1.
 - If $\mu_1 < \mu_2$, the difference between the two concentrations will decrease until they are identical, i.e. the two different families become one.
 - If $\mu_1 = \mu_2$, both concentrations could increase.

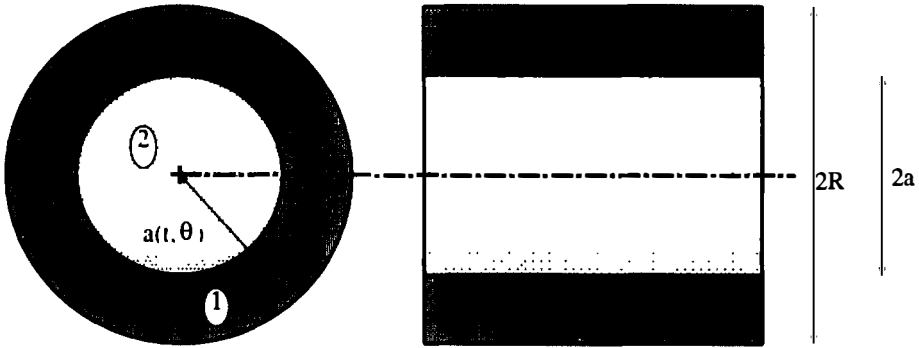


Fig. 12. Composite cylinder.

- When the whole volume is changed into material 2, the macroscopic behavior corresponds to the mechanical behavior of material 2.

5.3.3. *Conclusions.* Even if at time t the system is composed of only one family, the local response to the loading increment could be non-unique. In fact, there may exist many kinds of bifurcations. One part of the structure may be damaged (damage localization—no more disorder in the domain). Very well-ordered configurations may appear and specific space distributions of the constituent phases are obtained. One gets order among disorder. If $\mu_1 > \mu_2$, then a new, perfectly disordered, family may appear among the first family. In that case, there is more and more disorder in the structure.

Here, it is to be noted that the total dissipation is $D = G_c (dc/dt)$. So the macroscopic behavior is dissipative while the components are elastic. The transformation between 1 and 2 corresponds to a volume damage in some sense (Lemaitre and Chaboche, 1985).

5.4. Torsion of an elastic composite cylinder

In this section, the system is an infinite axisymmetric cylinder. The external radius is R and the radius of the core is a . Materials 1 and 2 occupy, respectively, the shell and the core. They are both linearly elastic. Two classic constants are useful in what follows :

$$K_i = \frac{E_i}{2(1 + \nu_i)(1 - 2\nu_i)} \quad \text{and} \quad \mu_i = \frac{E_i}{2(1 + \nu_i)},$$

where K_1 is assumed to be greater than K_2 .

At the beginning, the structure is axisymmetric, and the loading is axisymmetric and independent of the axial coordinate. Figures 12 and 13 summarize all the data. As in Berest (1989) and Fedelich and Berest (1988), the loading is as follows :

- On lateral surfaces ($r = R$) there is no loading $\lambda \mathbf{T}^d = \mathbf{0}$.
- The displacement field in a plane (r, θ) is such that : $u_r = 0$, $u_\theta = \lambda r z$ et $u_z = \lambda \eta(r)$.

The actual equilibrium state is defined by the displacement field in the structure and

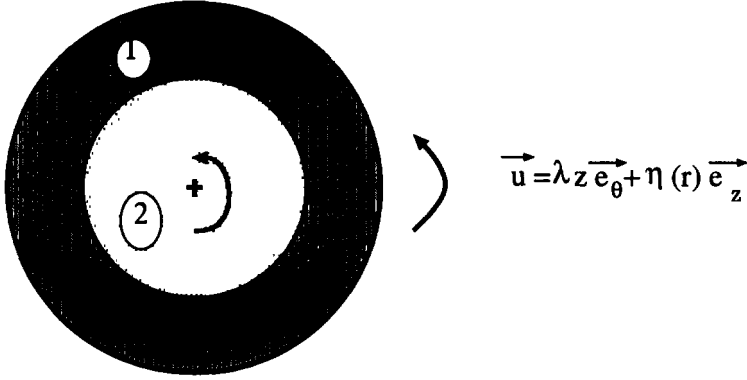


Fig. 13. Torsion of a composite cylinder.

the position of the interface Γ , given by a . The volume concentration of material 2 is denoted by $c = (a^2/R^2)$. The displacement fields are given in Appendix D. Correspondingly, the energy release rate on the interface is given by

$$G = (\mu_1 - \mu_2) \frac{\lambda^2 R^2}{2}.$$

Before the energy release rate equals G_C , no propagation may occur anywhere on the interface. When the Griffith's threshold is reached, then propagation is allowed on the whole interface. So, in order to study a case of general propagation, consider the velocity $\dot{\phi} = -\dot{a}(\theta)$ such that

$$\dot{a} = \alpha_0 + \sum_{n=1}^{\infty} [\alpha_n \cos n\theta + \beta_n \sin n\theta].$$

For the purpose of characterizing the actual state from the point of view of stability and bifurcation, it is necessary to get the displacement velocity at given velocity of the loading and given normal velocity of the interface. Then, according to the general demonstration, conclusions may be determined from the expression of the function $W(\dot{u}, \dot{a})$.

Using the velocity obtained in Appendix D, the characterization of the quasi-static evolution of the composite cylinder is given via the quadratic function :

$$\dot{a} W''_{aa} \dot{a} = -\lambda^2 \pi l a^2 (\mu_1 - \mu_2) \left[\alpha_0^2 + \sum_{n=2}^{\infty} \frac{[\alpha_n^2 + \beta_n^2]}{2} \frac{\{\mu_1 [(1+n)R^{2n} + (1-n)a^{2n}] + \mu_2 (R^{2n} - a^{2n})(1-n)\}}{[\mu_1 (R^{2n} + a^{2n}) + \mu_2 (R^{2n} - a^{2n})]} \right].$$

Whatever the propagation may be, because the quadratic functional W is non-positive, the evolution is unstable as soon as the Griffith's threshold is reached.

6. CONCLUSIONS

The modelling of partial damage is presented in this paper in order to characterize various structures in which surfaces of discontinuities propagate. Using the classic framework of standard materials, it appears necessary to consider the positions of the surfaces of discontinuity as internal variables whose evolution is governed by criteria and evolution laws similar to those used in fracture mechanics. Thus, in our analysis the surface can propagate when the stored free energy reaches a threshold value.

In order to characterize the evolution of the structure, the rate boundary value problem needs to be solved. One has to emphasize that the solution of this problem is analogous to that of the equilibrium problem: the surface of discontinuity is assumed to be known, but the normal velocity is unknown. Numerically, this assumption enables us to find the right velocity of the interface between the sound and damaged materials with one mesh.

Thanks to the global formulation, the solution of the rate boundary value problem is associated with a minimization or a maximization of a quadratic function, depending on the choice of the unknown fields. It is emphasized that this global formulation is helpful in characterizing the evolution of the structure in terms of existence and uniqueness of the solution to the rate boundary value problem, using more general results (Nguyen, 1984).

Links between the macroscopic model studied here and microscopic continuous damage (Pradeilles and Stolz, 1992) should be found via the relations between different behavior at various scales. For instance, the assemblage of Hashin composite spheres can create a macroscopic behavior with continuous mechanical characteristics in the sense of Kachanov, while at a microscopic scale the evolution is more dramatic as surfaces of discontinuities move in the structure.

This latter assemblage underlines the importance of the relations between microscopic behavior and macroscopic mechanical characteristics. In addition, this example focuses on the necessity of studying of the uniqueness of the response at microscopic scale, in order to describe what is a macroscopic behavior.

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APPENDIX A : DERIVATIONS ON A MOVING INTERFACE

A point M of the moving Γ , has coordinates x which are defined by the equation of Γ : $S(x, t) = 0$. Then, the propagation of the surface is related to the velocity \mathbf{V} of the point M by

$$\frac{\partial S}{\partial x} \cdot \mathbf{V} + \frac{\partial S}{\partial t} = 0 \quad \text{where} \quad \mathbf{V} = \frac{dx}{dt} = -\phi \mathbf{v} \quad \text{and} \quad \mathbf{v} \left\| \frac{\partial S}{\partial x} \right\| = \frac{\partial S}{\partial x}.$$

Thus,

$$\frac{\partial S}{\partial t} = -\mathbf{V} \cdot \mathbf{v} \left\| \frac{\partial S}{\partial \mathbf{x}} \right\| = \phi \left\| \frac{\partial S}{\partial \mathbf{x}} \right\| = \phi \frac{\partial S}{\partial \mathbf{x}} \cdot \mathbf{v}.$$

Let us consider now a geometrical or mechanical quantity f defined along Γ_r . Then its rate is defined by:

$$D_\phi(f) = \lim_{t \rightarrow 0} \frac{f(\mathbf{x} - \phi \, dt \mathbf{v}, t + dt) - f(\mathbf{x}, t)}{dt}.$$

Explicit formulations are obtained in the following cases:

- for continuous and differentiable fields quantities :

$$D_\phi(f) = \frac{\partial f}{\partial t} + \nabla f \cdot \frac{d\mathbf{x}}{dt} = \frac{\partial f}{\partial t} - \phi \nabla f \cdot \mathbf{v},$$

- for geometrical quantities : in the 2D case, with $(\mathbf{v}, \boldsymbol{\tau})$ a right-handed basis

$$D_\phi(\mathbf{v}) = \frac{d\phi}{ds} \boldsymbol{\tau} \text{ and } D_\phi(\boldsymbol{\tau}) = -\frac{d\phi}{ds} \mathbf{v};$$

in the 3D case, with $(\mathbf{v}, \boldsymbol{\tau}, \boldsymbol{\eta})$ a right-handed basis,

$$D_\phi(\boldsymbol{\tau}) = -(\nabla \phi \cdot \boldsymbol{\tau}) \mathbf{v}, \quad D_\phi(\boldsymbol{\eta}) = -(\nabla \phi \cdot \boldsymbol{\eta}) \mathbf{v}, \quad D_\phi(\mathbf{v}) = (\boldsymbol{\tau} \cdot \nabla \phi) \boldsymbol{\tau} + (\boldsymbol{\eta} \cdot \nabla \phi) \boldsymbol{\eta}$$

APPENDIX B: FROM THE GLOBAL FORMULATION TO THE LOCAL EQUATIONS

The following proof is given in the plane case. Obviously, the solution to the set of local equations is one to the variational inequality. Denoting the quantities w'_i [$\dot{\varepsilon}$, μ ($\partial f_i / \partial \mathbf{A}$)], μ ($\partial f / \partial \mathbf{A}$) in Ω , and c ($\partial g / \partial G$), respectively, by $[\dot{\sigma}$, $-\dot{\mathbf{A}}]$, $\dot{\alpha}$ and ϕ , then

$$\begin{aligned} \frac{\partial F}{\partial \mathbf{v}} (\mathbf{v}' - \mathbf{v}) + \frac{\partial F}{\partial c} (c' - c) + \frac{\partial F}{\partial \mu} (\mu' - \mu) &= \sum_{i=1}^2 \int_{\Omega} \left\{ \dot{\sigma} : (\dot{\varepsilon}' - \varepsilon) - \dot{\mathbf{A}} \left(\mu' \frac{\partial f_i}{\partial \mathbf{A}} - \dot{\alpha} \right) \right\} d\omega \\ &- \int_{\Gamma \Omega_r} \dot{\lambda} \mathbf{T}^d \cdot (\mathbf{v}' - \mathbf{v}) \, d\mathbf{a} \\ &- \int_{\Gamma} c \frac{\partial g}{\partial G} \{ [[\boldsymbol{\tau} \cdot \boldsymbol{\sigma}]]_{\Gamma} \cdot (\nabla(\mathbf{v}'_1 - \mathbf{v}_1) \cdot \boldsymbol{\tau}) + [[\boldsymbol{\eta} \cdot \boldsymbol{\sigma}]]_{\Gamma} \cdot (\nabla(\mathbf{v}'_1 - \mathbf{v}_1) \cdot \boldsymbol{\eta}) \} \, da \\ &+ \int_{\Gamma} c \frac{\partial g}{\partial G} \left\{ \left[\mathbf{A}(\mu' - \mu) \frac{\partial f}{\partial \mathbf{A}} \right]_{\Gamma} \right\} \, da \\ &- \int_{\Gamma} (c' - c) \frac{\partial g}{\partial G} \{ (\mathbf{v}' \cdot \dot{\sigma}_2) \cdot [[\nabla \mathbf{u} \cdot \mathbf{v}]]_{\Gamma} - [[\mathbf{A} \dot{\alpha}]]_{\Gamma} \} \, da \\ &- \int_{\Gamma} (c' - c) \frac{\partial g}{\partial G} D_\phi(G) \, da. \end{aligned}$$

But, thanks to the properties of the fields (\mathbf{v}, c) and (\mathbf{v}', c') , which are elements of K , this expression now becomes

$$\begin{aligned}
\frac{\partial F}{\partial \mathbf{v}}(\mathbf{v}' - \mathbf{v}) + \frac{\partial F}{\partial c}(c' - c) + \frac{\partial F}{\partial \mu}(\mu' - \mu) &= - \sum_{i=1}^2 \int_{\Omega_i} \{ \text{div } \dot{\sigma} \cdot (\mathbf{v}' - \mathbf{v}) \} d\omega - \int_{\partial \Omega_r} (\dot{\lambda} \mathbf{T}^d - \mathbf{n} \cdot \dot{\sigma}) \cdot (\mathbf{v}' - \mathbf{v}) da \\
&+ \int_{\Gamma} \left\{ [[\dot{\sigma} \cdot \mathbf{v}]]_{\Gamma} + \text{div}_{\Gamma} \left(\left[\left[c \frac{\partial g}{\partial \mathbf{G}} \sigma \right] \right]_{\Gamma} \right) \right\} \cdot (\mathbf{v}'_1 - \mathbf{v}_1) da \\
&- \int_{\Gamma} (c' - c) \frac{\partial g}{\partial \mathbf{G}} D_{\phi}(G) da.
\end{aligned}$$

Thus, all the local equations, (13), (16) and (17), are recovered when the solution $(\mathbf{v}, c) \in K$ is such that

$$\forall (\mathbf{v}', c') \in K, \quad \frac{\partial F}{\partial \mathbf{v}}(\mathbf{v}' - \mathbf{v}) + \frac{\partial F}{\partial c}(c' - c) + \frac{\partial F}{\partial \mu}(\mu' - \mu) \geq 0.$$

An addition inequality is obtained from the solution of the variational inequality. It amounts to a new condition on the interface. Propagation of the interface can never occur simultaneously with evolution of the internal variable on any area of the interface. This condition takes the form :

$$\phi \left[\left[\mathbf{A}(\mu' - \mu) \frac{\partial f}{\partial \mathbf{A}} \right] \right]_{\Gamma} \geq 0 \text{ on } \Gamma.$$

APPENDIX C : DISPLACEMENT IN THE COMPOSITE ELASTOPLASTIC SPHERE

When there is no plastic area in the composite sphere, the displacement is :

$$\mathbf{u}_1 = \frac{ur}{RD(c)} \left\{ 3k_2 + 4\mu_1 + 3 \frac{R^3}{r^3} (k_1 - k_2) \right\} \mathbf{e}_r \text{ and } \mathbf{u}_2 = \frac{ur}{RD(c)} (3k_1 + 4\mu_1) \mathbf{e}_r,$$

$$\text{with } D(c) = 3k_2 + 4\mu_1 + 3c(k_1 - k_2).$$

In the second step of the evolution of the elastoplastic composite sphere, the displacement in the three different areas are as follows :

- (1) The core in material 2 (inside the sphere of radius a) has the same behavior as in elastic case :

$$\begin{aligned}
\mathbf{u}_2 &= a_2 r \mathbf{e}_r, \\
\sigma_2 &= 3k_2 a_2 \{ \mathbf{e}_r \otimes \mathbf{e}_r + \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_\phi \otimes \mathbf{e}_\phi \}.
\end{aligned}$$

- (2) The plastic shell in material 1 is located between a sphere of radius a and a sphere of radius r_p . In what follows, $c_p = (r_p^3/R^3)$. The displacement, the plastic strain tensor and the stress tensor are given by the following relations :

$$\begin{aligned}
\mathbf{u}_{1p} &= u_{1p} \mathbf{e}_r, \\
\text{with } \frac{u_{1p}}{r} &= \frac{6\mu_1 \sigma_Y}{F_1} \ln \frac{r}{a} + \frac{c_1}{3k_1} + \frac{(C + 3\mu_1)(3k_1 + 4\mu_1) r_p^3}{6\mu_1 F_1 r^3}, \\
\varepsilon_p &= \frac{\sigma_Y (3k_1 + 4\mu_1)}{F_1} \left(1 - \frac{r_p^3}{r^3} \right), \\
\sigma_{rrp} &= \frac{6\mu_1 \sigma_Y}{F_1} \ln \frac{r}{a} + c_1 - \frac{2C(k_1 + 4\mu_1) \sigma_Y r_p^3}{3F_1 r^3}.
\end{aligned}$$

$$\sigma_{\theta\theta 1P} = \sigma_{\phi\phi 1P} = \frac{9\mu_1 k_1 \sigma_Y}{F_1} + c_1 + \frac{18\mu_1 k_1 \sigma_Y}{F_1} \ln \frac{r}{a} + \frac{C(3k_1 + 4\mu_1) \sigma_Y r_p^3}{3F_1 r^3},$$

where $F_i = 9\mu_i k_i + C(3k_i + 4\mu_i)$.

(3) The external shell is constituted by material 1, with no plastic deformation. It is bounded by the sphere of radius r_p and the external surface of radius R . The displacement and the stress tensor are :

$$\mathbf{u}_1 = \left(a_1 r + \frac{b_1}{r^2} \right) \mathbf{e}_r,$$

$$\sigma_1 = \left(3k_1 a_1 - \frac{4\mu_1 b_1}{r^3} \right) \mathbf{e}_r \otimes \mathbf{e}_r + \left(3k_1 a_1 + \frac{2\mu_1 b_1}{r^3} \right) \{ \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_\phi \otimes \mathbf{e}_\phi \}.$$

Because of the continuity of the displacement and the stress vector, on the surfaces $r = a$ and $r = r_p$, finally, we get :

$$a_2 = \frac{\sigma_Y(3k_1 + 4\mu_1) c_P}{18\mu_1(k_1 - k_2) c},$$

$$a_1 = \frac{2\sigma_Y \mu_1}{9F_1} \ln \frac{c_P}{c} + \frac{\sigma_Y(3k_1 + 4\mu_1) F_2 c_P}{18\mu_1 F_1 (k_1 - k_2) c} + \frac{2\mu_1 \sigma_Y}{F_1} + \frac{\sigma_Y c_P}{6\mu_1},$$

$$b_1 = \frac{\sigma_Y c_P R^3}{6\mu_1},$$

$$c_1 = \frac{(3k_1 + 4\mu_1) \sigma_Y F_2 k_1 c_P}{6\mu_1 F_1 (k_1 - k_2) c}.$$

When the composite sphere is constituted by a core of material 2 and the shell in material 1 with plastic strain, the displacement and stresses are as follows.

- Between $r = 0$ and $r = a$, the displacement is $\mathbf{u}_2 = a_2 r \mathbf{e}_r$. The stresses are $\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{\phi\phi} = 3k_2 a_2$.
- between $r = a$ and $r = R$, the actual state is defined by :

$$\sigma_{rr} = \frac{18\mu_1 k_1 \sigma_Y}{F_1} \ln \frac{r}{a} - \frac{2C\sigma_Y(3k_1 + 4\mu_1) R^3}{3F_1 r^3} + \frac{2Cd_1}{3r^3} + c_1,$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{18\mu_1 k_1 \sigma_Y}{F_1} \ln \frac{r}{a} + \frac{9\mu_1 k_1 \sigma_Y}{F_1} + c_1 - \frac{Cd_1}{3r^3} + \frac{C\sigma_Y(3k_1 + 4\mu_1) R^3}{3F_1 r^3},$$

$$\varepsilon_p = \frac{\sigma_Y(3k_1 + 4\mu_1)}{F_1} \left\{ 1 - \frac{R^3}{r^3} \right\} + \frac{d_1}{r^3},$$

$$\mathbf{u}_{1P} = \left\{ \frac{c_1}{3k_1} + \frac{6\mu_1 \sigma_Y}{F_1} \ln \frac{r}{a} + \frac{(C + 3\mu_1) R^3}{6\mu_1 r^3} \left\{ \frac{\sigma_Y(3k_1 + 4\mu_1)}{F_1} - \frac{d_1}{R^3} \right\} \right\} \mathbf{e}_r.$$

The continuity relations and the boundary conditions on Γ , give :

$$a_2 = \frac{F_1}{F_2 + 3(3\mu_1 + C)(k_1 - k_2) \frac{a^3}{R^3}} \left\{ \frac{u}{R} + 2 \frac{\mu_1 \sigma_Y}{3F_1} \ln \frac{a}{R} \right\},$$

$$c_1 = \frac{3F_2 k_1}{F_2 + 3(3\mu_1 + C)(k_1 - k_2) \frac{a^3}{R^3}} \left\{ \frac{u}{R} + \frac{2\mu_1 \sigma_Y}{F_1} \ln \frac{a}{R} \right\},$$

$$\frac{d_1}{R^3} = \frac{\sigma_Y(3k_1 + 4\mu_1)}{F_1} - \frac{18\mu_1(k_1 - k_2)a^3}{R^3 F_2 + 3(3\mu_1 + C)(k_1 - k_2)\frac{a^3}{R^3}} \left[\frac{u}{R} + \frac{2\mu_1\sigma_Y}{F_1} \ln \frac{a^3}{R^3} \right].$$

APPENDIX D: DISPLACEMENT IN THE COMPOSITE CYLINDER UNDER TORSION

In the elastic composite cylinder under torsion, the displacements are given as follows :

$$\mathbf{u} = \lambda r z \mathbf{e}_\theta + \lambda \eta(r) \mathbf{e}_z.$$

The equilibrium equations are given on η .

$$\begin{aligned} \Delta \eta &= 0, & \text{on } r \in [0, R], \\ \mu_1 \eta_{1,r} &= 0, & \text{on } r = R, \\ \left. \begin{aligned} [[\mu \eta_{,r}]]_{\Gamma} &= 0, \\ [[\eta]]_{\Gamma} &= 0, \end{aligned} \right\} & \text{on } r = a, \\ \mu_2 \eta_{2,r} &= 0 & \text{on } r = 0. \end{aligned}$$

This has the following solution in each area :

$$\begin{aligned} \mathbf{u} &= \lambda r z \mathbf{e}_\theta, \\ \sigma &= \lambda \mu_r (\mathbf{e}_\theta \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\theta). \end{aligned}$$

When the normal velocity of the interface is such that

$$\dot{\mathbf{a}}(\theta) = \delta a_0 + \sum_{n=1}^{\infty} \delta a_n \cos n\theta,$$

and the loading increasing is \dot{u} , the velocity displacement field is the solution of the local equations :

$$\begin{aligned} \Delta \dot{\eta}_i &= 0, & \text{in } \Omega_i, \\ \mu_1 \dot{\eta}_{1,r} &= 0, & \text{on } r = R, \\ \lim_{r \rightarrow 0} \dot{\eta}_2 &= 0, \\ \left. \begin{aligned} [[\mu \dot{\eta}_{,r}]]_{\Gamma} - [[\mu]]_{\Gamma} a \frac{\partial \dot{a}}{\partial s} &= 0, \\ [[\dot{\eta}]]_{\Gamma} &= 0, \end{aligned} \right\} & \text{on } r = a. \end{aligned}$$

Thanks to Parton and Perline (1981), the solution is :

$$\dot{\eta} = \sum_{k=0}^{\infty} f_k(r, \theta) = \sum_{k=0}^{\infty} r^k g_k(\theta).$$

Finally, one gets the general form of η , in each domain Ω , as

$$\dot{\eta}(r, \theta) = A_{i,0} + B_{i,0} \ln r + \sum_{n=1}^{\infty} [(A_{i,n} r^n + B_{i,n} r^{-n}) \cos n\theta + (C_{i,n} r^n + D_{i,n} r^{-n}) \sin n\theta].$$

The constants $A_{i,n}$, $B_{i,n}$, $C_{i,n}$ and $D_{i,n}$ are obtained through the boundary conditions and the Hadamard's relations on Γ_i . It follows that these are determined by :

for $i \in \{1, 2\}$,

$$A_{i,0} = B_{i,0} = 0.$$

$$A_{i,1} = B_{i,1} = 0,$$

$$C_{i,1} = D_{i,1} = 0,$$

$$\alpha_1 = \beta_1 = 0,$$

$$\forall n \geq 2, A_{1,n} = \beta_n \frac{(\mu_1 - \mu_2) a^{n+1}}{[\mu_1(R^{2n} + a^{2n}) + \mu_2(R^{2n} - a^{2n})]},$$

$$C_{1,n} = -\alpha_n \frac{(\mu_1 - \mu_2) a^{n+1}}{[\mu_1(R^{2n} + a^{2n}) + \mu_2(R^{2n} - a^{2n})]},$$

$$B_{1,n} = \beta_n \frac{(\mu_2 - \mu_1) a^{n+1} R^{2n}}{[\mu_1(R^{2n} + a^{2n}) + \mu_2(R^{2n} - a^{2n})]},$$

$$D_{1,n} = -\alpha_n \frac{(\mu_2 - \mu_1) a^{n+1} R^{2n}}{[\mu_1(R^{2n} + a^{2n}) + \mu_2(R^{2n} - a^{2n})]},$$

$$A_{2,n} = \beta_n \frac{(\mu_1 - \mu_2) a^{1-n} (R^{2n} - a^{2n})}{[\mu_1(R^{2n} + a^{2n}) + \mu_2(R^{2n} - a^{2n})]},$$

$$C_{2,n} = -\alpha_n \frac{(\mu_1 - \mu_2) a^{1-n} (R^{2n} - a^{2n})}{[\mu_1(R^{2n} + a^{2n}) + \mu_2(R^{2n} - a^{2n})]},$$

$$B_{2,n} = 0,$$

$$D_{2,n} = 0.$$

The functional F used in the study of the evolution can be reduced here to :

$$F(\mathbf{v}, \dot{a}, \dot{\lambda}) = \sum_{i=1}^2 \int_{\Omega} \frac{1}{2} \dot{\lambda}^2 \mu_i \left[(\dot{\eta}_{i,r})^2 + \left(\frac{\dot{\eta}_{i,\theta}}{r} \right)^2 \right] r dr d\theta dz - \int_{\Gamma} \frac{1}{2} \dot{a}^2 \lambda^2 (\mu_1 - \mu_2) a^2 d\theta dz.$$

So, using the previous results on the velocity displacement solution when \dot{a} and $\dot{\lambda}$ are assumed to be known, one gets :

$$\tilde{W}_{\dot{a}, \dot{\lambda}} = -\lambda^2 \pi l a^2 (\mu_1 - \mu_2) \left[\alpha_0^2 + \sum_{n=2}^{\infty} \frac{[\alpha_n^2 + \beta_n^2] \{ \mu_1 [(1+n) R^{2n} + (1-n) a^{2n}] + \mu_2 (R^{2n} - a^{2n}) (1-n) \}}{2 [\mu_1 (R^{2n} + a^{2n}) + \mu_2 (R^{2n} - a^{2n})]} \right].$$