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Explicit predictability and dispersion scaling exponents in fully developed turbulence

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Abstract

We apply a simple method to provide explicit expressions for different scaling exponents in intermittent fully developed turbulence, that before were only given through a Legendre transform. This includes predictability exponents for infinitesimal and non infinitesimal perturbations, Lagrangian velocity exponents, and dispersion exponents. We obtain also new results concerning inverse statistics corresponding to exit-time moments.

Key words: turbulence, intermittency, multifractal, scaling exponents

PACS: 47.27.-i, 47.53.+n, 47.27.Eq

1 Introduction

One of the main feature of homogeneous and isotropic fully developed turbulence, corresponding to very large Reynolds numbers, is the fact that statistical properties at small scales are scaling and universal [1], and possess intermittent fluctuations [2]. This is classically characterized using structure functions scaling exponents, describing the probability density of velocity fluctuations at all scales belonging to the inertial range [3]. This concerns the Eulerian velocity and its statistics, but the characteristic features of fully developed turbulence is not only intermittency and multiple scales, it is also the loss of predictability, chaotic behaviour, dispersion and mixing properties, and a large number of degrees of freedom. In the last ten years, many of these properties have been revisited taking into account intermittency to consider how different scaling laws related to these issues are modified (for a review of several results in this framework see [4]). The scaling exponents obtained have often been expressed as a Legendre transform of some function of velocity exponents, and numerical applications need a numerical Legendre transform involving the dimension function of velocity fluctuations. Here we present a simple method to

simplify Legendre transform expressions and to replace them with parametric functions that provide explicitly the scaling exponents when the Eulerian velocity scaling exponent structure functions are known. In the following the general framework is presented in section 2 and applied to different applications from predictability to ϵ -entropy studies in section 3. Let us note that the results presented here correspond to the ideal situation of homogeneous and isotropic turbulence; for realistic flows, this property cannot be expected to hold perfectly since the influence of boundary conditions may introduce statistical inhomogeneity.

2 The multifractal framework and Legendre transform

We recall first here the multifractal framework for velocity intermittency in turbulence, classically characterized using statistical moments of velocity increments. We recall the Legendre transform introduced to relate scaling moment function to the codimension function of velocity singularities. The latter is subsequently systematically used to simplify the Legendre transform of some general expression of the codimension function.

2.1 Singularities, moment functions and Legendre transform

In fully developed turbulence, intermittency is classically characterized by $\zeta(q)$, the scaling exponent of spatial structure functions. Denoting $\Delta V_\ell = |V(x + \ell) - V(x)|$ the increments of the velocity field at a spatial scale ℓ , their fluctuations are characterized, in the inertial range, using the scale invariant moment function $\zeta(q)$ (see [3,5] for reviews):

$$\langle \Delta V_\ell^q \rangle \sim \ell^{\zeta(q)} \quad (1)$$

where q is the order of moment. Kolmogorov's initial proposal [1], for a non-intermittent constant dissipation, leads to $\zeta(q) = q/3$. For intermittent turbulence, $\zeta(q)$ is a cumulant generating function, and is nonlinear and concave; only the third moment has no intermittency correction: $\zeta(3) = 1$ (see below). On the other hand, taking into account intermittency for the Eulerian velocity, one can write locally:

$$\Delta V_\ell \sim \ell^h \quad (2)$$

where h is a singularity of the velocity fluctuations. There is a continuous range of possible h values, and they are characterized through their codimension function $c(h)$ defined as [3,5]:

$$p(\Delta V_\ell) \sim \ell^{c(h)} \quad (3)$$

where $p(\Delta V_\ell)$ is the probability density of velocity fluctuations at scale ℓ . Here the codimension is used for convenience, instead of the more classical dimension $f(h) = d - c(h)$, where d is the dimension of the space ($d = 1$ or $d = 3$ in most studies), and $p(\Delta V_\ell)$ is the probability density of velocity fluctuations. Let us note that, while h can be both negative or positive (but it is most of the time positive), the codimension function $c(h)$ is positive and decreasing. This continuous range of h values justified the use of the term “multifractal” for such process [6].

The moments write:

$$\langle \Delta V_\ell^q \rangle = \int \Delta V_\ell^q p(\Delta V_\ell) \quad (4)$$

$$\sim \int \ell^{qh+c(h)} p(h) \sim \ell^{\zeta_E(q)} \quad (5)$$

using a saddle point argument [6], this gives the classical Legendre transform, between $\zeta_E(q)$ and the codimension function:

$$\zeta(q) = \min_h \{qh + c(h)\} \quad (6)$$

This can also be written in the following way, emphasizing the one-to-one relation between orders of moment q and singularities h :

$$\begin{cases} qh + c(h) = \zeta(q) \\ q = -c'(h) \end{cases} \quad (7)$$

The last relation can be replaced by $h = \zeta'(q)$. Equations (7) are used below.

2.2 *Estimates of structure function scaling exponents up to moments of order 8*

The scaling moment function $\zeta(q)$ is proportional to the cumulant generating function of the variable $\log |\Delta V_\ell|$; it is nonlinear and concave. There are two “fixed points” for this curve, corresponding to a priori exact values: $\zeta(0) = 0$ and $\zeta(3) = 1$. The latter relation is a consequence of the Kolmogorov 4/5 law [7] for the third order longitudinal structure function (without absolute values):

$$\langle \Delta V_\ell^3 \rangle = -\frac{4}{5} \epsilon \ell \quad (8)$$

The other moments do not have well-accepted theoretical values; the analytical curves depend on the stochastic model chosen for the statistics of velocity fluctuations. In agreement with Kolmogorov’s initial hypothesis concerning the universality of velocity fluctuations in the inertial range, experimental estimates are rather stable for small and medium orders of moments and do

not depend on the Reynolds number: the exponents $\zeta_E(q)$ have been estimated experimentally for more than twenty years, for different types of flows (see [8–13]), and are now considered as rather stable and almost universal until moments of order 7 or 8 [10,11]. Let us note that the moment of order 7 seems to be close to 2, for any reasons that did not receive up to now any theoretical founding.

Since these estimates are rather stable up to moments of order 8, we chose in the following for simplicity for $\zeta_E(q)$ average values of the estimates published in Ref. [9,11–13]: see Table 1. These are estimates obtained from many experiments with R_λ going from 35 to 5 000. The experimental average which is used here is quite close to all these experimental estimates.

3 Applications to different predictability and dispersion turbulence functions

3.1 Predictability of small perturbations in the intermediate dissipation range of turbulence

Let us first consider the predictability property of small perturbations in turbulence. It is well-known that the viscous cutoff, corresponding to the scale for which the Reynolds number is of order 1, is different from the Kolmogorov scale η when intermittency is taken into account: more intense singularities have a smaller local cutoff scale and weak fluctuations have larger cutoff scales. This range of scales is the intermediate dissipation range, where there is a mixture of inertial and dissipation effects. Using Eq. (2) and the Reynolds number $Re = VL/\nu$, this writes [14]:

$$\eta_h = LRe^{-1/(1+h)} \quad (9)$$

where L is a large scale and η_h is the cutoff scale associated to singularity h . This spatial scale is associated to a temporal cutoff scale τ_h through the velocity $v_h = (\eta_h)^h$: $\tau_h = \eta_h/v_h = (\eta_h)^{1-h}$ giving the singularity dependent cutoff time scale:

$$\tau_h = TRe^{\frac{h-1}{h+1}} \quad (10)$$

where T is the large time scale associated to L . This temporal cutoff scale is the smallest scale of the system: below these scales viscosity dominates. In fully developed turbulence, the positive maximum Lyapunov exponent λ , characterizing the exponential growth rate of an infinitesimal disturbance, is proportional to the inverse of the smallest characteristic time of the system. Since there is a range of time scales associated to the continuous range of h

values through Eq.(10), the Lyapunov exponent may be given by [15,16]:

$$\lambda \sim \int \frac{1}{\tau_h} dp(h) \sim \int Re^{\frac{1-h}{h+1}} Re^{\frac{-c(h)}{h+1}} dh \quad (11)$$

giving finally:

$$\lambda \sim Re^\alpha \quad (12)$$

with the exponent α given by a saddle-point argument since the Reynolds number is large [15,16]:

$$\alpha = \max_h \left\{ \frac{1-h-c(h)}{1+h} \right\} \quad (13)$$

This was solved numerically by Crisanti et al. [15] to provide $\alpha \simeq 0.46$ through a numerical Legendre transform using experimental estimates of the function $c(h)$. This result is slightly smaller than the one provided by Ruelle [17] for nonintermittent turbulence: $\alpha = 1/2$.

Let us go further and show here how to obtain this exponent, considering, for more generality, the moments of order $q > 0$ of the inverse times:

$$\left\langle \frac{1}{\tau_h^q} \right\rangle \sim Re^{\alpha(q)} \quad (14)$$

where, using as done above a saddle-point argument, the exponents $\alpha(q)$ are given by:

$$\alpha(q) = \max_h \left\{ \frac{q(1-h)-c(h)}{1+h} \right\} \quad (15)$$

In general, the singularity h_0 for which this condition is met verifies $G'(h_0) = 0$ where

$$G(h) = \frac{q(1-h)-c(h)}{1+h} \quad (16)$$

Estimating the derivative $G'(h_0) = 0$ and using Eq. (7) to introduce the moment q_0 associated to the singularity h_0 leads to:

$$\zeta(q_0) = 2q - q_0 \quad (17)$$

This provides a unique value of q_0 and hence of h_0 , for a given value of q . This equation, together with another use of the Legendre transform, gives the function $\alpha(q) = G(h_0)$:

$$\begin{aligned} \alpha(q) = G(h_0) &= \frac{q(1-h_0) - (\zeta(q_0) - q_0 h_0)}{1+h_0} \\ &= q_0 - q \end{aligned}$$

This gives finally the following parametric relation between $\alpha(q)$ and the moment function $\zeta(q)$:

$$\begin{cases} \zeta(q_0) = 2q - q_0 \\ \alpha(q) = q_0 - q \end{cases} \quad (18)$$

The nonintermittent curve obtained for $\zeta(q) = q/3$ is $\alpha(q) = q/2$. The explicit parametric relation (18) can be used to represent the curve $\alpha(q)$ for experimental estimates of $\zeta(q)$: this is shown in Fig. 1 using the average $\zeta(q)$ curve. One can see from Fig. 1 that up to moments of order 2.5 the linear non-intermittent curve is a very good approximation. The value of $\alpha(1) = \alpha$ is smaller than $1/2$ (close to 0.48) but the intermittent correction is very small. For $q = 1$ we have $\alpha = \alpha(1) = q_0 - 1$, where q_0 is given by $\zeta(q_0) = 2 - q_0$. For $q = 2$ one obtains $q_0 = 3$ since $\zeta(3) = 1$, and hence $\alpha(2) = 1$ is a fixed point, non affected by intermittency, recovering a result already given in Ref. [15]. One can also obtain the following estimates: $\alpha(3) \simeq 1.56$, $\alpha(4) \simeq 2.2$, $\alpha(4.5) \simeq 2.5$ (coming from $\zeta(7) \simeq 2$), and $\alpha(5) \simeq 2.85$.

3.2 Predictability of noninfinitesimal perturbations in turbulence

The same approach can be used to express scaling exponents characterizing time scale statistics associated to noninfinitesimal perturbations. As proposed in Ref. [18] and developed in Ref. [4], and using the notations of the latter reference, we consider a perturbation of size δ of the velocity field, for δ belonging to the inertial range (hence the term noninfinitesimal). The time scale associated to the eddy of typical velocity δ can also be considered as the decorrelation time associated to the perturbation. In this framework, the inverse time is $1/\tau \sim \delta/\ell$ and since using the singularity h , one has $\ell \sim \delta^{1/h}$, we obtain locally the relation between the time scale and the perturbation δ :

$$\frac{1}{\tau} \sim \delta^{\frac{h-1}{h}} \quad (19)$$

As before, this can be used to consider the moments of order $q > 0$ of the inverse times, but here we will consider their scaling property as a power-law function of the velocity perturbation δ . Using the same type of argument as above, the moments of the inverse time are expressed as an integral over all singularities h and we consider all functions as power-law functions of δ so that using a saddle-point argument, one has a scaling law:

$$\left\langle \frac{1}{\tau^q} \right\rangle \sim \delta^{-\beta(q)} \quad (20)$$

where the exponents $\beta(q)$ are given by the following relation (using the fact that the probability density writes $p(h) \sim \ell^{c(h)} \sim \delta^{c(h)/h}$):

$$-\beta(q) = \max_h \left\{ \frac{q(h-1) + c(h)}{h} \right\} \quad (21)$$

where the negative sign has been introduced for convenience to have positive final numerical values for $\beta(q)$ (see below). As done above for infinitesimal perturbations, this can be expressed explicitly introducing the singularity h_0 that maximises the expression into brackets. With the same procedure as above: involving a differentiation and the introduction of the moment of order q_0 associated to h_0 , we obtain the final result as a parametric relation between $\beta(q)$ and $\zeta(q)$:

$$\begin{cases} \zeta(q_0) = q \\ \beta(q) = q_0 - q = \zeta^{(-1)}(q) - q \end{cases} \quad (22)$$

The exponent $\beta(q)$ is simply linked to the reciprocal of ζ , denoted $\zeta^{(-1)}$. The nonintermittent curve obtained for $\zeta(q) = q/3$ is $\beta(q) = 2q$. Using Eq.(22) we can see that $\zeta(3) = 1$ gives $\beta(1) = 2$, which is a fixed point for $\beta(q)$, non affected by intermittency corrections [18]. Figure 2 shows the resulting function compared to the straight line of equation $2q$. It is visible that $\beta(q)$ grows very fast. Indeed, the relation $\zeta(7) \simeq 2$ gives the value $\beta(2) \simeq 5$. We can see that for $q < 1$, $\beta(q) < 2q$ and for $q > 1$, $\beta(q) > 2q$. The result $\beta(1) = 2$ was checked using shell-model simulations [4].

3.3 Lagrangian velocity structure functions' in turbulence

As an analogy with Kolmogorov's dimensional analysis in the Eulerian framework, Landau [19] proposed in 1944 a $1/2$ law for the temporal increments of the Lagrangian velocity $\Delta V_\tau = |V(t+\tau) - V(t)|$. This was later generalized by Novikov, with a Lagrangian intermittency framework for the velocity [20]:

$$\langle \Delta V_\tau^q \rangle \sim \tau^{\zeta_L(q)} \quad (23)$$

As for the Eulerian case, for a constant dissipation one obtains the ‘‘mean field’’ expression, neglecting intermittency: $\zeta_L(q) = q/2$. In this framework, the third order moment for the Eulerian velocity is analogous to the second order moment for the Lagrangian velocity: in case of intermittency $\zeta_L(q)$ is nonlinear and concave, and the non-intermittent function is valid only for $q = 2$: $\zeta_L(2) = 1$.

Recently some authors have proposed an hypothesis helping to relate Eulerian and Lagrangian structure functions [21–24], following an earlier proposal by Borgas [25]. They consider the velocity advecting Lagrangian trajectories as a

superposition of different velocity contributions coming from different eddies having different characteristic times. After a time τ the fastest eddies, of scale smaller than ℓ , are decorrelated so that at leading order, they assume [21–23]:

$$\Delta V_\tau \sim \Delta U_\ell \quad (24)$$

writing $\Delta U_\ell \sim \ell/\tau$ and using $\Delta U_\ell \sim \ell^h$, gives the time and space local correspondence $\tau \sim \ell^{1-h}$. Introducing this relation inside the integral of Eq.(7), one obtains for the moments of the Eulerian velocity increments:

$$\langle \Delta U_\ell^q \rangle \sim \int \tau^{\frac{qh+c(h)}{1-h}} dp(h) \quad (25)$$

Using a saddle point argument and using Eqs.(24) and (23), this gives the Lagrangian structure function as a Legendre transform [21]:

$$\zeta_L(q) = \min_h \left(\frac{qh + c(h)}{1-h} \right) \quad (26)$$

The authors that explored this relation did not go further to express $\zeta_L(q)$ in the general case [21–23].

In fact, as we showed elsewhere for a Lagrangian turbulence study [26], using an approach analogous to the ones above, one can obtain an explicit relation between Eulerian and Lagrangian structure functions' scaling exponents. We introduce the singularity h_0 that maximises the expression into brackets, and with the same procedure involving a differentiation and the introduction of the moment of order q_0 associated to h_0 we obtain the final result (see also [26]):

$$\begin{cases} \zeta_L(q) = \zeta_E(q_0) \\ q = q_0 - \zeta_E(q_0) \end{cases} \quad (27)$$

This relation provides $\zeta_L(q)$ when the function $\zeta_E(q)$ is known, the second line giving the link between q and q_0 . We can check that for $q_0 = 3$, the second line gives $q = 2$ and the first one, $\zeta_L(2) = \zeta_E(3) = 1$. We can also consider the case $q_0 = 7$ and using $\zeta_E(7) \simeq 2$, the approximate result $\zeta_L(5) \simeq 1$.

Let us note that an analogous expression is provided in [25] for singularities and (with our notations) codimension functions $c(h)$ in his Appendix A (Equation (A 4)). Borgas used for this other types of arguments, including a Reynolds number scaling expression for Eulerian and Lagrangian statistics; however, he did not go to the moment framework to provide an equivalent expression for structure functions scaling exponents. Let us also note that some comparisons are performed in [21] between their shell model Lagrangian values and Eulerian shell model estimates, transformed through a numerical Legendre transform using the analytical expression of a fit of $c(h)$. The same type of comparison is performed in [23] between their DNS estimates and a

numerical Legendre transform of Eulerian values. The approach proposed here is more direct since we test an explicit parametric relation corresponding to different hypothesis.

The Euler-Lagrange relation given here is non-linear, so that one cannot expect to obtain the same type of statistical models in Eulerian and Lagrangian frameworks. For example, as explained by Chevillard [24] using Borgas' relation, in this framework, the statistics cannot be lognormal for both Eulerian and Lagrangian velocity fields. Indeed, for $\zeta_E(q) = aq - bq^2$ with $a = (2 + \mu)/6$ and $b = \mu/18$, one obtains $\zeta_L(q) = (a - 1 + \sqrt{(1 - a)^2 + 4qb} - 2bq)/2b$. For more details see our paper [26] where different proposals for the Lagrangian structure functions scaling exponents are tested.

In Figure 3, the Lagrangian prediction is compared to Lagrangian estimates published in recent papers [21,27,22]. These papers report different types of such estimates: Ref. [27] gives the values of experimental estimates for four flows going from R_λ of 510 to 1 000, and Direct Numerical Simulations (DNS) estimates for $R_\lambda=75$ and 140. Ref. [21] reports shell model estimates and Ref. [23] DNS corresponding to $R_\lambda=280$. In order not to have a too heavy figure, we plotted in Figure 3 four different types of Lagrangian values: an average value of the 4 experiments shown in Ref. [27], the DNS simulation corresponding to $R_\lambda=140$ of the same publication, the shell model estimates of Ref. [21] and the DNS estimate of Ref. [23]. The average Eulerian curve is transformed and compared to these four Lagrangian estimates. Several comments can be proposed from Figure 3. First, let us note that these experimental and numerical estimations for Eulerian and Lagrangian scaling exponents come from different types of flows with different Reynolds numbers. For Eulerian estimates, the exponents have been carefully compared and are now considered as rather stable up to moments of order 8, but for Lagrangian estimates, this work is still to do; measurements are only available for a few years, and the scaling ranges are still small, so that the range of variability for these exponents is expected to narrow in the future. Indeed, as for the Eulerian case, one can expect that for high Reynolds numbers these exponents are rather stable and universal, up to a given order of moment that may be smaller than the Eulerian case, for which this critical order of moments is about 7-8.

With these comments in mind, we can provide some preliminary conclusions from the observation of Figure 3: (i) up to moment of order 3, the prediction are close to all Lagrangian data; (ii) for larger moments, there is too much scatter in the estimates to conclude about the validity of the transformation given by Eq.(27). A possible explanation of such scatter is the fact that scaling can be strongly affected by trapping in vortex filaments [23]. In this case, after a filtering of high frequencies, the data may be closer to the multifractal prediction.

3.4 Relative turbulent dispersion

We next consider the influence of intermittency on relative dispersion: this deals with the statistics of particle pair dispersion, advected by an intermittent Eulerian velocity field. This topic is close to the Lagrangian case. let us consider $R(t)$, the distance between a pair of particles advected by an intermittent turbulence. Following Ref. [20,28,29], one can introduce $\Delta V_\tau(R)$, the velocity difference along the Lagrangian trajectories. This can be written:

$$\frac{dR}{dt} = \Delta V_R \quad (28)$$

where ΔV_R is the increment of the Lagrangian velocity is evaluated for a distance R associated to a time scale $\tau = R/\Delta V_R$. Then using the same hypothesis as for the Lagrangian case (Eqs. (23) and (24)), we obtain the time evolution of the moments of the pair distance:

$$\langle R^q \rangle \sim \langle (\tau \Delta V_\tau)^q \rangle \sim \tau^{H(q)} \quad (29)$$

where $H(q)$ is the scaling exponent characterizing the intermittent pair dispersion. Using Eq.(26) the exponents $H(q)$ can be given as a Legendre transform of the Eulerian velocity, as provided by Boffetta et al. [28,29]:

$$H(q) = \min_h \left(\frac{q + c(h)}{1 - h} \right) \quad (30)$$

In fact Eq.(29) shows also directly that the dispersion exponent is simply related to the Lagrangian velocity exponent:

$$H(q) = q + \zeta_L(q) \quad (31)$$

Equations (27) and (31) then give:

$$\begin{cases} H(q) = q_0 \\ q = q_0 - \zeta_E(q_0) \end{cases} \quad (32)$$

This can also be written $H(q - \zeta(q)) = q$. This relation could also have been obtained from Eq.(30) using the same approach as above. With $q_0 = 3$ we obtain $q = 2$ and $H(2) = 3$, corresponding to Richardson's law $\langle R^2 \rangle \sim \tau^3$ [30], which is thus a fixed point for which there is no intermittency correction, as already noticed [20,28,29]. In Fig. 4, we represent the transform (32) compared to the non-intermittent line of equation $3q/2$, compared also to the shell model estimates published in Ref. [29]. This shows that the nonlinearity of $H(q)$ does not seem very strong. Furthermore, H is concave, and the values give by Eq. (32) are quite close to the ones reported in Ref. [29].

3.5 Exit-time moments and ϵ -entropy

We finally consider exit-time moments [31], also called inverse structure functions [32]. We consider a time series of a turbulent quantity $X(t)$ having intermittent fluctuations characterized by singularities h such as $\Delta X_\tau \sim \tau^h$ with codimension $c(h)$: $p(h) \sim \tau^{c(h)}$. The moments are assumed to scale as $\langle \Delta X_\tau^q \rangle \sim \tau^{\zeta(q)}$ with Eq.(9). One then studies the dynamics of such signal by considering exit times (also called distance structure functions or inverse turbulence functions): considering a threshold value $\delta = \Delta X_\tau$, let us denote $\tau(\delta)$ the first time needed to obtain an increment $\delta = \Delta X_\tau$. A new time series (τ_i) can be obtained this way, whose statistics will scale as the threshold value δ :

$$\langle \tau^q \rangle \sim \delta^{\chi(q)} \quad (33)$$

where $\chi(q)$ can be obtained, as before, as a Legendre transform involving $c(h)$. Writing $\tau \sim \delta^{1/h}$ and inserting this into the integral of the moment estimation gives [32,31]:

$$\chi(q) = \min_h \left(\frac{q + c(h)}{h} \right) \quad (34)$$

Here as before, we estimate this function explicitly. A derivative of $G(h) = (q + c(h))/h$ leads to $\chi(q) = G(h_0)$ with $\zeta(q_0) = -q$. We then obtain

$$\begin{cases} \chi(q) = -q_0 \\ \zeta(q_0) = -q \end{cases} \quad (35)$$

The result then writes simply:

$$\chi(-\zeta(q_0)) = -q_0 \quad (36)$$

which is exactly a result given in Ref. [33] based on an exact result for a special case (a multifractal Cantor set), and experimentally verified using shell model turbulence simulations, but with no formal proof of the formulae in the general case. A similar expression obtained in the very different context of Laplacian random walk is given in Ref. [34].

We must note here, as already noticed in [33], that Eq.(36) involves for either the original series or for the return time series, negative moments. Let us consider first positive moments of return times: $q > 0$ in Eq.(33). We need here the hypothesis of the existence of negative moments of the velocity structure functions, but it may not be the case, since this needs in practice extremely precise measurements. It is argued in [35] that negative moments for $q < -1$ do not exist since the pdf of velocity increments does not vanish at 0. The derivative writes $G'(h_0) = -(q + \zeta(q_0))/h_0^2$. For $q > 0$, if negative moments q_0 such that $\zeta(q_0) = -q < 0$ are not accessible in the experimental time series, then the derivative stays strictly negative, $G(h)$ does not reach a relative

minimum, and the minimal value selected by Eq.(34) will be obtained for the maximal value of h , which is a “minimal” singularity of the velocity fluctuations, and hence denoted here h_{\min} : $\chi(q) = G(h_{\min})$. Since this value of h is the same for all q 's, the result is a linear behaviour:

$$\chi(q) = \frac{q + c(h_{\min})}{h_{\min}} \quad (37)$$

The singularity h_{\min} is associated to small moments and the smallest fluctuations detectable in experimental time series; it is not clear whether this maximum singularity reaches a fixed value for Eulerian turbulence, and what is this value. This maximal singularity also depends on the precision of the experimental measurements which are analyzed. To check this, we consider numerical and experimental analysis of such inverse statistics for $q > 0$. We plot in Fig. 5 the data for $\chi(q)$ with $q > 0$ reported in Refs. [32,36]. The latter are experimental estimates, while the former are obtained from shell model calculations. Both are rather linear (the published experimental estimates contain only the first three moments), and seem to confirm a linear behaviour such as Eq.(37). Recently inverse statistics have been tested using wind-tunnel experimental data, confirming a linear behaviour for positive (relative) exponents [35].

Let us now focus on negative moments of the time statistics: $q > 0$ in Eq.(33). This corresponds to positive moments of inverse times. We have $\zeta(q_0) = -q$ and $\chi(q) = G(h_0) = -q_0$. The result will be simply written introducing $p = -q$ and $q_0(p) = -\chi(q)$:

$$\left\langle \left(\frac{1}{\tau} \right)^p \right\rangle \sim \delta^{-q_0(p)} \quad (38)$$

with:

$$\zeta(q_0) = p \quad (39)$$

The exponent obtained with the inverse statistics is in fact the inverse (in the sense of reciprocal) of the structure function:

$$q_0(p) = \zeta^{(-1)}(p) \quad (40)$$

For Eulerian velocity (with Taylor's hypothesis to interpret time statistics as being close to spatial statistics), a time series measured at a fixed location will be characterized by $\zeta = \zeta_E$. Thus $q = 1$ selects $q_0 = 3$, non affected by intermittency corrections. The non-intermittent curve is $q_0 = 3q$, and taking into account intermittency we have also the approximate value $q_0(2) \simeq 7$. The corresponding curves for Eulerian turbulence and for passive scalar turbulence (data from Ref. [37] corresponding to an average of several experimental estimates) are shown in Fig. 6: they are increasing fast, especially for the passive scalar case. Let us note also that the experimental results published in [35] do not confirm Eq.(40), except for $p = 1$.

Exit-time statistics have also been used in Ref. [38,39,4] to characterize the ϵ -entropy of turbulent signals. The concept of ϵ -entropy has been introduced to quantify the predictability and degree of complexity of continuously varying signals such as turbulent signals. The continuous signal is transformed into a symbolic time series using a grid in the phase space, of resolution ϵ . The classical Shannon entropy h of the resulting symbolic time series is then studied, as a function of the grid size ϵ (see [40,41]). Abel et al. [38,39] have proposed to study the ϵ -entropy of turbulent signals $h(\epsilon)$ using the exit-times $t(\epsilon)$, the time for the signal to undergo a fluctuation of size ϵ . They obtained the following expression

$$h(\epsilon) = h^\Omega(\epsilon) < \frac{1}{\tau} > \quad (41)$$

where $h^\Omega(\epsilon)$ is the exit-time ϵ -entropy, being bounded and having only a logarithmic variation with ϵ . The leading ϵ behaviour of $h(\epsilon)$ is then given by $< \frac{1}{\tau} >$, where the average is done here by considering a succession of exit time events, and is thus related to Eq. (38). This gives using Eq.(38) and $\epsilon = \delta$:

$$h(\epsilon) \sim \epsilon^{-\beta} \quad (42)$$

with $\beta = q_0(1)$ as given by Ref. [38,39]. But we can write also using Eq.(40):

$$\beta = \zeta^{(-1)}(1) \quad (43)$$

Hence for a turbulent time series recorded at a fixed location, but using Taylor's hypothesis, corresponding to Eulerian turbulence, $\beta = 3$ [38,39,4]. On the other hand, for Eulerian temperature turbulence (still with Taylor's hypothesis), one will have a larger value since $\zeta_\theta(3) \simeq 0.8$; using the values published in Ref. [37], we have $\zeta_\theta(4) \simeq 1$ and hence we obtain for passive scalar Eulerian turbulence $\beta_\theta \simeq 4$. For times series of Lagrangian velocity and passive scalar turbulence, both characterized by $\zeta(2) = 1$, we have $\beta = 2$. This recovers the result obtained in Ref. [40], which have been obtained using a dimensional argument, thus corresponding implicitly to Lagrangian velocity. Let us also note that Eq.(43) recovers the results of Kolmogorov [42]: for Brownian processes with scaling power spectra of the form $E(\omega) \sim \omega^{-(1+2H)}$, he gave $\beta = 1/H$. Equation (43) gives the same result, since for such process, $\zeta(q) = qH$, hence $\beta = \zeta^{(-1)}(1) = 1/H$.

Conclusion

We have introduced a simple procedure helping to express explicitly scaling exponents expressed as Legendre transforms in multifractal turbulence. We have applied this idea to several problems involving intermittent corrections to predictability and dispersion studies. Even if the basic idea is simple, this produced several interesting new results: several functions are obtained as

parametric transform of the Eulerian velocity structure function. We have confirmed and generalized some previously known results and provided some predictions.

More precisely, in this paper we have expressed the scaling moment function of infinitesimal perturbations, that characterize the inverse times statistics in the intermediate dissipation range, as function of the Reynolds number. For finite size perturbations, we have also provided the scaling moment function $\beta(q)$ of inverse times, as function of the size δ of perturbations, showing that this exponent is very simply related to the reciprocal of the velocity structure functions exponent $\zeta(q)$. We verified the known result $\beta(1) = 2$ and obtain the approximate value $\beta(2) \simeq 5$. We have also considered Lagrangian velocity structure functions $\zeta_L(q)$ and showed how to relate them explicitly to the Eulerian curve $\zeta_E(q)$. We compared the predicted Lagrangian curve to recent experimental values. We noticed the approximate value $\zeta_L(5) \simeq 1$. We also considered the scaling exponent $H(q)$ for pair dispersion which is very simply related to the Lagrangian scaling exponents. The obtained curve was compared to recent numerical estimates. We finally considered exit-time moments, and confirmed in the general case a result obtained recently on experimental grounds and analytically for a special case. We showed that, for exit-time positive moments, the scale invariant function $\chi(q)$ may often be linear, due to finite precision of experimental measurements. This is in agreement with several published experimental results. We then proposed to focus on inverse statistics considering negative moments, hence positive moments of inverse times. In this framework we showed that the resulting exponent $q_0(q)$ is the reciprocal of $\zeta(q)$. Using some recent results linking exist time statistics to ϵ -entropy expression, we obtain a new general result for the ϵ -entropy exponent of multi-affine signals $\beta = \zeta^{(-1)}(1)$ confirming $\beta = 3$ for Eulerian turbulence, and giving as prediction $\beta = 2$ for Lagrangian turbulence (velocity and passive scalar), $\beta \simeq 2$ for financial time series, and the approximate value $\beta \simeq 4$ for passive scalar Eulerian turbulence.

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Table 1

Some recent experimental estimations for $\zeta_E(q)$ and the average value used here. Ref. [9]: Re from 6,000 to 47,000; Ref. [11]: various experiments, R_λ going from 35 to 5,000; Ref. [12]: turbulent wake, $R_\lambda=230$; Ref. [13]: various experiments, R_λ going from 340 to 800.

q	[9]	[11]	[12]	[13]	Average value
1				0.368	0.37
2	0.71	0.70	0.69	0.694	0.70
3	1	1	1	1	1
4	1.28	1.25	1.29	1.282	1.28
5	1.53	1.50	1.55	1.541	1.53
6	1.78	1.75	1.79	1.782	1.78
7	2.01	1.97	2.01	2.007	2.00
8	2.22	2.10	2.22	2.217	2.19

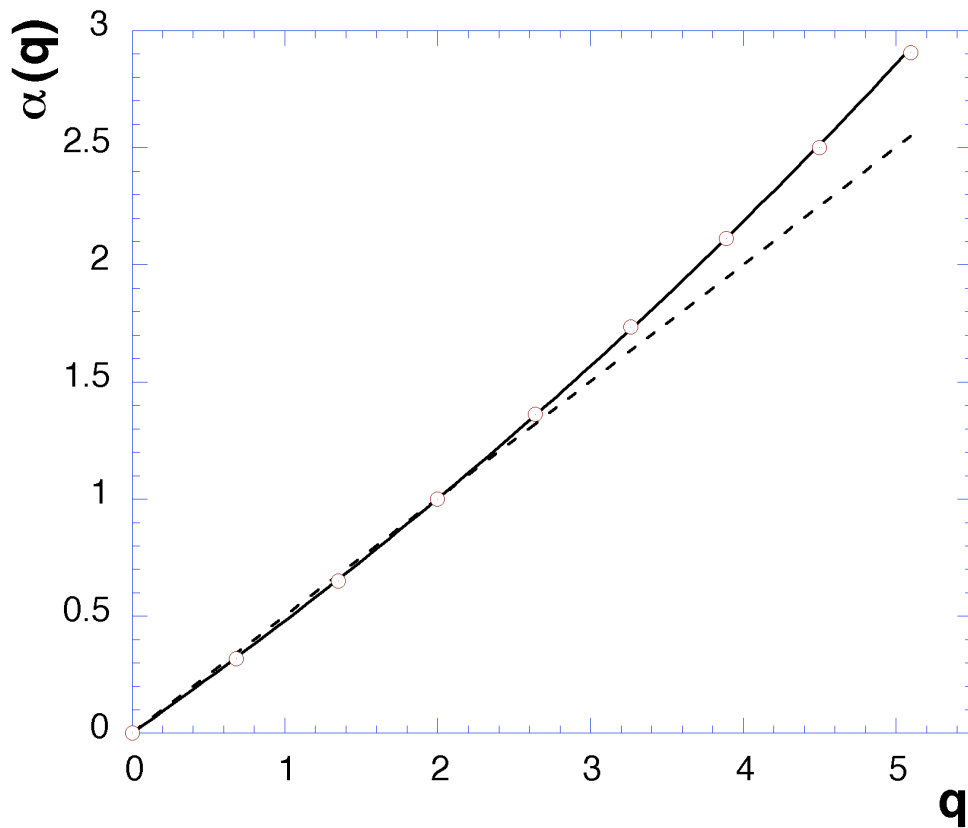


Fig. 1. The curve $\alpha(q)$ (open dots: experimental values; continuous line: lognormal fit) compared to the non-intermittent curve of equation $q/2$ (dotted line). It is visible that $\alpha(1) = \alpha < 1/2$, but the intermittency correction is very small. The value $\alpha(2) = 1$ is a fixed point not affected by intermittency. One obtains numerically $\alpha(3) \simeq 1.56$ and $\alpha(4) \simeq 2.2$.

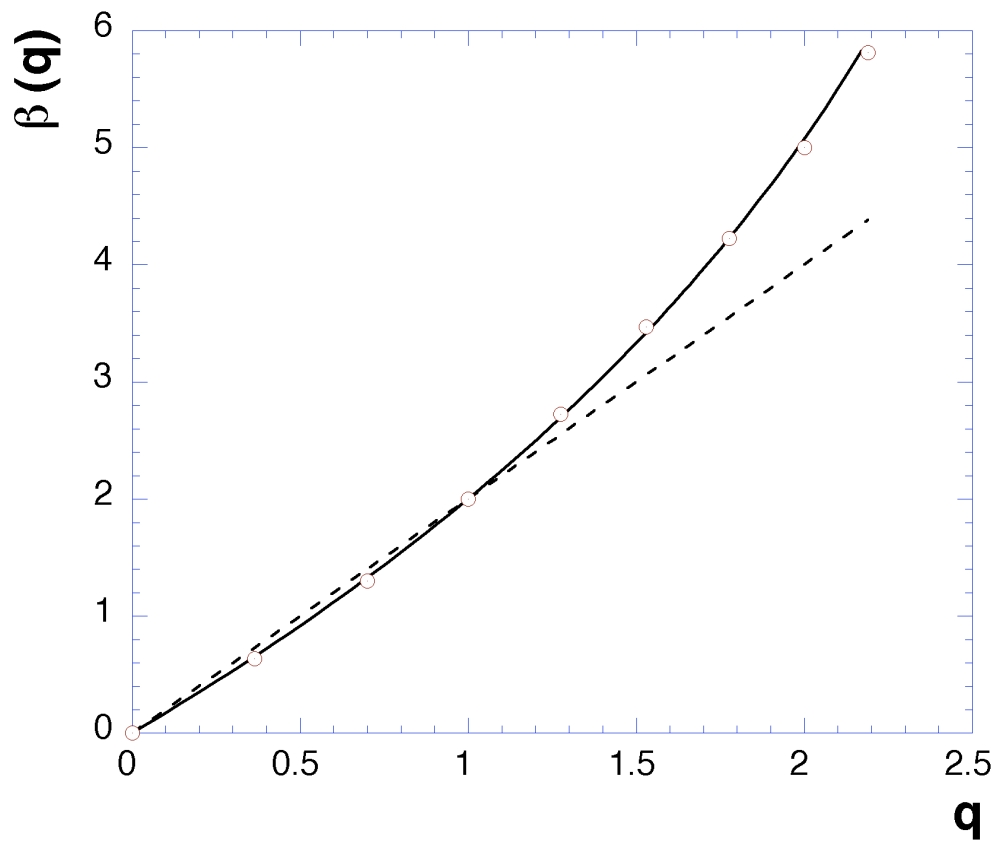


Fig. 2. The curve $\beta(q)$ (open dots: experimental values; continuous line: lognormal fit) compared to the non-intermittent curve of equation $2q$ (dotted line). This curve is strongly nonlinear and increasing very fast. The value $\beta(1) = 2$ is a fixed point and we may note also that $\beta(2) \simeq 5$.

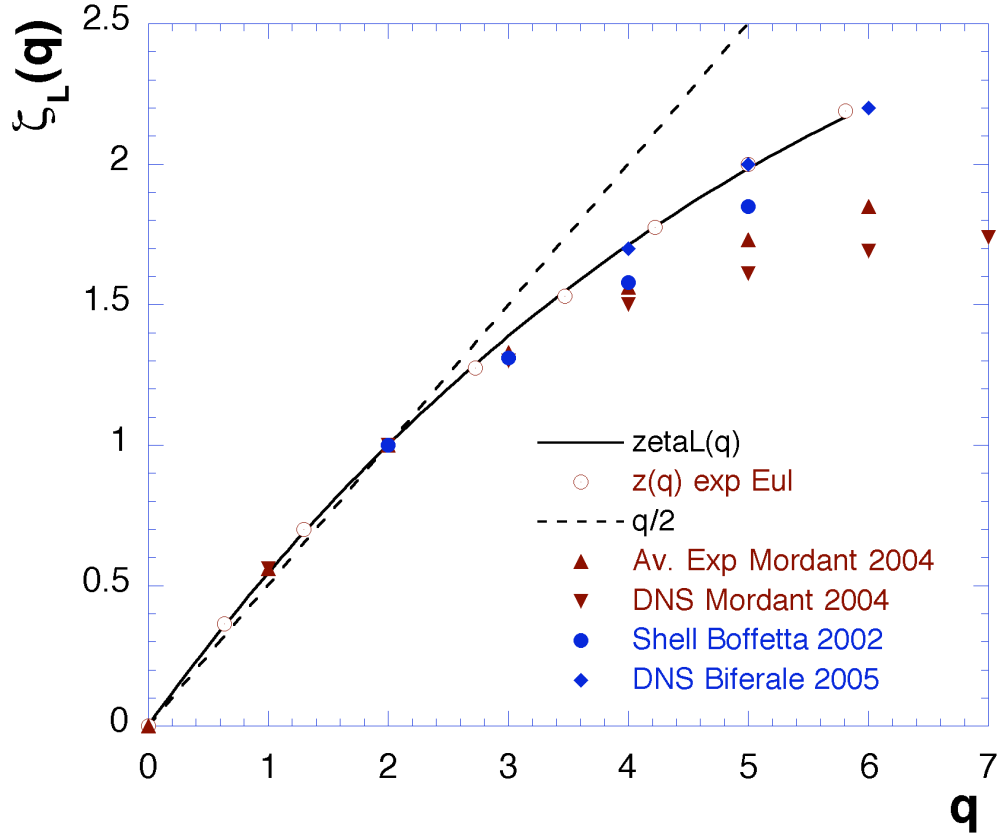


Fig. 3. The curve $\zeta_L(q)$ (open dots: experimental values; continuous line: lognormal fit) compared to the non-intermittent curve of equation $q/2$ (dotted line), and to four Lagrangian recent estimates: average of 4 experiments published in [27]; DNS from [27], with $R_\lambda=140$; shell model results from [21]; and DNS from [23]. The agreement is good for low orders of moment; for larger moments experimental and numerical estimates of scaling exponents have a larger scatter, and it is difficult to conclude concerning the validity of Eq. (27).

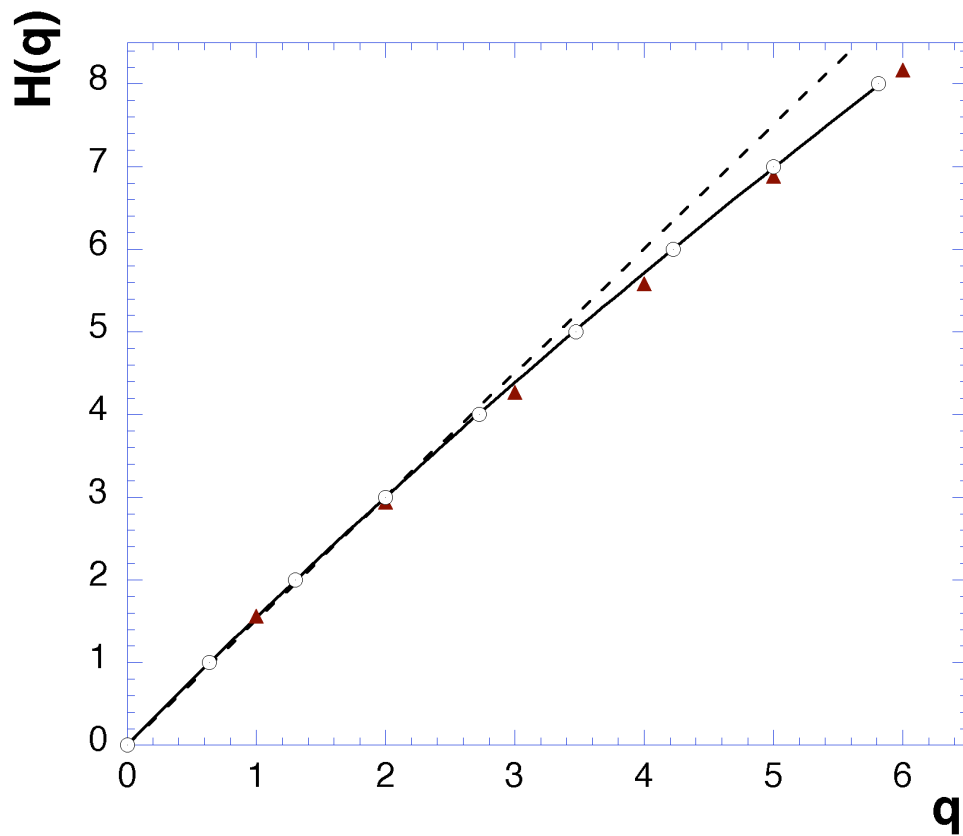


Fig. 4. The curve $H(q)$ (open dots: experimental values; continuous line: lognormal fit) compared to the non-intermittent curve of equation $3q/2$ (dotted line). This is also compared to the shell-model estimates published in [29]. The agreement is quite good. The curve H is only slightly nonlinear and concave.

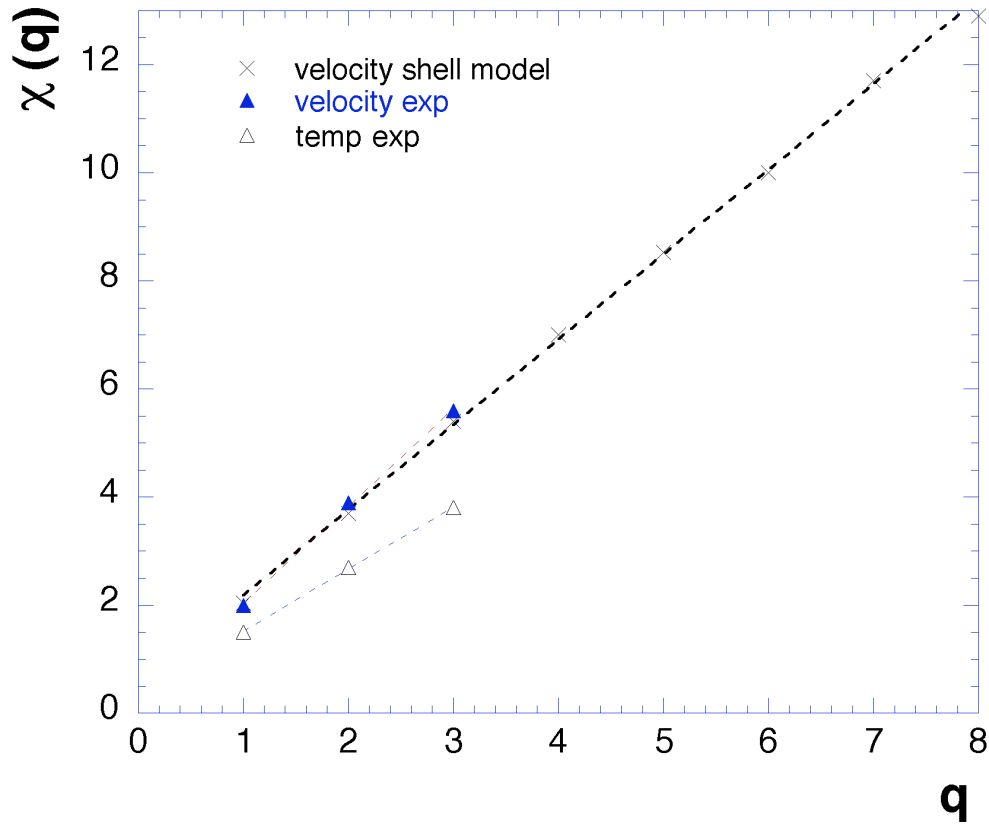


Fig. 5. Representation of data for $\chi(q)$ for $q > 0$ reported in Refs. [32,36]. These correspond to numerical estimates from a shell model [32] and experimental estimates with $R_\lambda = 582$ for velocity (longitudinal structure functions) and temperature turbulence [36]. These values are compatible with a linear behaviour (linear fits are displayed as dotted lines), as predicted by Eq.(37), although experimental estimates are provided here for only three values.

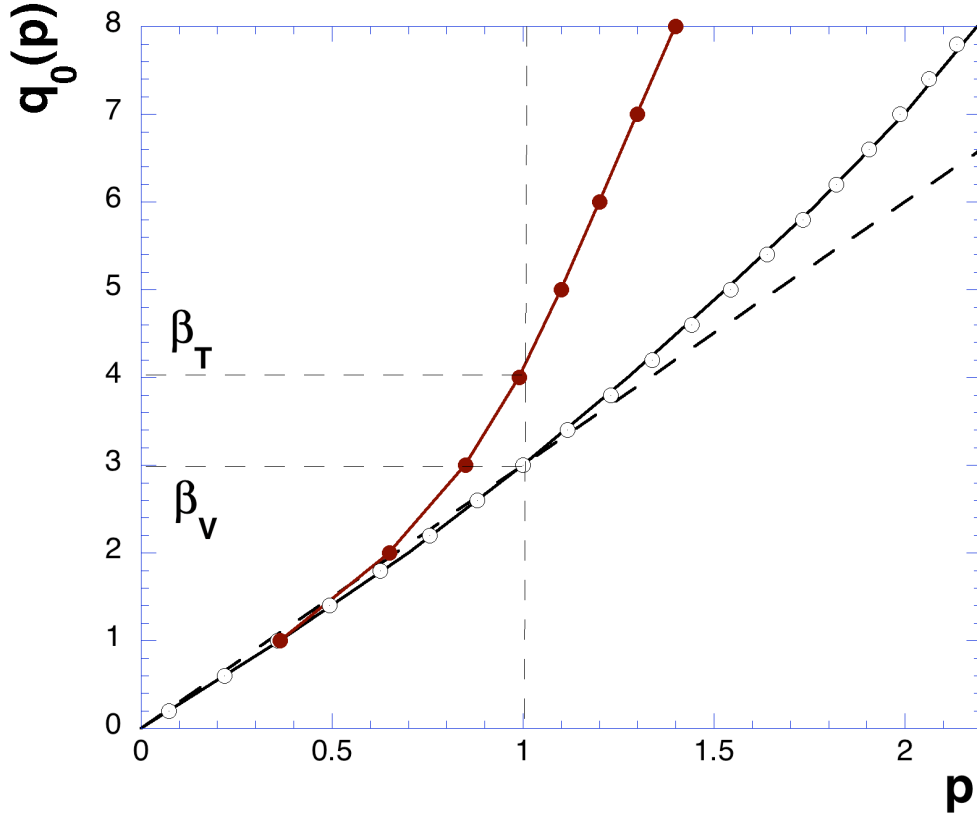


Fig. 6. The curve $q_0(p) = \zeta^{(-1)}(p)$ representing the scaling exponent function for the moments of order $p > 0$ of the inverse exit times, for the velocity (open dots: experimental values; continuous line: lognormal fit) compared to the non-intermittent curve of equation $3q$ (dotted line); also shown: the same curve for passive scalar turbulence (continuous line and closed dots, data from Ref. [37]). The passive scalar curve is very close to the velocity one for weak moments; it increases very quickly. We also represent the ϵ -entropy exponent $\beta = \zeta^{(-1)}(1)$, to emphasize the clear different values for velocity and passive scalar Eulerian turbulence