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# Energy methods in fracture mechanics: stability, bifurcation and second variations

Q. S. NGUYEN \*, C. STOLZ \* and G. DEBRUYNE \*\*

ABSTRACT. — The quasi-static evolution of a system of interacting linear cracks is considered in brittle fracture. Stability and bifurcation criteria are presented in terms of the second variation of the potential energy and with a formulation of the rate boundary value problem following Hill's method. A symmetric description is proposed for this problem involving as principal unknowns the crack propagation velocity and the displacement velocity defined on the current configuration. As a consequence, an explicit expression for the matrix of the second derivatives of energy with respect to the crack lengths is given in terms of new path-independent integrals. The numerical computation of these path-independent integrals by the f. e. m. is also considered and illustrated by some simple examples.

## 1. Introduction

The study of crack propagation and crack stability is an old problem in Fracture Mechanics. In brittle fracture, when the Griffith's criterion is assumed, stability analysis is very classical and most of the investigations of the literature have been concerned with the bi-dimensional problem of one linear crack in plane strain, plane stress or anti-plane shear; *cf.* for example ([Blum, 1969]); [Bui, 1977]). Its generalization to a system of linear cracks has been discussed by Bazant & Ohtsubo [1977] and by Nemat-Nasser, Sumi *et al.* [1980]. The case of a plane crack of arbitrary shape embedded in a three dimensional elastic solid has also been considered, *cf.*, for example, [Nguyen, 1980].

Since the Griffith's criterion has also an energetic significance, it has been shown that the quasi-static propagation of plane cracks is, in fact, a special case of a class of time-independent processes in Solid Mechanics which include most common problems of dry friction, brittle fracture, brittle damage and standard plasticity. For example, the quasi-static behaviour of a system of interacting linear cracks can be compared to the quasi-static behaviour of an elastic-plastic structure of finite degrees of freedom; the equations of evolution of these systems are of the same mathematical nature. The common feature in the modeling of these phenomena is the applicability of two simple and fundamental concepts: the notion of *energy* associated with a given state and the *normality law*

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\* Laboratoire de Mécanique des Solides, École Polytechnique, Centre commun X — Mines — Ponts — C.N.R.S.-U.R.A. n° 258, Palaiseau, France.

\*\* Électricité de France, Direction des Études et Recherches, Clamart, France.

expressed by Hill's maximum dissipation principle. The interested reader may refer to ([Hill, 1958], [Nguyen, 1984]) for a more complete presentation of the subject.

The objective of our discussion is to present a contribution to the aforementioned problem of a system of interacting linear cracks. Equations of evolution describing the variation of crack lengths with respect to a time-like parameter will be first given to illustrate the general mathematical framework. Our attention will be focussed on the formulation of bifurcation and stability criteria. As in plasticity, these two notions must be clearly separated ([H, 1958], [N, 1984]). This description points out the importance of the second variation of energy: its positivity is a sufficient condition of non bifurcation. As in plasticity, bifurcation and stability are conveniently obtained by considering the rate problem which consists of the determination of the rate of the crack lengths *versus* the load rate  $\dot{\lambda}$  when the current state is assumed to be known. For practical applications, the coupled rate problem is discussed here in terms of the two principal unknowns which are the crack length rate and the displacement rate. This *new formulation* gives us the possibility to obtain *an explicit expression about the second derivatives matrix of energy* with respect to cracks in terms of *new path independent integrals*.

The numerical computation of these path independent integrals by the f. e. m. is given in the last part of this paper. Its operational character is here illustrated by some simple examples.

## 2. Quasi-static evolution of a system of interacting linear cracks

Let us consider (Fig. 1) the following crack propagation problem in brittle fracture.

An elastic solid  $\Omega$  with  $n$  linear cracks of length  $l_i$ ,  $i = 1, n$ , is subjected to a loading path defined by load parameters  $\lambda_\mu$ ,  $\mu = 1, r$  in plane strain, plane stress or antiplane

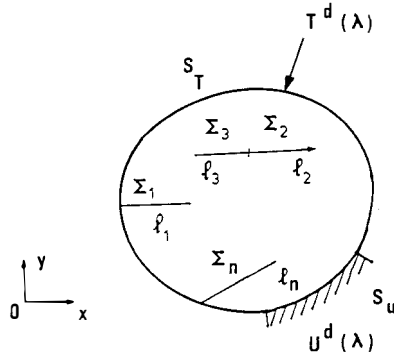


Fig. 1. — System of interacting linear cracks in brittle fracture.

shear. We want to study the response  $l(t)$  associated to a given loading path  $\lambda(t)$  from a given initial configuration  $l^0$ . The possibility of crack kinking is not considered here; the transformation is assumed to be infinitesimal and quasi-static; only stable crack growth

will be under consideration. As usual, the load consists of an applied force  $T^d(\lambda)$  on a portion  $S_T$  of the boundary  $\partial\Omega$  and a given displacement  $u^d(\lambda)$  on the complementary part  $S_u$ . Crack surfaces  $\Sigma_i, i=1, n$  are stress-free with unilateral contact condition without friction. As usual the body force is neglected and the materials is assumed to be homogeneous in order to obtain the simple property of invariant integrals.

If  $\varphi(\varepsilon)$  denotes the volumic elastic energy density the total potential energy of the solid is:

$$(2.1) \quad E(u, l, \lambda) = \int_{\Omega_l} \varphi(\varepsilon(u)) d\Omega - \int_{S_T} T^d(\lambda) \cdot u dS$$

where  $l \in \mathbb{R}_+^n, \lambda \in \mathbb{R}^r$  and the displacement field  $u$  must belong to the set of kinematically admissible displacements:

$$(2.2) \quad K(l, \lambda) = \left\{ u \left| \begin{array}{l} u \text{ defined on } \Omega_l, \text{ regular} \\ u = u^d(\lambda) \text{ on } S_u \\ [u_n] \geq 0 \text{ on } \Sigma_i, i=1, n \end{array} \right. \right\}.$$

At equilibrium, the displacement  $u$  must satisfy the virtual work equation which can be written in the form of a variational inequality to take account of unilateral contact without friction on crack surfaces:

$$(2.3) \quad E_{,u}(u, l, \lambda) \cdot \delta u = \int_{\Omega_l} \varphi_{,\varepsilon} \delta \varepsilon d\Omega - \int_{S_T} T^d \cdot \delta u dS \geq 0, \\ \forall \delta u = v - u, \quad v \in K(l, \lambda).$$

If elastic energy is also strictly convex, we obtain from (2.3) that the displacement at equilibrium  $u = u(l, \lambda)$  satisfies the minimum of energy and thus the value of energy at equilibrium  $W(l, \lambda)$  can be naturally introduced:

$$(2.4) \quad W(l, \lambda) = E(u(l, \lambda), l, \lambda) = \underset{\bar{u} \in K(l, \lambda)}{\text{Min}} E(\bar{u}, l, \lambda)$$

to eliminate, at least formally, the displacement.

In brittle fracture, Griffith's law of crack propagation can be expressed in the following way. If  $G_i$  is the energy release rate associated with the  $i$ -th crack length  $l_i$ :

$$(2.5) \quad G_i = -W_{,i}(l, \lambda)$$

then we obtain necessarily:

$$(2.6) \quad \begin{cases} \text{If } G_i < G_c, & l_i = 0 \quad (\text{no propagation}) \\ \text{If } G_i = G_c, & l_i \geq 0 \quad (\text{possible propagation}). \end{cases}$$

If the convex domain of admissible forces  $C$  is introduced:

$$(2.7) \quad C = \{ G \mid G_i \leq G_c, i=1, n \};$$

Equations (2.6) can also be written in the form:

$$(2.8) \quad \dot{l}_i(G_i - G_i^*) \geq 0, \quad \forall G^* \in C.$$

Thus Hill's maximum dissipation principle is satisfied as in standard plasticity [N, 1984]. Formally, the equations of evolution of the system are given by:

$$(2.9) \quad \left\{ \begin{array}{l} G_i = -W_{,i}(l, \lambda) \\ \dot{l}_i(G_i - G_i^*) \geq 0, \quad \forall G^* \in C \\ l_i(0) = l_i^0 \end{array} \right.$$

where the unknown  $l(t)$  must be computed in terms of the loading path  $\lambda(t)$ ,  $t \in [0, T]$ .

A better understanding of the process of evolution is given by the formulation of the rate problem. The current state  $(l, \lambda)$  is assumed to be known at time  $t$ ; the rate problem consists in determining the right-hand side (r.h.s.) derivative  $\dot{l}$  in terms of the r.h.s. derivative  $\dot{\lambda}$ .

It follows from (2.6) that  $(G_i - G_c)\dot{l}_i = 0$ . This equation gives after time-differentiation:

$$(2.10) \quad \dot{l}_i \dot{G}_i = 0, \quad \forall t \in [0, T], \quad i = 1, n.$$

These  $n$  equations permit the computation of  $n$  unknowns  $\dot{l}_i$  in the following way:

Assuming the present state to be known, let us consider the set

$$I = \{ i \in (1, \dots, n) \mid G_i = G_c \}$$

of indices  $i$  which correspond to possible crack growth at time  $t$ . Then the following proposition holds:

PROPOSITION 1. —  $\dot{l}_i \geq 0$ ,  $i \in I$  and satisfy the following variational inequality:

$$(2.11) \quad (W_{,ij} \dot{l}_j + W_{i\mu} \dot{\lambda}_\mu) (\beta_i - \dot{l}_i) \geq 0, \quad \forall \beta_i \geq 0, \quad i \in I$$

This proposition follows simply from the fact that  $\dot{G}_i \beta_i \leq 0$ ,  $\forall \beta_i \geq 0$ . From (2.10) one thus obtains  $\dot{G}_i (\beta_i - \dot{l}_i) \leq 0$  and hence (2.11).

The variational inequality (2.11) characterizes the first derivative  $\dot{l}$  as a function of the present state and load rate. Existence of *at least* one solution  $\dot{l}$  is ensured for arbitrary  $\dot{\lambda}$  when the matrix  $W_{,ij}$  satisfies a positivity condition (*cf.* for example [Lions, 1968] :

$$(2.12) \quad \beta_i W_{,ij} \beta_j > 0, \quad \forall \beta \neq 0, \quad \beta \in L = \{ \beta \mid \beta_i \geq 0, i \in I \}$$

The mechanical significance of (2.12) has been discussed [N, 1984]; (2.12) is a sufficient *condition of dynamic stability*. This condition is assumed to be satisfied in our discussion, and therefore stable propagation of our crack system will be considered.

If the matrix  $W_{,ij}$  satisfies another condition of positivity, more restrictive than (2.12):

$$(2.13) \quad \beta_i W_{,ij} \beta_j > 0, \quad \forall \beta \neq 0, \quad \beta \in \bar{L} = \{ \beta \mid \beta_i \text{ arbitrary}, i \in I \}$$

where  $\bar{L}$  denotes the vectorial space generated by  $L$ , then the solution  $\hat{l}$  is also unique. As has been shown previously [Nguyen, 1987], by the consideration of the problem of order  $m$ ,  $m=1,2$ , this *positivity not only ensures uniqueness* of the response  $\hat{l}$  but also the uniqueness of higher derivatives  $\hat{l}^{(m)}$ ,  $m=2, \dots$ . This positivity ensures *non-bifurcation* of any order and may be closely related to Hill's criterion of non-bifurcation in elastoplasticity [H, 1958].

### 3. Coupled displacement-propagation rate problem

The previous discussion outlines principal results concerning the quasi-static evolution relating the possibility of bifurcation and loss of stability. It underlines the fundamental role of the second variation  $\beta_i W_{,ij} \beta_j$  with respect to bifurcation and stability analysis of the crack propagation process. However, for practical applications, the second derivatives  $W_{,ij}$ ,  $W_{,i\mu}$  remain to be computed explicitly. On the other hand, displacement has an evident mechanical significance and it may be advantageous to study the coupled response of the system with displacements and crack lengths as principal unknowns.

The essential difficulty of the problem is the dependence of  $\Omega_t$  on the crack lengths and the presence of moving singularities when cracks propagate. One possibility has been investigated by Destuynder & Djaoua [1981] or by Ohtsuka [1981]; these authors have introduced a geometrical lagrangien description to formulate the problem on a fixed domain  $\Omega_0$ , in particular to compute the expression of  $G_1$  and  $G_{1,1}$  in terms of lagrangian variables. To our knowledge, this is the first attempt to obtain explicitly an expression for the second derivative  $W_{,11}$  in the case of one crack in brittle fracture. Our method is *a priori* different and based upon the concept of *transport of singularity*, cf. Nguyen [1980]. It may be compared to Cherepanov's considerations [1983] and consists in the introduction of a *moving reference frame in translation with the crack-tips*  $(AXY)_i$ . An interesting property is the conservation of the nature of the singularity in these coordinates. Far from crack-tips  $A_i$ ,  $i=1, n$  it is however convenient to remain in the fixed coordinates  $Oxy$ .

It is thus of interest to isolate the crack singularity by a closed curve  $\Gamma_i$  in translation with each crack (*Fig. 2*) delimiting a domain  $V_\Gamma^i$  surrounding the crack-tip  $A_i$ . All physical quantities will be expressed in terms of the fixed coordinates  $(x, y, t)$  outside  $V_\Gamma^i$  and in terms of the moving coordinates  $(X^i, Y^i, t)$  inside  $V_\Gamma^i$ . For example, the displacement  $u$  is described in  $\Omega$  by the following expressions:

$$(3.1) \quad u = \begin{cases} u(x, y, t), & x \in \Omega - \bigcup_i V_\Gamma^i \\ U(X^i, Y^i, t), & x \in V_\Gamma^i \end{cases} .$$

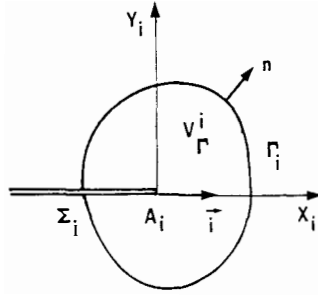


Fig. 2. – Fixed and moving references.

Clearly, this change of variables is related to the crack motion. If  $u$  is continuous across  $\Gamma_i$ , the time derivatives  $\dot{u} = u_{,t}(x, y, t)$  and  $\dot{u}^* = U_{,t}(X^i, Y^i, t)$  are not identical. They are connected to each other by the relation:

$$(3.2) \quad \dot{u} = \dot{u}^* - \dot{l}_i \cdot \nabla_i u$$

where  $\nabla_i u$  denotes the spatial derivative of  $u$  in the  $i$ -th direction of propagation.

It is important to note that the transport of singularity implies for any physical quantity  $g$  the asymptotic behaviour:

$$(3.3) \quad \dot{g} \sim -\dot{l}_i \cdot \nabla_i g$$

near the crack-tip  $A_i$  since  $\dot{g}^*$  has the same singularity as  $g$ .

Now, the expression of  $G_i$  can be readily derived. Indeed, to compute  $G_i$ , one can take a virtual time scale  $t = l_i$ ,  $l_j = 0$  if  $j \neq i$  and thus from (2.1) it follows that:

$$G_i = -\frac{\partial E}{\partial l_i} = -\int_{\Omega - v_{\Gamma}^i} \sigma \dot{\varepsilon} d\Omega + \int_{S_{\Gamma}} T^d \cdot \dot{u} dS + \int_{\Gamma_i} \varphi n_i d\Gamma - \int_{v_{\Gamma}^i} \sigma \varepsilon^* d\Omega.$$

Since

$$\int_{\Omega - v_{\Gamma}^i} \sigma \dot{\varepsilon} d\Omega = \int_{S_{\Gamma}} T^d \cdot \dot{u} dS - \int_{\Gamma_i} n \cdot \sigma \cdot \dot{u} d\Gamma,$$

from (3.3) and from the integrability property of the function  $\sigma \varepsilon^*$  which implies that:

$$\lim \int_{v_{\Gamma}^i} \sigma \varepsilon^* d\Omega = 0,$$

one obtains finally the classical expression:

$$(3.4) \quad G_i = \lim J_{\Gamma_i} \quad \text{with} \quad J_{\Gamma_i} = \int_{\Gamma_i} (\varphi n_i - n \cdot \sigma \cdot \nabla_i u) d\Gamma.$$

It is well known that  $J_{\Gamma_i}$  is path-independent if body forces are absent and if the material is homogeneous. Under these restrictions:

$$(3.5) \quad G_i = J_{\Gamma_i} = \int_{\Gamma_i} (\varphi n_i - n \cdot \sigma \cdot \nabla_i u) d\Gamma.$$

To formulate the rate problem, it is necessary to compute, as in (2.11), the expression for  $(d/dt) G_i(t) = \dot{G}_i$ . Since  $G_i$  is invariant with the choice of  $\Gamma_i$ , one obtains also  $\dot{G}_i = \dot{G}_i^*$  and the following proposition holds:

PROPOSITION 2. —  $\dot{G}_i$  is given by path-independent integrals:

$$(3.6) \quad \dot{G}_i = \int_{\Gamma_i} (n \cdot \nabla_i \sigma \cdot \dot{u} - n \cdot \dot{\sigma} \cdot \nabla_i u) d\Gamma.$$

The proof is obtained by time differentiation of (3.5) in the moving coordinates which first gives the expression:

$$(3.7) \quad \dot{G}_i = \int_{\Gamma_i} (\sigma \varepsilon n_i - n \cdot \sigma \cdot \nabla_i \dot{u} - n \cdot \dot{\sigma} \cdot \nabla_i u) d\Gamma,$$

and leads immediately to (3.6) according to the following lemma:

LEMMA. — Let  $\sigma, \dot{u}$  be independent fields defined on  $\Omega_i$  with arbitrary singularity at the vicinity of cracks such that  $\text{Div } \sigma = 0$ , with sufficient regularity on  $\Gamma$ . Then one obtains necessarily:

$$(3.8) \quad \int_{\Gamma_i} (\sigma \varepsilon n_i - n \cdot \sigma \cdot \nabla_i \dot{u} - n \cdot \dot{\sigma} \cdot \nabla_i u) d\Gamma = 0$$

Indeed, in (3.8) involve only the traces of these fields and their gradients on  $\Gamma_j$ . These fields can be regularized inside  $V_{\Gamma_i}$  by a regular prolongation, for example by regular fields  $\bar{\sigma}, \bar{\dot{u}}$  on a domain  $\bar{V}_{\Gamma_i}^i$ , contained in  $V_{\Gamma_i}^i$  such that Green's formula are valid in  $V_{\Gamma_i}^i$ .

For these regularized fields, the expression (3.8) follows immediately by Green's formula.

The path-independent property results naturally from the definition of  $G$ . This point can also be checked directly from the expression (3.6). If  $\Gamma_i$  and  $\Gamma_i'$  denote two different contours with  $V_{\Gamma_i'}^i \subset V_{\Gamma_i}^i$  for example, by Green's formula one obtains effectively:

$$\begin{aligned} \int_{\Gamma_i} (n \cdot \nabla_i \sigma \cdot \dot{u} - n \cdot \dot{\sigma} \cdot \nabla_i u) d\Gamma - \int_{\Gamma_i'} (n \cdot \nabla_i \sigma \cdot \dot{u} - n \cdot \dot{\sigma} \cdot \nabla_i u) d\Gamma \\ = \int_{V_{\Gamma_i}^i - V_{\Gamma_i'}^i} (\text{Div}(\nabla_i \sigma) \cdot \dot{u} + \nabla_i \sigma \cdot \varepsilon - \text{Div } \dot{\sigma} \cdot \nabla_i u - \dot{\sigma} \cdot \nabla_i \varepsilon) d\Gamma = 0, \end{aligned}$$



since  $\nabla_i \sigma = (\partial^2 \varphi / \partial \varepsilon \partial \varepsilon) \cdot \nabla_i \varepsilon$  and  $\dot{\sigma} = (\partial^2 \varphi / \partial \varepsilon \partial \varepsilon) \cdot \dot{\varepsilon}$ ,  $\text{Div}(\nabla_i \sigma) = 0$  and  $\text{Div} \dot{\sigma} = 0$ .

The expression (3.6) may be compared to a known formula of  $G$  in linear elasticity (cf. [Chen & Shield, 1977] or [Bui, 1983]). However, our result is new and the nature of our discussion is completely different.

This preliminary calculation enables us now to formulate the coupled rate problem in order to complete Proposition 1. Naturally, the material derivative  $\dot{u}$ , because of its strong singularity near the crack-tips, cannot be retained as a principal unknown. Our approach consists of the following choice:

The necessity to handle the rate  $\dot{\cdot}$  instead of  $\cdot$  near the crack-tips leads to the introduction of the rate fields  $u^0, \sigma^0, \dots$  with superscript  $0$  such that:

$$(3.9) \quad u^0 = \begin{cases} \dot{u} & \text{in } \Omega - \bigcup_i V_\Gamma^i \\ \dot{u}^* & \text{in } V_\Gamma^i \end{cases}.$$

Naturally the rates  $u^0, \sigma^0$  are not continuous across  $\Gamma_i$ .

The jump condition, according to (3.2), is:

$$(3.10) \quad \begin{cases} [u^0] + \dot{l}_i \cdot \nabla_i u^0 = 0 \\ [\sigma^0 n] + \dot{l}_i \nabla_i \sigma^0 \cdot n = 0 \end{cases} \quad \text{on } \Gamma_i, \quad i = 1, n.$$

Finally, the equations of evolution in terms of the rate  $\dot{u}, \dot{\sigma}, \dot{l}$ :

1. *Kinematic equations:*

$$\dot{\varepsilon} = (\nabla \dot{u}), \quad \dot{u} = \dot{u}^d \quad \text{on } S_u, \quad [\dot{u}_n] \geq 0 \quad \text{on } \Sigma_i \quad \text{if } [u_n] = 0$$

2. *Static equations:*

$$(3.11) \quad \begin{aligned} \text{Div} \dot{\sigma} &= 0 \quad \text{in } \Omega \\ \dot{\sigma} \cdot n &= \dot{T}^d \quad \text{on } S_T \\ \dot{\sigma} \cdot n &= 0 \quad \text{on } \Sigma_i \quad \text{if } [u_n] > 0 \end{aligned}$$

3. *Constitutive equations:*

$$\dot{\sigma} = \frac{\partial^2 \varphi}{\partial \varepsilon \partial \varepsilon} \cdot \dot{\varepsilon}$$

4. *Griffith's law:*

$$l_i \geq 0 \quad \text{if } G_i = G_c \quad \text{and if } \dot{G}_i = 0 \\ l_i = 0 \quad \text{otherwise}$$

lead to the following local equations in terms of the rate  $\overset{0}{u}$ ,  $\overset{0}{\sigma}$ ,  $\overset{0}{l}$ :

1. *Kinematic equations:*

$$\overset{0}{\varepsilon} = (\nabla \overset{0}{u}), \quad \overset{0}{u} = \overset{0}{u}^d \quad \text{on } S \\ [\overset{0}{u}_n] \geq 0 \quad \text{on } \Sigma_i \quad \text{if } [u] = 0 \\ [\overset{0}{u}_n] + \overset{0}{l}_i \nabla_i \overset{0}{u} = 0 \quad \text{on } \Gamma_i$$

2. *Static equations:*

$$(3.12) \quad \text{Div } \overset{0}{\sigma} = 0 \quad \text{in } \Omega \\ \overset{0}{\sigma} \cdot n = \overset{0}{T}^d \quad \text{on } S_T \\ \overset{0}{\sigma} \cdot n = 0 \quad \text{on } \Sigma_i \quad \text{if } [u_n] > 0 \\ [\overset{0}{\sigma} \cdot n] + \overset{0}{l}_i \nabla_i \overset{0}{\sigma} \cdot n = 0 \quad \text{on } \Gamma_i$$

3. *Constitutive equations:*

$$\overset{0}{\sigma} = \frac{\partial^2 \varphi}{\partial \varepsilon \partial \varepsilon} \cdot \overset{0}{\varepsilon}$$

4. *Griffith's law:*

$$\overset{0}{l}_i \geq 0 \quad \text{if } G_i = G_c \quad \text{and } \dot{G}_i = 0 \\ \overset{0}{l}_i = 0 \quad \text{otherwise}$$

In (3.12), equations (1)+(2)+(3) represent a *non-classic problem* of linear elasticity when the rates  $\overset{0}{l}$  are assumed to be known. Indeed, the displacement rate  $\overset{0}{u}$  is not continuous on  $\Gamma_i$  and surface forces  $\overset{0}{l}_i \nabla_i \overset{0}{\sigma} \cdot n$  are applied on  $\Gamma_i$ . However, the resolution of these equations does not raise any particular difficulty since it can be discussed theoretically in the classical framework of linear elasticity.

To the contrary, equations (1)+(2)+(3)+(4) define a non classical problem with unknowns  $(u, \sigma, \dot{l})$ . If the couple  $(u, \dot{l})$  is chosen as principal unknowns, one obtains the following associated variational formulation according to the proposition:

PROPOSITION 3. – Let  $F(u, \dot{l})$  and  $K$  be respectively:

– the functional:

$$(3.13) \quad F(u, \dot{l}) = \int_{\Omega} \frac{1}{2} \varepsilon \cdot \frac{\partial^2 \varphi}{\partial \varepsilon \partial \varepsilon} \cdot \varepsilon \, d\Omega - \int_{S_T} \mathbb{T}^d \cdot u \, dS - \int_{\Gamma_i} \dot{l}_i \nabla_i \sigma \cdot n u \, d\Gamma + \frac{\dot{l}_i^2}{2} \int_{\Gamma_i} \nabla_i \sigma \cdot n \cdot \nabla_i u \, d\Gamma$$

– and the convex domain of admissible rates:

$$(3.14) \quad K = \left\{ (u, \dot{l}) \left| \begin{array}{l} u = u^d \quad \text{on } S_u, \\ [u_n] \geq 0 \quad \text{on } \Sigma_i \quad \text{if } [u_n] = 0, \\ \dot{l}_i \geq 0, \\ [u] + \dot{l} \cdot \nabla_i u = 0 \quad \text{on } \Gamma_i \end{array} \right. \right\}$$

Then the rate  $(u, \dot{l})$  is a solution of the variational inequality:

$$(3.15) \quad (u, \dot{l}) \in K \text{ and satisfies:} \\ F_{,u} \cdot (\tilde{u} - u) + F_{,\dot{l}} \cdot (\tilde{l} - \dot{l}) \geq 0, \quad \forall (\tilde{u}, \tilde{l}) \in K$$

which is a classical problem of variational calculus since  $F$  is quadratic and  $K$  is convex.

The proof of the proposition follows immediately from (3.6) and (3.12). It should be remarked that the choice of  $\Gamma_i$  is arbitrary. This coupled formulation enables us to formulate again bifurcation and stability analyses in the same spirit as in Section 2. For example, a sufficient condition of stability of the present state is given by the positivity of  $F(u, \dot{l})$  on the convex set  $K_0 = K|_{u^d=0}$ . A sufficient condition of non-bifurcation is given by the positivity of  $F$  on the space  $\bar{K}_0$  generated by  $K_0$ .

#### 4. Expression of the second derivatives of energy

The objective of this section is the derivation of the expression of the second derivatives  $W_{,ij}$  and  $W_{i\mu}$  which appear in (2.11), as a direct application of the coupled rate problem described in Section 3.

To obtain  $W_{,ij}$  from the definition of  $G_i$ , one has to compute the variation of  $G_i$  when  $l_j$  varies while all the other lengths  $l_k$  are maintained constants,  $k \neq j$ . To obtain this, a fictitious time scale is introduced with  $\dot{l}_j = 1$  and  $\dot{l}_k = 0$  if  $k \neq j$ ,  $\dot{T}^d = 0$ ,  $u^d = 0$  in (3.12).

Let  $(u, \sigma)^{(j)}$  be the associated solution obtained by the resolution of the system of equations (1)+(2)+(3) of (3.12) with such given conditions.

Then, one obtains:

PROPOSITION 4. – *The second derivative  $W_{,ij}$  is given by:*

$$(4.1) \quad W_{ij}(l, \lambda) = \int_{\Gamma_i} (n \cdot \sigma_{-}^{(j)} \cdot \nabla_i u - n \cdot \nabla_i \sigma \cdot u_{-}^{(j)}) d\Gamma.$$

In (4.1), the second member is naturally path-independent. Note also that  $\sigma \cdot n$  and  $u$  are not continuous across  $\Gamma_i$ ; their internal traces are involved in this formula.

To obtain  $W_{,i\mu}$ , one has to compute the variation of  $G_i$  when  $\lambda_\mu$  varies and all the crack lengths  $l_i$ ,  $i=1, n$  are maintained fixed. To obtain this, a fictitious time scale is introduced with fictitious condition  $\dot{l}_i = 0$ ,  $\forall i=1, n$  and  $\lambda_\mu = 1$ ,  $\lambda_\nu = 0$  if  $\nu \neq \mu$ .

Let  $(u, \sigma)^{(\mu)}$  be the associated solution obtained by the resolution of the system of equations (1)+(2)+(3) of (3.12) with these given conditions.

Then the following proposition holds:

PROPOSITION 5. – *The second derivative  $W_{,i\mu}$  is given by:*

$$(4.2) \quad W_{,i\mu}(l, \lambda) = \int_{\Gamma_i} (n \cdot \sigma^{(\mu)} \cdot \nabla_i u - n \cdot \nabla_i \sigma \cdot u^{(\mu)}) d\Gamma$$

where the second member is naturally path-independent.

The symmetry  $W_{,ij} = W_{,ji}$  can be established directly as a consequence of the symmetry  $\partial^2 \varphi / \partial \varepsilon_{ij} \partial \varepsilon_{kl} = \partial^2 \varphi / \partial \varepsilon_{kl} \partial \varepsilon_{ij}$ .

Indeed,  $\sigma^{(i)}$  satisfies:

$$\text{Div } \sigma^{(i)} = 0 \quad \text{in } \Omega$$

$$[\sigma n] + \nabla_i \sigma n = 0 \quad \text{on } \Gamma_i.$$

Let  $(\delta u, \delta l_j)^{(i)}$  be a virtual displacement such that:

$$\delta u \varepsilon \mathbf{K}_j = \{ \delta u \mid [\delta u] + \delta l_j \nabla_j u = 0 \text{ on } \Gamma_j \}.$$

Then from Green's formula, it follows that

$$\int_{\Omega - \nu_{\Gamma_i}} \sigma^{(i)(j)} \delta \varepsilon d\Omega = - \int_{\Gamma_i} -\sigma_+^{(i)} \cdot n^{(j)} \partial u d\Gamma + \int_{\Gamma_j} [du] \sigma^{(j)(i)} \cdot n d\Gamma$$

$$\int_{\nu_{\Gamma_i}} \sigma^{(i)(j)} \delta \varepsilon d\Omega = \int_{\Gamma_i} \sigma_-^{(i)} \cdot n^{(j)} du d\Gamma$$

and thus the following expression holds  $\forall (du, \delta l_j)$ :

$$\int_{\Omega} \sigma^{(i)(j)} \delta \varepsilon d\Omega = \int_{\Gamma_i} n \cdot \nabla_i \sigma \cdot du d\Gamma + \int_{\Gamma_j} \delta l_j \cdot \nabla_j u \sigma n d\Gamma$$

in particular, if  $du = u^{(j)}$  and  $dl = 1$ , one obtains:

$$\int_{\Omega} \sigma \varepsilon^{(i)(j)} d\Omega = \int_{\Gamma_i} n \cdot \nabla_i \sigma \cdot u^{(j)} d\Gamma + \int_{\Gamma_j} \nabla_j u^{(i)} \sigma n d\Gamma$$

The same argument gives:

$$\int_{\Omega} \sigma \varepsilon^{(j)(i)} d\Omega = \int_{\Gamma_j} n \cdot \nabla_j \sigma \cdot u^{(i)} d\Gamma + \int_{\Gamma_i} \nabla_i u^{(j)} \sigma n d\Gamma$$

Since  $\sigma \varepsilon = \varepsilon \cdot (\partial^2 \varphi) / (\partial \varepsilon \partial \varepsilon) \cdot \varepsilon = \sigma \varepsilon$ , it follows that:

$$\int_{\Gamma_i} n \cdot \nabla_i \sigma \cdot u^{(j)} d\Gamma + \int_{\Gamma_j} \nabla_j u^{(i)} \sigma n d\Gamma = \int_{\Gamma_j} n \cdot \nabla_j \sigma \cdot u^{(i)} d\Gamma + \int_{\Gamma_i} \nabla_i u^{(j)} \sigma n d\Gamma$$

and thus the symmetry

$$W_{,ij} = \int_{\Gamma_i} \nabla_i u^{(j)} \cdot \sigma \cdot n - n \cdot \nabla_i \sigma \cdot u^{(j)} d\Gamma$$

$$= \int_{\Gamma_j} \nabla_j u^{(i)} \cdot \sigma \cdot n - n \cdot \nabla_j \sigma \cdot u^{(i)} d\Gamma = W_{,ji}$$

is obtained.

It is useful to give an interpretation of our results (3.12), (4.1), (4.2) in connection with Destuynder & Djaoua's lagrangian description [1981].

By a change of variable  $\underline{x} = \underline{x}(y, l)$ , these authors have written their equations on a fixed configuration  $\Omega_{t_0}$  and, in linear elasticity, obtained for  $\Omega_{t_0} = \Omega_t$ , *i.e.* when the reference configuration is the present one (updated lagrangian description):

$$(4.3) \quad W_{,1} = \int_{\Omega} (\sigma_{ij} u_{i,k} \theta_{k,j} - W \operatorname{div} \theta) d\Omega$$

$$(4.4) \quad W_{,11} = \int_{\Omega} (\sigma_{\underline{=}} \cdot \nabla u (\nabla \theta \cdot \nabla \theta) + \frac{1}{2} \sigma_{\underline{=}} \cdot \nabla u \det |\nabla \theta| - \sigma_{\underline{=}} \cdot (\nabla u \cdot \nabla \theta) \operatorname{div} \theta - \sigma_{\underline{=}} \cdot \nabla u \nabla \theta - \sigma_{\underline{=}} \cdot (\nabla u \cdot \nabla \theta) + \sigma_{\underline{=}} \cdot \nabla u \operatorname{div} \theta) d\Omega$$

where  $\theta = \underline{x}_{,i}(y, l)$  and  $(u, \sigma)$  denotes the solution of the rate problem in updated lagrangian description.

It is interesting to note that (4.3) is an alternative expression of energy release rate  $G$  and  $J$ -integral (3.5). As shown already by these authors and confirmed by numerous numerical discussions, the expression (4.3) leads to more accurate numerical results by the f.e.m. and seems to be better for practical applications.

In the same spirit, (4.4) is an alternative expression of the second derivative. Our result (4.1) corresponds to the particular choice  $\theta = \mathbf{i}$  in  $V_{\Gamma}^i$ , and  $\theta = 0$  elsewhere, which leads to the identification  $(u, \sigma) = (u, \sigma)$ . The connection between our approach with the updated lagrangian method is then clear.

It is also important to note that a more complete expression of the second derivatives  $W_{,ij}$  by lagrangian method for interacting linear cracks has been given recently by Suo and Combescure [Suo & Combescure, 1989].

Possible extension of our results into the context of anelasticity has been discussed for particular situations when crack propagation is still characterized by equations (2.10), (2.6) and when the material is such that  $\dot{\sigma} = \partial \varphi / \partial \dot{\varepsilon}(\dot{\varepsilon})$  i.e. if the material is hypoelastic with the existence of a strain rate potential. In this case, proposition 3 is still valid with a small modification on the functional  $F$ :

$$(4.5) \quad F(u, l) = \int_{\Omega - \sum_i V_{\Gamma}^i} \varphi_i(\varepsilon) d\Omega + \int_{V_{\Gamma}^i} \left( \varphi(\varepsilon - l_i \nabla_i \varepsilon) + l_i \nabla_i \sigma \cdot \varepsilon - \frac{l_i^2}{2} \nabla_i \sigma \nabla_i \varepsilon \right) - \int_{\Gamma_i} l_i \nabla_i \sigma \cdot n \cdot u_- d\Gamma + \frac{l_i^2}{2} \int_{\Gamma_i} n \cdot \Gamma_i \sigma \cdot \nabla_i u d\Gamma - \int_{S_{\Gamma}} \dot{\Gamma}^d \cdot u dS.$$

For example, in elastoplasticity and with a  $J$ -based criterion of propagation, the formulation of the coupled rate problem has been given [Nguyen & Stolz, 1985] in the same spirit.

## 5. Numerical illustration

The objective of this paragraph is to give a numerical illustration of the computation of the second derivatives of energy as defined by formula (4.1).

These expressions are here computed by standard technique of the finite element method in order to explore its validity in practical applications.

Only some academic examples will be considered for this purpose. The reader can also refer to the previous discussions of Bazant & Ohtsubo [1977] and of Nemat-Nasser *et al.* [1980] for more interesting examples related to the stability aspects.

Example 1: *Central crack in a rectangular plate*

A rectangular elastic plate with a central crack is submitted to a tensile load  $\sigma = 1$  Mpa in plane stress.

Since the crack length  $2a = 4$  mm is small compared to the plate dimension  $2b = 26$  mm, one obtains approximatly:

$$G = \frac{K_I^2}{E} = \frac{\pi a \sigma^2}{E} \quad \text{and} \quad W_{,11} = -\frac{\pi \sigma^2}{2E} = -7.855 \text{ Mpa}$$

Numerical results, using (4. 1) for different circular curves  $\Gamma$  of radius R, are:

- $\Gamma_1: R_1 = 0.5$  mm,  $W_{,11} = -7.32$  Mpa;
- $\Gamma_2: R_2 = 1.0$  mm  $W_{,11} = -7.85$  Mpa;
- $\Gamma_3: R_3 = 1.5$  mm,  $W_{,11} = -8.15$  Mpa;
- $\Gamma_4: R_4 = 5.0$  mm,  $W_{,11} = -7.85$  Mpa.

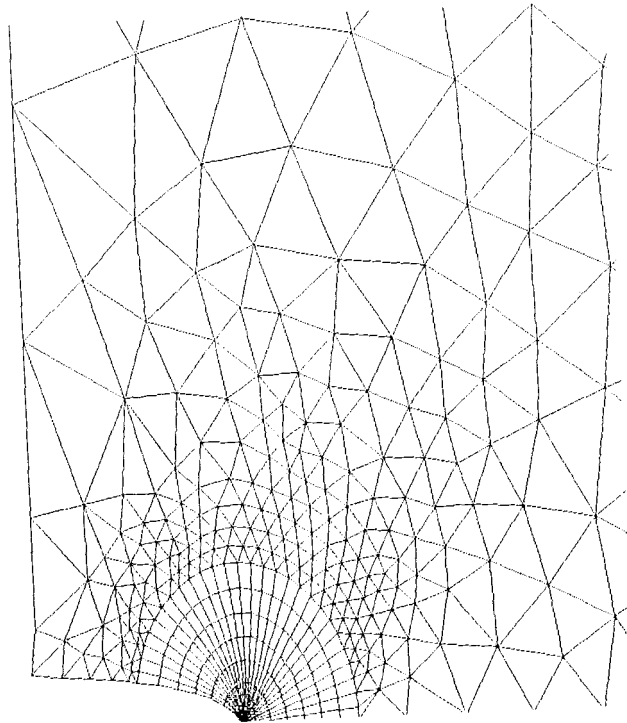


Fig. 3. – Central crack in a rectangular plate.

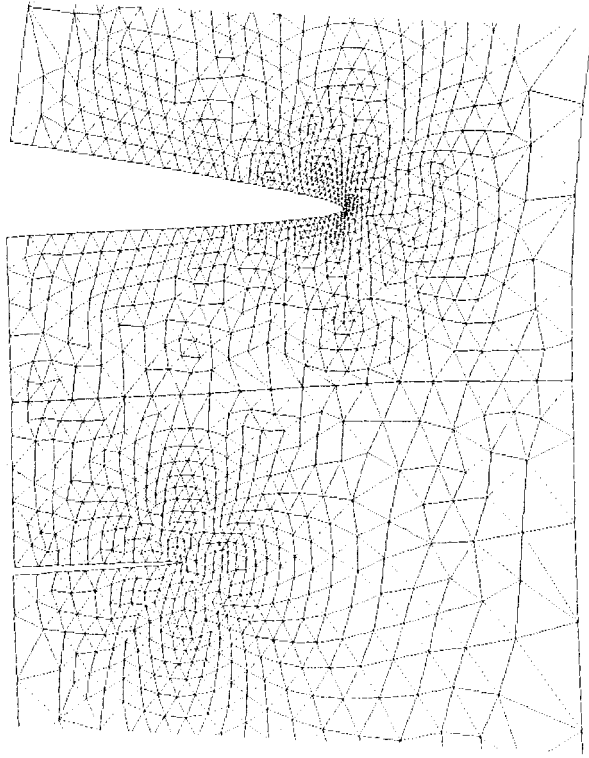


Fig. 4. – System of two parallel cracks.

It can be noted that, concerning the path independency, the numerical values are stable within 5% if  $\Gamma$  is not too closed to the crack tip. The f.e.m. mesh for this example consists of 1,312 triangles and 377 quadrangles, a view of this discretization near the crack is given by Figure 3.

*Example 2: System of two symmetric cracks*

A rectangular plate of dimensions  $2h=150$  mm and  $2b=20$  mm with two symmetric cracks of length  $a=4$  mm is now considered. When the plate is submitted to the same loading, the stress intensity factor  $K_I$  can be obtained approximately by:

$$K_I = \sigma \sqrt{\pi a} f(m), \quad m = \frac{a}{b}$$

$$f(m) = \left( \frac{2}{\pi m} \operatorname{tg} \frac{m\pi}{2} \right)^{1/2} \left( 1 + 0.122 \cos^4 \left( \frac{m\pi}{2} \right) \right)$$

following [Tada, 1973].

This expression gives after differentiation with respect to  $a$ :

$$W_{,11} + W_{,12} = -163 \text{ Mpa}$$



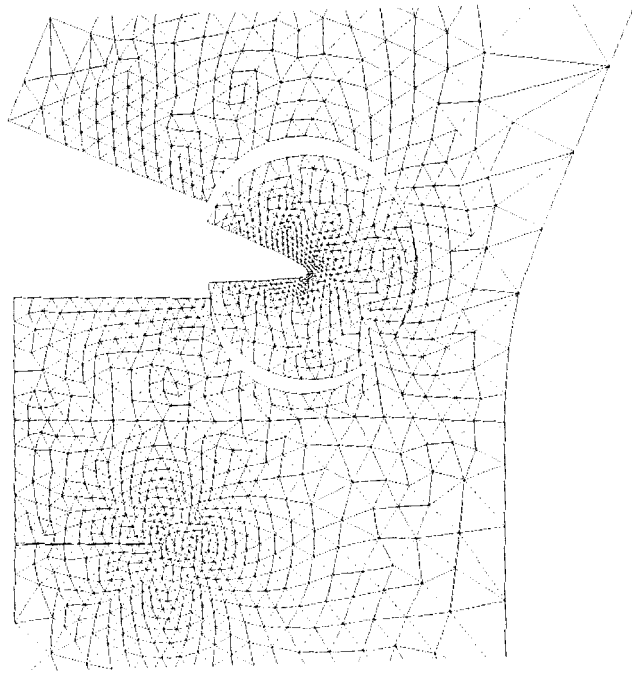


Fig. 5. – System of two parallel cracks. Rate of displacement  $u$ .

The numerical computation with a mesh consisting of 733 triangles and 1,544 nodal points leads to the following results:

$$\Gamma_1: R_1 = 1.0 \text{ cm}, W_{,11} = -369.9, W_{,22} = -371.5, W_{,12} = 214.5, W_{,21} = 214.5 \text{ Mpa};$$

$$\Gamma_2: R_2 = 1.5 \text{ cm}, W_{,11} = -377.9, W_{,22} = -357.3, W_{,12} = 206.6, W_{,21} = 221.4 \text{ Mpa};$$

$$\Gamma_3: R_3 = 2.0 \text{ cm}, W_{,11} = -367.6, W_{,22} = -371.0, W_{,12} = 216.6, W_{,21} = 216.7 \text{ Mpa}.$$

The variation between extreme values is about 5%. The mean value of  $W_{,11} + W_{,22}$  is about 15% greater than the preceding “semi-analytical” value. The symmetry  $W_{,12} = W_{,21}$  is respected within 1%. This symmetry is not automatically satisfied in the numerical computation because the numerical fields  $\sigma$  and  $\sigma_{,i}$  do not satisfy exactly equilibrium equations.

### Example 3: *System of two parallel cracks*

The same rectangular plate is now considered with two parallel cracks of length  $l_1 = 6 \text{ cm}$  and  $l_2 = 3 \text{ cm}$ . The deformed plate is shown on Figure 4. The computation of

the second derivatives following (4. 1) leads to:

$$\Gamma_1: R_1 = R_2 = 0.5 \text{ cm}, W_{,11} = -1,536., W_{,22} = -83., W_{,12} = 14., W_{,21} = 36. \text{ Mpa};$$

$$\Gamma_2: R_1 = R_2 = 1.0 \text{ cm}, W_{,11} = -1,338., W_{,22} = -75., W_{,12} = 2., W_{,21} = 2. \text{ Mpa};$$

$$\Gamma_3: R_1 = R_2 = 2.0 \text{ cm}, W_{,11} = -1,456., W_{,22} = -74., W_{,12} = 6., W_{,21} = 6. \text{ Mpa}.$$

Figure 5 illustrates the rate of displacement  $\dot{u}$  which must be computed in (4. 1). This field is not continuous on  $\Gamma_i$  according to (3. 10).

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