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► **To cite this version:**

Hongze Li, Jie Wu. The universality of symmetric power L-functions and their Rankin-Selberg L-functions. *Journal of the Mathematical Society of Japan*, 2007, 59 (2), pp.371–392. hal-00091802

**HAL Id: hal-00091802**

**<https://hal.science/hal-00091802>**

Submitted on 7 Sep 2006

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# The universality of symmetric power $L$ -functions and their Rankin-Selberg $L$ -functions

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**Abstract.** We establish the universality theorem for the first four symmetric power  $L$ -functions of automorphic forms and their associated Rankin-Selberg  $L$ -functions. This generalizes some results of Laurinćikas & Matsumoto and Matsumoto respectively.

## § 1. Introduction

The automorphic  $L$ -function is a powerful tool to study arithmetic, algebraic and geometric objects. Many results will follow from the known or conjectured analytic properties of automorphic  $L$ -functions. It is therefore important to explore an  $L$ -function in various analytic aspects. Here, we are concerned with the universality property. Roughly speaking, a function  $f$  has the universality property if every non-vanishing analytic function can be approximated uniformly on compact subsets in the half critical strip  $\mathbb{D}(\frac{1}{2})$  by translations of this function  $f$ , where  $\mathbb{D}(\frac{1}{2})$  denotes

$$(1.1) \quad \mathbb{D}(\sigma_0) := \{s \in \mathbb{C} : \sigma_0 < \Re s < 1\}$$

for any  $\sigma_0 < 1$ . According to Linnik-Ibragimov, it was conjectured that the universality property is intrinsic to all Dirichlet series which can be analytically continued to left of their abscissa of absolute convergence.

The universality of the Riemann zeta-function  $\zeta(s)$  was first discovered by Voronin [23]. More precisely he proved the following: Let  $\mathbb{K}$  be a closed disc of radius  $r < \frac{1}{4}$  centered at  $s = \frac{3}{4}$ , and  $\varphi(s)$  a non-vanishing analytic function in the interior of  $\mathbb{K}$  and continuous on  $\mathbb{K}$ . Then for any  $\varepsilon > 0$ , there is a real number  $t$  such that

$$(1.2) \quad \sup_{s \in \mathbb{K}} |\zeta(s + it) - \varphi(s)| < \varepsilon.$$

In 1981, Bagchi [1] developed a new method to deduce the universality property of  $\zeta(s)$  and obtained a result stronger than (1.2), as follows. Let  $\mathbb{K}$  be a compact subset of  $\mathbb{D}(\frac{1}{2})$  with connected complement and  $\varphi(s)$  a non-vanishing analytic function in the interior of  $\mathbb{K}$  and continuous on  $\mathbb{K}$ . Then for any  $\varepsilon > 0$ , we have

$$(1.3) \quad \liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ t \in [0, T] : \sup_{s \in \mathbb{K}} |\zeta(s + it) - \varphi(s)| < \varepsilon \right\} > 0,$$

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<sup>(1)</sup> Project supported by NSFC 10471090, the Scientific Research Foundation for the Returned Overseas Chinese Scholars and Shuguang Plan of Shanghai  
2000 *Mathematics Subject Classification*: 11F66  
*Key words and phrases*: Automorphic  $L$ -function, Universality

where  $\text{meas}(\cdot)$  is the Lebesgue measure. This result was generalized by different authors to many other  $L$ -functions such as Dirichlet  $L$ -functions, Dedekind  $L$ -functions, Hurwitz  $L$ -functions, Lerch  $L$ -functions, etc. A detailed historical account can be found in [15].

In this paper we are interested in the universality of automorphic  $L$ -functions. For a positive even integer  $k$  such that  $k = 12$  or  $k \geq 16$  <sup>(2)</sup>, we denote by  $H_k^*$  the set of all Hecke primitive eigencuspforms of weight  $k$  for the full modular group  $\text{SL}(2, \mathbb{Z})$ . The Fourier series expansion of  $f \in H_k^*$  at the cusp  $\infty$  is

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z} \quad (\Im z > 0),$$

where  $\lambda_f(n)$  is the  $n$ th (normalized) Fourier coefficient of  $f$ , verifying

$$(1.4) \quad \lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right)$$

for any integers  $m \geq 1$  and  $n \geq 1$ . In particular it is a multiplicative function of  $n$ . According to Deligne, for any prime number  $p$  there is  $\alpha_f(p)$  such that

$$(1.5) \quad \lambda_f(p^\nu) = \alpha_f(p)^\nu + \alpha_f(p)^{\nu-2} + \cdots + \alpha_f(p)^{-\nu} \quad (\nu \geq 1)$$

and

$$(1.6) \quad |\alpha_f(p)| = 1.$$

In particular  $\lambda_f(1) = 1$  and  $\lambda_f(n)$  is real.

For  $m \in \mathbb{N}$ , the  $m$ th symmetric power  $L$ -function attached to  $f \in H_k^*$  and its Rankin-Selberg  $L$ -function are defined as

$$(1.7) \quad L(s, \text{sym}^m f) := \prod_p \prod_{0 \leq j \leq m} (1 - \alpha_f(p)^{m-2j} p^{-s})^{-1}$$

and

$$(1.8) \quad L(s, \text{sym}^m f \times \text{sym}^m f) := \prod_p \prod_{0 \leq i, j \leq m} (1 - \alpha_f(p)^{2(m-i-j)} p^{-s})^{-1}$$

for  $\sigma > 1$ , respectively. The products over primes in (1.7) and (1.8) admit Dirichlet series representation

$$(1.9) \quad L(s, F) = \sum_{n=1}^{\infty} \lambda_F(n) n^{-s}$$

for  $\sigma > 1$ , where  $F = \text{sym}^m f$  or  $\text{sym}^m f \times \text{sym}^m f$ , and  $\lambda_F(n)$  is a multiplicative function. Following from (1.6), we have for  $n \geq 1$ ,

$$(1.10) \quad |\lambda_F(n)| \leq \begin{cases} d_{m+1}(n) & \text{if } F = \text{sym}^m f, \\ d_{(m+1)^2}(n) & \text{if } F = \text{sym}^m f \times \text{sym}^m f, \end{cases}$$

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<sup>(2)</sup> For  $k \in \{2, 4, 6, 8, 10, 14\}$ , there is no cusp forms of weight  $k$  for the full modular group  $\text{SL}(2, \mathbb{Z})$  (see [21])

where  $d_z(n)$  is the  $n$ th coefficient of the Dirichlet series  $\zeta(s)^z$ . The case  $F = \text{sym}^1 f$  in (1.10) is commonly known as Deligne's inequality.

According to [3, Section 3.2.1] and [10, Proposition 2.1], the gamma factors of  $L(s, \text{sym}^m f)$  and  $L(s, \text{sym}^m f \times \text{sym}^m f)$  are, respectively,

$$(1.11) \quad L_\infty(s, \text{sym}^m f) := \begin{cases} \prod_{\nu=0}^n \Gamma_{\mathbb{C}}(s + (\nu + \frac{1}{2})(k-1)) & \text{if } m = 2n + 1 \\ \Gamma_{\mathbb{R}}(s + \delta_{2 \nmid n}) \prod_{\nu=1}^n \Gamma_{\mathbb{C}}(s + \nu(k-1)) & \text{if } m = 2n \end{cases}$$

and

$$(1.12) \quad L_\infty(s, \text{sym}^m f \times \text{sym}^m f) = \begin{cases} \Gamma_{\mathbb{C}}(s)^{n+1} \prod_{\nu=1}^m \Gamma_{\mathbb{C}}(s + \nu(k-1))^{m-\nu+1} & \text{if } m = 2n + 1 \\ \Gamma_{\mathbb{R}}(s) \Gamma_{\mathbb{C}}(s)^n \prod_{\nu=1}^m \Gamma_{\mathbb{C}}(s + \nu(k-1))^{m-\nu+1} & \text{if } m = 2n, \end{cases}$$

where  $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2} \Gamma(s/2)$ ,  $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s} \Gamma(s)$  and

$$\delta_{2 \nmid n} = \begin{cases} 1 & \text{if } 2 \nmid n, \\ 0 & \text{otherwise.} \end{cases}$$

For  $F = \text{sym}^m f$  or  $F = \text{sym}^m f \times \text{sym}^m f$  where  $f \in \mathbb{H}_k^*$  and  $m = 1, 2, 3, 4$ , it is known that the function  $\Lambda(s, F) := L_\infty(s, F)L(s, F)$  is entire on  $\mathbb{C}$  and satisfies the functional equation

$$(1.13) \quad \Lambda(s, F) = \varepsilon_F \Lambda(1-s, F)$$

with  $\varepsilon_F = \pm 1$  (see [5, 7, 8, 9] for  $F = \text{sym}^m f$  and [10, 20] for  $F = \text{sym}^m f \times \text{sym}^m f$ ).

For the universality property of  $L(s, F)$ , we have the following result.

**Theorem 1.** *Let  $1 \leq m \leq 4$ ,  $2 \mid k$  such that  $k = 12$  or  $k \geq 16$ ,  $f \in \mathbb{H}_k^*$  and  $F = \text{sym}^m f$  or  $F = \text{sym}^m f \times \text{sym}^m f$ . Define*

$$(1.14) \quad \sigma_F := \begin{cases} 1 - (m+1)^{-1} & \text{if } F = \text{sym}^m f, \\ 1 - (m+1)^{-2} & \text{if } F = \text{sym}^m f \times \text{sym}^m f. \end{cases}$$

Let  $\mathbb{K}$  be a compact subset of  $\mathbb{D}(\sigma_F)$  with connected complement and  $\varphi(s)$  a non-vanishing analytic function in the interior of  $\mathbb{K}$  and continuous on  $\mathbb{K}$ . Then for any  $\varepsilon > 0$ , we have

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ t \in [0, T] : \sup_{s \in \mathbb{K}} |L(s + it, F) - \varphi(s)| < \varepsilon \right\} > 0.$$

**Remark.** (i) The particular case  $F = \text{sym}^1 f = f$  of Theorem 1 was first investigated by Kačėnas & Laurinćikas [6] and established completely by Laurinćikas & Matsumoto [13]. Another particular case  $F = \text{sym}^1 f \times \text{sym}^1 f = f \times f$  was considered by Matsumoto [16] recently.

(ii) Theorem 1 is established only for  $1 \leq m \leq 4$  due to the lack of knowledge about the high symmetric powers.

(iii) The reason why Theorem 1 holds only for  $\mathbb{D}(\sigma_F)$  instead of  $\mathbb{D}(\frac{1}{2})$  is that the estimate

$$\int_1^T |L(s, F)|^2 d\tau \ll_f T \quad (\forall T \geq 1)$$

is only achieved for  $\sigma > \sigma_F$  (see (5.3) below), where  $\sigma_F$  is defined as in Theorem 1. It seems interesting to improve this estimate further so that Theorem 1 can hold for  $\mathbb{D}(\frac{1}{2})$ .

(iv) It is possible to generalize (without too much difficulty) Theorem 1 to the case of the congruence subgroup  $\Gamma_0(N)$  with square-free  $N$ , as what Laurinćikas, Matsumoto & Steuding [15] did for  $L(s, f)$ .

Like [13] and [16], we shall use Bagchi's method to prove Theorem 1. (Interested readers are referred to [11] for an excellent paradigm on Bagchi's method.) One of their main tools is Rankin's asymptotic formula

$$\sum_{p \leq x} |\lambda_f(p)|^2 \sim \frac{x}{\log x}$$

for  $x \rightarrow \infty$  (see [18], theorem 2). However, such a prime number theorem for the symmetric  $m$ th power  $L$ -function with  $m \geq 2$  is not available. In Section 2, we shall establish this result based on [20] and [10], which is clearly of independent interest and may have many other applications.

As in [14], we can deduce the following as simple consequences of Theorem 1.

**Corollary 2.** *Let  $m, k, f, F$  and  $\sigma_F$  be as in Theorem 1. For  $\sigma_F < \sigma < 1$  and any positive integer  $J$ , define a mapping  $\psi : \mathbb{R} \rightarrow \mathbb{C}^J$  by*

$$\psi(\tau) := (L(\sigma + i\tau, F), L'(\sigma + i\tau, F), \dots, L^{(J-1)}(\sigma + i\tau, F)).$$

Then  $\psi(\mathbb{R})$  is dense in  $\mathbb{C}^J$ .

**Corollary 3.** *Let  $m, k, f, F$  and  $\sigma_F$  be as in Theorem 1, and  $J$  be a non negative integer. If the continuous functions  $g_j : \mathbb{C}^J \rightarrow \mathbb{C}$  ( $0 \leq j \leq J$ ) satisfy*

$$\sum_{j=0}^J s^j g_j(L(s, F), L'(s, F), \dots, L^{(J-1)}(s, F)) \equiv 0$$

for all  $s \in \mathbb{C}$ , then  $g_j \equiv 0$  ( $0 \leq j \leq J$ ).

**Acknowledgement.** We began working on this paper in April 2004 during the visit of the second author to Shanghai Jiaotong University, and finished in February 2005 when the first author visited l'Université Henri Poincaré (Nancy 1). We are indebted to both institutions for invitations and support. The second author would like to thank the GRD de Théorie des Nombres au CNRS for support. We would express our sincere gratitude to K. Matsumoto for his kind help in our study of his joint paper with Laurinćikas [13]. Finally the authors wish to thank the referee for pointing out a mistake in the earlier version.

## § 2. The prime number theorem for symmetric power $L$ -functions

Let  $m \in \mathbb{N}$ ,  $2 \mid k$  such that  $k = 12$  or  $k \geq 16$  and  $f \in \mathbf{H}_k^*$ . From (1.6), the product (1.8) is absolutely convergent for  $\sigma > 1$ . Thus we can define  $\Lambda_{\text{sym}^m f \times \text{sym}^m f}(n)$  by the relation

$$(2.1) \quad -\frac{L'}{L}(s, \text{sym}^m f \times \text{sym}^m f) = \sum_{n=1}^{\infty} \frac{\Lambda_{\text{sym}^m f \times \text{sym}^m f}(n)}{n^s}$$

for  $\sigma > 1$ . The aim of this section is to prove the following result.

**Proposition 2.1.** *Let  $1 \leq m \leq 4$ ,  $2 \mid k$  such that  $k = 12$  or  $k \geq 16$  and  $f \in \mathbb{H}_k^*$ . Then for  $x \rightarrow \infty$ , we have*

$$(2.2) \quad \sum_{n \leq x} \Lambda_{\text{sym}^m f \times \text{sym}^m f}(n) \sim x,$$

$$(2.3) \quad \sum_{p \leq x} |\lambda_f(p^m)|^2 \log p \sim x,$$

$$(2.4) \quad \sum_{p \leq x} |\lambda_f(p^m)|^2 \sim \frac{x}{\log x}.$$

This proposition will be referred as the *prime number theorem* for the coefficients of symmetric power  $L$ -functions associated with newforms. It plays a key role in our proof of Theorems 1 & 2, and is of independent interest. The case  $m = 1$  was considered by Rankin [18]. We shall prove this proposition with the non-vanishing property on  $\sigma = 1$  in standard way. To this end, we firstly prove two preliminary lemmas.

Let  $m \in \mathbb{N}$ ,  $2 \mid k$  such that  $k = 12$  or  $k \geq 16$  and  $f \in \mathbb{H}_k^*$ . We define

$$(2.5) \quad \Psi_{f,m}(s) := \prod_p \prod_{0 \leq \ell < m} \left\{ (1 - \alpha_f(p)^{2(m-\ell)} p^{-s}) (1 - \alpha_f(p)^{-2(m-\ell)} p^{-s}) \right\}^{-(\ell+1)}$$

for  $\sigma > 1$  and

$$(2.6) \quad \Lambda_{f,m}(n) = \begin{cases} 2 \sum_{j=1}^m (m+1-j) \cos[2j\theta_f(p)\nu] \log p & \text{if } n = p^\nu, \\ 0 & \text{otherwise,} \end{cases}$$

where  $\alpha_f(p)$  is determined by (1.5)–(1.6) and  $\theta_f(p) \in [0, \pi]$  is chosen such that  $\alpha_f(p) = e^{i\theta_f(p)}$ .

**Lemma 2.1.** *Let  $m \in \mathbb{N}$ ,  $2 \mid k$  such that  $k = 12$  or  $k \geq 16$  and  $f \in \mathbb{H}_k^*$ . Then for  $\sigma > 1$ , we have*

$$(2.7) \quad -\frac{\Psi'_{f,m}(s)}{\Psi_{f,m}(s)} = \sum_{n=1}^{\infty} \frac{\Lambda_{f,m}(n)}{n^s},$$

$$(2.8) \quad \log \Psi_{f,m}(s) = \sum_{n=2}^{\infty} \frac{\Lambda_{f,m}(n)}{n^s \log n}.$$

*Proof.* By the Deligne inequality, the Euler product  $\Psi_{f,m}(s)$  converges absolutely for  $\sigma > 1$ . Taking logarithmic derivative on both sides of (2.5), we have, for  $\sigma > 1$ ,

$$\begin{aligned} -\frac{\Psi'_{f,m}(s)}{\Psi_{f,m}(s)} &= \sum_p \sum_{0 \leq \ell < m} (\ell+1) \left( \frac{\alpha_f(p)^{2(m-\ell)} p^{-s} \log p}{1 - \alpha_f(p)^{2(m-\ell)} p^{-s}} + \frac{\alpha_f(p)^{-2(m-\ell)} p^{-s} \log p}{1 - \alpha_f(p)^{-2(m-\ell)} p^{-s}} \right) \\ &= \sum_p \sum_{\nu \geq 1} \sum_{0 \leq \ell < m} (\ell+1) \frac{[\alpha_f(p)^{2(m-\ell)\nu} + \alpha_f(p)^{-2(m-\ell)\nu}] \log p}{p^{s\nu}}, \end{aligned}$$

which is equivalent to (2.7).

Integrating (2.7) on the half-line  $\{s+t : t \geq 0\}$ , we obtain (2.8). □

**Lemma 2.2.** *Let  $1 \leq m \leq 4$ ,  $2 \mid k$  such that  $k = 12$  or  $k \geq 16$  and  $f \in \mathbb{H}_k^*$ . Then for  $\sigma \geq 1$  and  $s \neq 1$ , we have*

$$L(s, \text{sym}^m f \times \text{sym}^m f) \neq 0.$$

*Proof.* Noticing that

$$\sum_{\substack{0 \leq i, j \leq m \\ i+j=\ell}} 1 = \begin{cases} \ell + 1 & \text{if } 0 \leq \ell \leq m \\ 2m - \ell + 1 & \text{if } m < \ell \leq 2m \end{cases}$$

and using (1.8), we can write, for  $\sigma > 1$ ,

$$(2.9) \quad L(s, \text{sym}^m f \times \text{sym}^m f) = \zeta(s)^{m+1} \Psi_{f,m}(s).$$

As usual we denote by  $\Lambda(n)$  von Mangoldt's function. Following from (2.6), (2.7), (2.9) and the classical relations

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \quad \log \zeta(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s \log n} \quad (\sigma > 1),$$

we infer that

$$(2.10) \quad \Lambda_{\text{sym}^m f \times \text{sym}^m f}(n) = (m+1)\Lambda(n) + \Lambda_{f,m}(n) \quad (n \geq 1)$$

and

$$(2.11) \quad \log L(s, \text{sym}^m f \times \text{sym}^m f) = \sum_{n=2}^{\infty} \frac{\Lambda_{\text{sym}^m f \times \text{sym}^m f}(n)}{n^s \log n} \quad (\sigma > 1).$$

Next we calculate  $\Lambda_{\text{sym}^m f \times \text{sym}^m f}(p^\nu)$ . Write  $\vartheta_\nu = \theta_f(p)\nu$  for notational convenience, we get

$$\begin{aligned} \Lambda_{f,m}(p^\nu)(\log p)^{-1} &= 2 \sum_{1 \leq \ell \leq m} \sum_{1 \leq j \leq m-\ell+1} \cos(\ell 2\vartheta_\nu) \\ &= 2 \sum_{1 \leq j \leq m} \sum_{1 \leq \ell \leq m-j+1} \cos(\ell 2\vartheta_\nu), \end{aligned}$$

by (2.6). On the other hand, we have

$$\begin{aligned} \sum_{1 \leq \ell \leq m-j+1} \cos(\ell 2\vartheta_\nu) &= \Re e \left( \sum_{1 \leq \ell \leq m-j+1} e^{i\ell 2\vartheta_\nu} \right) \\ &= \Re e \left( \frac{e^{i(m-j+2)2\vartheta_\nu} - e^{i2\vartheta_\nu}}{e^{i2\vartheta_\nu} - 1} \right) \\ &= \Re e \left( e^{i(m-j+2)\vartheta_\nu} \frac{e^{i(m-j+1)\vartheta_\nu} - e^{-i(m-j+1)\vartheta_\nu}}{e^{i\vartheta_\nu} - e^{-i\vartheta_\nu}} \right) \\ &= \frac{\cos[(m-j+2)\vartheta_\nu] \sin[(m-j+1)\vartheta_\nu]}{\sin \vartheta_\nu}. \end{aligned}$$

Inserting it into the preceding formula and applying the identity

$$2 \cos \alpha \sin \beta = \sin(\alpha + \beta) - \sin(\alpha - \beta),$$

it follows that

$$\begin{aligned}\Lambda_{f,m}(p^\nu)(\log p)^{-1} &= \frac{1}{\sin \vartheta_\nu} \sum_{1 \leq j \leq m} (\sin\{[2(m-j+1)+1]\vartheta_\nu\} - \sin \vartheta_\nu) \\ &= \frac{1}{\sin \vartheta_\nu} \sum_{1 \leq \ell \leq m} \sin[(2\ell+1)\vartheta_\nu] - m.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\sum_{1 \leq \ell \leq m} \sin[(2\ell+1)\vartheta_\nu] &= \Im m \left( \sum_{1 \leq \ell \leq m} e^{i(2\ell+1)\vartheta_\nu} \right) \\ &= \Im m \left( e^{i\vartheta_\nu} \frac{e^{i(m+1)2\vartheta_\nu} - e^{i2\vartheta_\nu}}{e^{i2\vartheta_\nu} - 1} \right) \\ &= \Im m \left( e^{i(m+2)\vartheta_\nu} \frac{e^{im\vartheta_\nu} - e^{-im\vartheta_\nu}}{e^{i\vartheta_\nu} - e^{-i\vartheta_\nu}} \right) \\ &= \frac{\sin[(m+2)\vartheta_\nu] \sin(m\vartheta_\nu)}{\sin \vartheta_\nu} \\ &= \frac{\sin^2[(m+1)\vartheta_\nu] - \sin^2 \vartheta_\nu}{\sin \vartheta_\nu}.\end{aligned}$$

Combining this with the previous relation, we deduce that

$$\Lambda_{f,m}(p^\nu)(\log p)^{-1} = \left( \frac{\sin[(m+1)\vartheta_\nu]}{\sin \vartheta_\nu} \right)^2 - m - 1,$$

which implies, via (2.10),

$$(2.12) \quad \Lambda_{\text{sym}^m f \times \text{sym}^m f}(p^\nu) = \left( \frac{\sin[(m+1)\vartheta_\nu]}{\sin \vartheta_\nu} \right)^2 \log p.$$

In particular, we obtain with (1.5) that

$$\begin{aligned}(2.13) \quad \Lambda_{\text{sym}^m f \times \text{sym}^m f}(p) &= \left( \frac{\sin[(m+1)\theta_f(p)]}{\sin \theta_f(p)} \right)^2 \log p \\ &= \left( \sum_{0 \leq j \leq m} e^{i(m-2j)\theta_f(p)} \right)^2 \log p \\ &= |\lambda_f(p^m)|^2 \log p.\end{aligned}$$

Now we are ready to prove Lemma 2.2. Suppose  $L(s, \text{sym}^m f \times \text{sym}^m f)$  has a zero at  $1 + i\tau_0$  of order  $\ell \geq 1$ , where  $\tau_0 \neq 0$ . Consider the function

$$g(s) := L(s, \text{sym}^m f \times \text{sym}^m f)^3 L(s + i\tau_0, \text{sym}^m f \times \text{sym}^m f)^4 L(s + i2\tau_0, \text{sym}^m f \times \text{sym}^m f)^2.$$

Since  $L(s, \text{sym}^m f \times \text{sym}^m f)$  is holomorphic except for a simple pole at  $s = 1$ ,  $g(s)$  is holomorphic for  $\sigma \geq 1$  and the zero at  $s = 1$  is of order  $\geq 4\ell - 3 \geq 1$ .

But from (2.11), we have for  $\sigma > 1$ ,

$$\log g(s) = \sum_{n \geq 2} \frac{\Lambda_{\text{sym}^m f \times \text{sym}^m f}(n)}{n^s \log n} (3 + 4n^{-i\tau_0} + 2n^{-i2\tau_0}).$$

Together with (2.6), (2.10) and (2.12), we deduce, for  $\sigma > 1$ ,

$$\begin{aligned} \log |g(\sigma)| &= \sum_{n \geq 2} \frac{\Lambda_{\text{sym}^m f \times \text{sym}^m f}(n)}{n^\sigma \log n} (3 + 4 \cos(\tau_0 \log n) + 2 \cos(2\tau_0 \log n)) \\ &= \sum_{n \geq 2} \frac{\Lambda_{\text{sym}^m f \times \text{sym}^m f}(n)}{n^\sigma \log n} (1 + 2 \cos(\tau_0 \log n))^2 \\ &\geq 0. \end{aligned}$$

Thus  $|g(\sigma)| \geq 1$  for  $\sigma > 1$ , and  $g$  cannot have a zero of order  $4\ell - 3 (\geq 1)$  at  $\sigma = 1$ . This contradiction completes our proof.  $\square$

Next we shall apply Theorem II.7.11 of [22] to prove Proposition 2.1. Define

$$G(s) := -\frac{L'}{L}(s+1, \text{sym}^m f \times \text{sym}^m f) \frac{1}{s+1} - \frac{1}{s}.$$

Since  $\Lambda(s, \text{sym}^m f \times \text{sym}^m f)$  is holomorphic except for simple poles at  $s = 0, 1$ , the function  $G(s)$  is analytically continued to a meromorphic function on  $\mathbb{C}$ . By Lemma 2.2, we have

$$L(1 + i\tau, \text{sym}^m f \times \text{sym}^m f) \neq 0.$$

Thus  $G(s)$  is holomorphic in an open set containing the half-plane  $\sigma \geq 0$ . In particular we have

$$|G(2\sigma + i\tau) - G(\sigma + i\tau)| \leq \sigma \sup_{0 \leq \theta \leq 1, |\tau| \leq T} |G'(\theta + i\tau)|$$

for  $T > 0$ ,  $0 \leq \sigma \leq \frac{1}{2}$  and  $|\tau| \leq T$ . From this we deduce

$$\int_{-T}^T |G(2\sigma + i\tau) - G(\sigma + i\tau)| d\tau = o(1) \quad (\sigma \rightarrow 0+)$$

for each fixed  $T > 0$ . Now Theorem II.7.11 of [22] is applied with  $F = -L'/L$ ,  $a = c = 1$  and  $w = 0$  to yield the asymptotic formula (2.2).

From (2.13), we can write

$$\sum_{n \leq x} \Lambda_{\text{sym}^m f \times \text{sym}^m f}(n) = \sum_{p \leq x} |\lambda_f(p^m)|^2 \log p + R,$$

where we have, via (2.6) and (2.10),

$$\begin{aligned} R &:= \sum_{p^\nu \leq x, \nu \geq 2} \Lambda_{\text{sym}^m f \times \text{sym}^m f}(p^\nu) \\ &\leq \sum_{p \leq x^{1/2}} \sum_{\nu \leq \log x / \log p} (m+1)^2 \log p \\ &\leq (m+1)^2 \sum_{p \leq x^{1/2}} \log x \\ &\ll_m x^{1/2}. \end{aligned}$$

Thus (2.2) implies (2.3). Finally (2.4) follows from (2.3) by integration by parts.  $\square$

§ 3. Bagchi's method and proof of Theorem 1

In this section, we present Bagchi's method in our case and first formulate it as three propositions. At the end of this section, we shall apply Propositions 3.2 and 3.3 to prove Theorem 1. The proof of these three propositions will be given in sections 4, 5 and 6, respectively.

Let  $1 \leq m \leq 4$ ,  $2 \mid k$  such that  $k = 12$  or  $k \geq 16$ ,  $f \in H_k^*$ ,  $F = \text{sym}^m f$  or  $\text{sym}^m f \times \text{sym}^m f$ . Let  $\sigma_F$  and  $\mathbb{D}(\sigma_F)$  be defined as in (1.14) and (1.1), respectively. Denote by  $H_F$  the space of analytic functions on  $\mathbb{D}(\sigma_F)$  equipped with the topology of uniform convergence on compact subsets of  $\mathbb{D}(\sigma_F)$ .

Let  $\gamma := \{s \in \mathbb{C} : |s| = 1\}$  be the unit torus and

$$\Omega := \prod_p \gamma_p,$$

where  $\gamma_p = \gamma$  for all prime numbers  $p$ . With the product topology and componentwise multiplication,  $\Omega$  is a compact abelian topological group. Hence there is a unique probability Haar measure  $\mu_h$  on  $(\Omega, \mathcal{B}(\Omega))$  <sup>(3)</sup> and we have  $\mu_h = \prod_p \mu_{h,p}$ , where  $\mu_{h,p}$  is the Haar measure on  $(\gamma_p, \mathcal{B}(\gamma_p))$  (see [19], Theorem 5.14). For every  $\omega = \{\omega_p\} \in \Omega$ , we extend it to a completely multiplicative function, by defining

$$\omega_n := \prod_{p^\nu \parallel n} \omega_p^\nu.$$

In view of (1.10), we can prove, similar to Lemma 5.1.6 and Theorem 5.1.7 of [11], that there is a subset  $\tilde{\Omega} \subset \Omega$  with  $\mu_h(\tilde{\Omega}) = 1$  such that for any  $\tilde{\omega} \in \tilde{\Omega}$  the series

$$\sum_{n \geq 1} \tilde{\omega}_n \lambda_F(n) n^{-s}$$

and the product

$$\prod_p \sum_{\nu \geq 0} \tilde{\omega}_p^\nu \lambda_F(p^\nu) p^{-\nu s}$$

are uniformly convergent on compact subsets of the half-plane  $\sigma > \frac{1}{2}$ , and the equality

$$(3.1) \quad L(s, F; \tilde{\omega}) := \sum_{n \geq 1} \tilde{\omega}_n \lambda_F(n) n^{-s} = \prod_p \sum_{\nu \geq 0} \tilde{\omega}_p^\nu \lambda_F(p^\nu) p^{-\nu s}$$

holds. Clearly for  $\sigma > \frac{1}{2}$  and  $\tilde{\omega} \in \tilde{\Omega}$ , we have

$$L(s, F; \tilde{\omega}) = \prod_p (1 + \tilde{\omega}_p \lambda_F(p) p^{-s} + O(p^{-2\sigma})).$$

Therefore for any  $\tilde{\omega} \in \tilde{\Omega}$ , the series

$$(3.2) \quad L^\dagger(s, F; \tilde{\omega}) := - \sum_p \log(1 - \tilde{\omega}_p \lambda_F(p) p^{-s})$$

$$(3.3) \quad L_{p_0}^b(s, F; \tilde{\omega}) := - \sum_{p > p_0} \tilde{\omega}_p \log(1 - \lambda_F(p) p^{-s})$$

are uniformly convergent on compact subsets of the half-plane  $\sigma > \frac{1}{2}$ , where  $p_0 \geq 3$  is an arbitrarily fixed constant. Moreover, we introduce two subsets of  $H_F$ :

$$(3.4) \quad \mathcal{L}_F^\dagger := \{L^\dagger(s, F; \tilde{\omega}) : \tilde{\omega} \in \tilde{\Omega}\} \quad \text{and} \quad \mathcal{L}_{F, p_0}^b := \{L_{p_0}^b(s, F; \tilde{\omega}) : \tilde{\omega} \in \tilde{\Omega}\}.$$

The first auxiliary result of Bagchi's method is the denseness of  $\mathcal{L}_F^\dagger$ , which is important in the proof of Proposition 3.3 below.

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<sup>(3)</sup> For any space  $X$ , we denote by  $\mathcal{B}(X)$  the class of all Borel subsets of  $X$ .

**Proposition 3.1.** *Let  $1 \leq m \leq 4$ ,  $2 \mid k$  such that  $k = 12$  or  $k \geq 16$ ,  $f \in \mathbb{H}_k^*$  and  $F = \text{sym}^m f$  or  $\text{sym}^m f \times \text{sym}^m f$ .*

- (i) *For any fixed  $p_0 \geq 3$ , the set  $\mathcal{L}_{F,p_0}^b$  is dense in  $H_F$ .*
- (ii) *The set  $\mathcal{L}_F^\dagger$  is dense in  $H_F$ .*

Define three probability measures on  $(H_F, \mathcal{B}(H_F))$

$$(3.5) \quad P_{F,T}(A) := \frac{1}{T} \text{meas}\{t \in [0, T] : L(s + it, F) \in A\},$$

$$(3.6) \quad P_F(A) := \mu_h(\{\omega \in \Omega : L(s, F; \omega) \in A\}),$$

$$(3.7) \quad Q_F(A) := \mu_h(\{\omega \in \Omega : \log L(s, F; \omega) \in A\}),$$

for  $A \in \mathcal{B}(H_F)$ . The next limit theorem is one of the keys of Bagchi's method.

**Proposition 3.2.** *Let  $1 \leq m \leq 4$ ,  $2 \mid k$  such that  $k = 12$  or  $k \geq 16$ ,  $f \in \mathbb{H}_k^*$  and  $F = \text{sym}^m f$  or  $\text{sym}^m f \times \text{sym}^m f$ . Then the probability measure  $P_{F,T}$  converges weakly to  $P_F$  as  $T \rightarrow \infty$ .*

The third key step of Bagchi's method is to determine the support of the probability measure  $P_F$  on  $(H_F, \mathcal{B}(H_F))$ . By definition, a point  $s \in S$  is said to be in the support of a probability measure  $P$  on  $(S, \mathcal{B}(S))$  iff every open neighborhood of  $s$  has strictly positive measure. The set of all such points is called the support of  $P$ , denoted by  $S(P)$ . Clearly  $S(P)$  is the smallest closed subset of  $S$  such that  $P(S(P)) = 1$  (see [2], Chapter 1). The support of a  $S$ -valued random variable  $Y$  on the probability space  $(X, \mathcal{B}(X), \mu)$  is the support of the probability measure  $P_Y$  on  $(S, \mathcal{B}(S))$  where  $P_Y(A) = \mu(Y \in A)$  ( $A \in \mathcal{B}(S)$ ), called the distribution of  $Y$ .

**Proposition 3.3.** *Let  $1 \leq m \leq 4$ ,  $2 \mid k$  such that  $k = 12$  or  $k \geq 16$ ,  $f \in \mathbb{H}_k^*$  and  $F = \text{sym}^m f$  or  $\text{sym}^m f \times \text{sym}^m f$ . With the previous notation, we have the following results:*

- (i) *The support of the probability measure  $Q_F$  on  $(H_F, \mathcal{B}(H_F))$  is the whole space  $H_F$ .*
- (ii) *The support of the probability measure  $P_F$  on  $(H_F, \mathcal{B}(H_F))$  is*

$$S_0 := \{\varphi(s) \in H_F : \varphi(s) \neq 0 \text{ for any } s \in \mathbb{D}(\sigma_F) \text{ or } \varphi(s) \equiv 0\}.$$

Now we apply Propositions 3.1, 3.2 and 3.3 to prove Theorem 1.

Let  $\mathbb{K}$  be a compact subset of  $\mathbb{D}_\infty(\sigma_F)$  with connected complement. Let  $\varphi(s)$  be a non-vanishing continuous functions on  $\mathbb{K}$  which is analytic in the interior of  $\mathbb{K}$ . By Lemma 11 of [13], for any  $\varepsilon > 0$  we can find a polynomial  $p(s)$  such that  $p(s) \neq 0$  on  $\mathbb{K}$  and

$$(3.8) \quad \sup_{s \in \mathbb{K}} |\varphi(s) - p(s)| < \frac{1}{4}\varepsilon.$$

Since  $p(s)$  has only finitely many zeros, we can find a region  $G_1$  such that  $\mathbb{K} \subset G_1$  and  $p(s) \neq 0$  on  $G_1$ . We choose  $\log p(s)$  to be analytic in the interior of  $G_1$ . Applying Lemma 11 of [13] to  $\log p(s)$  again, we find another polynomial  $q(s)$  such that

$$(3.9) \quad \sup_{s \in \mathbb{K}} |p(s) - e^{q(s)}| < \frac{1}{4}\varepsilon.$$

From (3.8) and (3.9), we deduce, for any  $T > 0$ ,

$$(3.10) \quad \left\{t \in [0, T] : \sup_{s \in \mathbb{K}} |L(s + it, F) - e^{q(s)}| < \frac{\varepsilon}{2}\right\} \subset \left\{t \in [0, T] : \sup_{s \in \mathbb{K}} |L(s + it, F) - \varphi(s)| < \varepsilon\right\}.$$

On the other hand, the set

$$G := \left\{ g \in H_F : \sup_{s \in \mathbb{K}} |g(s) - e^{q(s)}| < \frac{1}{2}\varepsilon \right\}$$

belongs to  $G \in \mathcal{B}(H_F)$  and is open in  $H_F$ , thus we have

$$(3.11) \quad \begin{aligned} P_{F,T}(G) &= \frac{1}{T} \text{meas} \{ t \in [0, T] : L(s + it, F) \in G \} \\ &= \frac{1}{T} \text{meas} \left\{ t \in [0, T] : \sup_{s \in \mathbb{K}} |L(s + it, F) - e^{q(s)}| < \frac{1}{2}\varepsilon \right\}. \end{aligned}$$

By Proposition 3.1, the measure  $P_{F,T}(G)$  converges weakly to  $P_F(G)$  as  $T \rightarrow \infty$ . With (3.10) and (3.11), Theorem 1.1.8 of [11] leads to

$$\begin{aligned} &\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ t \in [0, T] : \sup_{s \in \mathbb{K}} |L(s + it, F) - \varphi(s)| < \varepsilon \right\} \\ &\geq \liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ t \in [0, T] : \sup_{s \in \mathbb{K}} |L(s + it, F) - e^{q(s)}| < \frac{1}{2}\varepsilon \right\} \\ &\geq P_F(G). \end{aligned}$$

Obviously  $e^{q(s)} \in S_0 = S(P_F)$  and  $G$  is a neighbourhood of  $e^{q(s)}$ . Therefore  $P_F(G) > 0$ . This completes the proof of Theorem 1.

#### § 4. Proof of Proposition 3.1

In order to prove Proposition 3.1, we first apply our result in Section 2 to establish a preliminary lemma, which is a generalization of the key lemma in Laurinćikas & Matsumoto [13].

**Lemma 4.1.** *Let  $1 \leq m \leq 4$ ,  $2 \mid k$  such that  $k = 12$  or  $k \geq 16$  and  $f \in H_k^*$ . For every  $\delta \in [0, 1)$ , define*

$$\mathcal{P}_\delta = \mathcal{P}_\delta(\text{sym}^m f) := \{ p : p \text{ is prime such that } |\lambda_f(p^m)| \geq \delta \}.$$

Let  $\eta > 0$  and  $c > 1 + \eta$  be two fixed constants. For any  $a \geq 2$  and  $(1 + \eta)a < b \leq ca$ , we have

$$\sum_{\substack{p \in \mathcal{P}_\delta \\ a < p \leq b}} \frac{1}{p} \geq \left\{ \frac{1 - \delta^2}{(m + 1)^2 - \delta^2} + o_{c,\delta,\eta}(1) \right\} \sum_{a(1+\eta) < p \leq b} \frac{1}{p},$$

where  $o_{c,\delta,\eta}(1)$  is a quantity tending towards 0 as  $a \rightarrow \infty$ .

*Proof.* Define

$$\pi_\delta(x) := \#\{ p \leq x : p \in \mathcal{P}_\delta \}.$$

In particular we have  $\mathcal{P}_0 = \mathcal{P}$  (the set of all prime numbers) and  $\pi_0(x) = \pi(x) := \#\{[1, x] \cap \mathcal{P}\}$ .

Clearly it is sufficient to prove that for any  $(1 + \eta)a < u \leq b$ ,

$$(4.1) \quad \pi_\delta(u) - \pi_\delta(a) \geq \left\{ \frac{1 - \delta^2}{(m + 1)^2 - \delta^2} + o_{c,\delta,\eta}(1) \right\} (\pi(u) - \pi(a)),$$

since the desired inequality follows from (4.1) via a simple integration by parts.

For  $a \leq u \leq b$ , the Deligne inequality  $|\lambda_f(p^m)| \leq m + 1$  allows us to write

$$(4.2) \quad \begin{aligned} \sum_{a < p \leq u} |\lambda_f(p^m)|^2 &\leq (m+1)^2 \sum_{a < p \leq u, p \in \mathcal{P}_\delta} 1 + \delta^2 \sum_{a < p \leq u, p \notin \mathcal{P}_\delta} 1 \\ &\leq [(m+1)^2 - \delta^2] [\pi_\delta(u) - \pi_\delta(a)] + \delta^2 [\pi(u) - \pi(a)]. \end{aligned}$$

According to (2.4) of Proposition 2.1, we have

$$\begin{aligned} \sum_{a < p \leq u} |\lambda_f(p^m)|^2 &= \pi(u)\{1 + o(1)\} - \pi(a)\{1 + o(1)\} \\ &= \pi(u) - \pi(a) + o(\pi(u)). \end{aligned}$$

Since  $(1 + \eta)a < u \leq ca$ , a simple calculation shows, via the prime number theorem, that

$$\begin{aligned} \pi(u) - \pi(a) &= \frac{u}{\log u} \{1 + o(1)\} - \frac{a}{\log a} \{1 + o(1)\} \\ &\geq \frac{\eta a}{\log a} \{1 + o_{c,\eta}(1)\} \\ &\geq \frac{\eta}{2c} \pi(u). \end{aligned}$$

Combining these two estimates yields

$$(4.3) \quad \sum_{a < p \leq u} |\lambda_f(p^m)|^2 = [\pi(u) - \pi(a)] \{1 + o_{c,\eta}(1)\}.$$

Now the desired inequality (4.1) follows from (4.2) and (4.3).  $\square$

Now we are ready to prove Proposition 3.1.

*Proof of Proposition 3.1.* Fix a  $\tilde{\omega}^0 = \{\tilde{\omega}_p^0\} \in \tilde{\Omega}$ , then the series

$$(4.4) \quad L_{p_0}^b(s, F; \tilde{\omega}^0) := - \sum_{p > p_0} \tilde{\omega}_p^0 \log(1 - \lambda_F(p)p^{-s})$$

converges in  $H_F$ . To prove assertion (i), we shall apply Lemma 4 of [13] to this series. In fact, it suffices to verify the condition (a) there, since conditions (b) and (c) are plainly satisfied.

Let  $\mu$  be a complex measure on  $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$  with compact support in  $\mathbb{D}(\sigma_F)$  such that

$$(4.5) \quad \sum_p \left| \int_{\mathbb{C}} \log(1 - \lambda_F(p)p^{-s}) d\mu(s) \right| < \infty.$$

Since  $\sigma_F > \frac{1}{2}$ , we see easily

$$(4.6) \quad \sum_p |\lambda_F(p)| |\rho(\log p)| < \infty$$

with

$$\rho(z) := \int_{\mathbb{C}} e^{-sz} d\mu(s).$$

We shall prove that (4.6) leads to

$$(4.7) \quad \rho(z) \equiv 0,$$

which implies the validity of condition (a) in Lemma 4 of [13] since for any non-negative integer  $r$ ,  $\int_{\mathbb{C}} s^r d\mu(s) = 0$  by differentiating (4.7)  $r$ -times with respect to  $z$  and taking  $z = 0$ . Noticing that

$$L_{p_0}^b(s, F; \tilde{\omega}) = L_{p_0}^b(s, F; (\tilde{\omega}/\tilde{\omega}^0)\tilde{\omega}^0) \quad \text{and} \quad \tilde{\omega}/\tilde{\omega}^0 = \{\tilde{\omega}_p/\tilde{\omega}_p^0\} \in \Omega,$$

Lemma 4 of [13] shows that  $\mathcal{L}_{F, p_0}^b$  is dense in  $H_F$ .

It remains to prove (4.7). Firstly we write

$$\rho(z) = \int_{\mathbb{C}} e^{zs} d\mu_-(s),$$

where the measure  $\mu_-$  is defined by  $\mu_-(A) = \mu(-A)$  for  $A \in \mathcal{B}(\mathbb{C})$  with  $-A := \{-a : a \in A\}$ . Clearly  $\mu_-$  supports in  $\{s \in \mathbb{C} : -1 < \sigma < -\frac{1}{2}\}$ . Thus  $\rho(z)$  verifies all conditions of Lemma 5 of [13]. If  $\rho(z) \not\equiv 0$ , then this lemma implies

$$(4.8) \quad \limsup_{r \rightarrow \infty} \frac{\log |\rho(r)|}{r} > -1.$$

Next we shall apply Lemma 7 of [13] to deduce an opposite inequality. This follows a contradiction, and hence (4.7) holds true.

Since the support of  $\mu$  is compact and is contained in  $\mathbb{D}(\sigma_F)$ , we have

$$|\rho(\pm iy)| \leq e^{My} \int_{\mathbb{C}} |d\mu(s)| \quad (y > 0),$$

where  $M = M_\mu$  is a positive constant such that  $\mu$  supports in  $(\sigma_F, 1) \times [-M, M]$ . Thus  $\rho(z)$  satisfies condition (a) of Lemma 7 of [13] with  $\alpha = M$ . Fix a positive number  $\beta < \pi/M$ , which assures condition (d) of Lemma 7 of [13].

A similar calculation to (2.13) allows us to obtain

$$(4.9) \quad \lambda_F(p) = \begin{cases} \lambda_f(p^m) & \text{if } F = \text{sym}^m f, \\ |\lambda_f(p^m)|^2 & \text{if } F = \text{sym}^m f \times \text{sym}^m f. \end{cases}$$

Define

$$\mathcal{L} := \{\ell \in \mathbb{N} : \exists r \in ((\ell - \frac{1}{4})\beta, (\ell + \frac{1}{4})\beta] \text{ such that } |\rho(r)| \leq e^{-r}\}.$$

By using (4.6) and (4.9), we can deduce, for any fixed  $\delta \in [0, 1)$ ,

$$\begin{aligned} \infty &> \sum_p |\lambda_F(p)| |\rho(\log p)| \geq \delta^2 \sum_{p \in \mathcal{P}_\delta} |\rho(\log p)| \\ &\geq \delta^2 \sum_{\ell \notin \mathcal{L}} \sum_{\substack{p \in \mathcal{P}_\delta \\ (\ell - \frac{1}{4})\beta < \log p \leq (\ell + \frac{1}{4})\beta}} |\rho(\log p)| \\ &\geq \delta^2 \sum_{\ell \notin \mathcal{L}} \sum_{\substack{p \in \mathcal{P}_\delta \\ (\ell - \frac{1}{4})\beta < \log p \leq (\ell + \frac{1}{4})\beta}} p^{-1}. \end{aligned}$$

Now we apply Lemma 4.1 with

$$a := \exp\{(\ell - \frac{1}{4})\beta\}, \quad b := \exp\{(\ell + \frac{1}{4})\beta\}, \quad c := \exp\{\frac{1}{2}\beta\}$$

and  $\eta > 0$  such that  $1 + \eta < c$ . It follows that

$$\begin{aligned} \infty &> \left\{ \frac{\delta^2(1-\delta^2)}{(m+1)^2-\delta^2} + o_{c,\delta,\eta}(1) \right\} \sum_{\ell \notin \mathcal{L}} \sum_{(1+\eta)a < p \leq b} \frac{1}{p} \\ &\geq \left\{ \frac{\delta^2(1-\delta^2)}{(m+1)^2-\delta^2} + o_{c,\delta,\eta}(1) \right\} \sum_{\ell \notin \mathcal{L}} \left\{ \left( \frac{1}{2} - \frac{\log(1+\eta)}{\beta} \right) \frac{1}{\ell} + O\left(\frac{1}{\ell^2}\right) \right\}, \end{aligned}$$

which implies

$$(4.10) \quad \sum_{\ell \notin \mathcal{L}} \frac{1}{\ell} < \infty.$$

If we write  $\mathcal{L} = \{a_1, a_2, \dots\}$  with  $a_1 < a_2 < \dots$ , it is easy to see that

$$(4.11) \quad \lim_{n \rightarrow \infty} \frac{a_n}{n} = 1.$$

In fact we have  $[x] = |\mathcal{L} \cap [1, x]| + |(\mathbb{N} \setminus \mathcal{L}) \cap [1, x]|$ . But (4.10) implies

$$|\mathbb{N} \setminus \mathcal{L} \cap [1, x]| \leq \sqrt{x} + \sum_{\sqrt{x} < \ell \leq x, \ell \notin \mathcal{L}} 1 \leq \sqrt{x} + x \sum_{\ell > \sqrt{x}, \ell \notin \mathcal{L}} 1/\ell = o(x).$$

Thus  $|\mathcal{L} \cap [1, x]| \sim x$ , which is equivalent to (4.11).

By the definition of  $\mathcal{L}$ , there exists a sequence  $\{r_n\}$  such that

$$(a_n - \frac{1}{4})\beta < r_n \leq (a_n + \frac{1}{4})\beta \quad \text{and} \quad |\rho(r_n)| \leq e^{-r_n}.$$

Then

$$\lim_{n \rightarrow \infty} \frac{r_n}{n} = \beta \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{\log |\rho(r_n)|}{r_n} \leq -1.$$

This shows that condition (c) of Lemma 7 of [13] is satisfied.

For any integers  $m$  and  $n$  such that  $m > n \geq 1$ , we have

$$r_m - r_n \geq (a_m - a_n - \frac{1}{2})\beta \geq \frac{1}{2}(a_m - a_n)\beta \geq \frac{1}{2}(m - n)\beta.$$

Thus the condition (b) of Lemma 7 of [13] is also satisfied.

Now we can apply Lemma 7 of [13] to write

$$\limsup_{r \rightarrow \infty} \frac{\log |\rho(r)|}{r} = \limsup_{n \rightarrow \infty} \frac{\log |\rho(r_n)|}{r_n} \leq -1.$$

This contradicts to (4.8), and the proof of assertion (i) completes.

Next we shall use the result in assertion (i) to prove (ii). Let  $\mathbb{K}$  be a compact subset of  $\mathbb{D}(\sigma_F)$  and  $\varphi \in H_F$ . For any  $\varepsilon > 0$ , we take  $p_0 \geq 3$  such that

$$(4.12) \quad \sup_{s \in \mathbb{K}} \sum_{p > p_0} \sum_{\nu \geq 2} \frac{|\lambda_F(p)|^\nu}{\nu p^{\nu\sigma}} < \frac{\varepsilon}{2}.$$

Since

$$\varphi(s) + \sum_{p \leq p_0} \log(1 - \lambda_F(p)p^{-s}) \in H_F,$$

assertion (i) shows that there is  $\tilde{\omega} = \{\tilde{\omega}_p\} \in \Omega$  such that

$$(4.13) \quad \sup_{s \in \mathbb{K}} \left| \varphi(s) + \sum_{p \leq p_0} \log(1 - \lambda_F(p)p^{-s}) - L_{p_0}^b(s, F; \tilde{\omega}) \right| < \frac{\varepsilon}{2}.$$

Taking

$$\tilde{\omega}'_p := \begin{cases} 1 & \text{if } p \leq p_0 \\ \tilde{\omega}_p & \text{if } p > p_0 \end{cases} \quad \text{and} \quad \tilde{\omega}' = \{\tilde{\omega}'_p\},$$

the inequalities (4.12) and (4.13) imply

$$\begin{aligned} \sup_{s \in \mathbb{K}} |\varphi(s) - L^\dagger(s, F; \tilde{\omega}')| &\leq \sup_{s \in \mathbb{K}} \left| \varphi(s) + \sum_{p \leq p_0} \log(1 - \lambda_F(p)p^{-s}) - L_{p_0}^b(s, F; \tilde{\omega}) \right| \\ &\quad + \sup_{s \in \mathbb{K}} \sum_{p > p_0} \left| \log(1 - \tilde{\omega}_p \lambda_F(p)p^{-s}) - \tilde{\omega}_p \log(1 - \lambda_F(p)p^{-s}) \right| \\ &< \frac{\varepsilon}{2} + \sup_{s \in \mathbb{K}} \sum_{p > p_0} \sum_{\nu \geq 2} \frac{|\lambda_F(p)|^\nu}{\nu p^{\nu\sigma}} \\ &< \varepsilon. \end{aligned}$$

This completes the proof.  $\square$

### § 5. Proof of Proposition 3.2

Obviously Proposition 3.2 is a particular case of Theorem 2 of [12]. Thus it suffices to verify all assumptions there, that is, to show that there is a positive constant  $c$  for which

$$(5.1) \quad L(s, F) \ll_f |\tau|^c \quad (\sigma > \sigma_F, |\tau| \geq 1),$$

and

$$(5.2) \quad \int_1^T |L(s, F)|^2 d\tau \ll_f T \quad (\sigma > \sigma_F, T \geq 1).$$

By using (1.11), (1.12) and (1.13), a standard Phragmén-Lindelöf argument allows us to obtain the convex bound for  $L(s, F)$ , i.e. (5.1) with  $c = (m + 1 + \delta_{2|m})/4$  if  $F = \text{sym}^m f$  and  $c = (m + 1)^2/4$  if  $F = \text{sym}^m f \times \text{sym}^m f$ . A detailed proof can be found in [10].

In order to verify (5.2), we can apply theorem 4 of Perelli [17], where an estimate of this type was established for a general class of  $L$ -functions. In view of (1.11) and (1.12), it is easy to see that  $L(s, F)$  lies in the class considered in Perelli [17] with evident choice of parameters. Therefore Theorem 4 of Perelli [17] gives

$$(5.3) \quad \int_1^T |L(s, F)|^2 d\tau \ll_{f, \varepsilon} T^{(1-\sigma)/(1-\sigma_F)+\varepsilon}$$

uniformly for  $\frac{1}{2} \leq \sigma < 1$  and  $T \geq 1$ , which implies (5.2). This completes the proof.  $\square$

### § 6. Proof of Proposition 3.3

By the definition,  $\{\omega_p\}$  is a sequence of independent random variables defined on the probability space  $(\Omega, \mathcal{B}(\Omega), \mu_h)$ , and the support of each  $\omega_p$  is the unit circle  $\gamma$ . Hence

$$\left\{ \log \left( \sum_{\nu \geq 0} \omega_p^\nu \lambda_F(p^\nu) p^{-\nu s} \right) \right\}$$

is a sequence of independent  $H_F$ -valued random elements, and the set

$$\left\{ \varphi \in H_F : \varphi(s) = \log \left( \sum_{\nu \geq 0} a^\nu \lambda_F(p^\nu) p^{-\nu s} \right), a \in \gamma \right\}$$

is the support of  $H_F$ -valued random element  $\log \left( \sum_{\nu \geq 0} \omega_p^\nu \lambda_F(p^\nu) p^{-\nu s} \right)$ . Consequently, by theorem 1.7.10 of [11] (see also [13], lemma 10), the support of the  $H_F$ -valued random element

$$\log L(s, F; \omega) = \sum_p \log \left( \sum_{\nu \geq 0} \omega_p^\nu \lambda_F(p^\nu) p^{-\nu s} \right)$$

is the closure of  $\mathcal{L}_F^\dagger$ , i.e. the whole space  $H_F$  by Proposition 3.1(ii). This proves the first assertion.

Now we consider any element  $\varphi = \varphi(s)$  of  $S_0 \setminus \{0\}$  and its neighbourhood  $G$  in  $S_0 \setminus \{0\}$ . Since the map  $\exp : H_F \rightarrow S_0 \setminus \{0\}$  is onto and continuous, we see that  $\exp^{-1}(\varphi) \in H_F$  exists, and  $\exp^{-1}(G)$  is a neighbourhood of  $\exp^{-1}(\varphi)$  in  $H_F$ . According to (i),  $H_F$  is the support of  $\log L(s, F; \omega)$ , so  $Q_F(\exp^{-1}(G)) > 0$ , where  $Q_F$  is the distribution of  $\log L(s, F; \omega)$ , defined by (3.7). But

$$Q_F(\exp^{-1}(G)) = P_F(G),$$

where  $P_F$  is the distribution of  $L(s, F; \omega)$  given by (3.6). Hence  $P_F(G) > 0$ . This implies that any  $\varphi \in S_0 \setminus \{0\}$  is an element of the support of  $L(s, F; \omega)$ . Thus

$$S_0 \setminus \{0\} \subset S(P_F).$$

By Lemma 9 of [13], we have  $\overline{S_0 \setminus \{0\}} = S_0$ . Since  $S(P_F)$  is closed, we deduce

$$(6.1) \quad S_0 \subset S(P_F).$$

Let  $\tilde{\Omega} \subset \Omega$  be described as in Section 3. Then for any  $\tilde{\omega} \in \tilde{\Omega}$ , we have

$$(6.2) \quad L(s, F; \tilde{\omega}) = \begin{cases} \prod_p \prod_{0 \leq j \leq m} (1 - \tilde{\omega}_p \alpha_f(p)^{m-2j} p^{-s})^{-1} & \text{if } F = \text{sym}^m f, \\ \prod_p \prod_{0 \leq i, j \leq m} (1 - \tilde{\omega}_p \alpha_f(p)^{2(m-i-j)} p^{-s})^{-1} & \text{if } F = \text{sym}^m f \times \text{sym}^m f. \end{cases}$$

Since every factor on the right-hand side of (6.1) is non-zero, the function  $L(s, F; \tilde{\omega})$  is also non-vanishing. Thus

$$\{L(s, F; \tilde{\omega}) : \tilde{\omega} \in \tilde{\Omega}\} \subset S_0 \setminus \{0\}$$

and

$$P_F(S_0 \setminus \{0\}) = \mu_h(\{\omega \in \Omega : L(s, F; \omega) \in S_0 \setminus \{0\}\}) \geq \mu_h(\tilde{\Omega}) = 1 \quad \Rightarrow \quad P_F(S_0) = 1.$$

Since  $S(P_F)$  is the smallest closed subset of  $H_F$  such that  $P_F(S(P_F)) = 1$  and  $S_0$  is closed, we must have

$$(6.3) \quad S(P_F) \subset S_0.$$

Now the required result follows from (6.1) and (6.3).  $\square$

§ 7. Proofs of Corollaries 2 and 3

The proofs of Corollaries 2 and 3 will follow closely those of Theorems 2 and 3 of [14], but we reproduce here the details for the convenience of readers.

Let  $s_0, \dots, s_{J-1}$  be complex numbers such that  $s_0 \neq 0$ . Inductively on  $J$ , we easily see that there is a polynomial  $p(s) = \sum_{j=0}^{J-1} b_j s^j$  such that

$$(e^{p(s)})^{(j)} \Big|_{s=0} = s_j \quad (0 \leq j \leq J-1).$$

Let  $\sigma_F < \sigma_1 < 1$ , and  $\mathbb{K}$  be a compact subset of  $\mathbb{D}(\sigma_F)$  with connected complement such that  $\sigma_1$  is contained in the interior of  $\mathbb{K}$ . We denote by  $\delta$  the distance of  $\sigma_1$  from the boundary of  $\mathbb{K}$ . Then for any  $\varepsilon > 0$ , Theorem 1 assures that we find a real  $\tau$  for which

$$\sup_{s \in \mathbb{K}} |L(s + i\tau, F) - e^{p(s-\sigma_1)}| < \frac{\varepsilon \delta^J}{2^J J!}$$

holds. Then, using Cauchy's integral formula we have

$$|L^{(j)}(\sigma_1 + i\tau, F) - s_j| = \frac{j!}{2\pi} \left| \int_{|s-\sigma_1|=\delta/2} \frac{L(s + i\tau, F) - e^{p(s-\sigma_1)}}{(s - \sigma_1)^{j+1}} ds \right| < \varepsilon$$

for  $0 \leq j \leq J-1$ , which implies Corollary 2. □

Next we prove Corollary 3. Without loss of generality, we suppose  $g_J \neq 0$ . Then there exists a bounded region  $\mathbb{G} \subset \mathbb{C}^J$  and a constant  $B_0 > 0$  such that  $|g_J| \geq B_0$  in  $\mathbb{G}$ .

Let  $\sigma \in \mathbb{D}(\sigma_F)$ . According to Corollary 2, we can find a sequence of real numbers  $\tau_n \rightarrow \infty$  such that

$$X_n = (L(\sigma + i\tau_n, F), L'(\sigma + i\tau_n, F), \dots, L^{(J-1)}(\sigma + i\tau_n, F)) \in \mathbb{G}.$$

By the assumption of Corollary 3, we have

$$\sum_{j=0}^{J-1} s^j g_j(L(s, F), L'(s, F), \dots, L^{(J-1)}(s, F)) = -s^J g_J(L(s, F), L'(s, F), \dots, L^{(J-1)}(s, F))$$

for all  $s \in \mathbb{C}$ . Letting  $s = \sigma + i\tau_n$  and dividing both sides by  $(\sigma + i\tau_n)^J$ , we obtain

$$\sum_{j=0}^{J-1} (\sigma + i\tau_n)^{j-J} g_j(X_n) = -g_J(X_n).$$

Since  $\mathbb{G}$  is bounded,  $|g_j(X_n)|$  is bounded ( $0 \leq j \leq J-1$ ). Hence the left-hand side of above tends to zero as  $n \rightarrow \infty$ . On the other hand,  $|g_J(X_n)| \geq B_0 > 0$ . This contradiction finishes the proof of Corollary 3. □

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