# A brody theorem for orbifolds

Frédéric Campana, Jörg Winkelmann

# A BRODY THEOREM FOR ORBIFOLDS

FREDERIC CAMPANA & JÖRG WINKELMANN

Abstract. We study the Kobayashi pseudodistance for orbifolds, proving an orbifold version of Brody's theorem and classifying which one-dimensional orbifolds are hyperbolic.

## 1. Introduction

We study orbifolds as introduced in [4], define morphisms and discuss hyperbolicity. For this purpose we establish a Brody theorem for orbifolds (see [2] for the Brody theorem for complex spaces (compare also [15] for a different approach)). Using this Brody theorem for orbifolds we then determine which one-dimensional orbifolds are hyperbolic.

We also show in the last section that some of the weakly special but not special projective threefolds constructed by Bogomolov-Tschinkel in [1] have a non-vanishing Kobayashi pseudometric, strengthening in these particular cases the result of [6].

There are two different classes of orbifold morphisms, baptised "classical" resp. "non-classcal".

In the "classical sense" many problems are easier to handle because "classical" orbifold morphisms behave very well with respect to étale orbifold morphisms. In particular, the classification of one-dimensional hyperbolic orbifolds can be obtained via "unfoldings".

In contrast, for determining which one-dimensional orbifolds are hyperbolic in the "non-classical" sense we really need our "Brody theorem for orbifolds".

## 2. Orbifolds

We always assume all complex spaces to be irreducible, reduced, normal, Hausdorff and paracompact.

We recall some notions introduced in [4].

Let $\mathbb{Q}_+ = \{x \in \mathbb{Q} : x > 0\}$. An effective $\mathbb{Q}$-Weil divisor $\Delta = \sum a_i[Z_i]$ is a Weil divisor with all its coefficients $a_i$ in $\mathbb{Q}_+$, the $Z_i$ being pairwise distinct irreducible reduced hypersurfaces on $X$. The support $|\Delta|$ of $\Delta$ is the union of all $Z_i's$. An orbifold $(X/\Delta)$ is a pair consisting of an irreducible complex space $X$ together with a Weil $\mathbb{Q}_+$-divisor

$\Delta = \sum a_i[Z_i]$ for which

$$a_i \in \{1\} \cup \{1 - \frac{1}{m} : m \in \mathbb{N}\} \ \forall i$$

In this case, $a_i = 1 - 1/m_i$ (resp $m_i = 1/(1 - a_i)$) is the weight (resp. the multiplicity) of $Z_i$ in $\Delta$. It is convenient to consider $\infty$ as the multiplicity for the weight $1 = 1 - \frac{1}{\infty}$.

If $\Delta$ is an empty divisor, we will frequently identify $(X/\Delta)$ with $X$. If $Z$ is a component of multiplicity 1 of $\Delta$, then $(X/\Delta)$ may (and frequently will be) identified with $(X'/\Delta')$ where $X' = X \setminus Z$ and $\Delta' = \Delta - [Z]$.

An orbifold $(X/\Delta)$ is called *compact* iff $X$ is compact and $\Delta$ contains no irreducible component of multiplicity 1 (=weight $\infty$).

An orbifold $(X/\Delta)$ is *smooth* (or non-singular) if $Y$ is smooth *and* $\Delta$ is a locally s.n.c. divisor.

## 3. Orbifold morphisms

Orbifold were introduced in [4] in the context of fibrations. For a reducible fiber of a fibration there are two ways to define its multiplicity: Classically one takes the greatest common divisor of the multiplicities of its irreducible components. Non-classically (and this is the point of view emphasized in [4]) one takes the infimum of these multiplicities. Correspondingly, we define two notions of orbifold morphisms, a "classical" one and a "non-classicial" one.

**Definition 1.** *Let $(X/\Delta)$ be an orbifold with $\Delta = \sum_i (1 - \frac{1}{m_i})Z_i$ where $m_i \in \mathbb{N} \cup \{\infty\}$ and where the $Z_i$ are distinct irreducible hypersurfaces.*

*A holomorphic map $h$ from the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ to $X$ is a (non-classical)* "orbifold morphism from $D$ to $(X/\Delta)$" *if $h(D) \not\subset |\Delta|$ and if, moreover $mult_x(h^*Z_i) \geq m_i$ for all $i$ and $x \in D$ with $h(x) \in |Z_i|$. If $m_i = \infty$, we require $h(D) \cap Z_i = \{\}$. The map $h$ is called a "classical orbifold morphism" if the condition "$mult_x(h^*Z_i) \geq m_i$" is replaced by the condition "$mult_x(h^*Z_i)$ is a multiple of $m_i$".*

**Definition 2.** *Let $(X/\Delta)$ and $(X'/\Delta')$ be orbifolds. Let $\Delta_1$ be the union of all irreducible components of $\Delta$ with multiplicity 1 (equivalently: weight $\infty$). An* "orbifold morphism" *(resp. "classical orbifold morphism") from $(X/\Delta)$ to $(X'/\Delta')$ is a holomorphic map $f : X \setminus \Delta_1 \to X'$ such that*

1. *$f(X) \not\subset |\Delta'|$.*
2. *For every orbifold morphism (resp. every classical orbifold morphism) in the sense of def. 1 $g : D \to (X/\Delta)$ with $g(D) \not\subset$*

*$f^{-1}(|\Delta'|)$ the composed map $f \circ g : D \to X'$ defines an orbifold morphism from $D$ to $(X'/\Delta')$.*

**Remark.** *If one would like to obtain a purely algebraic-geometric definition in the case where $X$, $X'$, $\Delta$ and $\Delta'$ are algebraic, one might use smooth algebraic curves instead of the unit disc.*

**Remark.** *Both notions ("classical" versus "non-classical") differ substantially. The class of "classical orbifold morphism" is a much more restricted one. Questions concerning "classical" morphism often can be handlel easily by using étale orbifold morphisms, which does not work in the non-classical setup. See ?? below. Similar differences occur also for questions related to function field and arithmetic versions, see* [5].

## 4. Examples of orbifold morphisms

4.1. **Elementary properties.** *Composition.* If $f : (X/\Delta) \to (X'/\Delta')$ and $g : (X'/\Delta') \to (X''/\Delta'')$ are orbifold morphisms, so is $g \circ f : (X/\Delta) \to (X''/\Delta'')$ unless $(g \circ f)(\Delta) \subset |\Delta''|$.

*Empty divisors.* If $\Delta$, $\Delta'$ are empty Weil divisors on complex spaces $X$ resp. $X'$, then every holomorphic map from $X$ to $X'$ defines an orbifold morphism from $(X/\Delta)$ to $(X'/\Delta')$.

*Majorisation/Minorisation.* If $f : (X/\Delta) \to (X'/\Delta')$ is an orbifold morphism and $\Delta''$ is a $\mathbb{Q}$-Weil divisor on $X'$ with $\Delta'' \leq \Delta'$, then $f$ is an orbifold morphism to $(X'/\Delta'')$, too.

Similarly: If $f : (X/\Delta) \to (X'/\Delta')$ is an orbifold morphism and $\Delta''$ is a $\mathbb{Q}$-Weil divisor on $X$ with $\Delta'' \geq \Delta$, then $f$ is an orbifold morphism from $(X/\Delta'')$, too.

If $f : X \to Y$ is a holomorphic map of complex spaces and $D$ is an irreducible reduced hypersurface on $Y$ such that

$$f : X \to \left( Y/(1 - \frac{1}{n})[D] \right)$$

is an orbifold morphism for all $n \in \mathbb{N}$, then $f(X) \cap |D| = \emptyset$.

4.2. **Curves.** Let $C$ and $C'$ be smooth complex curves, $p \in C$, $p' \in C'$, $n, n' \in \mathbb{N}$. Then a non-constant holomorphic map $f : C \to C'$ is an orbifold morphism from $(C/(1 - \frac{1}{n})\{p\})$ to $(C'/(1 - \frac{1}{n'})\{p'\})$ $mult_p\, f^*[\{p'\}] \geq \frac{n'}{n}$ and $mult_z\, f^*[\{p'\}] \geq n'$ for $z \in C \setminus \{p\}$.

In particular, if $C = C'$ and $p = p'$, then the identity map defines an orbifold morphism iff $n \geq n'$.

4.3. **Automorphisms.** Let $(X/\Delta)$ be an orbifold. An holomorphic automorphism $f$ of $X$ is an orbifold morphism iff $f^*\Delta = \Delta$.

4.4. **Blown up surface.** Let $S$ be a complex surface and $\pi : \hat{S} \to S$ the sigma-process centered at a point $c \in S$. Let $D_i$ be a finite family of irreducible reduced hypersurfaces (i.e. curves) on $S$ with total transforms $\pi^* D_i$ and strict transforms $\hat{D}_i$. Then $\pi^* D_i = \hat{D}_i + d_i E$ where $E = \pi^{-1}(c)$ and where $d_i$ denotes the multiplicity of $D_i$ at $c$. Let $\Delta = \sum_i (1 - \frac{1}{n_i}) D_i$ for some $n_i \in \mathbb{N}$ and let $\hat{\Delta}$ be a $\mathbb{Q}$-Weil divisor on $\hat{S}$. Let $m = \max_i \frac{n_i}{d_i}$. Then $\pi : \hat{S} \to S$ defines an orbifold morphism from $(\hat{S}/\hat{\Delta})$ to $(S/\Delta)$ iff each $\hat{D}_i$ occurs with multiplicity at least $(1 - \frac{1}{n_i})$ in $\hat{\Delta}$ and in addition $E$ occurs with multiplicity at least $1 - \frac{1}{m}$ in $\hat{\Delta}$.

4.5. **Quotients by group actions.** Let $G$ be a discrete group acting effectively on a complex curve $Y$. Such an action is called "proper" resp. "properly discontinuously" if the map $\mu : G \times Y \to Y \times Y$ given by $\mu(g, y) = (g \cdot y, y)$ is a proper map. In particular, if $G$ is finite, then every action of $G$ is proper. The quotient $X = Y/G$ has a the structure of a ringed topological space in a canonical way. If $G$ is acting properly, $Y/G$ is a complex space. If in addition $\dim(Y) = 1$, then $Y/G$ is smooth.

For $y \in Y$ let $G_y$ denote the isotropy group at $y$, i.e. $G_y = \{g : g \cdot y = y\}$. Assume that $\dim(Y) = 1$. In this case $Y/G$ is smooth and furthermore we can define a $\mathbb{Q}$-divisor $\Delta$ on $X = Y/G$ by $\Delta = \sum_{[y] \in Y/G} (1 - 1/\#G_y)\{[y]\}$.

Then $(X/\Delta)$ is an orbifold such that the natural projection from $Y$ onto $(X/\Delta)$ is a smooth orbifold morphism.

## 5. Ramification divisors

5.1. **Existence.**

**Theorem 1.** *Let $f : X \to Y$ be a surjective holomorphic map with constant fiber dimension between irreducible normal complex spaces.*

*Then there exists a unique Weil divisor $R_f$ on $X$ with the following properties:*

1. *If $D_0$ and $D_1$ are reduced irreducible hypersurfaces on $X$ resp. $Y$ and $D_0$ occurs with multplicity $m \geq 2$ in $f^* D_1$, then $D_0$ occurs with multiplicity $(m-1)$ in $R_f$.*
2. *$f(R_f)$ contains no open subset of $Y$.*

**Notation.** *This divisor $R_f$ is called "ramification divisor".*

**Remark.** *If the map is not surjective or the fibers are not equidimensional, then in general there is no such divisor with these properties.*

*Proof.* We simply define $R_f$ as the sum of all $D_0$ with respective multiplicities as required by the first property. There are two problems in doing so:

- Given an irreducible reduced hypersurface $D_0 \subset X$, we need that there is at most one irreducible reduced hypersurface $D_1 \subset Y$ such that $|D_0| \subset |f^*D_1|$.
- The sum must be locally finite.

The first property is a consequence of the assumption that $f$ is surjective with equidimensional fibers. For the second we observe that, for any such $D_0$ with multiplicity $\geq 2$, the support $|D_0|$ must not intersect the set $\Omega$ of all non-singular points $x \in X$ for which $f(x)$ is non-singular and $Df : T_xX \to T_{f(x)}Y$ is surjective. The complement of $\Omega$ is an analytic subset of $X$, hence it locally contains only finitely many hypersurfaces. For this reason the sum of all such $D_0$ is locally finite. □

**Proposition 1.** *If $f$ is a surjective finite morphism between smooth complex manifolds $X$ and $Y$, then $R_f$ is linearly equivalent to $K_X \otimes (f^*K_Y)^*$.*

This follows by pulling-back $n$-forms ($n = \dim(X) = \dim(Y)$).

There is no such statement in the case where the fibers are positive-dimensional: Let $C$ be a compact smooth curve and let $p_1$, $p_2$ be the projections from the product $X = C \times \mathbb{P}_1$ to its factors. Then $R_{p_1} = 0$, but $K_X \otimes (p_1^*K_C)^{-1} \sim p_2^*K_{\mathbb{P}_1}$.

### 5.2. **Composition rule.**

**Proposition 2.** *Let $f : X \to Y$ and $g : Y \to Z$ be surjective holomorphic maps with equidimensional fibers between normal complex spaces. Then*

$$R_{g\circ f} = R_f + f^*R_g - S_{f,g}$$

*where $S_{f,g}$ denotes the sum of those irreducible components of $R_f$ which are mapped dominantly on $Y$ by $g \circ f$.*

### 5.3. **Orbifold morphisms and ramification divisor.**

**Proposition 3.** *Let $(X/\Delta)$ and $(Y/\Delta')$ be smooth orbifolds. Let $f : X \to Y$ be a surjective holomorphic map with constant fibre dimension between irreducible complex spaces.*

*Then $f$ defines an orbifold morphism from $(X/\Delta)$ to $(Y/\Delta')$ if and only if $(R_f + \Delta - f^*\Delta') \geq 0$.*

*Proof.* We may check this for each irreducible component separately. Thus let $H$ be an irreducible component of $\Delta$ with multiplicity $(1-1/n)$

and let $H'$ be an irreducible component of $\Delta'$ with multiplicity $(1-1/m)$ such that $|H| \subset |f^*H'|$. Assume that $H$ occurs with multiplicity $d$ in $f^*H'$.

In order for to be an orbifold morphism, we need that $g^*f^*H'$ has multiplicity at least $m$ whenever $g : D \to X$ is a holomorphic map for which $g^*H$ has multiplicity $\geq n$. This is the case if $nd \geq m$.

On the other hand the multiplicity of $H$ in $R_f + \Delta - f^*\Delta'$ equals $(d-1) + (1-1/n) - (d(1-1/m))$. Now

$$(d-1) + (1-1/n) - (d(1-1/m)) = -1/n + d/m$$

and $(-1/n + d/m) \geq 0$ holds if and only if $nd \geq m$. □

**Lemma 1.** *Assume that there exists a non-constant orbifold morphism $f : (C/\Delta) \to (C'/\Delta')$ for some smooth compact Riemann surfaces $C$ and $C'$. Let $K_C$ and $K_{C'}$ denote the respective canonical line bundles on $C$ resp. $C'$.*

*Then*

$$\deg(K_C + \Delta) \geq d.\deg(K_{C'} + \Delta'),$$

*if $d$ is the geometric degree of $f$ (ie: the number of points of one of its generic fibres).*

*Proof.* Because $f$ is an orbifold morphism, we have $R_f + \Delta - f^*\Delta' \geq 0$. On the other hand, $R_f \sim K_C - f^*K_{C'}$. Therefore

$$\deg\left(K_C - f^*K_{C'} + \Delta - f^*\Delta'\right) \geq 0.$$

Hence

$$\deg\left(K_C + \Delta\right) \geq (\deg f)\deg\left(K_{C'} + \Delta'\right) \geq \deg\left(K_{C'} + \Delta'\right).$$

□

## 6. Orbifold base

**Lemma 2.** *Let $X$ and $Y$ be complex spaces, $\Delta$ an orbifold divisor on $X$ and $\Delta'$ and $\Delta''$ be orbifold divisors on $Y$.*

*Assume that $f : X \to Y$ is a holomorphic map which defines an orbifold morphism from $(X/\Delta)$ to both $(Y/\Delta')$ and $(Y/\Delta'')$.*

*Then $f$ likewise defines an orbifold morphism to $(Y/\max\{\Delta', \Delta''\})$.*

**Definition 3.** *Let $f : X \to Y$ be a holomorphic map of complex spaces. Then $(Y/\Delta)$ is an "orbifold base" for $f$ if $\Delta$ is a maximal $\mathbb{Q}$-Weil divisor for which $f$ defines an orbifold morphism from $(X/\emptyset)$ to $(Y/\Delta)$.*

In view of lemma 2 the following is immediate:

**Lemma 3.** *Let $f : X \to Y$ be a holomorphic map of complex spaces.*

*Either there exists an orbifold base or there is an infinite sequence of distinct irreducible reduced hypersurfaces $H_i$ on $Y$ such that $f : X \to (Y/\frac{1}{2}H_i)$ is an orbifold morphism.*

**Proposition 4.** *Let $X$ be an irreducible reduced complex space and let $f$ be a* surjective *holomorphic map from $X$ to an other reduced complex space $Y$.*

*Then there exists an orbifold base $(Y/\Delta)$.*

*Proof.* Let $H$ be an irreducible reduced hypersurface in $Y$ for which there exists a number $n \geq 2$ such that $f$ is an orbifold morphism to $(Y/(1-\frac{1}{n}))H$. Then for every $p \in X$, $q = f(p) \in H$ and every holomorphic map $g : D \to X$ with

$$g(0) = p$$

we have $mult_0((f \circ g)^* D \geq n \geq 2$.

Let

$$\Omega = \{x \in X_{reg} : Tf_x \text{ is surjective } \}$$

Then $\Omega$ can not intersect $f^{-1}(H_{reg})$. Therefore $|H| \subset Y \setminus f(\Omega)$. But $Y \setminus f(\Omega)$ is an analytic subset of $Y$. It follows that the family of all hypersurfaces $H_i$ for which there exists a number $n_i$ such that $f : X \to (Y/(1-\frac{1}{n_i})H_i)$ is a locally finite family. Hence

$$\Delta = \max(1 - \frac{1}{n_i})H_i$$

exists and $(Y/\Delta)$ is the orbifold base for $f : X \to Y$. □

**Remark.** *Surjectivity of $f$ is crucial, as shown by the following example of a curve $Q$ and a holomorphic map $i : Q \to \mathbb{P}_2$ for which there are infinitely many curves $L_s$ in $\mathbb{P}_2$ such that $f : Q \to (\mathbb{P}_2/\frac{1}{2}L_s)$ is an orbifold morphism.*

*Let $S$ be a finite subset of a smooth quadric $Q$ in $\mathbb{P}_2$. For each $s \in S$ let $L_s$ denote the line through $s$ which is tangent to $Q$ at $s$. Since $\deg(Q) = 2$, the two curves $Q$and $L_s$ intersect only at $s$ and there with multiplicity two. Then the embedding $i : Q \to \mathbb{P}_2$ defines an orbifold morphism from $Q = (Q/\emptyset)$ to $(\mathbb{P}_2/\Delta)$ with*

$$\Delta = \sum_{s \in S} \frac{1}{2}[L_s]$$

*Note that $S$ is an arbitrary finite subset, we do not need any bound on its cardinality.*

## 7. Canonical divisors

**Definition 4.** *For a smooth orbifold $(X/\Delta)$ we define the canonical divisor $K_{(X/\Delta)}$ as $K_X + \Delta$.*

*$(X/\Delta)$ is said to be of "general type" if $K_{(X/\Delta)} = K_X + \Delta$ is a big divisor on $X$.*

### 7.1. **Etale morphisms.**

**Definition 5.** *Let $(X/\Delta)$ and $(X'\Delta')$ be smooth orbifolds. An orbifold morphism $\pi : (X/\Delta) \to (X'\Delta')$ is called* étale *if $\pi$ induces a finite morphism from $X$ to $X'$ and $R_\pi = \Delta - \pi^*\Delta'$ where $R_\pi$ is the ramification divisor of $\pi : X \to X'$.*

If $(X/\Delta)$ and $(X'/\Delta')$ are compact, this is equivalent to the condition $\pi^* K_{(X'/\Delta')} \simeq K_{(X/\Delta)}$.

If in addition $X$ and $X'$ are one-dimensional, a finite morphism $f : X \to X'$ defines an étale orbifold morphism if and only if it is an orbifold morphism, and:

$$\deg(K_X + \Delta) = \deg(K_{(X/\Delta)} = d.\deg(K_{X'} + \Delta') = d.\deg(K_{(X'/\Delta')}.$$

A holomorphic map $f : D \to D$ gives an étale orbifold morphism from $(D/\left(1 - \frac{1}{n}\right)[\{0\}])$ to $(D/\left(1 - \frac{1}{m}\right)[\{0\}])$ iff $f'(z) \neq 0$ for $z \neq 0$ and $n\, mult_0(f) = m$.

Examples of étale orbifolds morphisms are given in §?? below.

## 8. Unfolding Orbicurves

**Theorem 2.** *Let $(C/\Delta)$ be a smooth orbifold with* $\dim(C) = 1$.

*Then there exists a finite étale (in th sense of def. 5) orbifold morphism from a curve $C'$ to $(C/\Delta)$, unless $(C/\Delta)$ is isomorphic to*

$$(\mathbb{P}_1/(1 - \frac{1}{m})[\{\infty\}]$$

*or*

$$(\mathbb{P}_1/(1 - \frac{1}{m})[\{\infty\}] + (1 - \frac{1}{n}[\{0\}]$$

*with $m \neq n$.*

As explained in [13] (p.??), this follows from group-theoretical work of Fox ([8]), Bundgaard and Nielsen ([3]).

### 8.1. **Examples.**

Consider the case $(2, 2, 2, 2)$ (meaning that the support of $\Delta$ consists of 4 distinct points with weights $1/2$ and multiplicity 2 each). For every four distinct points $p_i$ on $\mathbb{P}_1$ there exists an elliptic curve $E$ with a $2:1$-ramified covering $\pi : E \to \mathbb{P}_1$ which is ramified

precisely over the $p_1$. This covering is étale in the orbifold sense, and provides an unfolding of the given orbifold on $\mathbb{P}^1$.

Observe that $\mathrm{Aut}(\mathbb{P}_1)$ acts triply transitively on $\mathbb{P}_1$, so that if the support of $\Delta$ consists of three points, these can be assumed to be $0, 1, \infty$.

For the multiplicities $(2, 4, 4)$ we use the elliptic curve $C$ defined by $y^2 = x^3 - x$ with ramified covering $C \to \mathbb{P}_1$ given by the meromorphic function $x^2$. Then above 0 (resp. 1, $\infty$), there are 1 (resp. 2; 1) points with ramification multiplicities 4 (resp. 2; 4), and no other ramification. This ramified cover is thus an unfolding of this $(2, 4, 4)$ orbifold on $\mathbb{P}^1$.

For the multiplicities $(2, 3, 6)$ we use the elliptic curve $C$ defined by $y^2 = x^3 + 1$ with ramified covering $C \to \mathbb{P}_1$ given by the meromorphic function $y^2 = x^3 + 1$. Then above 0 (resp. 1, $\infty$), there are 3 (resp. 2; 1) points with ramification multiplicities 2 (resp. 3; 6), and no other ramification. This ramified cover is thus an unfolding of this $(2, 3, 6)$ orbifold on $\mathbb{P}^1$.

For the multiplicities $(3, 3, 3)$ we use the elliptic curve $C$ defined by $y^2 = x^3 + 1$ with ramified covering $C \to \mathbb{P}_1$ given by the meromorphic function $y$. Then above $-1$ (resp. 1, $\infty$), there is one single point with ramification multiplicity 3, and no other ramification. This ramified cover is thus an unfolding of this $(3, 3, 3)$ orbifold on $\mathbb{P}^1$.

## 9. Fundamental group

**Definition 6.** *Let $(X, \Delta)$ be an orbifold.*

*The* orbifold fundamental group *is the quotient of $\pi_1(X \setminus |\Delta|)$ by the normal subgroup $N$ generated by all loops who can be realized as the image of $t \mapsto \frac{1}{2}e^{2\pi i t}$ under some classical orbifold morphism from the unit disk $(D, \emptyset)$ to $(X, \Delta)$.*

**Lemma 4.** *Assume that $X$ is smooth. Then $N$ is generated by small loops around each connected component of the smooth part of $|\Delta|$.*

*Proof.* Let $H : [0, 1] \times D \to X$ be a homotopy between $f : D \to X$ and a constant map with value $p \in X \setminus |\Delta|$. Since $X$ is smooth we may assume by transversality arguments that $H$ stays away from $Sing(D)$. This implies the statement. □

**Corollary 1.** *Let $X$ be a complex space, locally irreducible at $x \in X$, $Z$ a closed subvariety. Then there exists an open neighbourhood $W$ of $x$ in $X$ such that $\pi_1(W \setminus Z)$ is generated by those elements which arises as the image of a loop in $D^*$ under a holomorphic map $f : D \to W$ with $f^{-1}(W) = \{0\}$.*

*Proof.* Desingularize $(X, Z)$ and then use the preceding observation that the fundamental group can be generated by small cycles around the smooth parts of the $\pi^{-1}(Z)$. □

**Proposition 5.** *A classical orbifold morphism induces a group homomorphism between the orbifold fundamental groups.*

**Remark.** *This statement is very false for the (non-classical) orbifold morphisms. For example, $z \mapsto z^d$ induces a non-classical orbifold morphism from $(D/(1-\frac{1}{n})[\{0\}])$ to $(D/(1-\frac{1}{m})[\{0\}])$ whenever $dn \geq m$ but there is no natural group homomorphism from $\pi_1(D/(1-\frac{1}{n})[\{0\}]) = \mathbb{Z}/n\mathbb{Z}$ to $\pi_1(D/(1-\frac{1}{m})[\{0\}]) = \mathbb{Z}/m\mathbb{Z}$ unless $n$ divides $m$.*

*Proof.* Each element $\gamma \in \pi_1(X \setminus |\Delta|)$ can be represented by a loop inside $X \setminus (|R_f + \Delta|)$. Let $\gamma_i$ $(I = 1, 2)$ be such loops homotopic to $\gamma$. Observe that $f(\gamma_i) \subset X \setminus |\Delta'|$. The $\gamma_i$ homotopic to each other in $X \setminus |\Delta|$. For $x \in X \setminus |\Delta|$ we have $f(x) \notin |\Delta'|$ unless $x \in R_f$. Hence the homotopy classes of $f \circ \gamma_i$ differ only by an element of $N'$.

It follows that there is a group homomorphism between the orbifold fundamental groups. □

**Proposition 6.** *Let $f : (X/\Delta) \to (X'/\Delta')$ be an étale orbifold morphism and let $g : (D, \emptyset) \to (X, \Delta)$ be a classical orbifold morphism.*

*Then there exists a classical orbifold morphism $\tilde{g} : D \to (X'/\Delta')$ such that $g = f \circ \tilde{g}$.*

*Proof.* Via considering fiber products it suffices to consider the case where $\dim(X) = 1$ in which case the claim follows from local calculations. □

**Remark.** *Again this is very false for non-classical orbifold morphisms: $h : z \mapsto z^n$ defines an étale orbifold morphism from $D$ to $(D/(1 - \frac{1}{n})[\{0\}])$, but for a given orbifold morphism $g : D \to (D/(1 - \frac{1}{n})[\{0\}])$ there exists a lift $\tilde{g}$ only if $g$ is in fact a classical orbifold morphism.*

**Proposition 7.** *Let $X/\Delta)$ be an orbifold for which $X$ is a smooth curve. Let $\Gamma$ be a subgroup of the orbifold fundamental group $\pi_1(X/\Delta)$.*

*Then there exists an orbifold $(X'/\Delta')$ and an étale orbifold map $f : (X'/\Delta') \to (X/\Delta)$ such that $(\pi_1(f))(\pi_1(X'/\Delta')) = \Gamma$.*

*Proof.* Take associate unramified covering of $X \setminus |\Delta|$ and compactify. □

**Remark.** *Thus classical orbifold morphisms from the unit disc to an orbifold curve $(C/\Delta)$ can be lifted to unfoldings of $(C/\Delta)$, while their non classical versions cannot. For this reason the study of these classical maps reduces to the non orbifold case on any unfolding, while the*

*study of the no classical version poses (seemingly) new problems. On the level of arithmetics, exactly the same situation appears: see* [7] *for the classical orbifold version of Mordell's conjecture on curves, and* [5] *for its non-classical version (which is presently only a conjecture).*

## 10. Uniformization

**Proposition 8.** *A smooth one-dimensional orbifold $(X/\Delta)$ has trivial fundamental group (as defined in ??) if and only if it is isomorphic to one of the following: $\mathbb{C}$, $D$, $\mathbb{P}_1$ $(\mathbb{P}_1/(1-\frac{1}{n})\{\infty\})$ or $(\mathbb{P}_1/(1-\frac{1}{n})\{\infty\}+(1-\frac{1}{m})\{0\})$ with $gcd(n,m)=1$.*

*For every smooth one-dimensional orbifold $(X/\Delta)$ there exists a smooth one-dimensional orbifold $(\tilde{X}/\tilde{\Delta})$ with trivial fundamental group and an étale orbifold morphism $\pi : (\tilde{X}/\tilde{\Delta}) \to (X/\Delta)$.*

## 11. Hyperbolicity and Kobayashi pseudodistance

We recall (and extend) from [4] the notion of orbifold Kobayashi pseudodistance by restricting to orbifold morphisms from the unit disc to $(X/\Delta)$.

More precisely:

**Definition 7.** *Let $(X/\Delta)$ be an orbifold with $\Delta = \sum_i a_i H_i$. Let $\Delta_1$ be the union of all $H_i$ with $a_i = 1$ (ie: weight one, or equivalently multiplicity infinite).*

*The orbifold Kobayashi pseudodistance of the orbifold $(X/\Delta)$ is the largest pseudodistance on $(X \setminus |\Delta_1|)$ such that every orbifold morphism from the unit disc $D$ to $(X/\Delta)$ is distance-decreasing with respect to the Poincaré distance on the unit disc.*

*One defines similarly the classical orbifold Kobayashi pseudodistance on $(X/\Delta)$ by replacing the above set of orbifold morphisms from the disc to $(X/\Delta)$ by their classical versions.*

**Remark.** *Let $d_X$ (resp. $d_{(X/\Delta)}$; resp. $d^*_{(X/\Delta)}$) be the usual (resp. orbifold; resp. classical orbifold) Kobayashi pseudodistance). Then we have:*

$$d_X \leq d_{(X/\Delta)} \leq d^*_{(X/\Delta)} \leq d_{X\setminus|\Delta|}.$$

*It is clear that $d_X$ and $d_{(X/\Delta)}$ are usually very different as well as $d^*_{(X/\Delta)}$ and $d_{X\setminus|\Delta}$. But we do not know a single example in which $d_{(X/\Delta)}$ and $d^*_{(X/\Delta)}$) differ.*

The definition implies immediately that the (classical) orbifold Kobayashi pseudodistance is distance-decreasing under (classical) orbifold morphisms between orbifolds.

As in the case of the usual Kobayashi pseudodistance for manifolds there is an equivalent definition using chains of disc:

*For $x, y \in X \setminus |\Delta_1|$ the (classical) Kobayashi pseudodistance $d_{(X/\Delta)}$ is the infimum over $\sum_i d_P(p_i, q_i)$ where $d_P$ is the distance function on the unit disc $D$ induced by the Poincaré metric and the infimum is taken over all finite families $f_1, \dots f_d$ of (classical) orbifold morphisms from $D$ to $(X/\Delta)$ with $f_1(p_1) = x$, $f_d(q_d) = y$ and $f_k(q_k) = f_{k+1}(p_{k+1})$.*

From this definition it is easily deduced that:

$$d_{(X/\Delta)} : X \setminus |\Delta_1| \times X \setminus |\Delta_1| \to \mathbb{R}$$

is continuous and that the set

$$E_x = \{y \in X \setminus \Delta : d_{(X/\Delta)}(x, y) = 0\}$$

is connected for every $x \in X \setminus |\Delta_1|$.

**Definition 8.** *An orbifold $(X/\Delta)$ is (classically) orbifold hyperbolic if the (classical) orbifold Kobayashi pseudodistance is a distance on $X \setminus \Delta_1$ where $\Delta_1$ is the union of the components of $\Delta$ with multiplicity one.*

**Corollary 2.** *Let $f : (X/\Delta) \to (X'/\Delta')$ be an étale orbifold morphism. Then $(X/\Delta)$ is classical orbifold hyperbolic if and only if $(X'/\Delta')$ has this property.*

## 12. Classical orbifold Kobayashi pseudodistance in dimension one

**Proposition 9.** *Let $(X/\Delta)$ be a one-dimensional smooth orbifold.*

*If there exists an étale orbifold morphism $\pi : D \to (X/\Delta)$, then*

$$d^*_{(X/\Delta)}(p, q) = \inf_{x \in \pi^{-1}(p); y \in \pi^{-1}(q)} d_D(x, y)$$

*If there is no étale orbifold morphism $\pi : D \to (X/\Delta)$, then $d^*_{(X/\Delta)} \equiv 0$.*

**Corollary 3.** *Let $(X/\Delta)$ be a compact smooth one-dimensional orbifold.*

*Then $(X/\Delta)$ is classically hyperbolic iff $\deg(K_{(X/\Delta)}) > 0$.*

12.1. **Examples.** We consider $X = D$, $\Delta = \left(1 - \frac{1}{n}\right)[\{0\}]$. Then $z \mapsto z^n$ yields an unfolding $D \to (X/\Delta)$ and consequently the classical Kobayashi pseudodistance on $(X/\Delta)$ is the distance function induced by the "push-forward" of the Poincaré metric on $D$ which is easily calculated as

$$\frac{4dzd\bar{z}}{n^2|z|^{2-\frac{2}{n}}\left(1 - |z|^{\frac{2}{n}}\right)^2}.$$

Note that for $n \to \infty$ this converges to

$$\frac{4dzd\bar{z}}{|z|^2 \left(\log |z^2|\right)^2}$$

which is the push-forward of the Poincaré metric under the universal covering map from $D$ to the punctured disc $D^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$.

## 13. An orbifold Brody theorem

Brody's theorem ([2]) is an important tool in the study of hyperbolicity questions for complex spaces. Here we will develop a version of this theorem for orbifolds.

As a first step we show:

**Proposition 10.** *Let $f_n : (X/\Delta) \to (X'/\Delta')$ be a sequence of orbifold morphisms. Assume that $(f_n)$, regarded as a sequence of holomorphic maps from $X$ to $X'$ converge locally uniformly to a holomorphic map $f : X \to X'$.*

*Then either $f(X) \subset |\Delta'|$ or $f$ is an orbifold morphism from $(X/\Delta)$ to $(X'/\Delta')$.*

*This statement, and its proof, hold both in the classical and non classical versions.*

*Proof.* Assume $f(X) \not\subset |\Delta'|$.

Fix an orbifold morphism $g : D \to (X/\Delta)$. By definition, $f_n \circ g$ are orbifold morphisms and we have to show that $f \circ g$ is an orbifold morphism as well.

Let $D_i$ be an irreducible component of $\Delta$ with multiplicity $\frac{m-1}{m}$. Let $p \in D$ with $q = f(g(p)) \in |D_i|$. We have to show that $(f \circ g)^* D_i$ has multiplicity at least $m$. In an open neighbourhood $U$ of $q$ in $X$ the divisor $D_i$ has a defining function $\rho$. Let $W$ be a relatively compact open neighbourhood of $p$ in $(f \circ g)^{-1}(U)$. The set of all maps $F : X \to X'$ with $F(g(\bar{W})) \subset U$ is open for the the topology of locally uniform convergence. Thus we have $f_n(g(W)) \subset U$ for all sufficiently large $n$. Now $\rho \circ f_n \circ g$ is a sequence of holomorphic functions on $W$ converging to $\rho \circ f \circ g$. Since we assumed that $f(X)$ is not contained in $|\Delta'|$, $\rho \circ f \circ g$ does not vanish identically. Hence there is a number $\epsilon > 0$ such that $S_\epsilon(p) = \{z \in \mathbb{C} : |z - p| = \epsilon\}$ is contained in $W$ and $\rho \circ f \circ g$ has no zero in $B_\epsilon(p) = \{z \in \mathbb{C} : |z - p| \leq \epsilon\}$ except at $p$.

The theorem of Rouché now implies that for all sufficiently large $n$ the multiplicity $\mu$ of $\rho \circ f \circ g$ at $p$ equals the sum of all multiplicities of all zeroes in $B_\epsilon(r)$ of $\rho \circ f_n \circ g$.

Hence there is at least one zero of $\rho \circ f_n \circ g$ in $B_\epsilon(r)$ for $n$ sufficiently large (since $f(p) \in |D_i|$). Furthermore each such zero has multiplicity at least $m$, because $f_n \circ g : D \to (X'/\Delta')$ are orbifold morphisms.

Therefore $\mu$ is at least $m$. Since this argument may be applied to all components $D_i$ of $|\Delta|$ and all points $p \in D$ with $f \circ g(p) \in |D_i|$ for every orbifold morphism $g : D \to (X/\Delta)$, we may conclude that $f$ is an orbifold morphism. □

**Remark.** *As said, this works as well for both "classical" and "non classical" orbifold morphisms: In the last case we use the ordinary ordering on* $\mathbb{N}$ *while in the first case we use the partial ordering of* $\mathbb{N}$ *by divisibility.*

**Proposition 11.** *Let* $(X/\Delta)$ *be an orbifold and let* $\Delta_1$ *be the union of components of* $\Delta$ *with weight one (or equivalently, multiplicity* $\infty$*). Assume that there are two distinct points* $p, q \in X \setminus |\Delta_1|$ *with orbifold Kobyashi pseudodistance zero. Let* $h$ *be a hermitian metric on* $X$ *and let* $d_h$ *be the induced distance function.*

*Then there exists a sequence of points* $p_n \in X \setminus |\Delta_1|$ *and orbifold morphisms* $f_n : D \to (X/\Delta)$ *such that* $f_n(0) = p_n$, $\lim p_n = p$ *and* $\lim ||f_n'(0)|| = +\infty$*, the latter calculated with respect to the Poincaré metric on* $D$ *and the hermitian metric* $h$ *on* $X$*.*

*Proof.* If not, there exists a neighbourhood $W$ of $p$ and a constant $C > 0$ such that $||f'(0)|| \leq C$ for all orbifold morphisms $f : D \to (X/\Delta)$ with $f(0) \in W$. Let us assume that this is the case. Since $D$ is homogeneous and the composition $f \circ \phi$ is an orbifold morphism for every orbifold morphism $f$ and every automorphism $\phi$ of $D$, this condition implies that $||f'(z)|| \leq C$ for every orbifold morphism $f : D \to (X/\Delta)$ and every $z \in D$ with $f(z) \in W$. By shrinking $W$, we may assume $q \notin W$. Now for every $\epsilon > 0$ there is a chain of orbifold discs as in §9 above with $\sum d_P(p_i, q_i) \leq \epsilon$. By taking geodesics in $D$ linking $p_i$ with $q_i$ and concatenating their images we obtain a piecewise smooth path $\gamma : [0, 1] \to X$ with $\gamma(0) = p$ and $\gamma(1) = q$. Let $\alpha = \inf\{t : \gamma(t) \notin W\}$. Then

$$\epsilon \geq d(p, \gamma(\alpha)) \geq C d_h(p, \partial W)$$

which leads to a contradiction since $d_h(p, \partial W) > 0$. □

We recall the "reparametrization lemma" of Brody which may be rephrased as follows:

**Proposition 12.** *Let* $X$ *be a compact complex manifold and* $f_n : D \to X$ *a sequence of holomorphic maps with* $\limsup ||f_n'(0)|| = +\infty$*.*

*Then there exists an increasing sequence of positive real numbers $r_n$ and a sequence of holomorphic maps $\alpha_n : D(r_n; 0) \to D$ such that $\lim r_n = +\infty$ and such that a subsequence of $f_n \circ \alpha_n$ converges locally uniformly to a holomorphic map $f : \mathbb{C} \to X$ with*

$$\sup_{z\in\mathbb{C}} ||f'(z)|| = ||f'(0)|| > 0.$$

**Theorem 3.** *Let $(X/\Delta)$ be a compact orbifold. Assume that the (classical) orbifold Kobayashi pseudodistance on $X \setminus |\Delta_1|$ is not a distance.*

*Then there exists a non-constant holomorphic map $f : \mathbb{C} \to X$ which is either a (classical) orbifold morphism or fulfills the property $f(\mathbb{C}) \subset |\Delta|$.*

*Furthermore*

$$\sup ||f'(z)|| = ||f'(0)|| > 0.$$

*Proof.* By prop. 11 there is a sequence of orbifold morphisms $f_n : D \to (X/\Delta)$ such that $\lim ||f'(0)|| = +\infty$. Due to "Brody reparametrization" (prop. 12) there are sequences $r_n \in \mathbb{R}^+$ and $\alpha_n : D(r_n; 0) \to D$ such that $\lim r_n = +\infty$ and such that a subsequence of $f_n \circ \alpha_n$ converges to a holomorphic map $f : \mathbb{C} \to X$ with $f'(0) \neq 0$. Now compositions of orbifold morphisms are orbifold morphisms, hence $f_n \circ \alpha_n$ are orbifold morphisms. Thus prop. 10 implies that for all $r > 0$ either $f|_{D_r} : D_r \to (X/\Delta)$ is an orbifold morphisms or $f(D_r) \subset |\Delta|$. As a consequence, either $f : \mathbb{C} \to (X/\Delta)$ is an orbifold morphism or $f(\mathbb{C}) \subset |\Delta|$. □

**Corollary 4.** *Let $(X/\Delta)$ be a one-dimensional compact orbifold.*

*Then either $(X/\Delta)$ is orbifold hyperbolic or there exists a non-constant orbifold morphism $f : \mathbb{C} \to (X/\Delta)$ with bounded derivative.*

*Proof.* Since $X$ is one-dimensional, $|\Delta|$ is discrete. As a consequence $f(\mathbb{C})$ can not be contained in $|\Delta|$ for a holomorphic map $f : \mathbb{C} \to X$ with $f'(0) \neq 0$. □

## 14. Nevanlinna theory

We use the usual notations of Nevanlinna theory (see e.g. [14]). In particular, if $D$ is a divisor on a complex space $X$ and $f : \mathbb{C} \to X$ is a holomorphic map, then

$$N_f(r, D) = \int_1^r \deg(f^*D|_{D_t})dt/t$$

and

$$N_f^1(r, D) = \int_1^r \deg((f^*D)_{red}|_{D_t})dt/t.$$

If furthermore $\omega$ is a $(1,1)$-form on $X$ (e.g. a Kähler form or $c_1(D)$), then

$$T_f(r,\omega) = \int_1^r \left( \int_{D_t} f^*\omega \right) dt/t.$$

**Proposition 13.** *Let $X$ be a compact complex manifold, $H$ an irreducible reduced hypersurface, $n \in \mathbb{N} \cup \{+\infty\}$, $\alpha = (1-1/n)$, $\Delta = \alpha H$ and $f : \mathbb{C} \to (X/\Delta)$ an orbifold morphism.*

*Then*

$$T_f(r, c_1(H)) - N_f^1(r,H) \geq \alpha T_f(r, c_1(H)).$$

*Proof.* By the First Main Theorem, we have

$$T_f(r, c_1(H)) \geq N_f(r,H) \geq 0$$

Now $N_f^1(r,H)$ is the "truncated counting function" which ignores multiplicities and $f^*H$ has multiplicity at least $n$ at every point of $f^{-1}|H|$. Hence

$$N_f(r,H) \geq n N_f^1(r,H)$$

Together these two inequalities imply

$$T_f(r, c_1(H)) - N_f^1(r,H) \geq \left(1 - \frac{1}{n}\right) T_f(r, c_1(H)) = \alpha T_f(r, c_1(H)).$$

□

**Definition 9.** *We say that the "S.M.T. with truncation level 1" holds for a holomorphic map $f$ from $\mathbb{C}$ to a compact complex manifold $X$ and a reduced effective divisor $D$ on $X$ if*

$$T_f(r, c_1(D+K)) - N_f^1(r,D) \leq \epsilon T_f(r,\omega)||_\epsilon$$

*for some positive $(1,1)$-form $\omega$ on $X$. (The notation $||_\epsilon$ means that the inequality holds for any $\epsilon > 0$, for $r$ outside a subset of finite measure depending on $\epsilon$).*

By a classical result of Nevanlinna ([14]), the "S.M.T. with truncation level one" holds for every non-constant holomorphic map to a one-dimensional compact complex manifold $X$ and every reduced effective divisor $D$.

**Proposition 14.** *Let $(X/\Delta)$ be a compact orbifold, and let $f : \mathbb{C} \to (X/\Delta)$ be an orbifold morphism such that the "S.M.T. with truncation level one" holds for the underlying holomorphic map $f : \mathbb{C} \to X$ and the divisor $H$ on $X$ which is obtained by replacing all mutiplicities by one. Then*

$$T_f(r, c_1(\Delta + K_X)) \leq \epsilon T_f(r,\omega)||_\epsilon$$

*for every positive $(1,1)$-form $\omega$ on $X$.*

*Proof.* Let $\Delta = \sum \alpha_i H_i$ with $\alpha_i = 1 - \frac{1}{n_i}$. Then $H = \sum H_i$. Due to the S.M.T. we have:

$$T_f(r, c_1(K_X)) + \sum_i \big(T_f(r, c_1(H_i)) - N_f^1(r, H_i)\big) \leq \epsilon T_f(r, \omega)||_\epsilon$$

By prop. 13:

$$T_f(r, c_1(H_i)) - N_f^1(r, H_i) \geq \alpha_i T_f(r, c_1(H_i))$$

Therefore:

$$\begin{aligned} T_f(r, c_1(\Delta + K_X)) &= T_f(r, c_1(K_X)) + \sum_i \alpha_i T_f(r, c_1(H_i)) \\ &\leq T_f(r, c_1(K_X)) + \sum_i \big(T_f(r, c_1(H_i)) - N_f^1(r, H_i)\big) \\ &\leq T_f(r, c_1(K_X)) + T_f(r, c_1(H)) - N_f^1(r, c_1(H)) \\ &\leq \epsilon T_f(r, \omega)||_\epsilon \end{aligned}$$

□

**Corollary 5.** *Let $X$ be a compact smooth complex curve (i.e. a compact Riemann surface) of genus $g$ such that there exists a non-constant orbifold morphism $f : \mathbb{C} \to (X/\Delta)$.*

*Then* $\deg(\Delta + K_X) \leq 0$, *i.e.* $\deg(\Delta) \leq 2 - 2g$.

*Proof.* For curves, the "S.M.T. with truncation level one" has already been established by Nevnalinna. It follows that $\deg(\Delta + K_X) \leq 0$ whenever there exists a non-constant orbifold morphism. But $\deg(\Delta + K_X) \leq 0$ is equivalent to $\deg(\Delta) \leq -\deg K_X = 2 - 2g$. □

**Corollary 6.** *Let $(X/\Delta)$ be a one-dimensional smooth compact orbifold*

*Then there exists a non-constant orbifold morphism $f : \mathbb{C} \to (X/\Delta)$ if and only if one of the following conditions is true:*

1. *$X$ is an elliptic curve and $\Delta$ is empty.*
2. *$X \simeq \mathbb{P}_1$ and $|\Delta|$ contains at most two points.*
3. *$X \simeq \mathbb{P}_1$ and there are numbers: $p \leq q \leq r \in \mathbb{N} \cup \{\infty\} \setminus \{1\}$ such that $(X/\Delta)$ is isomorphic to*
$$\left(\mathbb{P}_1/(1 - \frac{1}{p})\{0\} + (1 - \frac{1}{q})\{1\} + (1 - \frac{1}{r})\{\infty\}\right)$$
*and $1/p + 1/q + 1/r \geq 1$. (There are exactly 5 possibilities for $(p, q, r)$: $(2, 3, 4); (2, 3, 5); (2, 3, 6); (2, 4, 4); (3, 3, 3)$).*
4. *There is a point $\lambda \in \mathbb{C} \setminus \{0, 1\}$ such that $(X/\Delta)$ is isomorphic to*
$$\left(\mathbb{P}_1/(1 - \frac{1}{2})\{0\} + (1 - \frac{1}{2})\{1\} + (1 - \frac{1}{2})\{\infty\} + (1 - \frac{1}{2})\{\lambda\}\right)$$

*Proof.* A case-by-case check verifies that these are exactly the orbifold curves for which $\deg(\Delta + K_X) \leq 0$. Thus by the preceding corollary a non-constant orbifold morphism can exist only if the orbifold curve appears on the above list. On the other hand, using the uniformization theorem and the orbifold resolution as described above (see §), there exists a non-constant orbifold morphism for all these orbifold curves. □

## 15. Hyperbolicity of orbicurves

We characterize completely under which condition an orbifold of dimension one is orbifold hyperbolic.

**Theorem 4.** *Let $(X/\Delta)$ be an orbifold of dimension one.*

*Then $(X/\Delta)$ is orbifold hyperbolic if and only if it is classically orbifold hyperbolic.*

*If $X$ can be compactified to a smooth compact curve $\bar{X}$ by adding finitely many points and in addition the support $|\Delta|$ is finite, then the orbifold hyperbolicity of $(X/\Delta)$ is equivalent to $\deg(K_{\bar{X}} + \Delta) + \#(\bar{X} \setminus X) > 0$.*

*Otherwise (if there is no such compactification or the support $|\Delta|$ is infinite) the orbifold $(X/\Delta)$ is orbifold hyperbolic.*

*Proof.* We recall that a Riemann surface $X$ is hyperbolic unless it is an elliptic curve, $\mathbb{P}_1$, $\mathbb{C}$ or $\mathbb{C}^*$. In particular, if $X$ can not be compactified by adding finitely points, it must be hyperbolic and as a consequence $(X/\Delta)$ is orbifold hyperbolic and classically orbifold hyperbolic.

Now assume that $|\Delta|$ is finite and $X$ can be compactified by adding finitely many points. By adding these points to $\Delta$ we may assume that $X$ is already compact. If $(X/\Delta)$ is not hyperbolic, there is a orbifold morphism from $\mathbb{C}$ to $(X/\Delta)$. Using Nevanlinna theory, this implies $\deg(K_X + \Delta) \leq 0$. On the other hand, if $\deg(K_X + \Delta) \leq 0$, there are two possibilities: Either $X$ is an elliptic curve and $\Delta$ is empty or $X \simeq \mathbb{P}_1$. Evidently elliptic curves are not hyperbolic. Thus it remains to discuss the case $X = \mathbb{P}_1$. If $|\Delta|$ contains at most two points, $\mathbb{C}^*$ embedds into $(X/\Delta)$ which therefore can not be hyperbolic. Finally, if $|\Delta|$ contains at least three points, due to ?? there is an étale orbifold morphism from a compact curve $C$ to $(X/\Delta)$. Now $\deg(K_X + \Delta) \leq 0$ implies $\deg(K_C) \leq 0$ and thereby implies that is either $\mathbb{P}_1$ or an elliptic curve. In both cases the projection map from $C$ to $(X/\Delta)$ shows that the latter is not hyperbolic.

We still have to discuss the case where $X$ can be compactified by adding finitely many points, but $|\Delta|$ is infinite. Because $|\Delta|$ is infinite and the multiplicity at each point is at least $\frac{1}{2}$, we can find a finite

$\mathbb{Q}$-Weil divisor $\Delta'$ by taking finitely many components of $\Delta$ with the same multiplicities in such a way that $\deg(\Delta')$ is as large as desired. Therefore there is a finite restricted $\mathbb{Q}$-Weil divisor $\Delta'$ on $X$ such that

(1) the identity map of $X$ gives a strict orbifold morphism from $(X/\Delta)$ to $(X/\Delta')$
(2) $\deg(K_{\bar{X}} + \Delta') > 0$

It follows that $(X/\Delta)$ is hyperbolic and classically hyperbolic. □

**Remark.** *The situation is actually much less understood as may appear.*

*If $(X/\Delta)$ is an hyperbolic orbicurve, we do not know whether or not the classical and non classical Kobayashi pseudodistances coincide, not even in the most simple case where $(X/\Delta) = (D/\left(1 - \frac{1}{n}\right)[\{0\}]$ with $n \in \mathbb{N} \setminus \{1\}$.*

*In higher dimensions, is it still true that classical hyperbolicity coincides with (non classical) hyperbolicity ?*

*Is there any concrete example where one can calculate the (non-classical) orbifold Kobayashi pseudodistances (if these are not degenerate)?*

*What about the arithmetic counterpart? For the "classical" variant there is the work of Darmon ([7]), but nothing seems to be known about the "non-classical" variant.*

## 16. Weakly special projective threefolds with non-vanishing pseudometric

A compact Kähler manifold $X$ is said to be *weakly special* if there is no surjective meromorphic map $f' : X' \to Y$ in which $X'$ is a finite étale cover of $X$, and $Y$ is a variety of general type (ie: $\kappa(Y) = dim(Y) > 0$.

It is conjectured in [9] that weakly special manifolds are potentially dense (if defined over a number field).

Answering a question raised in [4], in [1] complex projective threefolds $X$ (defined over number fields if one so wishes) are constructed which are weakly special (and simply connected), but admit an elliptic fibration $\varphi : X \to S$, with $S$ an elliptic surface $g : S \to B = \mathbb{P}^1$ having $\kappa(S) = 1$, such that the base orbifold $(S/\Delta(\varphi))$ of the fibration $\varphi$ is of general type (ie: $\kappa(S, K_S + \Delta(\varphi)) = 2$).

In the terminology of [4], and because $\kappa(S) = 1 \geq 0$, a result of [4] shows that $\varphi$ is a *fibration of general type*, and so is the *core* of $X$, since its smooth fibres are elliptic curves, and so are *special*. Thus $X$ is *nonspecial*.

The conjectures made in [4] then claim that $X$ is not potentially dense (thus conflicting with the conjecture of [9]), and that its Kobayashi pseudometric $d_X = \varphi^*(\delta)$, for $\delta$ a suitable pseudometric on $S$, which is a metric on some dense Zariki open subset of $S$.

The arithmetic side of these conjectures remains open. But it was shown in [6] that for some $X$ of Bogomolov-Tschinkel, any entire curve $h : \mathbb{C} \to X$ has its image contained in a rational or elliptic curve of $X$, itself either contained in a fibre of $\varphi$, or in a fixed divisor $E$ of $X$ such that $\varphi(E)$ is a finite union of rational and elliptic curves of $S$.

We shall now show:

**Theorem 5.** *There exist Bogomolov-Tschinkel threefolds such that $d_X = \varphi^*(\delta)$, where $\delta$ is metric on $S$.*

For these manifolds $X$, the above theorem thus solves the hyperbolic version of the conjectures of [4].

**Proof:** The existence of $\delta$ as above is clear, since $d_X$ vanishes on the generic, and so all (by continuity of $d_X$ with respect to the complex topology of $X$) fibres of $\varphi$. We have to show that $\delta$ is a metric on a given $S$, for suitable choice of $X$. Recall from the constructions made in [1] and [6], that $\Delta(\varphi) = (1 - 1/m).D$, where $m > 1$ is an arbitrary integer, and that $D$ is the strict transform of $D_0$ under $\beta : S \to S_0$, the blow-up of the smooth surface $S_0$ in the finitely many intersection points of two generic (smooth) members $D_0, D_0'$ of an ample free linear system $|L_0|$ on $S_0$. More precisely, in [6], $S_0$ is choosen to be a double cover $u : S_0 \to P := \mathbb{P}^1 \times \mathbb{P}^1$ ramified along a certain divisor $T \subset P$. And $|L_0| = u^*(|M|)$, where $|M|$ is a very ample linear system on $P$, with bidegree $(a, b)$ which can be chosen to be arbitrarily big.

It was also shown in [6] that if $G' \subset X$ is a (complex) curve, and $G := \varphi(G') \subset S$, then $G$ is $m - tangent$ to $D$, which means that $G$ and $D$ have a contact of order at least $m$ at each of their intersection points. Thus $\delta \geq d_{(S/(1-1/m).D)}$, and we just need to show, by the above orbifold version of the theorem of Brody, and by the theorem in [6] asserting that for suitable $X$, the entire curves on $S$ which are $m$-tangent to $D$ are algebraically degenerate, that for suitable $m$ and $D_0$ (but $S$ being given as above), then $S$ does not contain any rational or elliptic curve $G$ which is m-tangent to $D$.

First, consider the case in which $G$ is contained in some fibre $F$ of $g : S \to B$, the elliptic fibration of $S$. If $m > L.F$, then $G$ cannot be m-tangent to $D$. We shall thus assume that $m > L.F = 2a$.

We now consider the second possible case, when $g(G) = B = \mathbb{P}^1$ (we say $G$ is $g$-horizontal). Because there are only countably many such rational or elliptic curves on $S$ (since $\kappa(S) = 1$), it is sufficient to

show that for each such fixed $G$, the general member $D_0$ of $L_0$ is not m-tangent to $G$. But this is clear, since $|L_0| = u^*(|M|)$, and $M$ is very ample on $P$. (Notice we have shown a much stronger property: the $g$-horizontal rational or elliptic curves on $S$ are not tangent to $D_0$, if $D_0$ is general in the linear system $|L_0|$, so that in fact, no $g$-horizontal rational curve $G$ on $S$ can be $m$-tangent to $D$ except maybe at 2 of their intersection points, for such a general choice of $D$, if $(a, b)$ are sufficiently big, so that $\beta(G).L_0) \geq 4$, for any curve $G$ on $S_0$. This property implies that $d_{(S/(1-1/m).D)}$ is a metric on $S$).

## References


[1] Bogomolov, F; Tschinkel, Y. Special elliptic fibrations. math AG/ 03.

[2] Brody, R.: Compact manifolds and hyperbolicity. *T.A.M.S.* **235** , 213–219 (1978)

[3] Bundgaard, S.; Nielsen, J.: On normal subgroups with finite index in $F$-groups. *Mat. Tidsskr. B.*1951, (1951). 56–58.

[4] Campana, F.: Orbifolds, Special Varieties and Classification Theory.Ann. Inst. Fourier 54 (2004), 499-665. (Also on mathAG/ 0110051).

[5] Campana, F.: Fibres multiples sur les surfaces: aspects arithmétiques et hyperboliques (sur arXiv (2004). A paraitre a Manu. Math.)

[6] Campana,F; Paun, M. Variétés faiblement spéciales à courbes entières dégénérées. math AG/0512.

[7] Darmon, H.: Faltings plus epsilon, Wiles plus epsilon and the generalized Fermat equation. *C.R. Math. Rep. Acad. Sci. Canada* **19**, no. 1, 3-14 (1997) Corrigendum no.2, p. 64.

[8] Fox, R.: On Fenchel's conjecture about $F$-groups. *Mat. Tidsskr. B.* 1952, (1952). 61–65.

[9] Harris, J; Tschinkel, Y. Rational points on quartics. Duke math. J; (2000)

[10] Kobayashi, S.: Hyperbolic complex spaces. Springer 1998.

[11] Lang, S.: Introduction to Complex hyperbolic spaces. Springer 1987.

[12] Lang, S. Number Theory III. Diophantine geometry. Encyclopaedia of Mathematical Sciences, 60. Springer-Verlag, Berlin, 1991.

[13] Namba, M.: Branched coverings and algebraic functions. Pitman Research Notes in Mathematics Series, 161. Longman; John Wiley & Sons, Inc., New York, 1987.

[14] Nevanlinna, R.: Zur Theorie der meromorphen Funktionen. *Acta Math.* **46**, 1-99 (1925)

[15] Zalcman, L.: Normal families: new perspectives. (English. English summary) *Bull. Amer. Math. Soc.* (N.S.) **35** (1998), no. 3, 215–230.



Frederic Campana & Jörg Winkelmann, Institut Elie Cartan (Mathématiques), Université Henri Poincaré Nancy 1, B.P. 239, F-54506 Vandœuvre-les-Nancy Cedex, France