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# The distribution of integers with at least two divisors in a short interval

Kevin Ford\*      Gérald Tenenbaum

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## 1 Introduction

Whereas, in usual cases, sieving by a set of primes may be fairly well controlled, through Buchstab's identity, sieving by a set of integers is a much more complicated task. However, some fairly precise results are known in the case where the set of integers is an interval. We refer to the recent work [1] of the first author for specific statements and references.

Define

$$\begin{aligned}\tau(n; y, z) &:= |\{d|n : y < d \leq z\}|, \\ H(x, y, z) &:= |\{n \leq x : \tau(n; y, z) \geq 1\}|, \\ H_r(x, y, z) &:= |\{n \leq x : \tau(n; y, z) = r\}|, \\ H_2^*(x, y, z) &:= |\{n \leq x : \tau(n; y, z) \geq 2\}| = \sum_{r \geq 2} H_r(x, y, z).\end{aligned}$$

Thus, the numbers  $H_r(x, y, z)$  ( $r \geq 1$ ) describe the local laws of the function  $\tau(n; y, z)$ . When  $y$  and  $z$  are close, it is expected that, if an integer has at least a divisor in  $(y, z]$ , then it usually has exactly one, in other words

$$H(x, y, z) \sim H_1(x, y, z). \tag{1.1}$$

In this paper, we address the problem of determining the exact range of validity of such behavior. In other words, we search a necessary and sufficient condition so that  $H_2^*(x, y, z) = o(H(x, y, z))$  as  $x$  and  $y$  tend to infinity.

As shown in [5], for given  $y$ , the threshold for the behavior of the function  $H(x, y, z)$  lies near the critical value

$$z = z_0(y) := y \exp\{(\log y)^{1-\log 4}\} \approx y + y/(\log y)^{\log 4 - 1}.$$

We concentrate on the case  $z_0(y) \leq z \leq 2y$ . Define

$$\begin{aligned}z &= e^\eta y, \quad \eta = (\log y)^{-\beta}, \quad \beta = \log 4 - 1 - \Xi/\sqrt{\log_2 y}, \quad \lambda = \frac{1 + \beta}{\log 2}, \\ Q(w) &= \int_1^w \log t \, dt = w \log w - w + 1.\end{aligned}$$

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Here  $\log_k$  denotes the  $k$ th iterate of the logarithm.

With the above notation, we have

$$\log(z/y) = \frac{e^{\Xi\sqrt{\log_2 y}}}{(\log y)^{\log 4 - 1}}, \quad \log\{z/z_0(y)\} = \frac{e^{\Xi\sqrt{\log_2 y}} - 1}{(\log y)^{\log 4 - 1}},$$

so

$$\begin{aligned} 0 \leq \Xi &\leq (\log 4 - 1)\sqrt{\log_2 y} + \frac{\log_2 2}{\sqrt{\log_2 y}}, \\ \frac{|\log_2 2|}{\log_2 y} &\leq \beta \leq \log 4 - 1, \\ \frac{1}{\log 2} + \frac{\kappa}{\log_2 y} &\leq \lambda \leq 2, \end{aligned}$$

with  $\kappa := |\log_2 2|/\log 2$ .

From Theorem 1 of [1], we know that, uniformly in  $10 \leq y \leq \sqrt{x}$ ,  $z_0(y) \leq z \leq 2y$ ,

$$H(x, y, z) \asymp \frac{\beta x}{(\Xi + 1)(\log y)^{Q(\lambda)}}. \quad (1.2)$$

By Theorems 5 and 6 of [1], for any  $c > 0$  and uniformly in  $y_0(r) \leq y \leq x^{1/2-c}$ ,  $z_0(y) \leq z \leq 2y$ , we have

$$\begin{aligned} \frac{H_1(x, y, z)}{H(x, y, z)} &\asymp_c 1, \\ \frac{\Xi + 1}{\sqrt{\log_2 y}} &\ll_{r,c} \frac{H_r(x, y, z)}{H(x, y, z)} \leq 1 \quad (r \geq 2). \end{aligned} \quad (1.3)$$

When  $0 \leq \Xi \leq o(\sqrt{\log_2 y})$  and  $r \geq 2$ , the upper and lower bounds above for  $H_r(x, y, z)$  have different orders. We show in this paper that the lower bound represents the correct order of magnitude.

**Theorem 1.** *Uniformly in  $10 \leq y \leq \sqrt{x}$ ,  $z_0(y) \leq z \leq 2y$ , we have*

$$\frac{H_2^*(x, y, z)}{H(x, y, z)} \ll \frac{\Xi + 1}{\sqrt{\log_2 y}}.$$

**Corollary 2.** *Let  $r \geq 2$  and  $c > 0$ . Uniformly in  $y_0(r, c) \leq y \leq x^{1/2-c}$ ,  $z_0(y) \leq z \leq 2y$ , we have*

$$\frac{H_r(x, y, z)}{H(x, y, z)} \asymp_{r,c} \frac{\Xi + 1}{\sqrt{\log_2 y}}.$$

**Corollary 3.** *Uniformly in  $10 \leq y \leq \sqrt{x}$ ,  $y < z \leq y + y(\log y)^{1-\log 4+o(1)}$ , we have*

$$H_1(x, y, z) \sim H(x, y, z).$$

Since we know from (1.3) that  $H_2^*(x, y, z) \gg H(x, y, z)$  when  $\beta \leq \log 4 - 1 - \varepsilon$  for any  $\varepsilon > 0$  we have therefore completely answered the question raised at the beginning of this introduction concerning the exact validity range for the asymptotic formula (1.1). This may be viewed as a complement a theorem of Hall (see [2], ch. 7; following a note mentioned by Hall in private correspondence, we slightly modify the statement) according to which

$$H(x, y, z) \sim F(-\Xi) \sum_{r \geq 1} r H_r(x, y, z) = F(-\Xi) \sum_{n \leq x} \tau(n; y, z) \quad (1.4)$$

in the range  $\Xi = o(\log_2 y)^{1/6}$ ,  $x > \exp\{\log z \log_2 z\}$  with

$$F(\xi) := \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\xi/\log 4} e^{-u^2} du.$$

It is likely that (1.4) still holds in the range  $(\log_2 y)^{1/6} \ll \Xi \leq o(\sqrt{\log_2 y})$ .

## 2 Auxiliary estimates

In the sequel, unless otherwise indicated, constants implied by  $O$ - and  $\ll$ - symbols are independent of any parameter.

Let  $m$  be a positive integer. We denote by  $P^+(m)$  and  $P^-(m)$  the smallest prime factor and largest prime factor of  $m$ , respectively, with the convention that  $P^-(1) = \infty$ ,  $P^+(1) = 1$ . We write  $\omega(m)$  for the number of distinct prime factors of  $m$  and  $\Omega(m)$  the number of prime power divisors of  $m$ . We further define

$$\omega(m; t, u) = \sum_{\substack{p^\nu \parallel m \\ t < p \leq u}} 1, \quad \Omega(m; t, u) = \sum_{\substack{p^\nu \parallel m \\ t < p \leq u}} \nu, \quad \bar{\Omega}(m; t) = \Omega(m; 2, t), \quad \bar{\Omega}(m) = \Omega(m; 2, m).$$

Also, we let  $\mathcal{P}(u, v)$  denote the set of integers all of whose prime factors are in  $(u, v]$  and write  $\mathcal{P}^*(u, v)$  for the set of squarefree members of  $\mathcal{P}(u, v)$ . By convention,  $1 \in \mathcal{P}^*(u, v)$ .

**Lemma 2.1.** *There is an absolute constant  $C > 0$  so that for  $\frac{3}{2} \leq u < v$ ,  $v \geq e^4$ ,  $0 \leq \alpha \leq 1/\log v$ , we have*

$$\sum_{\substack{m \in \mathcal{P}(u, v) \\ \omega(m) = k}} \frac{1}{m^{1-\alpha}} \leq \frac{(\log_2 v - \log_2 u + C)^k}{k!}.$$

*Proof.* For a prime  $p \leq v$ , we have  $p^\alpha \leq 1 + 2\alpha \log p$ , thus the sum in question is

$$\leq \frac{1}{k!} \left( \sum_{u < p \leq v} \frac{1}{p^{1-\alpha}} + \frac{1}{p^{2-2\alpha}} + \dots \right)^k \leq \frac{\{\log_2 v - \log_2 u + O(1)\}^k}{k!}.$$

□

**Lemma 2.2.** *Uniformly for  $u \geq 10$ ,  $0 \leq k \leq 2.9 \log_2 u$ , and  $0 \leq \alpha \leq 1/(100 \log u)$ , we have*

$$\sum_{\substack{P^+(m) \leq u \\ \bar{\Omega}(m) = k}} \frac{1}{m^{1-\alpha}} \ll \frac{(\log_2 u)^k}{k!}.$$

*Proof.* We follow the proof of Theorem 08 of [3]. Let  $w$  be a complex number with  $|w| \leq \frac{29}{10}$ . If  $p$  is prime and  $3 \leq p \leq u$ , then  $|w/p^{1-\alpha}| \leq \frac{99}{100}$  and  $p^\alpha \leq 1 + 2\alpha \log p$ . Thus,

$$S(w) := \sum_{P^+(m) \leq u} \frac{w^{\overline{\Omega}(m)}}{m^{1-\alpha}} = \left(1 - \frac{1}{2^{1-\alpha}}\right)^{-1} \prod_{3 \leq p \leq u} \left(1 - \frac{w}{p^{1-\alpha}}\right)^{-1} \ll e^{(\Re w) \log_2 u}.$$

Put  $r := k/\log_2 u$ . By Cauchy's formula and Stirling's formula,

$$\begin{aligned} \sum_{\substack{P^+(m) \leq u \\ \overline{\Omega}(m) = k}} \frac{1}{m^{1-\alpha}} &= \frac{1}{2\pi r^k} \int_{-\pi}^{\pi} e^{-ik\vartheta} S(re^{i\vartheta}) d\vartheta \\ &\ll \frac{(\log_2 u)^k}{k^k} \int_{-\pi}^{\pi} e^{k \cos \vartheta} d\vartheta \ll \frac{(\log_2 u)^k}{k!}. \end{aligned}$$

□

**Lemma 2.3.** *Suppose  $z$  is large,  $0 \leq a + b \leq \frac{5}{2} \log_2 z$  and*

$$\exp\{(\log x)^{9/10}\} \leq w \leq z \leq x, \quad xz^{-1/(10 \log_2 z)} \leq Y \leq x.$$

*The number of integers  $n$  with  $x - Y < n \leq x$ ,  $\overline{\Omega}(n; w) = a$  and  $\omega(n; w, z) = \Omega(n; w, z) = b$ , is*

$$\ll \frac{Y}{\log z} \frac{\{\log_2 w\}^a (b+1) \{\log_2 z - \log_2 w + C\}^b}{a! b!},$$

*where  $C$  is a positive absolute constant.*

*Proof.* There are  $\ll x^{9/10}$  integers with  $n \leq x^{9/10}$  or  $2^j | n$  with  $2^j \geq x^{1/10}$ . For other  $n$ , write  $n = rst$ , where  $P^+(r) \leq w$ ,  $s \in \mathcal{P}^*(w, z)$  and  $P^-(t) > z$ . Here  $\overline{\Omega}(r) = a$  and  $\omega(s) = b$ . We have either  $t = 1$  or  $t > z$ . In the latter case  $x/rs > z$ , whence  $Y/rs > \sqrt{z}$ . We may therefore apply a standard sieve estimate to bound, for given  $r$  and  $s$ , the number of  $t$  by

$$\ll \frac{Y}{rs \log z}.$$

By Lemmas 2.1 and 2.2,

$$\sum_{r,s} \frac{1}{rs} \ll \frac{(\log_2 w)^a (\log_2 z - \log_2 w + C)^b}{a! b!}.$$

If  $t = 1$ , then we may assume  $a + b \geq 1$ . Set  $p = P^+(n)$ . If  $b \geq 1$ , then  $p|s$  and we put  $r_1 := r$  and  $s_1 := s/p$ . Otherwise, let  $r_1 := r/p$  and  $s_1 := s = 1$ . Let  $A := \overline{\Omega}(r_1)$  and  $B := \omega(s_1)$ , so that  $A + B = a + b - 1$  in all circumstances. We have

$$p \geq x^{1/2\overline{\Omega}(n)} \geq x^{1/5 \log_2 z} \geq (x/Y)^2.$$

Define the non-negative integer  $h$  by  $z^{e^{-h-1}} < p \leq z^{e^{-h}}$ . By the Brun-Titchmarsh theorem, we see that, for each given  $r_1$  and  $s_1$ , the number of  $p$  is  $\ll Ye^h/(r_1 s_1 \log z)$ . Set  $\alpha := 0$  if  $h = 0$  and  $\alpha := e^h/(100 \log z)$  otherwise. For  $h \geq 1$ , we have  $r_1 s_1 > x^{3/4} z^{-1/e} > \sqrt{z}$ . Therefore, for  $h \geq 0$ ,

$$\frac{1}{r_1 s_1} \leq \frac{z^{-\alpha/2}}{(r_1 s_1)^{1-\alpha}} \ll \frac{e^{-e^h/200}}{(r_1 s_1)^{1-\alpha}}.$$

Now, Lemmas 2.1 and 2.2 imply that

$$\begin{aligned} \sum_{r_1, s_1} \frac{1}{(r_1 s_1)^{1-\alpha}} &\ll \frac{(\log_2 w)^A (\log_2 z - \log_2 w + C)^B}{A! B!} \\ &\ll (b+1) \frac{(\log_2 w)^a (\log_2 z - \log_2 w + C)^b}{a! b!}, \end{aligned}$$

where we used the fact that  $a \ll \log_2 w$ . Summing over all  $h$ , we derive that the number of those integers  $n > x^{9/10}$  satisfying the conditions of the statement is

$$\ll \frac{Y}{\log z} (b+1) \frac{(\log_2 w)^a (\log_2 z - \log_2 w + C)^b}{a! b!}.$$

Since  $a! b! \leq (3 \log_2 z)^{3 \log_2 z}$ , this last expression is  $> x^{9/10}$ . This completes the proof.  $\square$

Our final lemma is a special case of a theorem of Shiu (Theorem 03 of [3]).

**Lemma 2.4.** *Let  $f$  be a multiplicative function such that  $0 \leq f(n) \leq 1$  for all  $n$ . Then, for all  $x, Y$  with  $1 < \sqrt{x} \leq Y \leq x$ , we have*

$$\sum_{x-Y < n \leq x} f(n) \ll \frac{Y}{\log x} \exp \left\{ \sum_{p \leq x} \frac{f(p)}{p} \right\}.$$

### 3 Decomposition and outline of the proof

Throughout,  $\varepsilon$  will denote a very small positive constant. Since Theorem 1 holds trivially for  $\beta \leq \log 4 - 1 - \varepsilon$ , we henceforth assume that

$$\log 4 - 1 - \varepsilon \leq \beta \leq \log 4 - 1. \quad (3.1)$$

Let

$$K := \lfloor \lambda \log_2 z \rfloor,$$

so that  $(2 - \frac{3}{2}\varepsilon) \log_2 z \leq K \leq 2 \log_2 z$ . In light of (1.2), Theorem 1 reduces to

$$H_2^*(x, y, z) \ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}}. \quad (3.2)$$

At this stage, we notice for further reference that, by Stirling's formula, for  $k \leq K$  we have

$$\frac{\eta(2 \log_2 z)^k}{k! (\log z)^2} \leq \frac{\eta(2 \log_2 z)^K}{K! (\log z)^2} \asymp \frac{1}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}}. \quad (3.3)$$

Let  $\mathcal{H}$  denote the set of integers  $n \leq x$  with  $\tau(n; y, z) \geq 2$ . We count separately the integers  $n \in \mathcal{H}$  lying in 6 classes. In these definitions, we write  $k = \overline{\Omega}(n; z) = K - b$  and for brevity we put  $z_h = z^{e^{-h}}$ . Let

$$K_0 := (2 - 3\varepsilon) \log_2 z$$

and define

$$\begin{aligned}\mathcal{N}_0 &:= \{n \in \mathcal{H} : n \leq x/\log z \text{ or } \exists d > \log z : d^2 | n\}, \\ \mathcal{N}_1 &:= \{n \in \mathcal{H} \setminus \mathcal{N}_0 : k \notin (K_0, K]\}, \\ \mathcal{N}_2 &:= \bigcup_{1 \leq h \leq 5\varepsilon \log_2 z} \mathcal{N}_{2,h}, \\ \text{with } \mathcal{N}_{2,h} &:= \left\{n \in \mathcal{H} \setminus (\mathcal{N}_0 \cup \mathcal{N}_1) : \bar{\Omega}(n; z_h, z) \leq \frac{19}{10}h - \frac{1}{100}b\right\}.\end{aligned}$$

For integers  $n \in \mathcal{N}_2$ , we will only use the fact that  $\tau(n; y, z) \geq 1$ . Integers in other classes do not have too many small prime factors and it is sufficient to count pairs of divisors  $d_1, d_2$  of  $n$  in  $(y, z]$ . For each such pair, write  $v = (d_1, d_2)$ ,  $d_1 = vf_1$ ,  $d_2 = vf_2$ ,  $n = f_1 f_2 v u$  and assume  $f_1 < f_2$ . Let

$$F_1 = \bar{\Omega}(f_1), \quad F_2 = \bar{\Omega}(f_2), \quad V = \bar{\Omega}(v), \quad U = \bar{\Omega}(u, z), \quad (3.4)$$

and

$$Z := \exp\{(\log z)^{1-4\varepsilon}\}. \quad (3.5)$$

For further reference, we note that if  $n \notin \mathcal{N}_0$  and  $h \leq 5\varepsilon \log_2 z$ , then

$$\bar{\Omega}(n; z_h, z) = \omega(n; z_h, z).$$

Now we define  $\mathcal{H}^* := \mathcal{H} \setminus (\mathcal{N}_0 \cup \mathcal{N}_1 \cup \mathcal{N}_2)$  and

$$\begin{aligned}\mathcal{N}_3 &:= \{n \in \mathcal{H}^* : \min(u, f_2) \leq Z\}, \\ \mathcal{N}_4 &:= \{n \in \mathcal{H}^* : \min(u, f_2) > z^{1/10}\}, \\ \mathcal{N}_5 &:= \{n \in \mathcal{H}^* : Z < \min(u, f_2) \leq z^{1/10}\}.\end{aligned}$$

In the above decomposition, the main parts are  $\mathcal{N}_2$  and  $\mathcal{N}_5$ . We expect  $\mathcal{N}_2$  to be small since, conditionally on  $\bar{\Omega}(n; z) = k$ , the normal value of  $\bar{\Omega}(n; z_h, z)$  is  $hk/\log_2 z > \frac{19}{10}h$ . It is more difficult to see that  $\mathcal{N}_5$  is small too. This follows from the fact that we count integers in this set according to their number of factorizations in the form  $n = uvf_1 f_2$  with  $y < vf_1 < vf_2 \leq z$ . Suppose for instance that  $f_1, f_2 \leq z_j$ . For  $\bar{\Omega}(n; z) = k$  and  $\bar{\Omega}(n; z_j, z) = G$ , then, ignoring the localization of  $vf_1$  and  $vf_2$  in  $(y, z]$ , there are  $4^{k-G}2^G = 4^k 2^{-G}$  such factorizations. Thus, larger  $G$  means fewer factorizations. On probabilistic grounds, larger  $G$  should also mean fewer factorizations with the localization of  $vf_1$  and  $vf_2$ .

We now briefly consider the cases of  $\mathcal{N}_0$  and  $\mathcal{N}_1$ .

Trivially,

$$|\mathcal{N}_0| \leq \frac{x}{\log z} + \sum_{d > \log z} \frac{x}{d^2} \ll \frac{x}{\log z} \ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}}, \quad (3.6)$$

since  $Q(\lambda) \leq Q(2) = \log 4 - 1$  in the range under consideration.

By the argument on pages 40–41 of [3],

$$\sum_{\substack{n \leq x \\ \bar{\Omega}(n; z) > K}} 1 \ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}}.$$

Setting  $t := 1 - \frac{3}{2}\varepsilon$ , Lemma 2.4 gives

$$\begin{aligned} \sum_{\substack{n \leq x \\ \tau(n; y, z) \geq 1 \\ \bar{\Omega}(n; z) \leq K_0}} 1 &\leq t^{-(2-3\varepsilon)\log_2 z} \sum_{\substack{dm \leq x \\ y < d \leq z}} t^{\bar{\Omega}(d) + \bar{\Omega}(m; z)} \ll x(\log z)^{2t-2-\beta-(2-3\varepsilon)\log t} \\ &\ll x(\log y)^{-\beta-2\varepsilon^2} \ll x(\log y)^{-Q(\lambda)-\varepsilon^2/2}. \end{aligned}$$

Therefore,

$$|\mathcal{N}_1| \ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}}. \quad (3.7)$$

In the next four sections, we show that

$$|\mathcal{N}_j| \ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}} \quad (2 \leq j \leq 5). \quad (3.8)$$

Together with (3.6) and (3.7), this will complete the proof of Theorem 1.

## 4 Estimation of $|\mathcal{N}_2|$

We plainly have  $|\mathcal{N}_2| \leq \sum_h |\mathcal{N}_{2,h}|$ . For  $1 \leq h \leq 5\varepsilon \log_2 z$ , the numbers  $n \in \mathcal{N}_{2,h}$  satisfy

$$\begin{cases} x/\log z < n \leq x, \\ k := \bar{\Omega}(n; z) = K - b, \quad 0 \leq b \leq 3\varepsilon \log_2 z, \\ \bar{\Omega}(n; z_h, z) \leq \frac{19}{10}h - \frac{1}{100}b, \end{cases}$$

We note at the outset that  $\mathcal{N}_{2,h}$  is empty unless  $h \geq b/190$ .

Write  $n = du$  with  $y < d \leq z$  and  $u \leq x/y$ . Let

$$\bar{\Omega}(d; z_h) = D_1, \quad \Omega(d; z_h, z) = D_2, \quad \bar{\Omega}(u; z_h) = U_1, \quad \Omega(u; z_h, z) = U_2,$$

so that  $D_1 + D_2 \geq 1$ ,  $D_2 + U_2 \leq \frac{19}{10}h - \frac{1}{100}b$  and  $D_1 + D_2 + U_1 + U_2 = k$ .

Fix  $k = K - b$ ,  $h$ ,  $D_1$ ,  $D_2$ ,  $U_1$  and  $U_2$ . By Lemma 2.3 (with  $w = z_h$ ,  $a = U_1$ ,  $b = U_2$ ), the number of  $u$  is

$$\ll \frac{x}{y \log z} \frac{(\log_2 z - h)^{U_1}}{U_1!} (U_2 + 1) \frac{(h + C)^{U_2}}{U_2!}.$$

A second application of Lemma 2.3 yields that the number of  $d$  is

$$\ll \frac{\eta y}{\log z} \frac{(\log_2 z - h)^{D_1}}{D_1!} (D_2 + 1) \frac{(h + C)^{D_2}}{D_2!}.$$

Since  $D_2 + U_2 < 2h$ , we have  $(h + C)^{U_2 + D_2} \leq e^{2C} h^{U_2 + D_2}$ . Summing over  $D_1, D_2, U_1, U_2$  with  $G = D_2 + U_2$  fixed and using the binomial theorem, we find that the number of  $n$  in question is

$$\ll \frac{\eta x}{(\log z)^2} (\log_2 z - h)^{k-G} h^G (G + 1)^2 \sum_{\substack{U_1 + D_1 = k - G \\ D_2 + U_2 = G}} \frac{1}{U_1! D_1! D_2! U_2!} \ll \frac{\eta x 2^k}{(\log z)^2} A(h, G),$$



where

$$A(h, G) = (G + 1)^2 \frac{(\log_2 z - h)^{k-G} h^G}{(k - G)! G!}.$$

Since  $G + 1 \leq G_h := \lfloor \frac{19}{10} h \rfloor$ , we have

$$\frac{A(h, G + 1)}{A(h, G)} \geq \frac{h(k - G)}{(G + 1)(\log_2 z - h)} \geq \frac{k - 10\varepsilon \log_2 z}{1.9(1 - 5\varepsilon) \log_2 z} > \frac{21}{20}$$

if  $\varepsilon$  is small enough. Next,

$$\begin{aligned} A(h, G_h) &\leq (G_h + 1)^2 \frac{(\log_2 z - h)^{k-G_h} (hk)^{G_h}}{k! (G_h/e)^{G_h}} \\ &\ll (h + 1)^2 \frac{(\log_2 z)^k}{k!} \left(\frac{20}{19}e\right)^{19h/10} e^{-h(k-G_h)/\log_2 z} \\ &\ll \frac{(\log_2 z)^k}{k!} e^{-h/500}, \end{aligned}$$

since  $(k - G_h)/\log_2 z > 2 - 13\varepsilon$  and  $\frac{19}{10} \log(\frac{20}{19}e) < 2 - 1/400$ . Thus,

$$\sum_{b/190 \leq h \leq 5\varepsilon \log_2 z} \sum_{0 \leq G \leq G_h} A(h, G) \ll \sum_{b/190 \leq h \leq 5\varepsilon \log_2 z} A(h, G_h) \ll \frac{(\log_2 z)^k}{k!} e^{-b/95000}$$

and so

$$\sum_{\substack{n \in \mathcal{N}_2 \\ \Omega(n; z) = k}} 1 \ll \frac{\eta x (2 \log_2 z)^k}{(\log z)^2 k!} e^{-(K-k)/95000} \ll \frac{x e^{-(K-k)/95000}}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}},$$

by (3.3). Summing over the range  $K_0 \leq k \leq K$  furnishes the required estimate (3.8) for  $j = 2$ .

## 5 Estimation of $|\mathcal{N}_3|$

All integers  $n = f_1 f_2 uv$  counted in  $\mathcal{N}_3$  verify

$$\begin{cases} x/\log z < n \leq x, \\ \bar{\Omega}(n; z) \leq K, \\ y < v f_1 < v f_2 \leq z, \quad \min(u, f_2) \leq Z, \end{cases}$$

where  $Z$  is defined in (3.5). This is all we shall use in bounding  $|\mathcal{N}_3|$ .

Let  $\mathcal{N}_{3,1}$  be the subset corresponding to the condition  $f_2 \leq Z$  and let  $\mathcal{N}_{3,2}$  comprise those  $n \in \mathcal{N}_3$  such that  $u \leq Z$ .

If  $f_2 \leq Z$ , then  $v > z^{1/2}$  and  $u > x/\{v Z^2 \log z\} > x^{1/3}$ . For  $\frac{1}{2} \leq t \leq 1$  we have

$$\begin{aligned} |\mathcal{N}_{3,1}| &\leq \sum_{f_1, f_2, v, u} t^{\bar{\Omega}(f_1 f_2 uv; z) - K} \\ &= t^{-K} \sum_{f_1 \leq Z} t^{\bar{\Omega}(f_1)} \sum_{f_1 < f_2 \leq e^\eta f_1} t^{\bar{\Omega}(f_2)} \sum_{y/f_1 < v \leq z/f_1} t^{\bar{\Omega}(v)} \sum_{u \leq x/f_1 f_2 v} t^{\bar{\Omega}(u; z)}. \end{aligned}$$

Apply Lemma 2.4 to the three innermost sums. The  $u$ -sum is

$$\ll \frac{x}{f_1 f_2 v} (\log z)^{t-1} \leq \frac{x}{f_1 y} (\log z)^{t-1}.$$

and the  $v$ -sum is

$$\ll \frac{\eta y}{f_1} (\log z)^{t-1}.$$

The  $f_2$ -sum is  $\ll \eta f_1 (\log f_1)^{t-1}$  if  $f_1 > \eta^{-3}$  and otherwise is  $\ll \eta f_1$  trivially (note that  $\eta f_1 \gg 1$  follows from the fact that  $(f_1 + 1)/f_1 \leq f_2/f_1 \leq e^\eta$ ). Next

$$\begin{aligned} \sum_{f_1 \leq \eta^{-3}} \frac{1}{f_1} + \sum_{2 \leq f_1 \leq Z} \frac{t^{\bar{\Omega}(f_1)}}{f_1} (\log f_1)^{t-1} &\ll \log_2 z + (\log_2 z) \max_{j \leq \log_2 Z} e^{j(t-1)} \sum_{f_1 \leq \exp\{e^j\}} \frac{t^{\bar{\Omega}(f_1)}}{f_1} \\ &\ll (\log_2 z) (\log Z)^{2t-1}. \end{aligned}$$

Thus,

$$|\mathcal{N}_{3,1}| \ll x (\log_2 x) (\log x)^E$$

with  $E = -2\beta - \lambda \log t + 2t - 2 + (2t - 1)(1 - 4\varepsilon)$ . We select optimally  $t := \frac{1}{4}\lambda/(1 - 2\varepsilon)$ , and check that  $t \geq \frac{1}{2}$  since  $\lambda \geq 2 - \varepsilon/\log 2$ . Then

$$\begin{aligned} E &= -Q(\lambda) + \lambda \log(1 - 2\varepsilon) + 4\varepsilon \leq -Q(\lambda) + (2 - \varepsilon/\log 2)(-2\varepsilon - 2\varepsilon^2) + 4\varepsilon \\ &< -Q(\lambda) - \varepsilon^2. \end{aligned}$$

Next, we consider the case when  $u \leq Z$ . We observe that this implies

$$\frac{1}{4}vz^2 \leq vx \leq vn \log z = u f_1 v f_2 v \log z \leq Z z^2 \log z$$

hence  $v \leq 4Z \log z \leq Z^2$ , and therefore

$$\min(f_1, f_2) > z^{1/2}.$$

Also,  $z > x^{1/3}$  since  $x/\log z < n = uv f_1 f_2 \leq Z z^2$ . Thus, for  $\frac{1}{2} \leq t \leq 1$ , we have

$$\begin{aligned} |\mathcal{N}_{3,2}| &\leq \sum_{f_1, f_2, v, u} t^{\bar{\Omega}(f_1 f_2 uv; z) - K} \\ &= t^{-K} \sum_{v \leq Z^2} t^{\bar{\Omega}(v)} \sum_{u \leq xv/y^2} t^{\bar{\Omega}(u)} \sum_{y/v < f_1 \leq z/v} t^{\bar{\Omega}(f_1)} \sum_{y/v < f_2 \leq z/v} t^{\bar{\Omega}(f_2)}. \end{aligned}$$

The sums upon  $f_1$  and  $f_2$  are each

$$\ll \frac{\eta y}{v} (\log z)^{t-1}$$

and the  $u$ -sum is

$$\ll \frac{xv}{y^2} (\log 2xv/y^2)^{t-1} \leq \frac{xv}{y^2} (\log 2v)^{t-1}.$$

Thus, selecting the same value  $t := \frac{1}{4}\lambda/(1 - 2\varepsilon)$ , we obtain

$$\begin{aligned} |N_{3,2}| &\ll t^{-K} x \eta^2 (\log z)^{2t-1} \sum_{v \leq Z^2} \frac{t^{\bar{\Omega}(v)} (\log 2v)^{t-1}}{v} \\ &\ll x (\log_2 z) (\log z)^E \leq x (\log_2 z) (\log z)^{-Q(\lambda) - \varepsilon^2}. \end{aligned}$$

This completes the proof of (3.8) with  $j = 3$ .

## 6 Estimation of $|\mathcal{N}_4|$

We now consider those integers  $n = f_1 f_2 u v$  such that

$$\begin{cases} x/\log z < n \leq x, \\ k := \bar{\Omega}(n; z) = K - b, \quad 0 \leq b \leq 3\varepsilon \log_2 z, \\ y < v f_1 < v f_2 \leq z, \quad \min(u, f_2) > z^{1/10}. \end{cases}$$

With the notation (3.4), fix  $k, F_1, F_2, U$  and  $V$ . Here  $u, f_1$  and  $f_2$  are all  $> \frac{1}{2}z^{1/10}$ . By Lemma 2.3 (with  $w = z$ ), for each triple  $f_1, f_2, v$  the number of  $u$  is

$$\ll \frac{x}{f_1 f_2 v \log z} \frac{(\log_2 z)^U}{U!}.$$

Using Lemma 2.3 two more times, we obtain, for each  $v$ ,

$$\sum_{y/v < f_1 \leq z/v} \frac{1}{f_1} \sum_{y/v < f_2 \leq z/v} \frac{1}{f_2} \ll \frac{\eta^2}{(\log z)^2} \frac{(\log_2 z)^{F_1+F_2}}{F_1! F_2!}.$$

Now, Lemma 2.2 gives

$$\sum_v \frac{1}{v} \ll \frac{(\log_2 z)^V}{V!}.$$

Gathering these estimates and using (3.3) yields

$$\begin{aligned} |\mathcal{N}_4| &\ll \frac{x\eta^2}{(\log z)^3} \sum_{(2-3\varepsilon)\log_2 z \leq k \leq K} \sum_{F_1+F_2+U+V=k} \frac{(\log_2 z)^k}{F_1! F_2! U! V!} \\ &= \frac{x\eta^2}{(\log z)^3} \sum_{(2-3\varepsilon)\log_2 z \leq k \leq K} \frac{(2\log_2 z)^k}{k!} 2^k \\ &\ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y} \log z} \frac{2^K \eta}{\sqrt{\log_2 y} \log z} \ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}}. \end{aligned}$$

Thus (3.8) holds for  $j = 4$ .

## 7 Estimation of $|\mathcal{N}_5|$

It is plainly sufficient to bound the number of those  $n = f_1 f_2 u v$  satisfying the following conditions

$$\begin{cases} x/\log z < n \leq x, \\ k := \bar{\Omega}(n; z) = K - b, \quad 0 \leq b \leq 3\varepsilon \log_2 z, \\ \bar{\Omega}(n; z_h, z) > \frac{19}{10}h - \frac{1}{100}b \quad (1 \leq h \leq 5\varepsilon \log_2 z) \\ y < v f_1 < v f_2 \leq z, \quad Z < \min(u, f_2) \leq z^{1/10}. \end{cases}$$

Define  $j$  by  $z_{j+2} < \min(u, f_2) \leq z_{j+1}$ . We have  $1 \leq j \leq 5\varepsilon \log_2 z$ . Let  $\mathcal{N}_{5,1}$  be the set of those  $n$  satisfying the above conditions with  $u \leq z_{j+1}$  and let  $\mathcal{N}_{5,2}$  be the complementary set, for which  $f_2 \leq z_{j+1}$ .

If  $u \leq z_{j+1}$ , then  $v \leq (z^2 u \log z)/x \leq 4u \log z \leq z_j$  and  $f_2 > f_1 > z^{1/2}$ . Recall notation (3.4) and write

$$F_{11} := \overline{\Omega}(f_1; z_j), \quad F_{12} := \Omega(f_1; z_j, z), \quad F_{21} := \overline{\Omega}(f_2; z_j), \quad F_{22} := \Omega(f_2; z_j, z),$$

so that the initial condition upon  $\overline{\Omega}(n; z_h, z)$  with  $h = j$  may be rewritten as

$$F_{12} + F_{22} \geq G_j := \max(0, \lfloor \frac{19}{10}j - b/100 \rfloor).$$

We count those  $n$  in a dyadic interval  $(X, 2X]$ , where  $x/(2 \log z) \leq X \leq x$ . Fix  $k, j, X, U, V, F_{rs}$  and apply Lemma 2.3 to sums over  $u, f_1, f_2$ . The number of  $n$  is question is

$$\begin{aligned} &\leq \sum_{v \leq z_j} \sum_{vX/z^2 \leq u \leq 2vX/y^2} \sum_{y/v < f_1 \leq z/v} \sum_{y/v < f_2 \leq z/v} 1 \\ &\ll \frac{\eta^2 X e^j}{(\log z)^3} \frac{(\log_2 z - j)^{U+F_{11}+F_{21}}}{U!F_{11}!F_{21}!} (F_{12} + 1)(F_{22} + 1) \frac{(j + C)^{F_{12}+F_{22}}}{F_{12}!F_{22}!} \sum_{v \leq z_j} \frac{1}{v}. \end{aligned}$$

Bounding the  $v$ -sum by Lemma 2.2, and summing over  $X, U, V, F_{rs}$  with  $F_{12} + F_{22} = G$  yields

$$|\mathcal{N}_{5,1}| \ll \frac{\eta^2 x}{(\log z)^3} \sum_{(2-3\varepsilon) \log_2 z \leq k \leq K} 4^k \sum_{1 \leq j \leq 5\varepsilon \log_2 z} \sum_{G_j \leq G \leq k} M(j, G),$$

where

$$M(j, G) := e^j (G + 1)^2 \frac{(\log_2 z - j)^{k-G} (j + C)^G}{2^G (k - G)! G!}.$$

Let  $j_b = \lfloor \frac{1}{2}b + 100C + 100 \rfloor$ . If  $j \leq j_b$ , then  $j + C \leq \frac{99}{100}(j + C_b)$  with  $C_b := 3C + 2 + \frac{b}{100}$  and, introducing  $R := \max_{G \geq 0} \{(G + 1)^2 (\frac{99}{100})^G\}$ , we have

$$\begin{aligned} \sum_{1 \leq j \leq j_b} \sum_{G_j \leq G \leq k} M(j, G) &\leq R \sum_{1 \leq j \leq j_b} e^j \sum_{0 \leq G \leq k} \frac{(\log_2 z - j)^{k-G} (j + C_b)^G}{2^G G! (k - G)!} \\ &\ll \frac{1}{k!} \sum_{1 \leq j \leq j_b} e^j (\log_2 z - \frac{1}{2}j + \frac{1}{2}C_b)^k \\ &\ll \frac{(\log_2 z)^k}{k!} \sum_{1 \leq j \leq j_b} e^{j+(b/200-j/2)k/\log_2 z} \\ &\ll \frac{(\log_2 z)^k}{k!} e^{b/100+2\varepsilon j_b} \ll \frac{(\log_2 z)^k}{k!} e^{b/50}. \end{aligned}$$

When  $j > j_b$ , then

$$G_j \geq \frac{9}{5}(j + C) + \frac{1}{10}(j_b + C + 1) - \frac{1}{100}b - 1 \geq \frac{9}{5}(j + C) + 9 \geq 189.$$

Thus, for  $G \geq G_j$  we have

$$\frac{M(j, G+1)}{M(j, G)} = \left( \frac{G+2}{G+1} \right)^2 \frac{j+C}{2(G+1)} \frac{k-G}{\log_2 z - j} \leq \frac{4}{7}.$$

Therefore,

$$\begin{aligned} \sum_{G_j \leq G \leq k} M(j, G) &\ll M(j, G_j) \ll \frac{j^2 e^j (\log_2 z - j)^{k-G_j} (jk)^{G_j}}{k! 2^{G_j} G_j!} \\ &\leq \frac{j^2 e^j (\log_2 z)^k}{k!} e^{-j(k-G_j)/\log_2 z} \left( \frac{e j k}{2 G_j \log_2 z} \right)^{G_j} \ll \frac{(\log_2 z)^k}{k!} e^{-j/5}, \end{aligned}$$

since  $k - G_j \geq (2 - 10\varepsilon) \log_2 z$ ,  $e j k / (2 G_j \log_2 z) \leq \frac{5}{9} e$ , and  $-1 + \frac{19}{10} \log(\frac{5}{9} e) < -\frac{1}{5}$ . We conclude that

$$\sum_{1 \leq j \leq 5\varepsilon \log_2 z} \sum_{G_j \leq G \leq k} M(j, G) \ll \frac{(\log_2 z)^k}{k!} e^{b/50} \quad (7.1)$$

and hence, by (3.3),

$$|\mathcal{N}_{5,1}| \ll \frac{\eta^2 x}{(\log z)^3} \sum_{k \leq K} \frac{(2 \log_2 z)^k}{k!} 2^{K-b/2} \ll \frac{\eta^2 2^K x}{(\log z)^3} \frac{(2 \log_2 z)^K}{K!} \ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}}.$$

Now assume  $f_2 \leq z_{j+1}$ . Then  $\min(u, v) > \sqrt{z}$ . Fix  $F_1, F_2$  and

$$\bar{\Omega}(v; z_j) = V_1, \quad \Omega(v; z_j, z) = V_2, \quad \bar{\Omega}(u; z_j) = U_1, \quad \Omega(u; z_j, z) = U_2.$$

By Lemma 2.3, given  $f_1, f_2$  and  $v$ , the number of  $u$  is

$$\ll \frac{x}{f_1 f_2 v \log z} \frac{(\log_2 z - j)^{U_1} (U_2 + 1) (j + C)^{U_2}}{U_1! U_2!}.$$

Applying Lemma 2.3 again, for each  $f_1$  we have

$$\sum_{\substack{f_1 < f_2 \leq e^\eta f_1 \\ y/f_1 < v \leq z/f_1}} \frac{1}{f_2 v} \ll \frac{\eta^2 e^j}{(\log z)^2} \frac{(V_2 + 1) (\log_2 z - j)^{V_1 + F_2} (j + C)^{V_2}}{V_1! V_2! F_2!}.$$

By Lemma 2.2,

$$\sum_{f_1 \leq z_j} \frac{1}{f_1} \ll \frac{(\log_2 z - j)^{F_1}}{F_1!}.$$

Combine these estimates, and sum over  $F_1, F_2, U_1, U_2, V_1, V_2$  with  $V_2 + U_2 = G$ . As in the estimation of  $|\mathcal{N}_{5,1}|$ , sum over  $k, j, G$  using (3.3) and (7.1). We obtain

$$\begin{aligned} |\mathcal{N}_{5,2}| &\ll \frac{\eta^2 x}{(\log z)^3} \sum_{(2-3\varepsilon) \log_2 z \leq k \leq K} 4^k \sum_{1 \leq j \leq 5\varepsilon \log_2 z} \sum_{G_j < G \leq k} M(j, G) \\ &\ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}}. \end{aligned}$$

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