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The distribution of integers with at least two divisors in a short interval

Kevin Ford* Gérald Tenenbaum

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1 Introduction

Whereas, in usual cases, sieving by a set of primes may be fairly well controlled, through Buchstab's identity, sieving by a set of integers is a much more complicated task. However, some fairly precise results are known in the case where the set of integers is an interval. We refer to the recent work [1] of the first author for specific statements and references.

Define

$$\begin{aligned}\tau(n; y, z) &:= |\{d|n : y < d \leq z\}|, \\ H(x, y, z) &:= |\{n \leq x : \tau(n; y, z) \geq 1\}|, \\ H_r(x, y, z) &:= |\{n \leq x : \tau(n; y, z) = r\}|, \\ H_2^*(x, y, z) &:= |\{n \leq x : \tau(n; y, z) \geq 2\}| = \sum_{r \geq 2} H_r(x, y, z).\end{aligned}$$

Thus, the numbers $H_r(x, y, z)$ ($r \geq 1$) describe the local laws of the function $\tau(n; y, z)$. When y and z are close, it is expected that, if an integer has at least a divisor in $(y, z]$, then it usually has exactly one, in other words

$$H(x, y, z) \sim H_1(x, y, z). \tag{1.1}$$

In this paper, we address the problem of determining the exact range of validity of such behavior. In other words, we search a necessary and sufficient condition so that $H_2^*(x, y, z) = o(H(x, y, z))$ as x and y tend to infinity.

As shown in [5], for given y , the threshold for the behavior of the function $H(x, y, z)$ lies near the critical value

$$z = z_0(y) := y \exp\{(\log y)^{1-\log 4}\} \approx y + y/(\log y)^{\log 4 - 1}.$$

We concentrate on the case $z_0(y) \leq z \leq 2y$. Define

$$\begin{aligned}z &= e^\eta y, \quad \eta = (\log y)^{-\beta}, \quad \beta = \log 4 - 1 - \Xi/\sqrt{\log_2 y}, \quad \lambda = \frac{1 + \beta}{\log 2}, \\ Q(w) &= \int_1^w \log t \, dt = w \log w - w + 1.\end{aligned}$$

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Here \log_k denotes the k th iterate of the logarithm.

With the above notation, we have

$$\log(z/y) = \frac{e^{\Xi\sqrt{\log_2 y}}}{(\log y)^{\log 4 - 1}}, \quad \log\{z/z_0(y)\} = \frac{e^{\Xi\sqrt{\log_2 y}} - 1}{(\log y)^{\log 4 - 1}},$$

so

$$\begin{aligned} 0 &\leq \Xi \leq (\log 4 - 1)\sqrt{\log_2 y} + \frac{\log_2 2}{\sqrt{\log_2 y}}, \\ \frac{|\log_2 2|}{\log_2 y} &\leq \beta \leq \log 4 - 1, \\ \frac{1}{\log 2} + \frac{\kappa}{\log_2 y} &\leq \lambda \leq 2, \end{aligned}$$

with $\kappa := |\log_2 2|/\log 2$.

From Theorem 1 of [1], we know that, uniformly in $10 \leq y \leq \sqrt{x}$, $z_0(y) \leq z \leq 2y$,

$$H(x, y, z) \asymp \frac{\beta x}{(\Xi + 1)(\log y)^{Q(\lambda)}}. \quad (1.2)$$

By Theorems 5 and 6 of [1], for any $c > 0$ and uniformly in $y_0(r) \leq y \leq x^{1/2-c}$, $z_0(y) \leq z \leq 2y$, we have

$$\begin{aligned} \frac{H_1(x, y, z)}{H(x, y, z)} &\asymp_c 1, \\ \frac{\Xi + 1}{\sqrt{\log_2 y}} &\ll_{r,c} \frac{H_r(x, y, z)}{H(x, y, z)} \leq 1 \quad (r \geq 2). \end{aligned} \quad (1.3)$$

When $0 \leq \Xi \leq o(\sqrt{\log_2 y})$ and $r \geq 2$, the upper and lower bounds above for $H_r(x, y, z)$ have different orders. We show in this paper that the lower bound represents the correct order of magnitude.

Theorem 1. *Uniformly in $10 \leq y \leq \sqrt{x}$, $z_0(y) \leq z \leq 2y$, we have*

$$\frac{H_2^*(x, y, z)}{H(x, y, z)} \ll \frac{\Xi + 1}{\sqrt{\log_2 y}}.$$

Corollary 2. *Let $r \geq 2$ and $c > 0$. Uniformly in $y_0(r, c) \leq y \leq x^{1/2-c}$, $z_0(y) \leq z \leq 2y$, we have*

$$\frac{H_r(x, y, z)}{H(x, y, z)} \asymp_{r,c} \frac{\Xi + 1}{\sqrt{\log_2 y}}.$$

Corollary 3. *Uniformly in $10 \leq y \leq \sqrt{x}$, $y < z \leq y + y(\log y)^{1-\log 4+o(1)}$, we have*

$$H_1(x, y, z) \sim H(x, y, z).$$

Since we know from (1.3) that $H_2^*(x, y, z) \gg H(x, y, z)$ when $\beta \leq \log 4 - 1 - \varepsilon$ for any $\varepsilon > 0$ we have therefore completely answered the question raised at the beginning of this introduction concerning the exact validity range for the asymptotic formula (1.1). This may be viewed as a complement a theorem of Hall (see [2], ch. 7; following a note mentioned by Hall in private correspondence, we slightly modify the statement) according to which

$$H(x, y, z) \sim F(-\Xi) \sum_{r \geq 1} r H_r(x, y, z) = F(-\Xi) \sum_{n \leq x} \tau(n; y, z) \quad (1.4)$$

in the range $\Xi = o(\log_2 y)^{1/6}$, $x > \exp\{\log z \log_2 z\}$ with

$$F(\xi) := \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\xi/\log 4} e^{-u^2} du.$$

It is likely that (1.4) still holds in the range $(\log_2 y)^{1/6} \ll \Xi \leq o(\sqrt{\log_2 y})$.

2 Auxiliary estimates

In the sequel, unless otherwise indicated, constants implied by O - and \ll -symbols are independent of any parameter.

Let m be a positive integer. We denote by $P^+(m)$ and $P^-(m)$ the smallest prime factor and largest prime factor of m , respectively, with the convention that $P^-(1) = \infty$, $P^+(1) = 1$. We write $\omega(m)$ for the number of distinct prime factors of m and $\Omega(m)$ the number of prime power divisors of m . We further define

$$\omega(m; t, u) = \sum_{\substack{p^\nu \parallel m \\ t < p \leq u}} 1, \quad \Omega(m; t, u) = \sum_{\substack{p^\nu \parallel m \\ t < p \leq u}} \nu, \quad \bar{\Omega}(m; t) = \Omega(m; 2, t), \quad \bar{\Omega}(m) = \Omega(m; 2, m).$$

Also, we let $\mathcal{P}(u, v)$ denote the set of integers all of whose prime factors are in $(u, v]$ and write $\mathcal{P}^*(u, v)$ for the set of squarefree members of $\mathcal{P}(u, v)$. By convention, $1 \in \mathcal{P}^*(u, v)$.

Lemma 2.1. *There is an absolute constant $C > 0$ so that for $\frac{3}{2} \leq u < v$, $v \geq e^4$, $0 \leq \alpha \leq 1/\log v$, we have*

$$\sum_{\substack{m \in \mathcal{P}(u, v) \\ \omega(m) = k}} \frac{1}{m^{1-\alpha}} \leq \frac{(\log_2 v - \log_2 u + C)^k}{k!}.$$

Proof. For a prime $p \leq v$, we have $p^\alpha \leq 1 + 2\alpha \log p$, thus the sum in question is

$$\leq \frac{1}{k!} \left(\sum_{u < p \leq v} \frac{1}{p^{1-\alpha}} + \frac{1}{p^{2-2\alpha}} + \dots \right)^k \leq \frac{\{\log_2 v - \log_2 u + O(1)\}^k}{k!}.$$

□

Lemma 2.2. *Uniformly for $u \geq 10$, $0 \leq k \leq 2.9 \log_2 u$, and $0 \leq \alpha \leq 1/(100 \log u)$, we have*

$$\sum_{\substack{P^+(m) \leq u \\ \bar{\Omega}(m) = k}} \frac{1}{m^{1-\alpha}} \ll \frac{(\log_2 u)^k}{k!}.$$

Proof. We follow the proof of Theorem 08 of [3]. Let w be a complex number with $|w| \leq \frac{29}{10}$. If p is prime and $3 \leq p \leq u$, then $|w/p^{1-\alpha}| \leq \frac{99}{100}$ and $p^\alpha \leq 1 + 2\alpha \log p$. Thus,

$$S(w) := \sum_{P^+(m) \leq u} \frac{w^{\bar{\Omega}(m)}}{m^{1-\alpha}} = \left(1 - \frac{1}{2^{1-\alpha}}\right)^{-1} \prod_{3 \leq p \leq u} \left(1 - \frac{w}{p^{1-\alpha}}\right)^{-1} \ll e^{(\Re w) \log_2 u}.$$

Put $r := k/\log_2 u$. By Cauchy's formula and Stirling's formula,

$$\begin{aligned} \sum_{\substack{P^+(m) \leq u \\ \bar{\Omega}(m) = k}} \frac{1}{m^{1-\alpha}} &= \frac{1}{2\pi r^k} \int_{-\pi}^{\pi} e^{-ik\vartheta} S(re^{i\vartheta}) d\vartheta \\ &\ll \frac{(\log_2 u)^k}{k^k} \int_{-\pi}^{\pi} e^{k \cos \vartheta} d\vartheta \ll \frac{(\log_2 u)^k}{k!}. \end{aligned}$$

□

Lemma 2.3. Suppose z is large, $0 \leq a + b \leq \frac{5}{2} \log_2 z$ and

$$\exp\{(\log x)^{9/10}\} \leq w \leq z \leq x, \quad xz^{-1/(10 \log_2 z)} \leq Y \leq x.$$

The number of integers n with $x - Y < n \leq x$, $\bar{\Omega}(n; w) = a$ and $\omega(n; w, z) = \Omega(n; w, z) = b$, is

$$\ll \frac{Y}{\log z} \frac{\{\log_2 w\}^a (b+1) \{\log_2 z - \log_2 w + C\}^b}{a! b!},$$

where C is a positive absolute constant.

Proof. There are $\ll x^{9/10}$ integers with $n \leq x^{9/10}$ or $2^j | n$ with $2^j \geq x^{1/10}$. For other n , write $n = rst$, where $P^+(r) \leq w$, $s \in \mathcal{P}^*(w, z)$ and $P^-(t) > z$. Here $\bar{\Omega}(r) = a$ and $\omega(s) = b$. We have either $t = 1$ or $t > z$. In the latter case $x/rs > z$, whence $Y/rs > \sqrt{z}$. We may therefore apply a standard sieve estimate to bound, for given r and s , the number of t by

$$\ll \frac{Y}{rs \log z}.$$

By Lemmas 2.1 and 2.2,

$$\sum_{r,s} \frac{1}{rs} \ll \frac{(\log_2 w)^a (\log_2 z - \log_2 w + C)^b}{a! b!}.$$

If $t = 1$, then we may assume $a + b \geq 1$. Set $p = P^+(n)$. If $b \geq 1$, then $p|s$ and we put $r_1 := r$ and $s_1 := s/p$. Otherwise, let $r_1 := r/p$ and $s_1 := s = 1$. Let $A := \bar{\Omega}(r_1)$ and $B := \omega(s_1)$, so that $A + B = a + b - 1$ in all circumstances. We have

$$p \geq x^{1/2\bar{\Omega}(n)} \geq x^{1/5 \log_2 z} \geq (x/Y)^2.$$

Define the non-negative integer h by $z^{e^{-h-1}} < p \leq z^{e^{-h}}$. By the Brun-Titchmarsh theorem, we see that, for each given r_1 and s_1 , the number of p is $\ll Ye^h/(r_1 s_1 \log z)$. Set $\alpha := 0$ if $h = 0$ and $\alpha := e^h/(100 \log z)$ otherwise. For $h \geq 1$, we have $r_1 s_1 > x^{3/4} z^{-1/e} > \sqrt{z}$. Therefore, for $h \geq 0$,

$$\frac{1}{r_1 s_1} \leq \frac{z^{-\alpha/2}}{(r_1 s_1)^{1-\alpha}} \ll \frac{e^{-e^h/200}}{(r_1 s_1)^{1-\alpha}}.$$

Now, Lemmas 2.1 and 2.2 imply that

$$\begin{aligned} \sum_{r_1, s_1} \frac{1}{(r_1 s_1)^{1-\alpha}} &\ll \frac{(\log_2 w)^A (\log_2 z - \log_2 w + C)^B}{A! B!} \\ &\ll (b+1) \frac{(\log_2 w)^a (\log_2 z - \log_2 w + C)^b}{a! b!}, \end{aligned}$$

where we used the fact that $a \ll \log_2 w$. Summing over all h , we derive that the number of those integers $n > x^{9/10}$ satisfying the conditions of the statement is

$$\ll \frac{Y}{\log z} (b+1) \frac{(\log_2 w)^a (\log_2 z - \log_2 w + C)^b}{a! b!}.$$

Since $a! b! \leq (3 \log_2 z)^{3 \log_2 z}$, this last expression is $> x^{9/10}$. This completes the proof. \square

Our final lemma is a special case of a theorem of Shiu (Theorem 03 of [3]).

Lemma 2.4. *Let f be a multiplicative function such that $0 \leq f(n) \leq 1$ for all n . Then, for all x, Y with $1 < \sqrt{x} \leq Y \leq x$, we have*

$$\sum_{x-Y < n \leq x} f(n) \ll \frac{Y}{\log x} \exp \left\{ \sum_{p \leq x} \frac{f(p)}{p} \right\}.$$

3 Decomposition and outline of the proof

Throughout, ε will denote a very small positive constant. Since Theorem 1 holds trivially for $\beta \leq \log 4 - 1 - \varepsilon$, we henceforth assume that

$$\log 4 - 1 - \varepsilon \leq \beta \leq \log 4 - 1. \quad (3.1)$$

Let

$$K := \lfloor \lambda \log_2 z \rfloor,$$

so that $(2 - \frac{3}{2}\varepsilon) \log_2 z \leq K \leq 2 \log_2 z$. In light of (1.2), Theorem 1 reduces to

$$H_2^*(x, y, z) \ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}}. \quad (3.2)$$

At this stage, we notice for further reference that, by Stirling's formula, for $k \leq K$ we have

$$\frac{\eta(2 \log_2 z)^k}{k! (\log z)^2} \leq \frac{\eta(2 \log_2 z)^K}{K! (\log z)^2} \asymp \frac{1}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}}. \quad (3.3)$$

Let \mathcal{H} denote the set of integers $n \leq x$ with $\tau(n; y, z) \geq 2$. We count separately the integers $n \in \mathcal{H}$ lying in 6 classes. In these definitions, we write $k = \overline{\Omega}(n; z) = K - b$ and for brevity we put $z_h = z^{e^{-h}}$. Let

$$K_0 := (2 - 3\varepsilon) \log_2 z$$

and define

$$\begin{aligned}\mathcal{N}_0 &:= \{n \in \mathcal{H} : n \leq x/\log z \text{ or } \exists d > \log z : d^2 | n\}, \\ \mathcal{N}_1 &:= \{n \in \mathcal{H} \setminus \mathcal{N}_0 : k \notin (K_0, K]\}, \\ \mathcal{N}_2 &:= \bigcup_{1 \leq h \leq 5\varepsilon \log_2 z} \mathcal{N}_{2,h}, \\ \text{with } \mathcal{N}_{2,h} &:= \left\{n \in \mathcal{H} \setminus (\mathcal{N}_0 \cup \mathcal{N}_1) : \overline{\Omega}(n; z_h, z) \leq \frac{19}{10}h - \frac{1}{100}b\right\}.\end{aligned}$$

For integers $n \in \mathcal{N}_2$, we will only use the fact that $\tau(n; y, z) \geq 1$. Integers in other classes do not have too many small prime factors and it is sufficient to count pairs of divisors d_1, d_2 of n in $(y, z]$. For each such pair, write $v = (d_1, d_2)$, $d_1 = vf_1$, $d_2 = vf_2$, $n = f_1 f_2 v u$ and assume $f_1 < f_2$. Let

$$F_1 = \overline{\Omega}(f_1), \quad F_2 = \overline{\Omega}(f_2), \quad V = \overline{\Omega}(v), \quad U = \overline{\Omega}(u, z), \quad (3.4)$$

and

$$Z := \exp\{(\log z)^{1-4\varepsilon}\}. \quad (3.5)$$

For further reference, we note that if $n \notin \mathcal{N}_0$ and $h \leq 5\varepsilon \log_2 z$, then

$$\overline{\Omega}(n; z_h, z) = \omega(n; z_h, z).$$

Now we define $\mathcal{H}^* := \mathcal{H} \setminus (\mathcal{N}_0 \cup \mathcal{N}_1 \cup \mathcal{N}_2)$ and

$$\begin{aligned}\mathcal{N}_3 &:= \{n \in \mathcal{H}^* : \min(u, f_2) \leq Z\}, \\ \mathcal{N}_4 &:= \{n \in \mathcal{H}^* : \min(u, f_2) > z^{1/10}\}, \\ \mathcal{N}_5 &:= \{n \in \mathcal{H}^* : Z < \min(u, f_2) \leq z^{1/10}\}.\end{aligned}$$

In the above decomposition, the main parts are \mathcal{N}_2 and \mathcal{N}_5 . We expect \mathcal{N}_2 to be small since, conditionally on $\overline{\Omega}(n; z) = k$, the normal value of $\overline{\Omega}(n; z_h, z)$ is $hk/\log_2 z > \frac{19}{10}h$. It is more difficult to see that \mathcal{N}_5 is small too. This follows from the fact that we count integers in this set according to their number of factorizations in the form $n = uvf_1 f_2$ with $y < vf_1 < vf_2 \leq z$. Suppose for instance that $f_1, f_2 \leq z_j$. For $\overline{\Omega}(n; z) = k$ and $\overline{\Omega}(n; z_j, z) = G$, then, ignoring the localization of vf_1 and vf_2 in $(y, z]$, there are $4^{k-G}2^G = 4^k 2^{-G}$ such factorizations. Thus, larger G means fewer factorizations. On probabilistic grounds, larger G should also mean fewer factorizations with the localization of vf_1 and vf_2 .

We now briefly consider the cases of \mathcal{N}_0 and \mathcal{N}_1 .

Trivially,

$$|\mathcal{N}_0| \leq \frac{x}{\log z} + \sum_{d > \log z} \frac{x}{d^2} \ll \frac{x}{\log z} \ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}}, \quad (3.6)$$

since $Q(\lambda) \leq Q(2) = \log 4 - 1$ in the range under consideration.

By the argument on pages 40–41 of [3],

$$\sum_{\substack{n \leq x \\ \overline{\Omega}(n; z) > K}} 1 \ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}}.$$

Setting $t := 1 - \frac{3}{2}\varepsilon$, Lemma 2.4 gives

$$\begin{aligned} \sum_{\substack{n \leq x \\ \tau(n; y, z) \geq 1 \\ \bar{\Omega}(n; z) \leq K_0}} 1 &\leq t^{-(2-3\varepsilon)\log_2 z} \sum_{\substack{dm \leq x \\ y < d \leq z}} t^{\bar{\Omega}(d) + \bar{\Omega}(m; z)} \ll x(\log z)^{2t-2-\beta-(2-3\varepsilon)\log t} \\ &\ll x(\log y)^{-\beta-2\varepsilon^2} \ll x(\log y)^{-Q(\lambda)-\varepsilon^2/2}. \end{aligned}$$

Therefore,

$$|\mathcal{N}_1| \ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}}. \quad (3.7)$$

In the next four sections, we show that

$$|\mathcal{N}_j| \ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}} \quad (2 \leq j \leq 5). \quad (3.8)$$

Together with (3.6) and (3.7), this will complete the proof of Theorem 1.

4 Estimation of $|\mathcal{N}_2|$

We plainly have $|\mathcal{N}_2| \leq \sum_h |\mathcal{N}_{2,h}|$. For $1 \leq h \leq 5\varepsilon \log_2 z$, the numbers $n \in \mathcal{N}_{2,h}$ satisfy

$$\begin{cases} x/\log z < n \leq x, \\ k := \bar{\Omega}(n; z) = K - b, \quad 0 \leq b \leq 3\varepsilon \log_2 z, \\ \bar{\Omega}(n; z_h, z) \leq \frac{19}{10}h - \frac{1}{100}b, \end{cases}$$

We note at the outset that $\mathcal{N}_{2,h}$ is empty unless $h \geq b/190$.

Write $n = du$ with $y < d \leq z$ and $u \leq x/y$. Let

$$\bar{\Omega}(d; z_h) = D_1, \quad \Omega(d; z_h, z) = D_2, \quad \bar{\Omega}(u; z_h) = U_1, \quad \Omega(u; z_h, z) = U_2,$$

so that $D_1 + D_2 \geq 1$, $D_2 + U_2 \leq \frac{19}{10}h - \frac{1}{100}b$ and $D_1 + D_2 + U_1 + U_2 = k$.

Fix $k = K - b$, h , D_1 , D_2 , U_1 and U_2 . By Lemma 2.3 (with $w = z_h$, $a = U_1$, $b = U_2$), the number of u is

$$\ll \frac{x}{y \log z} \frac{(\log_2 z - h)^{U_1}}{U_1!} (U_2 + 1) \frac{(h + C)^{U_2}}{U_2!}.$$

A second application of Lemma 2.3 yields that the number of d is

$$\ll \frac{\eta y}{\log z} \frac{(\log_2 z - h)^{D_1}}{D_1!} (D_2 + 1) \frac{(h + C)^{D_2}}{D_2!}.$$

Since $D_2 + U_2 < 2h$, we have $(h + C)^{U_2+D_2} \leq e^{2C} h^{U_2+D_2}$. Summing over D_1, D_2, U_1, U_2 with $G = D_2 + U_2$ fixed and using the binomial theorem, we find that the number of n in question is

$$\ll \frac{\eta x}{(\log z)^2} (\log_2 z - h)^{k-G} h^G (G + 1)^2 \sum_{\substack{U_1+D_1=k-G \\ D_2+U_2=G}} \frac{1}{U_1! D_1! D_2! U_2!} \ll \frac{\eta x 2^k}{(\log z)^2} A(h, G),$$

where

$$A(h, G) = (G+1)^2 \frac{(\log_2 z - h)^{k-G} h^G}{(k-G)!G!}.$$

Since $G+1 \leq G_h := \lfloor \frac{19}{10}h \rfloor$, we have

$$\frac{A(h, G+1)}{A(h, G)} \geq \frac{h(k-G)}{(G+1)(\log_2 z - h)} \geq \frac{k - 10\varepsilon \log_2 z}{1.9(1-5\varepsilon) \log_2 z} > \frac{21}{20}$$

if ε is small enough. Next,

$$\begin{aligned} A(h, G_h) &\leq (G_h+1)^2 \frac{(\log_2 z - h)^{k-G_h} (hk)^{G_h}}{k! (G_h/e)^{G_h}} \\ &\ll (h+1)^2 \frac{(\log_2 z)^k}{k!} \left(\frac{20}{19}e\right)^{19h/10} e^{-h(k-G_h)/\log_2 z} \\ &\ll \frac{(\log_2 z)^k}{k!} e^{-h/500}, \end{aligned}$$

since $(k - G_h)/\log_2 z > 2 - 13\varepsilon$ and $\frac{19}{10} \log(\frac{20}{19}e) < 2 - 1/400$. Thus,

$$\sum_{b/190 \leq h \leq 5\varepsilon \log_2 z} \sum_{0 \leq G \leq G_h} A(h, G) \ll \sum_{b/190 \leq h \leq 5\varepsilon \log_2 z} A(h, G_h) \ll \frac{(\log_2 z)^k}{k!} e^{-b/95000}$$

and so

$$\sum_{\substack{n \in \mathcal{N}_2 \\ \bar{\Omega}(n; z) = k}} 1 \ll \frac{\eta x (2 \log_2 z)^k}{(\log z)^2 k!} e^{-(K-k)/95000} \ll \frac{x e^{-(K-k)/95000}}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}},$$

by (3.3). Summing over the range $K_0 \leq k \leq K$ furnishes the required estimate (3.8) for $j = 2$.

5 Estimation of $|\mathcal{N}_3|$

All integers $n = f_1 f_2 uv$ counted in \mathcal{N}_3 verify

$$\begin{cases} x/\log z < n \leq x, \\ \bar{\Omega}(n; z) \leq K, \\ y < v f_1 < v f_2 \leq z, \quad \min(u, f_2) \leq Z, \end{cases}$$

where Z is defined in (3.5). This is all we shall use in bounding $|\mathcal{N}_3|$.

Let $\mathcal{N}_{3,1}$ be the subset corresponding to the condition $f_2 \leq Z$ and let $\mathcal{N}_{3,2}$ comprise those $n \in \mathcal{N}_3$ such that $u \leq Z$.

If $f_2 \leq Z$, then $v > z^{1/2}$ and $u > x/\{v Z^2 \log z\} > x^{1/3}$. For $\frac{1}{2} \leq t \leq 1$ we have

$$\begin{aligned} |\mathcal{N}_{3,1}| &\leq \sum_{f_1, f_2, v, u} t^{\bar{\Omega}(f_1 f_2 uv; z) - K} \\ &= t^{-K} \sum_{f_1 \leq Z} t^{\bar{\Omega}(f_1)} \sum_{f_1 < f_2 \leq e^\eta f_1} t^{\bar{\Omega}(f_2)} \sum_{y/f_1 < v \leq z/f_1} t^{\bar{\Omega}(v)} \sum_{u \leq x/f_1 f_2 v} t^{\bar{\Omega}(u; z)}. \end{aligned}$$

Apply Lemma 2.4 to the three innermost sums. The u -sum is

$$\ll \frac{x}{f_1 f_2 v} (\log z)^{t-1} \leq \frac{x}{f_1 y} (\log z)^{t-1}.$$

and the v -sum is

$$\ll \frac{\eta y}{f_1} (\log z)^{t-1}.$$

The f_2 -sum is $\ll \eta f_1 (\log f_1)^{t-1}$ if $f_1 > \eta^{-3}$ and otherwise is $\ll \eta f_1$ trivially (note that $\eta f_1 \gg 1$ follows from the fact that $(f_1 + 1)/f_1 \leq f_2/f_1 \leq e^\eta$). Next

$$\begin{aligned} \sum_{f_1 \leq \eta^{-3}} \frac{1}{f_1} + \sum_{2 \leq f_1 \leq Z} \frac{t^{\bar{\Omega}(f_1)}}{f_1} (\log f_1)^{t-1} &\ll \log_2 z + (\log_2 z) \max_{j \leq \log_2 Z} e^{j(t-1)} \sum_{f_1 \leq \exp\{e^j\}} \frac{t^{\bar{\Omega}(f_1)}}{f_1} \\ &\ll (\log_2 z) (\log Z)^{2t-1}. \end{aligned}$$

Thus,

$$|\mathcal{N}_{3,1}| \ll x (\log_2 x) (\log x)^E$$

with $E = -2\beta - \lambda \log t + 2t - 2 + (2t - 1)(1 - 4\varepsilon)$. We select optimally $t := \frac{1}{4}\lambda/(1 - 2\varepsilon)$, and check that $t \geq \frac{1}{2}$ since $\lambda \geq 2 - \varepsilon/\log 2$. Then

$$\begin{aligned} E &= -Q(\lambda) + \lambda \log(1 - 2\varepsilon) + 4\varepsilon \leq -Q(\lambda) + (2 - \varepsilon/\log 2)(-2\varepsilon - 2\varepsilon^2) + 4\varepsilon \\ &< -Q(\lambda) - \varepsilon^2. \end{aligned}$$

Next, we consider the case when $u \leq Z$. We observe that this implies

$$\frac{1}{4}vz^2 \leq vx \leq vn \log z = u f_1 v f_2 v \log z \leq Z z^2 \log z$$

hence $v \leq 4Z \log z \leq Z^2$, and therefore

$$\min(f_1, f_2) > z^{1/2}.$$

Also, $z > x^{1/3}$ since $x/\log z < n = uv f_1 f_2 \leq Z z^2$. Thus, for $\frac{1}{2} \leq t \leq 1$, we have

$$\begin{aligned} |\mathcal{N}_{3,2}| &\leq \sum_{f_1, f_2, v, u} t^{\bar{\Omega}(f_1 f_2 uv; z) - K} \\ &= t^{-K} \sum_{v \leq Z^2} t^{\bar{\Omega}(v)} \sum_{u \leq xv/y^2} t^{\bar{\Omega}(u)} \sum_{y/v < f_1 \leq z/v} t^{\bar{\Omega}(f_1)} \sum_{y/v < f_2 \leq z/v} t^{\bar{\Omega}(f_2)}. \end{aligned}$$

The sums upon f_1 and f_2 are each

$$\ll \frac{\eta y}{v} (\log z)^{t-1}$$

and the u -sum is

$$\ll \frac{xv}{y^2} (\log 2xv/y^2)^{t-1} \leq \frac{xv}{y^2} (\log 2v)^{t-1}.$$

Thus, selecting the same value $t := \frac{1}{4}\lambda/(1 - 2\varepsilon)$, we obtain

$$\begin{aligned} |N_{3,2}| &\ll t^{-K} x \eta^2 (\log z)^{2t-1} \sum_{v \leq Z^2} \frac{t^{\bar{\Omega}(v)} (\log 2v)^{t-1}}{v} \\ &\ll x (\log_2 z) (\log z)^E \leq x (\log_2 z) (\log z)^{-Q(\lambda) - \varepsilon^2}. \end{aligned}$$

This completes the proof of (3.8) with $j = 3$.

6 Estimation of $|\mathcal{N}_4|$

We now consider those integers $n = f_1 f_2 u v$ such that

$$\begin{cases} x/\log z < n \leq x, \\ k := \overline{\Omega}(n; z) = K - b, \quad 0 \leq b \leq 3\varepsilon \log_2 z, \\ y < v f_1 < v f_2 \leq z, \quad \min(u, f_2) > z^{1/10}. \end{cases}$$

With the notation (3.4), fix k , F_1 , F_2 , U and V . Here u, f_1 and f_2 are all $> \frac{1}{2}z^{1/10}$. By Lemma 2.3 (with $w = z$), for each triple f_1, f_2, v the number of u is

$$\ll \frac{x}{f_1 f_2 v \log z} \frac{(\log_2 z)^U}{U!}.$$

Using Lemma 2.3 two more times, we obtain, for each v ,

$$\sum_{y/v < f_1 \leq z/v} \frac{1}{f_1} \sum_{y/v < f_2 \leq z/v} \frac{1}{f_2} \ll \frac{\eta^2}{(\log z)^2} \frac{(\log_2 z)^{F_1+F_2}}{F_1! F_2!}.$$

Now, Lemma 2.2 gives

$$\sum_v \frac{1}{v} \ll \frac{(\log_2 z)^V}{V!}.$$

Gathering these estimates and using (3.3) yields

$$\begin{aligned} |\mathcal{N}_4| &\ll \frac{x\eta^2}{(\log z)^3} \sum_{(2-3\varepsilon)\log_2 z \leq k \leq K} \sum_{F_1+F_2+U+V=k} \frac{(\log_2 z)^k}{F_1! F_2! U! V!} \\ &= \frac{x\eta^2}{(\log z)^3} \sum_{(2-3\varepsilon)\log_2 z \leq k \leq K} \frac{(2\log_2 z)^k}{k!} 2^k \\ &\ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y} \log z} \frac{2^K \eta}{\log z} \ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}}. \end{aligned}$$

Thus (3.8) holds for $j = 4$.

7 Estimation of $|\mathcal{N}_5|$

It is plainly sufficient to bound the number of those $n = f_1 f_2 u v$ satisfying the following conditions

$$\begin{cases} x/\log z < n \leq x, \\ k := \overline{\Omega}(n; z) = K - b, \quad 0 \leq b \leq 3\varepsilon \log_2 z, \\ \overline{\Omega}(n; z_h, z) > \frac{19}{10}h - \frac{1}{100}b \quad (1 \leq h \leq 5\varepsilon \log_2 z) \\ y < v f_1 < v f_2 \leq z, \quad Z < \min(u, f_2) \leq z^{1/10}. \end{cases}$$

Define j by $z_{j+2} < \min(u, f_2) \leq z_{j+1}$. We have $1 \leq j \leq 5\varepsilon \log_2 z$. Let $\mathcal{N}_{5,1}$ be the set of those n satisfying the above conditions with $u \leq z_{j+1}$ and let $\mathcal{N}_{5,2}$ be the complementary set, for which $f_2 \leq z_{j+1}$.

If $u \leq z_{j+1}$, then $v \leq (z^2 u \log z)/x \leq 4u \log z \leq z_j$ and $f_2 > f_1 > z^{1/2}$. Recall notation (3.4) and write

$$F_{11} := \overline{\Omega}(f_1; z_j), \quad F_{12} := \Omega(f_1; z_j, z), \quad F_{21} := \overline{\Omega}(f_2; z_j), \quad F_{22} := \Omega(f_2; z_j, z),$$

so that the initial condition upon $\overline{\Omega}(n; z_h, z)$ with $h = j$ may be rewritten as

$$F_{12} + F_{22} \geq G_j := \max(0, \lfloor \frac{19}{10}j - b/100 \rfloor).$$

We count those n in a dyadic interval $(X, 2X]$, where $x/(2 \log z) \leq X \leq x$. Fix k, j, X, U, V, F_{rs} and apply Lemma 2.3 to sums over u, f_1, f_2 . The number of n in question is

$$\begin{aligned} &\leq \sum_{v \leq z_j} \sum_{vX/z^2 \leq u \leq 2vX/y^2} \sum_{y/v < f_1 \leq z/v} \sum_{y/v < f_2 \leq z/v} 1 \\ &\ll \frac{\eta^2 X e^j}{(\log z)^3} \frac{(\log_2 z - j)^{U+F_{11}+F_{21}}}{U! F_{11}! F_{21}!} (F_{12} + 1)(F_{22} + 1) \frac{(j + C)^{F_{12}+F_{22}}}{F_{12}! F_{22}!} \sum_{v \leq z_j} \frac{1}{v}. \end{aligned}$$

Bounding the v -sum by Lemma 2.2, and summing over X, U, V, F_{rs} with $F_{12} + F_{22} = G$ yields

$$|\mathcal{N}_{5,1}| \ll \frac{\eta^2 x}{(\log z)^3} \sum_{(2-3\varepsilon) \log_2 z \leq k \leq K} 4^k \sum_{1 \leq j \leq 5\varepsilon \log_2 z} \sum_{G_j \leq G \leq k} M(j, G),$$

where

$$M(j, G) := e^j (G + 1)^2 \frac{(\log_2 z - j)^{k-G} (j + C)^G}{2^G (k - G)! G!}.$$

Let $j_b = \lfloor \frac{1}{2}b + 100C + 100 \rfloor$. If $j \leq j_b$, then $j + C \leq \frac{99}{100}(j + C_b)$ with $C_b := 3C + 2 + \frac{b}{100}$ and, introducing $R := \max_{G \geq 0} \{(G + 1)^2 (\frac{99}{100})^G\}$, we have

$$\begin{aligned} \sum_{1 \leq j \leq j_b} \sum_{G_j \leq G \leq k} M(j, G) &\leq R \sum_{1 \leq j \leq j_b} e^j \sum_{0 \leq G \leq k} \frac{(\log_2 z - j)^{k-G} (j + C_b)^G}{2^G G! (k - G)!} \\ &\ll \frac{1}{k!} \sum_{1 \leq j \leq j_b} e^j \left(\log_2 z - \frac{1}{2}j + \frac{1}{2}C_b \right)^k \\ &\ll \frac{(\log_2 z)^k}{k!} \sum_{1 \leq j \leq j_b} e^{j + (b/200 - j/2)k / \log_2 z} \\ &\ll \frac{(\log_2 z)^k}{k!} e^{b/100 + 2\varepsilon j_b} \ll \frac{(\log_2 z)^k}{k!} e^{b/50}. \end{aligned}$$

When $j > j_b$, then

$$G_j \geq \frac{9}{5}(j + C) + \frac{1}{10}(j_b + C + 1) - \frac{1}{100}b - 1 \geq \frac{9}{5}(j + C) + 9 \geq 189.$$

Thus, for $G \geq G_j$ we have

$$\frac{M(j, G+1)}{M(j, G)} = \left(\frac{G+2}{G+1} \right)^2 \frac{j+C}{2(G+1)} \frac{k-G}{\log_2 z - j} \leq \frac{4}{7}.$$

Therefore,

$$\begin{aligned} \sum_{G_j \leq G \leq k} M(j, G) &\ll M(j, G_j) \ll \frac{j^2 e^j (\log_2 z - j)^{k-G_j} (jk)^{G_j}}{k! 2^{G_j} G_j!} \\ &\leq \frac{j^2 e^j (\log_2 z)^k}{k!} e^{-j(k-G_j)/\log_2 z} \left(\frac{e j k}{2 G_j \log_2 z} \right)^{G_j} \ll \frac{(\log_2 z)^k}{k!} e^{-j/5}, \end{aligned}$$

since $k - G_j \geq (2 - 10\varepsilon) \log_2 z$, $e j k / (2 G_j \log_2 z) \leq \frac{5}{9} e$, and $-1 + \frac{19}{10} \log(\frac{5}{9} e) < -\frac{1}{5}$. We conclude that

$$\sum_{1 \leq j \leq 5\varepsilon \log_2 z} \sum_{G_j \leq G \leq k} M(j, G) \ll \frac{(\log_2 z)^k}{k!} e^{b/50} \quad (7.1)$$

and hence, by (3.3),

$$|\mathcal{N}_{5,1}| \ll \frac{\eta^2 x}{(\log z)^3} \sum_{k \leq K} \frac{(2 \log_2 z)^k}{k!} 2^{K-b/2} \ll \frac{\eta^2 2^K x}{(\log z)^3} \frac{(2 \log_2 z)^K}{K!} \ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}}.$$

Now assume $f_2 \leq z_{j+1}$. Then $\min(u, v) > \sqrt{z}$. Fix F_1, F_2 and

$$\bar{\Omega}(v; z_j) = V_1, \quad \Omega(v; z_j, z) = V_2, \quad \bar{\Omega}(u; z_j) = U_1, \quad \Omega(u; z_j, z) = U_2.$$

By Lemma 2.3, given f_1, f_2 and v , the number of u is

$$\ll \frac{x}{f_1 f_2 v \log z} \frac{(\log_2 z - j)^{U_1} (U_2 + 1) (j + C)^{U_2}}{U_1! U_2!}.$$

Applying Lemma 2.3 again, for each f_1 we have

$$\sum_{\substack{f_1 < f_2 \leq e^\eta f_1 \\ y/f_1 < v \leq z/f_1}} \frac{1}{f_2 v} \ll \frac{\eta^2 e^j}{(\log z)^2} \frac{(V_2 + 1) (\log_2 z - j)^{V_1 + F_2} (j + C)^{V_2}}{V_1! V_2! F_2!}.$$

By Lemma 2.2,

$$\sum_{f_1 \leq z_j} \frac{1}{f_1} \ll \frac{(\log_2 z - j)^{F_1}}{F_1!}.$$

Combine these estimates, and sum over $F_1, F_2, U_1, U_2, V_1, V_2$ with $V_2 + U_2 = G$. As in the estimation of $|\mathcal{N}_{5,1}|$, sum over k, j, G using (3.3) and (7.1). We obtain

$$\begin{aligned} |\mathcal{N}_{5,2}| &\ll \frac{\eta^2 x}{(\log z)^3} \sum_{(2-3\varepsilon) \log_2 z \leq k \leq K} 4^k \sum_{1 \leq j \leq 5\varepsilon \log_2 z} \sum_{G_j < G \leq k} M(j, G) \\ &\ll \frac{x}{(\log y)^{Q(\lambda)} \sqrt{\log_2 y}}. \end{aligned}$$

References

- [1] K. Ford, The distribution of integers with a divisor in a given interval, (2004).
- [2] R.R Hall, *Sets of multiples*, Cambridge Tracts in Mathematics 118, Cambridge University Press, Cambridge 1996.
- [3] R. R. Hall and G. Tenenbaum, *Divisors*, Cambridge Tracts in Mathematics, vol. 90, Cambridge University Press, Cambridge, 1988.
- [4] R. R. Hall and G. Tenenbaum, The set of multiples of a short interval, in : D.V. Chudnovsky *et al.* (eds), *Number Theory* (New York Seminar 1989–90), 119–128 (Springer Verlag 1991).
- [5] G. Tenenbaum, Sur la probabilité qu'un entier possède un diviseur dans un intervalle donné, *Compositio Math.* **51** (1984), 243–263.