A generic characterization of direct summands for orthogonal involutions

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Abstract. — The 'transcendental methods' in the algebraic theory of quadratic forms are based on two major results, proved in the 60's by Cassels and Pfister, and known as the representation and the subform theorems. A generalization of the representation theorem was proven by Jean-Pierre Tignol in 1996, in the setting of central simple algebras with involution. This paper studies the subform question for orthogonal involutions. A generic characterization of direct summands is given; an analogue of the subform theorem is proven for division algebras and algebras of index at most 2.

Introduction

The 'transcendental methods' in the algebraic theory of quadratic forms are based on two major results, proved in the 60's by Cassels and Pfister, and known as the representation and the subform theorems (see [**Sch85**, Ch. 4 §3] or [**Lam05**, CH. 9 §1 and 2]). J.-P. Tignol gave in [**Tig96**] a generalization of the representation theorem for algebras with involution (of any kind), which implies the corresponding statement for quadratic forms.

In this paper, the subform question is studied in the context of algebras with orthogonal involutions. The main results, which give partial answers to this question, are theorems 3.1 and 4.1, stated and proved in §3 and §4. The first one gives a generic characterization of direct summands, which is valid for any algebra with orthogonal involution, but which is much weaker than the subform theorem in the split case. The second one is an analogue of the subform theorem, but only for division algebras and algebras of index at most 2. Before proving these theorems, we define in §1 direct summands of an algebra with involution, using the direct sum of [**Dej95**], and we restate the subform theorem in a convenient way for our purpose in §2.

We assume throughout the paper that the base field F has characteristic different from 2, and refer the reader to [**KMRT98**] for basic facts on algebras with involution.

1. Direct summands of an algebra with involution

Consider two central simple algebras with involution (B, τ) and (B', τ') over F, which are Morita equivalent, i.e. B and B' are Brauer equivalent and τ and τ' are of the same type. They can be represented as $(B, \tau) = (\mathrm{End}_D(N), \mathrm{ad}_{h_N})$ and $(B', \tau') = (\mathrm{End}_D(N'), \mathrm{ad}_{h_{N'}})$ for some hermitian modules (N, h_N) and $(N', h_{N'})$ over a central division algebra with involution (D, γ) over F. The direct sum for hermitian modules gives rise to an algebra with involution, $(\mathrm{End}_D(N \oplus N'), \mathrm{ad}_{h_N \oplus h_{N'}})$, which is Morita equivalent to (B, τ) and (B', τ'). But the involutions τ and τ' only determine the hermitian forms h_N and $h_{N'}$ up to a scalar factor, and modifying this factors independently may lead to a hermitian form which is not similar to $h_N \oplus h_{N'}$.

In [**Dej95**], Dejaiffe defined a notion of direct sum for algebras with involution which extends direct sum of hermitian modules, i.e. such that $(\mathrm{End}_D(N \oplus N'), \mathrm{ad}_{h_N \oplus h_{N'}})$ is a direct sum of $(\mathrm{End}_D(N), \mathrm{ad}_{h_N})$ and $(\mathrm{End}_D(N'), \mathrm{ad}_{h_{N'}})$. Precisely, given any Morita equivalence data between two algebras with involution (B, τ) and (B', τ') she defines the corresponding direct sum of (B, τ) and (B', τ'). Note that different Morita equivalence datas (which amount to modifying scalars as in the previous paragraph) may lead to non isomorphic direct sums of the same (B, τ) and (B', τ').

We say that (B, τ) is a *direct summand* of (A, σ) if there exist (B', τ'), Morita equivalent to (B, τ) and a direct sum of (B, τ) and (B', τ') which is isomorphic to (A, σ). By [**Dej95**, Prop. 2.2], this leads to the following definition:

Definition 1.1. — The algebra with involution (B, τ) is a direct summand of (A, σ) if there exist a σ-symmetric idempotent $e \in A$ such that $(eAe, \sigma_{|eAe})$ is isomorphic to (B, τ).

This condition can be translated in terms of hermitian modules as follows. Fix representations $(A, \sigma) = (\mathrm{End}_D(M), \mathrm{ad}_{h_M})$ and $(B, \tau) = (\mathrm{End}_D(N), \mathrm{ad}_{h_N})$, for some hermitian modules over a central division algebra with orthogonal involution (D, γ) over F. Denote by $M_0 \subset M$ the image of the idempotent e, and by h_{M_0} the restriction of h_M to M_0. The algebra with involution $(eAe, \sigma_{|eAe})$ is $(\mathrm{End}_D(M_0), \mathrm{ad}_{h_{M_0}})$. Hence, (B, τ) is a direct summand of (A, σ) if and only if M contains a submodule M_0 such that h_{M_0} is similar to h_M.

In the split orthogonal case, that is $(A, \sigma) = (\mathrm{End}_F(V), \mathrm{ad}_{q_V})$ and $(B, \tau) = (\mathrm{End}_F(W), \mathrm{ad}_{q_W})$, for some quadratic spaces (V, q_V) and (W, q_W) over F, we get that (B, τ) is a direct summand of (A, σ) if and only if there exists a scalar $\lambda \in F^\times$ such that λq_W is a subform of q_V. A 'generic' condition under which this is satisfied is given by the subform theorem, at least in a version up to similarities. For further use, we give in the next section a projective version of this statement.

2. A projective version of the subform theorem up to similarities

2.1. Definition of $q_W{}^{proj}$. — Denote by $F(W)$ the function field of the affine variety W and by $F(\mathbb{P}W)$ the field of rational functions on the projective variety $\mathbb{P}W$. The generic point of the projective space $\mathbb{P}W$, viewed as an $F(\mathbb{P}W)$ rational point, gives rise to a line $L_W \subset W_{F(\mathbb{P}W)}$, which we call the generic line. We define the *projective class of q_W* to be the square class in $F(\mathbb{P}W)$ of the value of q_W at any point of the generic line. It is a well defined element of $F(\mathbb{P}W)^\times / F(\mathbb{P}W)^{\times 2}$, and the notation $q_W{}^{proj}$ stands for an element in $F(\mathbb{P}W)$ who belongs to the projective class of q_W.

If we identify $F(\mathbb{P}W)$ with the subset $F(W)_0 \subset F(W)$ of degree 0 homogeneous functions, we may describe the projective class of q_W as the quotient $\frac{q_W}{f^2}$, where q_W is viewed as an element of $F(W)$ and f is an arbitrary degree 1 homogeneous element so that the quotient is in $F(W)_0$. The square class of $\frac{q_W}{f^2}$ clearly does not depend on the choice of f.

Let $(e_1, \ldots, e_n)$ be any basis of W over F, with dual basis $t_1, \ldots t_n$, so that $F(W) \simeq F(t_1, \ldots, t_n)$. We may then identify $F(\mathbb{P}W)$ with $F(\theta_2, \ldots \theta_n)$, where $\theta_2, \ldots, \theta_n$ are indeterminates, by $f \mapsto f(1, \theta_2, \ldots, \theta_n)$. Assume moreover that the basis $(e_1, \ldots e_n)$ is orthogonal for q_W, and let $b_i = q_W(e_i)$. Using these identifications, one may check that the element $b_1 + b_2\theta_2^2 + \cdots + b_n\theta_n^2$ belongs to the projective class of q_W. Indeed, we have $b_1 + b_2\theta_2^2 + \cdots + b_n\theta_n^2 = q_W(\delta)$, for $\delta = e_1 + e_2\theta_2 + \cdots + e_n\theta_n \in L_W \subset W_{F(\mathbb{P}W)}$.

2.2. A projective version of the subform theorem up to similarities. — The classical subform theorem ([**Sch85**, Ch. 4, Th. 3.7], [**Lam05**, Ch. 9, Th. 2.8]) has the following easy consequence:

Proposition 2.1. — *Let (V, q_V) and (W, q_W) be two quadratic spaces, with q_V anisotropic. There exists $\lambda \in F^\times$ such that λq_W is a subform of q_V if and only if*

(dep) $\qquad\qquad\qquad \exists \lambda \in F^\times$ *such that* $q_{V\,F(\mathbb{P}W)}$ *represents* $\lambda q_W{}^{proj}$.

Remark 2.2. — This statement does not depend on the choice of $q_W{}^{proj}$, since the set of values represented by a quadratic form is stable under multiplication by a square.

Proof. — Since $q_{W\,F(\mathbb{P}W)}$ represents q_W^{proj} (see §2.1), condition (dep) clearly is necessary. Let us prove it is also sufficient. Pick a vector $v \in V_{F(\mathbb{P}W)}$ satisfying $q_{V\,F(\mathbb{P}W)}(v) = \lambda(b_1 + b_2\theta_2^2 + \cdots + b_n\theta_n^2)$. If we identify $F(\mathbb{P}W)(t) = F(\theta_2, \ldots, \theta_n)(t)$ with $F(W)$ by $f \mapsto f(\frac{t_2}{t_1}, \ldots, \frac{t_n}{t_1})(t_1)$, we get $q_{V\,F(W)}(tv) = \lambda(b_1 t_1^2 + \cdots + b_n t_n^2)$. By the classical subform theorem, this implies that λq_W is a subform of q_V. $\qquad\square$

Remark 2.3. — There are two differences between this statement and the classical subform theorem. Using $F(\mathbb{P}W)$ instead of $F(W)$ and q_W^{proj} instead of q_W is not a serious one. As opposed to this, one should notice that it does not seem easy to prove the subform theorem up to similarities (even in an affine version) without using the subform theorem.

3. The generic ideal and a characterization of direct summands

We now go back to the setting of algebras with involution, and for simplicity, we restrict ourselves to the orthogonal case. Hence (A, σ) and (B, τ) are Brauer equivalent central simple algebras over F, with orthogonal involutions.

We denote by X_B the Brauer-Severi variety of B and by F_B the field of rational functions on X_B. It is a generic splitting field for the algebra B. By definition of X_B, its generic point, viewed as an F_B-rational point, corresponds to a right ideal of reduced dimension 1 of the split algebra $B_{F_B} := B \otimes_F F_B$. We call it the *generic ideal* of B and denote it by I_B. By [**KMRT98**, (1.12)], given a representation $B_{F_B} = \mathrm{End}_{F_B}(W)$, there exists a unique line L_B in the F_B-vector space W such that $I_B = \mathrm{Hom}_{F_B}(W, L_B)$; we call it the *generic line* of B.

If B is split and $(B, \tau) = (\mathrm{End}_F(W), \mathrm{ad}_{q_W})$ for some quadratic space (W, q_W) over F, then X_B is isomorphic to $\mathbb{P}W$, and with the notations of §2.1, we have $L_B = L_W$ and $I_B = Hom_{F(\mathbb{P}W)}(W, L_W)$.

In this section, we prove the following theorem, which gives a necessary and sufficient condition under which (A, σ) contains (B, τ) as a direct summand in terms of the generic ideal of B. Note that we do not view this as an analogue of the subform theorem (see remark 4.4 below). In particular, the involution σ may be isotropic.

Theorem 3.1. — *Let (A, σ) and (B, τ) be two Brauer-equivalent central simple algebras, each endowed with an orthogonal involution. Then (A, σ) contains (B, τ) as a direct summand if and only if*

(iso)   *there exist a σ-symmetric idempotent $e \in A$*

$$\text{and an isomorphism } \Psi : eAe \overset{\sim}{\to} B \text{ such that}$$

$$\forall g \in I_B, \ \exists f \in A_{F_B}, \ \Psi_{F_B}(e\sigma(f)fe) = \tau(g)g.$$

Remark 3.2. — (i) The idea of considering elements of the type $\tau(g)g$, with g of rank 1, is borrowed from Tignol's paper [**Tig96**]. Since B_{F_B} is split, such an element may give us some information on the value of the underlying F_B-quadratic form on the image of g, namely the generic line (see lemma 3.3 below). In the split case, this value precisely is the projective class of q_W.

(ii) Since A and B are Brauer equivalent, there exist in general many σ-symmetric idempotents $e \in A$ such that eAe is isomorphic to B. Indeed, one may take for e any orthogonal projection on a submodule of M of dimension over D equal to $\dim_D(N)$. Given such an idempotent e, Theorem 3.1 actually gives a criterion of isomorphism between the involutions $\sigma_{|eAe}$ and τ.

Proof. — Condition (iso) is clearly necessary. Indeed, if (B, τ) is a direct summand of (A, σ), there exist $e \in A$ such that $e^2 = \sigma(e) = e$ and an isomorphism of algebras with

involution $\Psi : (eAe, \sigma_{|eAe}) \xrightarrow{\sim} (B, \tau)$. For any $g \in I_B \subset B_{F_B}$, the element $f = \psi_{F_B}^{-1}(g)$ clearly satisfies the required condition.

We have to prove condition (iso) is also sufficient. Consider (A, σ) and (B, τ) as in the theorem, and take representations $(A, \sigma) = (\mathrm{End}_D(M), \mathrm{ad}_{h_M})$ and $(B, \tau) = (\mathrm{End}_D(N), \mathrm{ad}_{h_N})$, for some hermitian modules over a central division algebra with orthogonal involution (D, γ) over F. Let $M_0 \subset M$ be the image of e, so that e is the orthogonal projection on M_0. The isomorphim $eAe = \mathrm{End}_D(M_0) \simeq B$ is given by some isomorphism of D-modules $\psi : M_0 \xrightarrow{\sim} N$. Let us denote by h_{M_0} the restriction of h_M to M_0. We will prove that h_{M_0} is similar to h_N, which in turn implies that (B, τ) is a direct summand of (A, σ).

Let $n_1, \ldots, n_s$ be an orthogonal basis of (N, h_N). The elements $m_k = \psi^{-1}(n_k)$ for $1 \le k \le s$ form a basis of the D-module M_0. We will actually compute h_{M_0} in this basis, using condition (iso).

Since A and D are Brauer equivalent to B, they both split over F_B. Let (E, q_γ) be a quadratic space over F_B such that $(D_{F_B}, \gamma) \simeq (\mathrm{End}_{F_B}(E), \mathrm{ad}_{q_\gamma})$. By Morita theory, we then have (see for instance [**BFP95**, §1.4])

$$B_{F_B} \simeq \mathrm{End}_{F_B}(W), \text{ where } W = (N \otimes_F F_B) \otimes_{D \otimes_F F_B} E,$$

and similarly,

$$A_{F_B} \simeq \mathrm{End}_{F_B}(V), \text{ where } V = (M \otimes_F F_B) \otimes_{D \otimes_F F_B} E, \text{ and}$$

$$eA_{F_B}e \simeq \mathrm{End}_{F_B}(V_0), \text{ where } V_0 = (M_0 \otimes_F F_B) \otimes_{D \otimes_F F_B} E.$$

Moreover, the involutions σ, τ and $\sigma_{|eAe}$ are respectively adjoint, after scalar extension to F_B to the quadratic forms q_V, q_W and q_{V_0} defined by

$$b_{q_V}((m \otimes \lambda) \otimes e, (m' \otimes \lambda') \otimes e') = b_{q_\gamma}(e, (h_M(m, m') \otimes \lambda\lambda')(e')),$$

and similarly for q_W and q_{V_0}.

Let $e_1, \ldots, e_d$ be an orthogonal basis of (E, q_γ). We denote by $d_i = h_N(n_i, n_i)$ and $a_j = q_\gamma(e_j)$. The elements $n_i \otimes 1 \otimes e_j$ for $1 \le i \le s$ and $1 \le j \le d$ form a basis of the F_B vector space W. Let δ be any non trivial element in the generic line $L_B \subset W$ defined at the beginning of this section.

The proof is based on the following computation, which the reader may easily check:

Lemma 3.3. — *Consider the element $g \in I_B$ defined by $g(n_i \otimes 1 \otimes e_j) = \alpha_{i,j}\delta$. Then, $\tau(g)g$ maps $n_i \otimes 1 \otimes e_j$ to $\alpha_{i,j}\, q_W(\delta)(\sum_{1 \le k \le s\,,\,1 \le l \le d} \alpha_{k,l}(n_k d_k^{-1} \otimes 1 \otimes \frac{e_l}{a_l}))$.*

For any couple $1 \le i \le s$ and $1 \le j \le d$, let us first apply this lemma to the element $g_{i,j} \in I_B$ which maps $n_i \otimes 1 \otimes e_j$ to δ, and any other element of the basis to 0, so that $\tau(g_{i,j})g_{i,j}$ maps $n_i \otimes 1 \otimes e_j$ to $n_i d_i^{-1} \otimes q_W(\delta) \otimes \frac{e_j}{a_j}$ and any other element of the basis to 0. Denote by $f_{i,j}$ the corresponding element of $A_{F_B} = \mathrm{End}_{F_B}(V)$ given by condition (iso). It satisfies

$$\psi_{F_B} e\sigma(f_{i,j}) f_{i,j} e\psi_{F_B}^{-1} = \tau(g_{i,j})g_{i,j}.$$

from which we deduce that $e\sigma(f_{i,j})f_{i,j}e$ maps $m_i \otimes 1 \otimes e_j$ to $m_i d_i^{-1} \otimes q_W(\delta) \otimes \frac{e_j}{a_j}$ and any other element of the basis to 0. Now, we can compute

$$b_{q_{V_0}}(f_{i,j}(m_k \otimes 1 \otimes e_l), f_{i,j}(m_p \otimes 1 \otimes e_q))$$

for any $k, p \in \{1, \ldots, s\}$ and $l, q \in \{1, \ldots, d\}$ in two different ways. First, it is equal to

$$(1) \quad b_{q_{V_0}}(m_k \otimes 1 \otimes e_l, e\sigma(f_{i,j})f_{i,j}e(m_p \otimes 1 \otimes e_q))$$

$$= \begin{cases} 0 \text{ if } p \neq i \text{ or } q \neq j \\ q_W(\delta)b_{q_\gamma}(e_l, (h_{M_0}(m_k, m_i)d_i^{-1})(\frac{e_j}{a_j})) \text{ if } (p, q) = (i, j). \end{cases}$$

By symmetry, it is also equal to $\begin{cases} 0 \text{ if } k \neq i \text{ or } l \neq j \\ q_W(\delta)b_{q_\gamma}(e_q, (h_{M_0}(m_p, m_i)d_i^{-1})(\frac{e_j}{a_j})) \text{ if } (k, l) = (i, j). \end{cases}$

From this, we deduce that if $k \neq i$, then for any l,

$$b_{q_\gamma}(e_l, (h_{M_0}(m_k, m_i)d_i^{-1})(e_j)) = 0.$$

Hence $(h_{M_0}(m_k, m_i)d_i^{-1})(e_j) = 0$. This is valid for any value of j, and we finally get

$$(2) \quad\quad\quad\quad\quad h_{M_0}(m_k, m_i) = 0 \text{ if } k \neq i$$

Let us take now $k = i$. For any $l \neq j$, we have

$$b_{q_\gamma}(e_l, (h_{M_0}(m_i, m_i)d_i^{-1})(e_j)) = 0.$$

Since the basis $(e_1, \ldots, e_d)$ is orthogonal for q_γ, this implies that there exists an element $\lambda_{i,j} \in F_B^\times$ such that

$$(h_{M_0}(m_i, m_i)d_i^{-1})(e_j)) = \lambda_{i,j}e_j.$$

In other words, the element $h_{M_0}(m_i, m_i)d_i^{-1} \in D \subset D \otimes_F F_B \simeq \mathrm{End}_{F_B}(E)$ corresponds in the basis $e_1, \ldots, e_d$ to the diagonal matrix with coefficients $\lambda_{i,j}$, $1 \leq j \leq d$. Let us now prove that the coefficients $\lambda_{i,j}$ are all equal. Consider the element $g_i \in I_B$ which maps $n_i \otimes 1 \otimes e_j$ to δ for all $j = 1, \ldots, d$ and any other element of the basis to 0. Again by the previous lemma, $\tau(g_i)g_i$ maps $n_i \otimes 1 \otimes e_j$ to $n_i d_i^{-1} \otimes q_W(\delta) \otimes (\frac{e_1}{a_1} + \cdots + \frac{e_d}{a_d})$ for any j, and any other element of the basis to 0. Let f_i be the corresponding element of $A_{F_B} \simeq \mathrm{End}_{F_B}(V)$ given by condition (iso). We get that $e\sigma(f_i)f_ie$ maps any $m_i \otimes 1 \otimes e_j$ to $m_i d_i^{-1} \otimes q_W(\delta) \otimes (\frac{e_1}{a_1} + \cdots + \frac{e_d}{a_d})$. The same computation as above for

$$b_{q_{V_0}}(f_i(m_i \otimes 1 \otimes e_j), f_i(m_i \otimes 1 \otimes e_l)),$$

now gives $\lambda_{i,j} = \lambda_{i,l}$, which proves that $h_{M_0}(m_i, m_i)d_i^{-1}$ is actually central, $h_{M_0}(m_i, m_i)d_i^{-1} = \lambda_i \in F_B^\times$. Since we know from the very beginning it lies in $D \subset D \otimes_F F_B$, λ_i actually belongs to $F^\times$.

To finish the proof, consider the element $g \in I_B$ which maps any element of the basis to δ. Then, $\tau(g)g$ maps any $n_i \otimes 1 \otimes e_j$ to

$$(n_1 d_1^{-1} + \cdots + n_s d_s^{-1}) \otimes q_W(\delta) \otimes (\frac{e_1}{a_1} + \cdots + \frac{e_d}{a_d}).$$

Again, let f be the corresponding element of $A_{F_B} \simeq \mathrm{End}_{F_B}(V)$ given by condition (iso). We get that $e\sigma(f)fe$ maps any $m_i \otimes 1 \otimes e_j$ to

$$(m_1 d_1^{-1} + \cdots + m_s d_s^{-1}) \otimes q_W(\delta) \otimes (\frac{e_1}{a_1} + \cdots + \frac{e_d}{a_d}).$$

Computing as before $b_q(f(m_i \otimes 1 \otimes e_j), f(m_k \otimes 1 \otimes e_l))$ in two different ways, we get $\lambda_i = \lambda_k = \lambda \in F^\times$. Hence we have proven that

$$(3) \qquad h_{M_0}(m_i, m_i) = \lambda d_i = \lambda h_N(n_i, n_i) \text{ for any } 1 \leq i \leq s,$$

and combined with 2, this finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \blacksquare$

4. A subform theorem in some particular cases

Under some assumption on the algebra, we may improve the previous theorem. Note that conditions (ii) and (iii) in the following statement are direct consequences of condition (iso) of theorem 3.1. Also, the involution σ now is supposed to be anisotropic.

Theorem 4.1. — *Let (A, σ) and (B, τ) be two Brauer-equivalent central simple algebras, each endowed with an orthogonal involution, with (A, σ) anisotropic. We assume moreover that either the index d of the algebras A and B is at most 2 or B is a division algebra. Then the following assertions are equivalent :*

(i)   *(B, τ) is a direct summand of (A, σ);*

(ii)   *For any left ideal $J \subset B$ of reduced dimension d with $\tau(J)$ anisotropic, there exists a symmetric idempotent $e \in A$ and an isomorphism $\Psi : eAe \simeq B$ such that $\forall g \in I_B \cap J_{F_B}$, $\exists f \in A_{F_B}$, with $\Psi_{F_B}(e\sigma(f)fe) = \tau(g)g$;*

(iii)   *There exists a left ideal $J \subset B$ of reduced dimension d with $\tau(J)$ anisotropic, a symmetric idempotent $e \in A$, and an isomorphism $\Psi : eAe \simeq B$, such that $\forall g \in I_B \cap J_{F_B}$, $\exists f \in A_{F_B}$, with $\Psi_{F_B}(e\sigma(f)fe) = \tau(g)g$.*

Remark 4.2. — It follows from [**KMRT98**, (1.12) and (6.2)] that the algebra B always contains left ideals J of reduced dimension d such that $\tau(J)$ is anisotropic. Indeed, $J = \mathrm{Hom}_D(N/(n.D)^\perp, N)$ satisfies these conditions as soon as the vector $n \in N$ is anisotropic.

Moreover, for any such ideal J, after scalar extension J_{F_B} coincides with the set of endomorphisms of the split algebra $B_{F_B} = \mathrm{End}_{F_B}(W)$ whose kernel contains some particular subspace $W_1 \subset W$ of codimension d. Hence, since d is non zero and L_B

has dimension 1, there exists non trivial elements in J_{F_B} with image included in L_B, that is non trivial elements in the intersection $J_{F_B} \cap I_B$. This guarantees conditions (ii) and (iii) are non empty.

Clearly, (i) implies (ii) and (ii) implies (iii). We now prove the theorem in the split and division cases. The index 2 case will be proven in § 5.

4.1. The division case. — In the particular case when B is a division algebra, there exists a unique left ideal of reduced dimension d, B itself, which is necessarily anisotropic. Hence conditions (ii) and (iii) in that case both reduce to condition (iso) of theorem 3.1. This already proves the result is true if B is a division algebra. Note that the anisotropy hypothesis is not necessary in that case.

4.2. The split case. — We assume now that A and B are split, and take representations $(A, \sigma) = (\mathrm{End}_F(V), \mathrm{ad}\, q_V)$ and $(B, \tau) = (\mathrm{End}_F(W), \mathrm{ad}_{q_W})$. The involution σ is anisotropic if and only if q_V is anisotropic, and (A, σ) contains (B, τ) as a direct summand if and only if q_V contains λq_W as a subform for some $\lambda \in F^\times$.
Hence the theorem follows from the following proposition :

Proposition 4.3. — *If A and B are split, then condition (iii) is equivalent to condition (dep) of Theorem 2.1.*

Proof. — Recall the isomorphism $B = \mathrm{End}_F(W) \simeq W \otimes W$ given by $(x \otimes y)(z) = x b_{q_W}(y, z)$. It is an isomorphism of algebras with involution if we endow $W \otimes W$ with the product $(x \otimes y)(z \otimes w) = b_{q_W}(y, z) x \otimes w$ and the involution $\tau(x \otimes y) = y \otimes x$.
Under this isomorphism, a left ideal $J \subset B = \mathrm{End}_F(W)$ of reduced dimension 1 corresponds to $\{x \otimes w,\ x \in W\}$ for some non trivial $w \in W$, uniquely defined up to a scalar factor. Moreover, $\sigma(J) = \{w \otimes x,\ x \in W\}$ is anisotropic if and only if the vector w itself is anisotropic. On the other hand, after scalar extension to F_B, the ideal I_B corresponds under the same isomorphism to $L_B \otimes_{F_B} W_{F_B}$.
Let us first assume (iii). Denote by V_0 the image of the idempotent e. The isomorphims $\Psi : eAe = \mathrm{End}_F(V_0) \to B$ is given by $\Psi(f) = \psi f \psi^{-1}$, for some isomorphism $\psi : V_0 \simeq W$.
Fix a vector $w \in W$ such that $J = \{x \otimes w,\ x \in W\}$. Any $g \in I_B \bigcap J_{F_B}$ can be written as $g = \delta \otimes w$ for some $\delta \in L_B$. We then have $\tau(g)g = q_W(\delta) w \otimes w$. Hence, $\tau(g)g$ maps w to $q_W(\delta) q_W(w) w$, and any element of the orthogonal of w to 0.
Let us now consider the element $v = \psi^{-1}(w) \in V_0$, and denote by f the element of $\mathrm{End}_{F_B}(V_{F_B})$ corresponding to g given by condition (iii). We have
$q_V(f(v)) = q_V(fe(v)) = b_{q_V}(e\sigma(f)fe\psi^{-1}(w), v) = b_{q_V}(\psi^{-1}(\tau(g)g(w)), v) = q_W(\delta) q_W(w) q_V(v)$. This proves (dep) is satisfied. Indeed, we have already noticed in §3 that $q_W(\delta)$ belongs to the projective class of q_W. On the other hand, v and w are both defined over F, w is anisotropic and q_V is anisotropic. Hence $q_W(w) q_V(v) \in F^\times$.

Let us assume conversely that (dep) is satisfied, and let $\nu \in V_{F_B}$ be a vector satisfying $q_V(\nu) = \lambda q_W^{proj} = \lambda q_W(\delta_0)$ for some fixed non trivial $\delta_0 \in L$. Specializing this equality, one may find anisotropic vectors $w \in W$ and $v \in V$ such that $q_V(v) = \lambda q_W(w)$. We let $v_1 = v, v_2, \ldots, v_m$ be an orthogonal basis of (V, q_V) and $w_1 = w, w_2, \ldots, w_n$ be an orthogonal basis of (W, q_W). Let $V_0 \subset V$ be the sub-vector space generated by $v_1, \ldots v_n$. We define e to be the orthogonal projection on V_0, $\psi : V_0 \tilde{\to} W$ the isomorphism defined by $\psi(v_i) = w_i$ for $i = 1, \ldots, n$, and $\Psi : eAe \tilde{\to} B$ the isomorphism defined by $\Psi(f) = \psi f \psi^{-1}$.

Any element $g \in I_B \cap J_{F_B}$ can be written as $g = l\delta_0 \otimes w$ for some $l \in F_B$. Take the corresponding element in A_{F_B} to be $f = \frac{l}{\lambda} \nu \otimes v$. We then have $\tau(g)g = l^2 q_W(\delta_0) w \otimes w$ while $\sigma(f)f = \frac{l^2}{\lambda^2} q_V(\nu) v \otimes v$. An easy computation shows that $\psi e \sigma(f) f e \psi^{-1}(w) = \frac{l^2}{\lambda^2} q_V(\nu) q_V(v) w = l^2 q_W(\delta_0) q_W(w) w = \tau(g)g(w)$. Moreover, both maps are trivial on the orthogonal of w, since by definition of ψ, the image under ψ^{-1} of the orthogonal of w is included in the orthogonal of v. Hence $\psi e \sigma(f) f e \psi^{-1} = \tau(g)g$, and this proves (iii). □

Remark 4.4. — Note that proposition 4.3 actually shows that, in the split case, theorem 4.1 is a reformulation of theorem 2.1. Hence, we may consider it as an analogue for algebras with involution of the subform theorem up to similarities. It would be nice to have a proof which does not use any version of the subform theorem. As opposed to this, we do not consider theorem 3.1 as an analogue of the subform theorem. One should notice, in particular, that condition (iii) do imply that (A, σ) contains a direct summand isomorphic to (B, τ), but this need not be $(eAe, \sigma_{|eAe})$ as the proof above shows. While under condition (iso) $(eAe, \sigma_{|eAe})$ is isomorphic to (B, τ), where e precisely is the idempotent mentioned in the condition. Hence, theorem 3.1 actually is a criterion of isomorphism rather than a subform theorem.

5. Hermitian forms and the index 2 case

Before proving the theorem for algebras of index 2, we translate the conditions of theorem 4.1 in terms of hermitian forms.

5.1. Conditions (ii) and (iii) in terms of hermitian forms. —

Proposition 5.1. — *Let (A, σ) and (B, τ) be two Brauer equivalent central simple algebras both endowed with an orthogonal involution. Fix representations $(A, \sigma) = (\text{End}_D(M), \text{ad}_{h_M})$ and $(B, \tau) = (\text{End}_D(N), \text{ad}_{h_N})$ and let q_V and q_W denote as in §3 the quadratic forms corresponding via Morita theory to the hermitian forms $h_{M F_B}$ and $h_{N F_B}$. We denote by δ a fixed non zero element in the generic line L_B.*

The algebras with involution (A, σ) and (B, τ) satisfy condition (iii) if and only if there exists an anisotropic vector $n \in N$, a vector $m \in M$ and a scalar $\lambda \in F^\times$ such that $h_M(m) = \lambda h_N(n)$, and q_V represents $\lambda q_W(\delta)$.

Remark 5.2. — Note that in particular, if (A, σ) and (B, τ) satisfy condition (iii), then the hermitian forms h_M and h_N have a common value up to a central scalar $\lambda \in F^\times$.

Proof. — Let us assume first that (A, σ) and (B, τ) satisfy condition (iii). We use the computations made in the proof of theorem 3.1, and keep the same notations. Consider J, e and Ψ as given by condition (iii). We denote again by M_0 the image of e and we let $\psi : M_0 \xrightarrow{\sim} M$ be an isomorphism such that $\Psi(f) = \psi f \psi^{-1}$ for any $f \in A$. As recalled in remark 4.2, the ideal $J \subset B = \mathrm{End}_D(N)$ can be written as $J = \mathrm{Hom}_D(N/(n.D)^\perp, N)$, for some anisotropic vector $n \in N$. Define m to be $m = \psi^{-1}(n)$. Since n is anisotropic, we may choose an orthogonal basis $n_1, \ldots, n_2$ of (N, h_N) over D such that $n_1 = n$. One may then easily check that the elements $g_{1,j}$ and g_1 defined in the proof of theorem 3.1 belongs to $J_{F_B} \cap I_B$, and condition (iii) proves the existence of corresponding elements $f_{1,j}$ and f_1 in A_{F_B}. By the computations made in the proof of theorem 3.1 we get that $h_M(m) = \lambda h_N(n)$ for some $\lambda \in F^\times$.

Moreover, if we let $\nu = f_{1,1}(m_1 \otimes 1 \otimes e_1) \in V_0$, equation (1) proves that $q_V(\nu) = b_{q_{V_0}}(f_{1,1}(m_1 \otimes 1 \otimes e_1)) = \lambda q_W(\delta)$.

Let us now prove the converse. Since n is anisotropic, there exists orthogonal basis $m_1, \ldots m_r$ of M and $n_1, \ldots, n_s$ of N such that $n_1 = n$ and $m_1 = m$. We let J be $J = \mathrm{Hom}_D(N/(n.D)^\perp, N)$, and take e to be the orthogonal projection on the submodule M_0 of M generated by $m_0, \ldots, m_s$ and ψ to be the isomorphism $\psi : M_0 \xrightarrow{\sim} N$ given by $\psi(m_i) = n_i$.

Any endomorphism $g \in J_{F_B} \cap I_B$ is defined by

$$g(n_1 \otimes 1 \otimes e_j) = \alpha_j \delta \text{ for some } \alpha_j \in F_B \text{ and } g(n_i \otimes 1 \otimes e_j) = 0 \text{ for } i \geq 2.$$

We then define $f \in A_{F_B} = \mathrm{End}_{F_B}(V)$ by

$$f(m_1 \otimes 1 \otimes e_j) = \alpha_j \nu \text{ and } f(m_i \otimes 1 \otimes e_j) = 0 \text{ for } i \geq 2,$$

where $\nu \in V$ satisfies $q_V(\nu) = \lambda q_W(\delta)$. As in lemma 3.3, one may check that

$$\sigma(f)f(m_1 \otimes 1 \otimes e_j) = \alpha_j q_V(\nu)(m_1 h_M(m_1)^{-1} \otimes 1 \otimes \sum_{l=1}^{d} \alpha_l \frac{e_l}{a_l}),$$

$$\text{and } \sigma(f)f(m_i \otimes 1 \otimes e_j) = 0 \text{ for } i \geq 2.$$

Hence we get

$$\psi e \sigma(f) f e \psi^{-1}(n_1 \otimes 1 \otimes e_j) = \alpha_j \lambda q_W(\delta)(n_1 \lambda^{-1} h_N(n_1)^{-1} \otimes 1 \otimes \sum_{l=1}^{d} \alpha_l \frac{e_l}{a_l}),$$

$$\text{and } \psi e \sigma(f) f e \psi^{-1}(n_i \otimes 1 \otimes e_j) = 0 \text{ for } i \geq 2$$

which by lemma 3.3 implies $\psi e \sigma(f) f e \psi^{-1} = \tau(g)g$ and hence finishes the proof. $\square$

From the previous proposition and the correspondence between left ideals J of reduced dimension D such that $\tau(J)$ is anisotropic and anisotropic vectors in (N, h_N) recalled in remark 4.2, we deduce a translation of condition (ii) in terms of hermitian forms:

Corollary 5.3. — *Let (A, σ), (B, τ), (M, h_M), (N, h_N) and δ be as in proposition 5.1.*
The algebras with involution (A, σ) and (B, τ) satisfy condition (ii) if and only if for any anisotropic vector $n \in N$, there exist a vector $m \in M$ and a scalar $\lambda \in F^\times$ such that $h_M(m) = \lambda h_N(n)$ and q_V represents $\lambda q_W(\delta)$.

5.2. Proof of theorem 4.1 in the index 2 case. — The theorem in the index 2 case now follows from the following proposition :

Proposition 5.4. — *Let (D, γ) be a division algebra of index at most 2 with orthogonal involution and (M, h_M) and (N, h_N) two hermitian forms over (D, γ), with h_M anisotropic. Then h_M contains h_N as a subform if and only if q_V represents $q_W(\delta)$.*

Proof. — If D is split, this statement is the projective version of the classical subform theorem which is already known. Hence we assume D is a division quaternion algebra. The condition is clearly necessary.
To prove it also is sufficient, let us first extend scalars to F_D, so that the situation is split. By a result due both to I. Dejaiffe [**Dej01**] and Parimala-Sridharan-Suresh [**PSS01**], we know that the involution σ remains anisotropic after scalar extension to F_D, so that the quadratic form q_V is anisotropic. Hence we can apply the subform theorem to q_V and q_W. Since the generic point of a variety maps under scalar extension to the generic point of the extended variety, the condition implies that $q_{V F_D(\mathbb{P}W)}$ represents q_W^{proj}, and hence q_V contains q_W as a subform. By Morita equivalence, this precisely means that $(h_M)_{F_D}$ contains $(h_N)_{F_D}$ as a subform, and it only remains to go down using the injectivity of the natural map from the Witt group of D to the Witt group of F_D and the excellence result of [**PSS01**] $\qquad \square$

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