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Products of ratios of consecutive integers

Régis de la Bretèche, Carl Pomerance & Gérald Tenenbaum

For Jean-Louis Nicolas, on his sixtieth birthday

1. Introduction

Let \( \{ \varepsilon_n \}_{1 \leq n < N} \) be a finite sequence with each \( \varepsilon_n \in \{ 0, \pm 1 \} \), and write

\[
\frac{a}{b} = \prod_{1 \leq n < N} \left( \frac{n}{n+1} \right)^{\varepsilon_n},
\]

where the fraction is in its smallest terms. Now, define \( A(N) \) as the maximal value of \( a \) as \( \{ \varepsilon_n \}_{1 \leq n < N} \) runs through all possible \( 3^{N-1} \) sequences of \( 0, \pm 1 \), and let \( B(N) \) denote the corresponding value of \( b \). (Note that maximizing \( b \) instead of \( a \) would lead to \( B(N)/A(N) \).) We obviously have \( A(N) \leq N! \), hence \( \log A(N) \leq N \log N \) for all \( N \). In [7], it is shown by an elegant “near-tiling” of the integers in \([1, N]\) with triples \( n, 2n, 2n + 1 \) that

\[
\log A(N) \leq \left\{ \frac{2}{3} + o(1) \right\} N \log N.
\]

Further, a brief argument of M. Langevin is presented that

\[
\log A(N) \geq \{ \log 4 + o(1) \} N.
\]

Our aim in this article is to establish the true order of magnitude for \( \log A(N) \).

Put

\[
k(c) := 1 + 2 \log(1 - 2c) - \frac{2}{c} \log \left( 1 + \frac{2c^2}{1 - 3c} \right),
\]

\[
K(c) := 2 \int_0^c k(u) \, du, \quad K := \max_{0 < c < 1/5} K(c) \approx 0.107005.
\]

Theorem 1.1. For large \( N \), we have

(1-1) \quad \log A(N) \geq \{ K + o(1) \} N \log N.

Let \( P(n) \) denote the largest prime factor of a positive integer \( n \) with the convention that \( P(1) = 1 \). The lower bound (1-1) is an easy consequence of the estimate stated in the following result.

Theorem 1.2. For \( c \in [0, 1] \), \( x \geq 1 \), let \( S(x, c) \) denote the number of those integers \( n \) not exceeding \( x \) such that \( \min\{ P(n), P(n+1) \} > x^{1-c} \). Then, for any fixed \( c_0 \in [0, \frac{1}{5}] \) and uniformly for \( c \in [0, c_0] \), \( x \to \infty \), we have

(1-2) \quad S(x, c) \leq 2x \int_0^c \log \left( \frac{1 - v}{1 - v - 2c} \right) \frac{dv}{1 - v} + o(x).
Remark. Under a suitably strong form of the Elliott–Halberstam hypothesis, we get the better bound
\[(1-3) \quad S(x, c) \leq x \left\{ \log (1 - c) \right\}^2 + o(x)\]
for \(0 \leq c \leq \frac{1}{2}\). This would yield the value \(K = 4/\sqrt{e} - 2 \approx 0.426123\) in Theorem 1.1. See Section 3 for further methodological remarks.

The bound (1-3) is probably optimal, in fact it is likely to be the case that
\(S(x, c) = x \left\{ 1 - g(1/(1 - c)) \right\}^2 + o(x)\)
for \(0 \leq c < 1\), where \(g\) is the Dickman–de Bruijn function. An even stronger statement is suggested in [3].

Note that (1-1) follows from (1-2) by selecting \(\varepsilon_n = 1\) if \(P(n) > N^{1-c}\) and \(P(n) > P(n+1), \varepsilon_n = -1\) if \(P(n+1) > N^{1-c}\) and \(P(n+1) > P(n)\) and \(\varepsilon_n = 0\) in all other cases. Indeed, with these choices for \(\varepsilon_n\), we obtain that for each prime \(p > N^{1-c} \geq N^{1/2}\), the exponent on \(p\) in the prime factorization of the rational number \(A(N)/B(N)\) is
\[
\sum_{n < N} 2 - \sum_{n < N} \frac{2}{P(n) > P(n+1)} - \sum_{n < N} \frac{2}{P(n)} = 2.
\]

Thus,
\[
\log A(N) \geq \sum_{n < N, P(n) > N^{1-c}} 2 \log P(n) - \sum_{n < N, P(n), P(n+1) > N^{1-c}} 2 \log \min\{P(n), P(n+1)\}.
\]

We have
\[
\sum_{n < N, P(n), P(n+1) > N^{1-c}} 2 \log \min\{P(n), P(n+1)\} = \int_0^c (1 - u) \log N \ dS(N, u)
\]
\[
= (\log N) \left\{ (1 - c)S(N, c) + \int_0^c S(N, u) \ du \right\},
\]
and, since the number of \(n < N\) with \(P(n) > N^{1-c}\) is \(N \log \{1/(1 - c)\} + o(N)\) uniformly for \(0 \leq c \leq 1/2\),
\[
\sum_{n < N, P(n) > N^{1-c}} \log P(n) = cN \log N + o(N).
\]

We thus obtain
\[
\log A(N) \geq 2(\log N) \left\{ cN - (1 - c)S(N, c) - \int_0^c S(N, u) \ du + o(N) \right\}
\]
\[
\geq 2N(\log N) \left\{ g(c) + o(1) \right\},
\]
where we have set
\[
g(c) := c - (1 - c)f(c) - \int_0^c f(u) \ du, \quad \text{with} \quad f(u) := 2 \int_0^u \log \left( \frac{1 - v}{1 - v - 2u} \right) \frac{dv}{1 - v}.
\]

We check by computation that \(g'(c) = k(c)\). This implies the desired estimate.
2. Proof of Theorem 1.2

We employ the Rosser–Iwaniec sieve. A slightly better numerical bound could be obtained from a more sophisticated sieve method. We do not pursue such an improvement here but provide precise indications in Section 3. We refer to [5], [6] for a complete reference of the Rosser-Iwaniec coefficients and merely recall the property we shall use. We denote by \( \gamma \) the Euler constant, and we let \( p \) run over primes.

**Lemma 2.1.** Let \( \mathcal{Q} \) denote a set of primes, let \( z \geq 2 \) and write \( Q(z) := \prod_{p \leq z, p \in \mathcal{Q}} p \). There exists a sequence \( \{ \lambda_d \}_{d=1}^{\infty} \) of real numbers, vanishing for \( d > z \) or \( \mu(d) = 0 \), satisfying \( \lambda_1 = 1, \ |\lambda_d| \leq 1, \ \mu \ast 1 \leq \lambda \ast 1 \), and such that for any number \( \alpha > 0 \),

\[
\sum_{d|Q(z)} \frac{\lambda_d w(d)}{d} \leq \prod_{p \leq z, p \in \mathcal{Q}} \left( 1 - \frac{w(p)}{p} \right) \left\{ 2e^\gamma + O\left( \frac{1}{(\log z)^{1/3}} \right) \right\},
\]

uniformly for all multiplicative functions \( w \) satisfying

(i) \( 0 < w(p) < p \quad (p \in \mathcal{Q}); \)

(ii) \( \prod_{u < p < v, p \in \mathcal{Q}} \left( 1 - \frac{w(p)}{p} \right)^{-1} \leq \frac{\log v}{\log u} \left( 1 + \frac{\alpha}{\log u} \right) \quad (2 \leq u \leq v \leq z). \)

If \( n \) is counted by \( S(x, c) \), then \( n = ap_1 = bp_2 - 1 \), where \( p_1 \) and \( p_2 \) are primes greater than \( x^{1-c} \). Then \( a \) and \( b \) are obviously coprime, and moreover \( 2|ab \). We need an upper bound for the number \( Z(a, b) \) of admissible pairs \( (p_1, p_2) \) for given \( a, b \).

Let \( C \) be a sufficiently large constant and set \( z := \left( \frac{x}{a} \right)^{1/2} b^{-1} (\log x)^{-C} \). If \( \mathcal{Q} \) is the set of all primes not dividing \( a \) and with \( \{ \lambda_d \}_{d=1}^{\infty} \) the sequence from Lemma 2.1, we plainly have

\[
Z(a, b) \leq \sum_{p_1 \leq x/a \atop ap_1 \equiv -1 (\text{mod } b)} \mu \ast 1((ap_1 + 1)/b, Q(z))
\]

(2.1)

\[
\leq \sum_{d|Q(z)} \lambda_d \sum_{p_1 \leq x/a \atop ap_1 \equiv -1 (\text{mod } bd)} 1.
\]

Let us put, for real \( y \geq 2 \) and integers \( q, r \) with \( q \geq 1 \),

\[
\pi(y; q, r) := \sum_{p \leq y \atop p \equiv r (\text{mod } q)} 1, \quad E(y; q) := \max_{(r, q) = 1} |\pi(y; q, r) - \text{li}(y) / \phi(q)|.
\]

We apply Lemma 2.1 to the multiplicative function \( d \mapsto d\phi(b)/\phi(bd) \). Using the fact that \( (a, bd) = 1 \) for each \( d \mid Q(z) \), and noticing that \( c \) bounded below \( 1/5 \) ensures that \( z \geq b \) when \( x \) is large enough, we deduce that

\[
Z(a, b) \leq M(a, b) + R(a, b)
\]
with
\[ R(a, b) := \sum_{d \leq z} E(x/a; bd) \]
and
\[ M(a, b) := \sum_{d \mid Q(z)} \frac{\lambda_d \text{li}(x/a)}{\varphi(bd)} \]
\[ \leq \{2e^\gamma + o(1)\} \frac{\text{li}(x/a)}{\varphi(b)} \prod_{\substack{p \leq z \atop p \mid ab}} \left(1 - \frac{1}{p - 1}\right) \prod_{\substack{p \leq z \atop p \nmid b}} \left(1 - \frac{1}{p}\right) \]
\[ = \{2e^\gamma + o(1)\} \frac{\text{li}(x/a)}{b} \prod_{\substack{p \leq z \atop p \nmid ab}} \left(1 + \frac{1}{p(p - 2)}\right). \]

Now we observe that, uniformly as \( x \) tends to \( \infty \) and \( a, b \) vary in the specified ranges,
\[ \prod_{\substack{p \leq z \atop p > 2}} \left(\frac{p - 2}{p - 1}\right) = 2 \prod_{\substack{p \leq z \atop p > 2}} \frac{p(p - 2)}{(p - 1)^2} \prod_{\substack{p \leq z \atop p > 2}} \left(1 - \frac{1}{p}\right) \sim \frac{2e^{-\gamma}}{A \log z} \]
where
\[ A := \prod_{p > 2} \left(1 + \frac{1}{p(p - 2)}\right). \]

Therefore, writing
\[ h(n) := \prod_{\substack{p \mid n \atop p > 2}} \left(\frac{p - 1}{p - 2}\right), \]
we obtain that the estimate
\[ (2.3) \quad M(a, b) \leq \frac{\{8 + o(1)\} h(ab) x}{Aab \log(x/a) \log(x/ab^2)} \]
holds uniformly for \( a \leq x^\varepsilon, b \leq x^\varepsilon, (a, b) = 1 \), as \( x \to \infty \).

Let \( \tau(m) \) denote the number of divisors of \( m \). By the Bombieri–Vinogradov theorem, we have, with \( X_a := (x/a)^{1/2}(\log x)^{-C} \),
\[ \sum_{b \leq x^\varepsilon} R(a, b) \leq \sum_{m \leq X_a} \tau(m) E(x/a; m) \]
\[ \ll \left\{ \sum_{m \leq X_a} E(x/a; m) \sum_{m \leq X_a} \tau(m)^2 E(x/a; m) \right\}^{1/2} \ll \frac{x}{a(\log x)^2}. \]
where we have used the trivial estimate $E(x/a; m) \ll x/am$ and the well-known fact that $\sum_{m \leq x} \tau(m)^2/m \ll (\log x)^4$. Therefore, we obtain from (2·2) and (2·3)

\[
S(x, c) \leq \sum_{a \leq x^c; b \leq x^c; (a, b) = 1} Z(a, b)
\]

(2·4)

\[
\leq \frac{8 + o(1)}{A} \sum_{a \leq x^c} \frac{h(a)}{a \log(x/a)} \sum_{\substack{b \leq x^c \geq 1 \atop (b,a)=1}} \frac{h(b)}{b \log(x/ab^2)} + O\left(\frac{x}{\log x}\right).
\]

We have for $\nu = 0$ or 1

\[
\sum_{\substack{b \geq 1 \atop (b,a)=1}} h(2^\nu b) \sim A y^{2\nu/2} \varepsilon(a)^{-c} (y \to \infty),
\]

where

\[
H(s) := \prod_{p > 2} \left(1 + \frac{1}{p^s(p-2)}\right), \quad G_a(s) := \left(1 - \frac{\varepsilon(a)}{2^s}\right) \prod_{\substack{p \mid a \atop p > 2}} \left(\frac{1 - p^{-s}}{1 + p^{-s}/(p-2)}\right),
\]

with $\varepsilon(a) = 1$ if $a$ is even, $\varepsilon(a) = 0$ if $a$ is odd. The functions $H$ and $G_a$ can be analytically continued to the half-plane $\Re s > 0$. Note that $H(1) = A$, $G_a(1) = 2^{-\varepsilon(a)} h(a)^{-1}$. By estimates of Selberg–Delange type (see [8], chap. II.5), (2·5) yields in turn

\[
\sum_{\substack{b \leq y \atop (b,a)=1}} h(2^\nu b) \sim A y^{2\nu/2} \varepsilon(a)^{-c} (y \to \infty),
\]

and

\[
\sum_{\substack{b \leq x^c \geq 1 \atop (a,b)=1, 2 \mid ab}} \frac{h(b)}{b \log(x/ab^2)} = \frac{A}{4h(a)} \log\left(\frac{1 - v_a}{1 - 2c - v_a}\right) + o(1) \quad (x \to \infty)
\]

and $v_a := (\log a)/\log x$. Substituting this back into (2·4), we arrive at

\[
S(x, c) \leq \{2 + o(1)\} x \sum_{a \leq x^c} \frac{1}{a \log(x/a)} \log\left(\frac{1 - v_a}{1 - 2c - v_a}\right)
\]

\[
= \{2 + o(1)\} x \int_0^{x^c} \log\left(\frac{1 - v}{1 - 2c - v}\right) \frac{dv}{1 - v}.
\]

We remark that with a little more care, the bound $1/5$ in the theorem may be replaced with $1/3$. 
3. Further remarks

In [3] it is shown that if \( N \) is large, then for at least \( 0.0099 N \) values of \( n \leq N \) we have \( P(n) > P(n + 1) \), and for at least \( 0.0099 N \) values of \( n \leq N \) we have \( P(n) < P(n + 1) \). It follows from Theorem 1.2 that each inequality occurs on a set of integers \( n \) of lower asymptotic density

\[
\log \left( \frac{1}{1 - c} \right) - 2 \int_0^c \log \left( \frac{1 - v}{1 - v - 2c} \right) \frac{dv}{1 - v}
\]

for each value of \( c \) with \( 0 < c < 1/5 \). The maximum of this expression is greater than \( 0.05544 \) so we have majorized the result from [3]. Presumably, the set \( E \) of integers \( n \) with \( P(n) > P(n + 1) \) has asymptotic density \( 1/2 \). A general theorem of Hildebrand [4] also implies that \( E \) has positive lower asymptotic density, but we did not check the numerical value that can be derived from this result.

In [3] it is shown that \( P(n) < P(n + 1) < P(n + 2) \) holds infinitely often, and it was conjectured that so too \( P(n) > P(n + 1) > P(n + 2) \) holds infinitely often. This conjecture was recently proved by Balog in [1].

We observe that the maximal value \( A(N) \) corresponds to a sequence \( \varepsilon = \{ \varepsilon_n \}_{1 \leq n < N} \) where \( \varepsilon_n \in \{-1,1\} \).

**Proposition 3.1.** Let \( N \geq 1 \). There exists \( \{ \varepsilon_n \}_{1 \leq n < N} \in \{-1,1\}^{N-1} \) such that

\[
\frac{A(N)}{B(N)} = \prod_{1 \leq n < N} \left( \frac{n}{n+1} \right)^{\varepsilon_n}.
\]

**Remark.** Let \( A_{0,1}(N) \) (respectively \( A_{-1,1}(N), A_{-1,0}(N) \)) the maximum of numerators where the exponents \( \varepsilon_n \) are restricted to \( \{0,1\} \) (respectively \( \{-1,1\}, \{-1,0\} \)). By the proposition, we have \( A_{-1,1}(N) = A(N) \) and

\[
\log A_{0,1}(N) = \frac{1}{2} \log A(N) + O(\log N) = \log A_{-1,0}(N) + O(\log N).
\]

For example, if \( \{ \varepsilon_n \}_{1 \leq n < N} = \{0,1\}^{N-1} \), we have \( \{2\varepsilon_n - 1\}_{1 \leq n < N} = \{-1,1\}^{N-1} \). Since the constant sequence \( -1 \) gives the numerator \( N \), we deduce the result.

**Proof.** Take a sequence \( \{ \varepsilon_n \}_{1 \leq n < N} \in \{-1,0,1\}^{N-1} \) where some \( \varepsilon_n = 0 \). Write the associated product as \( A/B \) with \( (A,B) = 1 \). If we let \( \varepsilon_n = 1 \), the new numerator is

\[
\frac{A}{(A, n + 1)} \times \frac{n}{(B, n)}
\]

while if we let \( \varepsilon_n = -1 \), the new numerator is

\[
\frac{A}{(A,n)} \times \frac{n + 1}{(B,n + 1)}.
\]
Assuming both of these expressions are smaller than $A$, we obtain

$$n < (A, n+1)(B, n) \quad \text{and} \quad n+1 < (A, n)(B, n+1).$$

Multiplying these inequalities and using $(A, B) = (n, n+1) = 1$ we obtain

$$n(n+1) < (AB, n(n+1)),$$

a contradiction. So we may choose $\varepsilon_n \in \{\pm 1\}$ without decreasing the associated numerator. With this method we can replace each 0 value with $\pm 1$ and the value of the associated numerator will not decrease. $\Box$

As pointed out in Section 2, the result of Theorem 1.2 can be improved by using more sophisticated sieve methods. Indeed, in the present treatment, we sift integers of the form $(a p_1 + 1)/b$ for fixed $a$ and $b$, and then sum over $a$ after having disposed of the $b$-sum by the Bombieri–Vinogradov theorem, thus applying this result to a sequence of length $x/a$ and hence of distribution level roughly $\sqrt{x/a}$. We could, alternatively, sift, for fixed $b$, the sequence comprising all $(a p_1 + 1)/b$ of a piece. We could then restrict the study to, say, $x/\log x < a p_1 \leq x$. This corresponds to the Dirichlet convolution of two characteristic functions that are of Siegel–Walfisz type, and thus has distribution level $x^{1/2+o(1)}$ — see, e.g., [2] theorem 0(b). Such an approach yields, for any given $c_0$, $0 < c_0 < \frac{1}{4}$, the upper bound

$$S(x, c) \leq 2x \log (1 - 2c) \log (1 - c) + o(x)$$

uniformly for $0 \leq c \leq c_0$. In turn, this improvement allows the better value $K \approx 0.112945$ in Theorem 1.1, and ups the density in the first paragraph of this section to 0.05866. We warmly thank Étienne Fouvry for these observations.

References