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Products of ratios of consecutive integers

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For Jean-Louis Nicolas, on his sixtieth birthday

1. Introduction

Let $\{\varepsilon_n\}_{1\leqslant n<N}$ be a finite sequence with each $\varepsilon_n \in \{0, \pm 1\}$, and write

$$\frac{a}{b} = \prod_{1\leqslant n<N} \Big(\frac{n}{n+1}\Big)^{\varepsilon_n},$$

where the fraction is in its smallest terms. Now, define $A(N)$ as the maximal value of a as $\{\varepsilon_n\}_{1\leqslant n<N}$ runs through all possible 3^{N-1} sequences of $0, \pm 1$, and let $B(N)$ denote the corresponding value of b. (Note that maximizing b instead of a would lead to $B(N)/A(N)$.) We obviously have $A(N) \leqslant N!$, hence $\log A(N) \leqslant N \log N$ for all N. In [7], it is shown by an elegant "near-tiling" of the integers in $[1, N]$ with triples $n, 2n, 2n+1$ that

$$\log A(N) \leqslant \big\{\tfrac{2}{3} + o(1)\big\} N \log N.$$

Further, a brief argument of M. Langevin is presented that

$$\log A(N) \geqslant \{\log 4 + o(1)\} N.$$

Our aim in this article is to establish the true order of magnitude for $\log A(N)$. Put

$$k(c) := 1 + 2\log(1-2c) - \frac{2}{c}\log\Big(1 + \frac{2c^2}{1-3c}\Big),$$

$$K(c) := 2\int_0^c k(u)\,\mathrm{d}u, \qquad K := \max_{0<c<1/5} K(c) \approx 0.107005.$$

Theorem 1.1. *For large N, we have*

$$\log A(N) \geqslant \{K + o(1)\} N \log N. \tag{1·1}$$

Let $P(n)$ denote the largest prime factor of a positive integer n with the convention that $P(1) = 1$. The lower bound (1·1) is an easy consequence of the estimate stated in the following result.

Theorem 1.2. *For $c \in [0, 1]$, $x \geqslant 1$, let $S(x, c)$ denote the number of those integers n not exceeding x such that $\min\{P(n), P(n+1)\} > x^{1-c}$. Then, for any fixed $c_0 \in]0, \frac{1}{5}[$ and uniformly for $c \in [0, c_0]$, $x \to \infty$, we have*

$$S(x, c) \leqslant 2x\int_0^c \log\Big(\frac{1-v}{1-v-2c}\Big)\frac{\mathrm{d}v}{1-v} + o(x). \tag{1·2}$$

Remark. Under a suitably strong form of the Elliott–Halberstam hypothesis, we get the better bound

$$S(x,c) \leqslant x\big\{\log(1-c)\big\}^2 + o(x) \tag{1·3}$$

for $0 \leqslant c \leqslant \frac{1}{2}$. This would yield the value $K = 4/\sqrt{\mathrm{e}} - 2 \approx 0.426123$ in Theorem 1.1. See Section 3 for further methodological remarks.

The bound (1·3) is probably optimal, in fact it is likely to be the case that $S(x,c) = x\big\{1 - \varrho\big(1/\{1-c\}\big)\big\}^2 + o(x)$ for $0 \leqslant c < 1$, where ϱ is the Dickman–de Bruijn function. An even stronger statement is suggested in [3].

Note that (1·1) follows from (1·2) by selecting $\varepsilon_n = 1$ if $P(n) > N^{1-c}$ and $P(n) > P(n+1)$, $\varepsilon_n = -1$ if $P(n+1) > N^{1-c}$ and $P(n+1) > P(n)$ and $\varepsilon_n = 0$ in all other cases. Indeed, with these choices for ε_n, we obtain that for each prime $p > N^{1-c} \geqslant N^{1/2}$, the exponent on p in the prime factorization of the rational number $A(N)/B(N)$ is

$$\sum_{\substack{n<N\\ P(n)=p}} 2 \;-\; \sum_{\substack{n<N\\ P(n+1)>P(n)=p}} 2 \;-\; \sum_{\substack{n<N\\ P(n)>P(n+1)=p}} 2.$$

Thus,

$$\log A(N) \geqslant \sum_{\substack{n<N\\ P(n)>N^{1-c}}} 2\,\log P(n) \;-\; \sum_{\substack{n<N\\ P(n),P(n+1)>N^{1-c}}} 2\,\log\min\{P(n),P(n+1)\}.$$

We have

$$\begin{aligned}\sum_{\substack{n\leqslant N\\ P(n),P(n+1)>N^{1-c}}} 2\,\log\min\{P(n),P(n+1)\} &= \int_0^c (1-u)\log N\;\mathrm{d}S(N,u)\\ &= (\log N)\Big\{(1-c)S(N,c) + \int_0^c S(N,u)\,\mathrm{d}u\Big\},\end{aligned}$$

and, since the number of $n < N$ with $P(n) > N^{1-c}$ is $N\log\{1/(1-c)\} + o(N)$ uniformly for $0 \leqslant c \leqslant 1/2$,

$$\sum_{\substack{n<N\\ P(n)>N^{1-c}}} \log P(n) = cN\log N + o(N).$$

We thus obtain

$$\begin{aligned}\log A(N) &\geqslant 2(\log N)\Big\{cN - (1-c)S(N,c) - \int_0^c S(N,u)\,\mathrm{d}u + o(N)\Big\}\\ &\geqslant 2N(\log N)\big\{g(c) + o(1)\big\},\end{aligned}$$

where we have set

$$g(c) := c - (1-c)f(c) - \int_0^c f(u)\,\mathrm{d}u, \quad \text{with} \quad f(u) := 2\int_0^u \log\Big(\frac{1-v}{1-v-2u}\Big)\frac{\mathrm{d}v}{1-v}.$$

We check by computation that $g'(c) = k(c)$. This implies the desired estimate.

2. Proof of Theorem 1.2

We employ the Rosser–Iwaniec sieve. A slightly better numerical bound could be obtained from a more sophisticated sieve method. We do not pursue such an improvement here but provide precise indications in Section 3. We refer to [5], [6] for a complete reference of the Rosser-Iwaniec coefficients and merely recall the property we shall use. We denote by γ the Euler constant, and we let p run over primes.

Lemma 2.1. *Let $\mathcal{Q}$ denote a set of primes, let $z \geqslant 2$ and write $Q(z) := \prod_{p \leqslant z,\, p \in \mathcal{Q}} p$. There exists a sequence $\{\lambda_d\}_{d=1}^{\infty}$ of real numbers, vanishing for $d > z$ or $\mu(d) = 0$, satisfying $\lambda_1 = 1$, $|\lambda_d| \leqslant 1$, $\mu * \mathbf{1} \leqslant \lambda * \mathbf{1}$, and such that for any number $\alpha > 0$,*

$$\sum_{d \mid Q(z)} \frac{\lambda_d w(d)}{d} \leqslant \prod_{\substack{p \leqslant z \\ p \in \mathcal{Q}}} \Big(1 - \frac{w(p)}{p}\Big)\Big\{2\mathrm{e}^{\gamma} + O_\alpha\Big(\frac{1}{(\log z)^{1/3}}\Big)\Big\},$$

uniformly for all multiplicative functions w satisfying

$$\text{(i)} \qquad 0 < w(p) < p \qquad (p \in \mathcal{Q}),$$

$$\text{(ii)} \qquad \prod_{u < p \leqslant v,\, p \in \mathcal{Q}} \Big(1 - \frac{w(p)}{p}\Big)^{-1} \leqslant \frac{\log v}{\log u}\Big(1 + \frac{\alpha}{\log u}\Big) \qquad (2 \leqslant u \leqslant v \leqslant z).$$

If n is counted by $S(x, c)$, then $n = ap_1 = bp_2 - 1$, where p_1 and p_2 are primes greater than x^{1-c}. Then a and b are obviously coprime, and moreover $2|ab$. We need an upper bound for the number $Z(a, b)$ of admissible pairs (p_1, p_2) for given a, b. Let C be a sufficiently large constant and set $z := (x/a)^{1/2} b^{-1} (\log x)^{-C}$. If $\mathcal{Q}$ is the set of all primes not dividing a and with $\{\lambda_d\}_{d=1}^{\infty}$ the sequence from Lemma 2.1, we plainly have

$$\begin{aligned} Z(a,b) &\leqslant \sum_{\substack{p_1 \leqslant x/a \\ ap_1 \equiv -1 \,(\mathrm{mod}\, b)}} \mu * \mathbf{1}\big((ap_1+1)/b, Q(z)\big) \\ &\leqslant \sum_{d \mid Q(z)} \lambda_d \sum_{\substack{p_1 \leqslant x/a \\ ap_1 \equiv -1 \,(\mathrm{mod}\, bd)}} 1. \end{aligned} \tag{2·1}$$

Let us put, for real $y \geqslant 2$ and integers q, r with $q \geqslant 1$,

$$\pi(y; q, r) := \sum_{\substack{p \leqslant y \\ p \equiv r \,(\mathrm{mod}\, q)}} 1, \quad E(y; q) := \max_{(r,q)=1} |\pi(y; q, r) - \mathrm{li}(y)/\varphi(q)|.$$

We apply Lemma 2.1 to the multiplicative function $d \mapsto d\varphi(b)/\varphi(bd)$. Using the fact that $(a, bd) = 1$ for each $d \mid Q(z)$, and noticing that c bounded below $1/5$ ensures that $z \geqslant b$ when x is large enough, we deduce that

$$Z(a, b) \leqslant M(a, b) + R(a, b) \tag{2·2}$$

with

$$R(a,b) := \sum_{d \leqslant z} E(x/a; bd)$$

and

$$\begin{aligned} M(a,b) &:= \sum_{d \mid Q(z)} \frac{\lambda_d \operatorname{li}(x/a)}{\varphi(bd)} \\ &\leqslant \{2\mathrm{e}^\gamma + o(1)\} \frac{\operatorname{li}(x/a)}{\varphi(b)} \prod_{\substack{p \leqslant z \\ p \nmid ab}} \Big(1 - \frac{1}{p-1}\Big) \prod_{\substack{p \leqslant z \\ p \mid b}} \Big(1 - \frac{1}{p}\Big) \\ &= \{2\mathrm{e}^\gamma + o(1)\} \frac{\operatorname{li}(x/a)}{b} \prod_{\substack{p \leqslant z \\ p \nmid ab}} \Big(\frac{p-2}{p-1}\Big). \end{aligned}$$

Now we observe that, uniformly as x tends to ∞ and a, b vary in the specified ranges,

$$\prod_{\substack{p \leqslant z \\ p > 2}} \Big(\frac{p-2}{p-1}\Big) = 2 \prod_{\substack{p \leqslant z \\ p > 2}} \frac{p(p-2)}{(p-1)^2} \prod_{p \leqslant z} \Big(1 - \frac{1}{p}\Big) \sim \frac{2\mathrm{e}^{-\gamma}}{A \log z}$$

where

$$A := \prod_{p > 2} \Big(1 + \frac{1}{p(p-2)}\Big).$$

Therefore, writing

$$h(n) := \prod_{\substack{p \mid n \\ p > 2}} \Big(\frac{p-1}{p-2}\Big),$$

we obtain that the estimate

$$M(a,b) \leqslant \frac{\{8 + o(1)\} h(ab) x}{Aab \log(x/a) \log(x/ab^2)} \tag{2·3}$$

holds uniformly for $a \leqslant x^c$, $b \leqslant x^c$, $(a,b) = 1$, as $x \to \infty$.

Let $\tau(m)$ denote the number of divisors of m. By the Bombieri–Vinogradov theorem, we have, with $X_a := (x/a)^{1/2} (\log x)^{-C}$,

$$\begin{aligned} \sum_{b \leqslant x^c} R(a,b) &\leqslant \sum_{m \leqslant X_a} \tau(m) E(x/a; m) \\ &\ll \Big\{ \sum_{m \leqslant X_a} E(x/a; m) \sum_{m \leqslant X_a} \tau(m)^2 E(x/a; m) \Big\}^{1/2} \ll \frac{x}{a (\log x)^2}, \end{aligned}$$

where we have used the trivial estimate $E(x/a;m) \ll x/am$ and the well-known fact that $\sum_{m\leqslant x} \tau(m)^2/m \ll (\log x)^4$. Therefore, we obtain from (2·2) and (2·3)

$$\begin{aligned} S(x,c) &\leqslant \sum_{\substack{a\leqslant x^c,\, b\leqslant x^c \\ (a,b)=1,\, 2|ab}} Z(a,b) \\ &\leqslant \frac{8+o(1)}{A} x \sum_{a\leqslant x^c} \frac{h(a)}{a\log(x/a)} \sum_{\substack{b\leqslant x^c \\ 2|ab \\ (b,a)=1}} \frac{h(b)}{b\log(x/ab^2)} + O\Big(\frac{x}{\log x}\Big). \end{aligned} \tag{2·4}$$

We have for $\nu = 0$ or 1

$$\sum_{\substack{b\geqslant 1 \\ (b,a)=1}} \frac{h(2^\nu b)}{b^s} = H(s)G_a(s)\zeta(s) \qquad (\Re e\, s > 1) \tag{2·5}$$

where

$$H(s) := \prod_{p>2} \Big(1 + \frac{1}{p^s(p-2)}\Big), \qquad G_a(s) := \Big(1 - \frac{\varepsilon(a)}{2^s}\Big) \prod_{\substack{p|a \\ p>2}} \Big(\frac{1-p^{-s}}{1+p^{-s}/(p-2)}\Big),$$

with $\varepsilon(a) = 1$ if a is even, $\varepsilon(a) = 0$ if a is odd. The functions H and G_a can be analytically continued to the half-plane $\Re e\, s > 0$. Note that $H(1) = A$, $G_a(1) = 2^{-\varepsilon(a)}h(a)^{-1}$. By estimates of Selberg–Delange type (see [8], chap. II.5), (2·5) yields in turn

$$\sum_{\substack{b\leqslant y \\ (b,a)=1}} h(2^\nu b) \sim \frac{Ay}{2^{\varepsilon(a)}h(a)} \qquad (y \to \infty),$$

and

$$\sum_{\substack{b\leqslant x^c \\ (a,b)=1,\, 2|ab}} \frac{h(b)}{b\log(x/ab^2)} = \frac{A}{4h(a)} \log\Big(\frac{1-v_a}{1-2c-v_a}\Big) + o(1) \qquad (x \to \infty)$$

and $v_a := (\log a)/\log x$. Substituting this back into (2·4), we arrive at

$$\begin{aligned} S(x,c) &\leqslant \{2+o(1)\}x \sum_{a\leqslant x^c} \frac{1}{a\log(x/a)} \log\Big(\frac{1-v_a}{1-2c-v_a}\Big) \\ &= \{2+o(1)\}x \int_0^c \log\Big(\frac{1-v}{1-2c-v}\Big)\frac{\mathrm{d}v}{1-v}. \end{aligned}$$

□

We remark that with a little more care, the bound 1/5 in the theorem may be replaced with 1/3.

3. Further remarks

In [3] it is shown that if N is large, then for at least $0.0099N$ values of $n \leqslant N$ we have $P(n) > P(n+1)$, and for at least $0.0099N$ values of $n \leqslant N$ we have $P(n) < P(n+1)$. It follows from Theorem 1.2 that each inequality occurs on a set of integers n of lower asymptotic density

$$\log\Big(\frac{1}{1-c}\Big) - 2\int_0^c \log\Big(\frac{1-v}{1-v-2c}\Big)\frac{\mathrm{d}v}{1-v}$$

for each value of c with $0 < c < 1/5$. The maximum of this expression is greater than 0.05544 so we have majorized the result from [3]. Presumably, the set E of integers n with $P(n) > P(n+1)$ has asymptotic density $1/2$. A general theorem of Hildebrand [4] also implies that E has positive lower asymptotic density, but we did not check the numerical value that can be derived from this result.

In [3] it is shown that $P(n) < P(n+1) < P(n+2)$ holds infinitely often, and it was conjectured that so too $P(n) > P(n+1) > P(n+2)$ holds infinitely often. This conjecture was recently proved by Balog in [1].

We observe that the maximal value $A(N)$ corresponds to a sequence $\varepsilon = \{\varepsilon_n\}_{1\leqslant n<N}$ where $\varepsilon_n \in \{-1,1\}$.

Proposition 3.1. *Let $N \geqslant 1$. There exists $\{\varepsilon_n\}_{1\leqslant n<N} \in \{-1,1\}^{N-1}$ such that*

$$\frac{A(N)}{B(N)} = \prod_{1\leqslant n<N}\Big(\frac{n}{n+1}\Big)^{\varepsilon_n}.$$

Remark. Let $A_{0,1}(N)$ (respectively $A_{-1,1}(N)$, $A_{-1,0}(N)$) the maximum of numerators where the exponents ε_n are restricted to $\{0,1\}$ (respectively $\{-1,1\}$, $\{-1,0\}$). By the proposition, we have $A_{-1,1}(N) = A(N)$ and

$$\log A_{0,1}(N) = \tfrac{1}{2}\log A(N) + O(\log N) = \log A_{-1,0}(N) + O(\log N).$$

For example, if $\{\varepsilon_n\}_{1\leqslant n<N} \in \{0,1\}^{N-1}$, we have $\{2\varepsilon_n - 1\}_{1\leqslant n<N} \in \{-1,1\}^{N-1}$. Since the constant sequence -1 gives the numerator N, we deduce the result.

Proof. Take a sequence $\{\varepsilon_n\}_{1\leqslant n<N} \in \{-1,0,1\}^{N-1}$ where some $\varepsilon_n = 0$. Write the associated product as A/B with $(A,B) = 1$. If we let $\varepsilon_n = 1$, the new numerator is

$$\frac{A}{(A,n+1)} \times \frac{n}{(B,n)},$$

while if we let $\varepsilon_n = -1$, the new numerator is

$$\frac{A}{(A,n)} \times \frac{n+1}{(B,n+1)}.$$

Assuming both of these expressions are smaller than A, we obtain

$$n < (A, n+1)(B, n) \quad \text{and} \quad n+1 < (A, n)(B, n+1).$$

Multiplying these inequalities and using $(A, B) = (n, n+1) = 1$ we obtain

$$n(n+1) < (AB, n(n+1)),$$

a contradiction. So we may choose $\varepsilon_n \in \{\pm 1\}$ without decreasing the associated numerator. With this method we can replace each 0 value with ± 1 and the value of the associated numerator will not decrease. □

As pointed out in Section 2, the result of Theorem 1.2 can be improved by using more sophisticated sieve methods. Indeed, in the present treatment, we sift integers of the form $(ap_1+1)/b$ for fixed a and b, and then sum over a after having disposed of the b-sum by the Bombieri–Vinogradov theorem, thus applying this result to a sequence of length x/a and hence of distribution level roughly $\sqrt{x/a}$. We could, alternatively, sift, for fixed b, the sequence comprising all $(ap_1+1)/b$ of a piece. We could then restrict the study to, say, $x/\log x < ap_1 \leqslant x$. This corresponds to the Dirichlet convolution of two characteristic functions that are of Siegel–Walfisz type, and thus has distribution level $x^{1/2+o(1)}$ — see, e.g., [2] theorem 0(b). Such an approach yields, for any given c_0, $0 < c_0 < \frac{1}{4}$, the upper bound

$$S(x, c) \leqslant 2x \log(1-2c) \log(1-c) + o(x) \tag{3·1}$$

uniformly for $0 \leqslant c \leqslant c_0$. In turn, this improvement allows the better value $K \approx 0.112945$ in Theorem 1.1, and ups the density in the first paragraph of this section to 0.05866. We warmly thank Étienne Fouvry for these observations.

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