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AN OBSTRUCTION FOR THE MEAN CURVATURE OF A CONFORMAL IMMERSION $S^n \rightarrow \mathbb{R}^{n+1}$

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ABSTRACT. We prove a Pohozaev type identity for non-linear eigenvalue equations of the Dirac operator on Riemannian spin manifolds with boundary. As an application, we obtain that the mean curvature H of a conformal immersion $S^n \rightarrow \mathbb{R}^{n+1}$ satisfies $\int \partial_X H = 0$ where X is a conformal vector field on S^n and where the integration is carried out with respect to the Euclidean volume measure of the image. This identity is analogous to the Kazdan-Warner obstruction that appears in the problem of prescribing the scalar curvature on S^n inside the standard conformal class.

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Let (M, g) be a compact Riemannian manifold with a conformal vector field X . Given a function s on M , then it is a classical question to ask whether s is the scalar curvature of a metric \tilde{g} conformal to g . The determination of the set of all such functions s is still open, although several partial results are known, in particular there are necessary conditions that s has to satisfy in order to be a scalar curvature.

On the one hand there are topological obstructions. If for example M is spin and has non-vanishing \hat{A} genus, then the scalar curvature of any metric on M has either to be negative somewhere or the Ricci curvature vanishes everywhere on M .

However, if one fixes the conformal class $[g]$ as described above, there are further obstructions that arise from conformal vector fields. For example if M is S^n with the standard conformal structure, Kazdan and Warner [KW75] derived a famous obstruction. A slightly stronger version of this obstruction due to Bourguignon and Ezin [BE87] is described in the following theorem.

Theorem 1. *Let X be a conformal vector field on the compact manifold (M, g) . If s is the scalar curvature of a metric $\tilde{g} = u^{4/(n-2)}g$, then*

$$\int_M \partial_X s \, dv_{\tilde{g}} = 0$$

where $dv_{\tilde{g}} = u^{\frac{2n}{n-2}} dv_g$ is the volume measure associated to \tilde{g} .

Tightly related to the Kazdan-Warner obstruction is the Pohozaev identity. Let Ω be a star-shaped open set of \mathbb{R}^n ($n \in \mathbb{N}$) with smooth boundary. We denote by $\Delta = -\sum_{i=1}^n \partial_{ii}$ the Laplacian on \mathbb{R}^n . Let $u \in C^2(\bar{\Omega})$ be a positive solution of $\Delta u = u^{p-1}$ on Ω with $u|_{\partial\Omega} \equiv 0$. The vector field $X = \sum_{i=1}^n x^i \partial_i$ is conformal. If one uses similar methods as in the proof of the Kazdan-Warner obstruction, then one obtains the Pohozaev identity ([Po65]) which asserts that:

$$\left(1 - \frac{n}{2} + \frac{n}{p}\right) \int_{\Omega} u^p = \frac{1}{2} \int_{\partial\Omega} \langle \nu, X \rangle (\partial_{\nu} u)^2 \quad (1)$$

where ν resp. ∂_{ν} is the outer normal vector resp. the outer normal derivative on $\partial\Omega$. One among many important consequences of this inequality is that no non-trivial solutions exist if $p \geq \frac{2n}{n-2}$. Another application is an alternative proof of the Kazdan-Warner obstructions in the case that (M, g) is the sphere with the standard conformal structure [DR99].

In the present short article, we establish a similar identity for the classical Dirac operator D . We derive this identity on manifolds with boundary in order to admit future Pohozaev type applications. Then,

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we will specialize to compact manifolds without boundary, where we will derive a Kazdan-Warner type obstruction for the mean curvature of a conformal immersion $S^2 \rightarrow \mathbb{R}^3$.

Our main theorem is:

Theorem 2. *Let (M, g, χ) be a compact Riemannian spin manifold of dimension n with boundary ∂M (possibly equal to \emptyset) and with Dirac operator D . We assume that there exists a smooth spinor field ψ which satisfies for some $p > 1$,*

$$D\psi = H|\psi|^{p-2}\psi, \quad H \in C^\infty(M). \quad (2)$$

Furthermore, we assume that X is a conformal vector field on M . Then, we have the following Pohozaev type identity:

$$\int_{\partial M} \langle \nu \cdot \mathcal{L}_X \psi, \psi \rangle = \frac{p-2}{p} \int_{\partial M} H|\psi|^p g(X, \nu) + \left(\frac{1}{n} - \frac{p-2}{p} \right) \int_M H \operatorname{div} X |\psi|^p + \frac{2}{p} \int_M (\partial_X H) |\psi|^p,$$

where ν denotes the outward pointing normal vector along ∂M , and where $\langle \cdot, \cdot \rangle$ denotes the real scalar product on spinors.

Proof: The flow associated to the conformal vector field X will be denoted as α^t . If p is in the interior of M , then $\alpha^t(p)$ exists for times t close to 0. For any $t \in \mathbb{R}$ let f^t be the conformal scaling function of α^t , i.e. $(d\alpha^t)_p$ is $f^t(p)$ times an isometry from $T_p M$ to $T_{\alpha^t(p)} M$. Let $\alpha^t_* : \Sigma_p M \rightarrow \Sigma_{\alpha^t(p)} M$ be the spinor identification map as constructed in [Ht74, Hi86, BG92]. In particular, this map has the pointwise properties that

$$|\alpha^t_*(\psi)| = |\psi|$$

and the following transformation formula for conformal changes of the metric. Let $\varphi \in \Gamma(\Sigma M)$ be a spinor field. For t close to 0, we then define the map $\alpha^t_\# : \Gamma(\Sigma M) \rightarrow \Gamma(\Sigma \tilde{M})$, $\alpha^t_\#(\varphi) := \alpha^t_* \circ \varphi \circ \alpha^{-t}$, where \tilde{M} is M without an open neighborhood of the boundary.

Then

$$D\alpha^t_\# \left((f^t)^{-\frac{n-1}{2}} \psi \right) = \alpha^t_\# \left((f^t)^{-\frac{n+1}{2}} D\psi \right).$$

Now we assume that ψ satisfies (2), and we obtain

$$D\alpha^t_\# \left((f^t)^{-\frac{n-1}{2}} \psi \right) = \alpha^t_\# \left((f^t)^{-\frac{n+1}{2}} H|\psi|^{p-2}\psi \right).$$

Deriving with respect to t at $t = 0$ yields

$$-\frac{n-1}{2} D\beta\psi + D \frac{d}{dt} \Big|_{t=0} \alpha^t_\# \psi = -\frac{n+1}{2} H\beta|\psi|^{p-2}\psi + H|\psi|^{p-2} \frac{d}{dt} \Big|_{t=0} \alpha^t_\# \psi \quad (3)$$

$$+ (p-2)H \left\langle \frac{d}{dt} \Big|_{t=0} \alpha^t_\# \psi, \psi \right\rangle |\psi|^{p-4}\psi - (\partial_X H) |\psi|^{p-2}\psi. \quad (4)$$

where $\beta := \frac{d}{dt} \Big|_{t=0} f^t$. We reformulate using definition of the *Lie derivative of spinor fields in the direction X* [BG92], i.e.

$$\mathcal{L}_X(\psi) = -\frac{d}{dt} \Big|_{t=0} \alpha^t_\#(\psi). \quad (5)$$

Together with $D\beta\psi = \beta D\psi + \nabla\beta \cdot \psi$ and (2) one then concludes that

$$\frac{n-1}{2} \nabla\beta \cdot \psi + D\mathcal{L}_X\psi = H\beta|\psi|^{p-2}\psi + H|\psi|^{p-2}\mathcal{L}_X\psi \quad (6)$$

$$+ (p-2)H \langle \mathcal{L}_X\psi, \psi \rangle |\psi|^{p-4}\psi + (\partial_X H) |\psi|^{p-2}\psi. \quad (7)$$

After multiplication with ψ , the $\nabla\beta \cdot \psi$ -term vanishes, and we obtain

$$\langle D\mathcal{L}_X\psi, \psi \rangle = (p-1)H|\psi|^{p-2} \langle \mathcal{L}_X\psi, \psi \rangle + H\beta|\psi|^p + (\partial_X H) |\psi|^p.$$

The product rule for the Lie derivative tells us that

$$|\psi|^{p-2} \langle \mathcal{L}_X\psi, \psi \rangle = \frac{1}{2} |\psi|^{p-2} \partial_X |\psi|^2 = |\psi|^{p-1} \partial_X |\psi| = \frac{1}{p} \partial_X |\psi|^p.$$

Hence, we obtain

$$\langle D\mathcal{L}_X\psi, \psi \rangle = \frac{p-1}{p} H \partial_X |\psi|^p + H\beta |\psi|^p + (\partial_X H) |\psi|^p.$$

Strictly speaking, this equation is valid in the interior, but it extends to the boundary by continuity. Now, we integrate over M . With partial integration for the Dirac operator one obtains

$$\int_M \langle D\mathcal{L}_X\psi, \psi \rangle = \int_M \langle \mathcal{L}_X\psi, D\psi \rangle + \int_{\partial M} \langle \nu \cdot \mathcal{L}_X\psi, \psi \rangle = \int_M H \underbrace{\langle \mathcal{L}_X\psi, |\psi|^{p-2}\psi \rangle}_{=\frac{1}{p}\partial_X |\psi|^p} + \int_{\partial M} \langle \nu \cdot \mathcal{L}_X\psi, \psi \rangle.$$

This yields

$$\int_{\partial M} \langle \nu \cdot \mathcal{L}_X\psi, \psi \rangle = \frac{p-2}{p} \int_M H \partial_X |\psi|^p + \int_M H\beta |\psi|^p + \int_M (\partial_X H) |\psi|^p.$$

Using $\operatorname{div}(H|\psi|^p X) = (\partial_X H) |\psi|^p + H \partial_X |\psi|^p + H |\psi|^p \operatorname{div} X$ and $\operatorname{div} X = n\beta$ we obtain

$$\int_{\partial M} \langle \nu \cdot \mathcal{L}_X\psi, \psi \rangle = \frac{p-2}{p} \int_{\partial M} H |\psi|^p g(X, \nu) + \left(\frac{1}{n} - \frac{p-2}{p} \right) \int_M H \operatorname{div} X |\psi|^p + \frac{2}{p} \int_M (\partial_X H) |\psi|^p,$$

Examples 3.

1.) Let Ω be domain in \mathbb{R}^n with smooth boundary, let $X = r\partial_r = \sum x^i \partial_i$, and we will assume that $H = \lambda$ is constant. Then $\beta \equiv 1$ and we obtain

$$\int_{\partial\Omega} \langle \nu \cdot \mathcal{L}_X\psi, \psi \rangle = \lambda \frac{p-2}{p} \int_{\partial\Omega} \langle X, \nu \rangle |\psi|^p + \lambda \left(1 - \frac{p-2}{p} n \right) \int_{\Omega} |\psi|^p.$$

This inequality bears many analogies to equation (1). In particular, the constant $1 - \frac{p-2}{p}$ before the integral over Ω vanishes if p takes the value $p = 2n/(n-1)$. This value plays the same role in non-linear Dirac equations as the value $p = 2n/(n-2)$ does for the Laplace operator.

2.) If M is a closed manifold and X is a conformal vector field, then for $p = 2n/(n-1)$ we obtain

$$\int_M (\partial_X H) |\psi|^p = 0.$$

Corollary 4. [Kazdan-Warner type obstructions] *Let $f : S^n \rightarrow \mathbb{R}^{n+1}$, $n \geq 2$, be a conformal immersion (possibly with branching points of even order in the case $n = 2$). We denote by $H : S^n \rightarrow \mathbb{R}$ the mean curvature of this immersion. Then, for any conformal vector field X we have*

$$\int_{S^n} (\partial_X H) f^*(d\mu) = 0$$

where $d\mu$ is the volume element on $f(S^n)$ induced from the Euclidean metric on \mathbb{R}^{n+1} . In particular, $\partial_X H$ changes sign.

The corollary is particularly interesting in dimension $n = 2$. If $f : S^2 \rightarrow \mathbb{R}^3$ is any immersion, then after possibly composing with a diffeomorphism $S^2 \rightarrow S^2$, we can assume that f is conformal.

The corollary is analogous to results in [KW75], [BE87] and [DR99].

Proof: Let ψ be parallel spinor on \mathbb{R}^{n+1} . Then, as proven in [KS96, Ba98, Fr98], the restriction of ψ on Σ satisfies equation (2) with $p = 2n/(n-1)$, and $|\psi|^p d\nu = f^*(d\mu)$ where $d\nu$ is the standard volume element on S^n . Since this equation is conformally invariant we obtain a solution of (2) on S^n equipped with the standard metric. The corollary then immediately follows from example (2) above.

Example 5. Let $x_3 : S^2 \rightarrow \mathbb{R}$ be the third component of the standard inclusion. One shows that $X := \operatorname{grad} x_3$ is a conformal vector field on S^2 , where the gradient is taken with respect to the standard metric on S^2 . Then for any $\varepsilon \in \mathbb{R} \setminus \{0\}$ one has $\partial_X(\varepsilon x_3 + 1) = \varepsilon g(\operatorname{grad} x_3, \operatorname{grad} x_3)$ which is of constant sign. Hence $\varepsilon x_3 + 1 : S^2 \rightarrow \mathbb{R}$ is not the mean curvature of a conformal immersion.

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