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**HAL Id: hal-00091044**

**<https://hal.science/hal-00091044v1>**

Preprint submitted on 5 Sep 2006

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# New Proofs of Davies-Simon's Theorems about Ultracontractivity and Logarithmic Sobolev Inequalities related to Nash Type Inequalities

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September 4, 2006

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## Abstract

We present new proofs of two theorems of E.B. Davies and B. Simon (Thm. 2.2.4 and Cor. 2.2.8 in [D]) about ultracontractivity property (*Ult* for short) of semigroups of operators and logarithmic Sobolev inequalities with parameter (*LSIWP* for short) satisfied by the generator of the semigroup. In our proof, we use neither the  $L^p$  version of the *LSIWP* (Theorem 2.3) nor Stein's interpolation. Our tool is Nash type inequality (*NTI* for short) as an intermediate step between *Ult* and *LSIWP*. We also present new results. First, a new formulation about the implication  $LSIWP \Rightarrow Ult$  using a result of T.Coulhon ([C]). Second, we show that *LSIWP* and *NTI* are equivalent. We discuss different approaches to get Nash type inequalities from an ultracontractivity property. We give some examples of one-exponential and double-exponential ultracontractivity and also discuss the general theory for the second case.

*Mathematics Subject Classification (2000)*: 39B62.

*Key words* : ultracontractivity property, logarithmic Sobolev inequality with parameter, Nash type inequality, semigroups of operators, Dirichlet form, Heat kernel, infinite Torus.

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\*Research partially supported by the European Commission (IHP Network " Harmonic Analysis and Related Problems" 2002-2006, contract HPRN-CT-2001-00273-HARP)

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# 1 Introduction

In this paper, we give new proofs of theorems due to E.B.Davies and B.Simon (see [D] Thm. 2.2.4 and Cor. 2.2.8 (see also [DS])). They proved that an ultracontractivity property (*Ult* for short),

$$\|T_t f\|_\infty \leq e^{M(t)} \|f\|_1, \quad \forall t > 0 \quad (Ult)_M \tag{1.1}$$

of a semigroup  $T_t = e^{-tA}$  (under some additional assumptions) implies a logarithmic Sobolev inequality with parameter (*LSIWP* for short),

$$\int f^2 \log(f/\|f\|_2) d\mu \leq t(Af, f) + \beta(t) \|f\|_2^2, \quad \forall t > 0 \quad (LSIWP)_\beta \tag{1.2}$$

satisfied by the generator  $A$  with  $\beta = M$ . They also proved some converse results with some additional assumptions on the function  $\beta$ . Indeed, it is not always true that a  $(LSIWP)_\beta$  satisfied by a generator  $A$  implies an ultracontractivity property  $(Ult)_M$  of the corresponding semigroup  $T_t = e^{-tA}$ . In [DS] (see Rmk 1 p.359), the authors give an example of generator  $A$  satisfying  $(LSIWP)_\beta$  with  $\beta(t) = ce^{1/t}$  but with no ultracontractivity. So the converse implication doesn't hold in general. But under some conditions on the function  $\beta$ , it can be proved that  $(LSIWP)_\beta$  implies  $(Ult)_{\tilde{M}}$  with some function  $\tilde{M}$  (which may differ of the function  $M$  in (1.1)). An interesting situation is when  $\tilde{M}(t) = c_1 M(c_2 t) + c_3$ . In that case, the two statements  $(LSIWP)_\beta$  and  $(Ult)_M$  are equivalent in the sense that  $M$  and  $\tilde{M}$  behave in the same way. For example,  $e^{M(t)} = C e^{-\lambda t} t^{-d} e^{c/t^\gamma}$  i.e  $M(t) = k_1 - \lambda t - d \ln t + c/t^\gamma$  with  $k_1 = \ln C, \lambda, d, c, \gamma > 0$ . But we are unable to prove this relation between  $\tilde{M}(t)$  and

$M(t)$  in the general situation. For instance, if  $\beta(t) = e^{c/t^\gamma}$  with  $0 < \gamma < 1$ , we are only able to prove that  $\tilde{M}(t) = e^{c/t^{\gamma'}}$  with  $\gamma' = \frac{\gamma}{1-\gamma}$  as far as the author knows. It could be conjectured that  $\gamma' = \frac{\gamma}{1-\gamma}$  is optimal. In particular, the singularity of the behavior of  $\gamma'$  clearly appears when  $\gamma$  goes to 1.

Let us recall briefly the interest of LSIWP and ultracontractivity property. The ultracontractivity property is equivalent for (symmetric) semigroup to the following on-diagonal estimate of the heat kernel:

$$\sup_x h_t(x, x) = \sup_{x, y} h_t(x, y) \leq e^{M(t)}, \quad \forall t > 0 \quad (1.3)$$

with

$$T_t f(x) = \int h_t(x, y) d\mu(y) \quad (1.4)$$

( $h_t(x, y)$  is the so-called heat kernel).

So, when we are able to prove a LSIWP and we have at hand a theorem saying that LSIWP implies an ultracontractivity property with an explicit bound, we immediately deduce the same bound on the heat kernel. Note that, if we replace LSIWP by a Nash type inequality, the same remark holds true (see [C]). Recall that a Nash type inequality (NTI for short) is the control by a function  $\Theta$  of the  $L^2$ -norm by the quadratic form associated to the generator of the semigroup when the  $L^1$ -norm is bounded. More precisely,

$$\Theta \left( \|f\|_2^2 \right) \leq (Af, f), \quad \forall f \in \mathcal{D}(A), \|f\|_1 \leq 1. \quad (1.5)$$

Since LSIWP and Nash type inequality have the main goal (To prove ultracontractivity), we may ask for relations between these two inequalities. In that paper, we show that LSIWP and Nash type inequality are equivalent. Moreover, we use Nash type inequality to give a new proof of the statement  $(LSIWP) \Rightarrow (Ult)$ . We use again Nash type inequality for the converse implication. This sheds new lights on relationships between these three inequalities.

We now describe the contents of this paper.

In Section 2, we describe well-known results about the relationship between ultracontractivity property and logarithmic Sobolev inequalities with parameter following [DS]. We also recall part of the proofs for the convenience of the reader and also to be compared with the new methods developed in that paper.

In Section 3, we prove the implication  $(LSIWP)_\beta \Rightarrow (Ult)_M$  under the usual additional assumptions on the function  $\beta$ . We use a new approach for the proof: we introduce an intermediate step with Nash type inequality. In fact, we give two different results. The first result is a new proof of a corollary of a general result of Davies and Simon (see [DS]). The second proof gives another ultracontractive

bound using a result of T.Coulhon [C]. This last result has more general applications.

In Section 4, we study the converse:  $(Ult)_M \Rightarrow (LSIWP)_\beta$ . The new proof has two steps. We deduce a Nash type inequality from the ultracontractivity property by using again a result of ([C]). This Nash type inequality is equivalent to a Nash type inequality with parameter already close to  $(LSIWP)_\beta$ . The second step consists in applying a method of truncation for Dirichlet forms developed in [BCLS] to obtain  $(LSIWP)_\beta$ .

In Section 5, we prove the equivalence between LSIWP and Nash type inequality using ideas developed in the preceding sections.

In Section 6, we discuss different well-known approaches to prove Nash type inequalities. Such discussion arises naturally since Nash type inequality is the main tool of our proofs.

In Section 7, we focus on the polynomial ultracontractivity property i.e  $e^{M(t)} = ct^{-\nu}$ ,  $\nu > 0$  in (1.1) which is equivalent to the usual  $L^2$ -Sobolev inequality. We show how this ultracontractivity property can be expressed in terms of different functional inequalities. In particular, we recall the weak-Sobolev inequality introduced earlier by D. Bakry (see [B]). This section essentially collects these information.

Section 8. To show how this general theory can be applied outside of the polynomial ultracontractivity property setting, we mention two families of examples of heat kernels on the infinite dimensional torus  $\mathbb{T}^\infty$  coming from [B2]. The generator of the semigroup is an infinite dimensional Laplacian with constant coefficients. The sequence of these coefficients has to go to infinity. Depending on the speed of this sequence, the corresponding heat kernel bound has a different behavior. The first family of examples is the one-exponential ultracontractivity behavior i.e

$$e^{M(t)} = c_1 e^{c_2/t^\gamma}, \quad (\gamma > 0).$$

in (1.1). In this situation,  $(Ult)_M$  and the corresponding  $(LSWIP)_\beta$  are equivalent (for the abstract theory) in the sense that  $M(t) = c/t^\gamma$  and  $\beta(t) = c/t^\gamma$  with possibly different constants  $c$ .

The second family of examples is the double-exponential ultracontractivity behavior i.e

$$e^{M(t)} = c_1 e^{e^{c_2/t^\gamma}}, \quad (\gamma > 0).$$

in (1.1). In this case, the general theory doesn't give equivalence between  $(Ult)_M$  and the corresponding  $(LSWIP)_\beta$ . More examples of heat kernel behaviors (i.e ultracontractivity ) can be found in [B2].

In Section 9, we discuss in the general frame work the relationship between double-exponential ultracontractivity property and the corresponding  $(LSWIP)_\beta$  (or the corresponding Nash type inequality).

## 2 Relations between ultracontractivity and LSIWP

Let  $(e^{-At})_{t \geq 0} = (T_t)_{t \geq 0}$  be a symmetric Markov semigroup on  $L^2(X, d\mu)$  with generator  $A$  defined on a  $\sigma$ -finite measure space  $(X, d\mu)$ . We say that  $(T_t)_{t \geq 0}$  is ultracontractive if for any  $t > 0$ , there exists a finite positive number  $a(t)$  such that, for all  $f \in L^1$  :

$$\|T_t f\|_\infty \leq a(t) \|f\|_1. \quad (2.1)$$

An equivalent formulation (by interpolation) of ultracontractivity is as follows: For any  $t > 0$ , there exists a finite positive number  $c(t)$  such that, for all  $f \in L^2$ ,

$$\|T_t f\|_\infty \leq c(t) \|f\|_2 \quad (2.2)$$

Also by duality, the inequality (2.2) is equivalent to

$$\|T_t f\|_2 \leq c(t) \|f\|_1 \quad (2.3)$$

It is known that, under the assumptions on the semigroup  $(T_t)_{t \geq 0}$ , (2.2) implies (2.1) with  $a(t) \leq c^2(t/2)$  and (2.1) implies (2.2) with  $c(t) \leq \sqrt{a(t)}$ .

We say that the generator  $A$  satisfies LSIWP (logarithmic Sobolev inequality with parameter) if there exist a monotonically decreasing continuous function  $\beta : (0, +\infty) \rightarrow (0, +\infty)$  such that

$$\int f^2 \log f \, d\mu \leq \epsilon Q(f) + \beta(\epsilon) \|f\|_2^2 + \|f\|_2^2 \log \|f\|_2 \quad (2.4)$$

for all  $\epsilon > 0$  and  $0 \leq f \in \text{Quad}(A) \cap L^1 \cap L^\infty$  where  $\text{Quad}(A)$  is the domain of  $\sqrt{A}$  in  $L^2$  and  $Q(f) = (\sqrt{A}f, \sqrt{A}f)$ .

This inequality is modeled on the celebrated Gross inequality [G].

In [DS],[D], the authors show that LSIWP implies ultracontractivity property under an integrability condition on  $\beta$ . This condition can be enlarged and be stated as follows:

**Theorem 2.1** (Cor. 2.2.8 [D]). *Let  $\beta(\epsilon)$  be a monotonically decreasing continuous function of  $\epsilon$  such that*

$$\int f^2 \log f \, d\mu \leq \epsilon Q(f) + \beta(\epsilon) \|f\|_2^2 + \|f\|_2^2 \log \|f\|_2 \quad (2.5)$$

for all  $\epsilon > 0$  and  $0 \leq f \in \text{Quad}(A) \cap L^1 \cap L^\infty$ . Suppose that for one  $\eta > -1$ ,

$$M_\eta(t) = (\eta + 1)t^{-(\eta+1)} \int_0^t s^\eta \beta\left(\frac{s}{\eta + 1}\right) ds \quad (2.6)$$

is finite for all  $t > 0$ . Then  $e^{-At}$  is ultracontractive and

$$\|e^{-At}\|_{\infty,2} \leq e^{M_\eta(t)} \quad (2.7)$$

for all  $0 < t < \infty$ .

Before recalling the proof of Davies and Simon, we make some comments.

Corollary 2.2.8 of [D] is Theorem 2.1 with  $\eta = 0$ . In the literature, Corollary 2.2.8 of [D] is used, for instance, to deal with  $\beta(t) = ct^{-\alpha}$  for  $0 < \alpha < 1$  and we had to go back to Theorem 2.2.7 to deal with the case  $\alpha \geq 1$  (see [FL] for such an instance of application). Theorem 2.1 unify these two cases in one case just by an appropriate choice of  $\eta$ . Indeed it is easy to obtain the bound of ultracontractivity in the theorem above with the parameter  $\eta$  by the same argument used to treat the example 2.3.4 p.72 of [D]. The proof will be recalled below. Note that, in general for our applications,  $\beta$  is non-increasing so that for any  $\eta > -1$ , we have  $\beta(\frac{t}{\eta+1}) \leq M_\eta(t), t > 0$ .

So the interest of such result relies on the fact that we can choose the parameter  $\eta$ . Indeed for some parameter  $\eta$  the integral (2.6) may not converge at the origin but it may converge for some other parameters  $\eta$ . For instance when  $\beta(t) = c/t^\alpha$  ( $\alpha > 0$ ), we obtain  $M_\eta(t) = c'/t^\alpha$  with the same index  $\alpha$  but we have to choose  $\eta > \alpha - 1$ . The weight  $s^\eta$  is used to remove the singularity of the integral at the origine. So for this example of class of functions, with an appropriate choice of  $\eta$ , the integral (2.6) converges and we recover the function  $\beta$  (up to a multiplicative constant).

It may also happened that, for some function  $\beta$ , the integral doesn't converge for any choice of  $\eta$ . For instance,  $\beta(t) = \exp(c/t^\alpha), \alpha > 0$ .

The aim of this paper is to give a different proof of this result (see Section 3).

We now recall the main steps of the proof of Theorem 2.1 for the case  $\eta = 0$  and give the proof of the general case  $\eta > -1$  (which can be deduce from Example 2.3.4 p.72 of [D]).

The first step is the following lemma. This lemma says that if an  $L^2$ -version of LSIWP is satisfied then an  $L^p$ -version is also satisfied for any  $p \in (2, +\infty)$ .

**Lemma 2.2** (*Lemma 2.2.6 [D]*) *Assume that the LSIWP (2.4) is satisfied. Then*

$$\int g^p \log g \, d\mu \leq \epsilon(Ag, g^{p-1}) + 2\beta(\epsilon)p^{-1} \|g\|_p^p + \|g\|_p^p \log \|g\|_p \quad (2.8)$$

for all  $2 < p < \infty$ , all  $\epsilon > 0$  and all  $g \in \mathcal{D}_+ = \bigcup_{t>0} e^{-At}(L^1 \cap L^\infty)_+$ .

For the next step, the parameter  $\epsilon$  can be chosen as a function of  $p$  in the  $L^p$ -inequality (2.8). Then we can deduce the ultracontractivity property from this family of  $L^p$ -inequalities.

**Theorem 2.3** (*Thm 2.2.7 [D]*) *Let  $\epsilon(p) > 0$  and  $\Gamma(p)$  be two continuous functions defined for  $2 < p < \infty$  such that*

$$\int f^p \log f \, d\mu \leq \epsilon(p) \langle Af, f^{p-1} \rangle + \Gamma(p) \|f\|_p^p + \|f\|_p^p \log \|f\|_p \quad (2.9)$$

for all  $2 < p < \infty$  and all  $f \in \mathcal{D}_+ = \bigcup_{t>0} e^{-At}(L^1 \cap L^\infty)_+$ . If

$$t = \int_2^\infty p^{-1} \epsilon(p) dp, \quad M = \int_2^\infty p^{-1} \Gamma(p) dp \quad (2.10)$$

are both finite then  $e^{-At}$  maps  $L^2$  into  $L^\infty$  and

$$\|e^{-At}\|_{\infty,2} \leq e^M \quad (2.11)$$

**Proof of Theorem 2.1:** Let  $\eta > -1$  and set  $\nu^{-1} = \eta + 1 > 0$ . We apply Lemma 2.2 and Theorem 2.3 with

$$\epsilon(p) = t\nu 2^\nu p^{-\nu}, \quad \Gamma(p) = 2\beta(\epsilon(p))p^{-1},$$

then  $M(t) = \int_2^{+\infty} 2\beta(\epsilon(p))p^{-2} dp = M_\eta(t)$  as defined above. This completes the proof.

Now we consider the converse statement. We recall the following result due to Davies and Simon. In this statement, we note that there is no restriction as the integrability condition of Theorem 2.1. Thus an ultracontractivity property always implies LSIWP.

**Theorem 2.4** (*Thm 2.2.3 [D]*) Assume that  $e^{-At}$  is ultracontractive i.e.

$$\|e^{-At}\|_{\infty,2} \leq e^{M(t)} \quad (2.12)$$

for all  $t > 0$ , where  $M(t)$  is a monotonically decreasing continuous function of  $t$ .

Then  $0 \leq f \in \text{Quad}(A) \cap L^1 \cap L^\infty$  implies  $f^2 \log f \in L^1$ , and the logarithmic Sobolev inequality

$$\int f^2 \log f d\mu \leq \epsilon Q(f) + M(\epsilon) \|f\|_2^2 + \|f\|_2^2 \log \|f\|_2 \quad (2.13)$$

for all  $\epsilon > 0$ .

**Proof:** We just recall the main arguments. We consider  $Q_z = e^{-tzA}$  for  $0 \leq \Re z \leq 1$  for a fixed  $t$ . For any  $y \in \mathbb{R}$ , we have

$$\|Q_{iy} f\|_2 \leq \|f\|_2$$

and

$$\|Q_{1+iy}\|_\infty \leq e^{M(t)} \|f\|_2.$$

By Stein's complex interpolation Theorem, for any  $0 < s < t$ , with  $\theta = s/t$ , we have

$$\|Q_\theta\|_{p(s)} \leq e^{\theta M(t)} \|f\|_2 = e^{sM(t)/t} \|f\|_2$$



with  $p(s) = 2t/(t-s)$ . Note that at this stage, the dependance of the bound in  $s$  is very simple and have the value 1 at  $s = 0$ .

The second idea is to obtain the expression under the integral with  $\ln f$  by deriving at  $s = 0$  the  $L^{p(s)}$ -norm of  $f_s = T_s f$  (here we skip the details). Let  $\phi(s) = \|f_s\|_{p(s)}^{p(s)}$ ,

$$\phi'(s) = p(s) \langle -A f_s, f_s^{p(s)-1} \rangle + p'(s) \int f_s^{p(s)} \ln f_s d\mu.$$

Let  $\psi(s) = e^{sM(t)/t}$  and assume  $\|f\|_2 = 1$  thus  $\phi(0) = \psi(0) = 1!$  Consequently,

$$\phi'(0) \leq \psi'(0)$$

that is

$$-2 \langle A f, f \rangle + \frac{2}{t} \int f^2 \ln f d\mu \leq 2M(t)/t.$$

The proof is completed.

Our new proof in Section 4 avoid the interpolation argument .

The main applications we have in mind for Theorem 2.1 is with

$$\beta(\epsilon) = \ln(a(\epsilon)) = \ln c_1 - \lambda \epsilon - d \ln \epsilon + c_2/\epsilon^\gamma$$

with  $c_1, c_2 > 0$  and  $\lambda, d, \gamma \geq 0$  (i.e.  $a(t) = c_1 e^{-\lambda t} t^{-d} \exp(c_2/t^\gamma)$ ). For a suitable choice of  $\eta$  in Theorem 2.1, we obtain for  $M_\eta$  a function of the same type as  $\beta$ . More precisely  $M_\eta(t) = \ln c'_1 - \lambda' t - d \ln t + c'_2/t^\gamma$ . So  $e^{M_\eta(t)} = c'_1 \exp(-\lambda' t) t^{-d} \exp(c'_2/t^\gamma)$  is of the same type as  $a(t)$  above (that is up to constants  $c_1, \lambda, c_2$ ). Note that the exponents  $d$  and  $\gamma$  are preserved in this transformation. For this class of function, Theorem 2.1 and Theorem 2.4 are converse of each other.

Of course, other classes of functions  $\beta$  can be considered but we are not always able to pass from LSIWP to ultracontractivity. Indeed, it is worth noting that there exists also a semigroup which is not ultracontractive but satisfies LSIWP (2.5) with  $\beta(\epsilon) = c_1 \exp(c_2/\epsilon)$  (see p.359 and also Section 4 p.355 and section 5 p.357 in [DS]). At the end of this paper, we give an alternative proof for an explicit bound for ultracontractivity when  $\beta(\epsilon) = c_1 \exp(c_2/\epsilon^\gamma)$ ,  $\gamma \in ]0, 1[$ .

To finish this section, we mention the following result. If we only suppose  $0 \leq f \in Quad(A)$  then it is not obvious that  $f^2 \log f \in L^1$  (at least when  $\mu$  is not finite) and we have a slight variation of Theorem 2.4.

**Theorem 2.5** (*Thm 2.2.4 [D]*). *Let  $e^{-At}$  be ultracontractive with*

$$\|e^{-At}\|_{\infty,2} \leq e^{M(t)} \tag{2.14}$$

for all  $t > 0$ , where  $M(t)$  is a monotonically decreasing continuous function of  $t$ . Then  $0 \leq f \in \text{Quad}(A)$  implies

$$\int f^2 \log_+ f \, d\mu \leq \epsilon Q(f) + \beta(\epsilon) \|f\|_2^2 + \|f\|_2^2 \log \|f\|_2 \quad (2.15)$$

for all  $\epsilon > 0$  where  $\beta(\epsilon) = M(\epsilon/4) + 2$ .

In the next section, we give a new proof of the theorem 2.1 in the form of Theorem 2.5.

### 3 LSWIP implies ultracontractivity

#### 3.1 Davies-Simon result

Assuming LSIWP is satisfied by the generator, we give a new proof of ultracontractivity property of the associated semigroup (under some integrability conditions). We do not use  $L^p$  version of LSWIP as in [DS]. We only use of the  $L^2$  inequalities. There are three steps in our proof. First step: from LSIWP we deduce a (relaxed) Nash type inequality for the generator using a convexity argument (Lemma 3.1). Second step: we derive a differential inequality satisfied by the associated semigroup. Third step: we prove a universal bound on all solutions of this differential inequality (Lemma 3.3) and, as a consequence, we deduce the ultracontractivity property .

The first lemma depends on a convexity argument (Jensen inequality). This lemma will also be used in section 4 and 6. This lemma comes from [BiMa].

**Lemma 3.1** *If  $f \in L^1 \cap L^\infty$  with  $f \geq 0$  and  $\|f\|_1 = 1$  then*

$$\|f\|_2^2 \log \|f\|_2 \leq \int f^2 \log(f/\|f\|_2) \, d\mu \quad (3.1)$$

and, more generally, if  $f \in L^1 \cap L^\infty$  with  $f \geq 0$ ,

$$\|f\|_2^2 \log \|f\|_2 \leq \int f^2 \log f \, d\mu - \|f\|_2^2 \log \|f\|_2 + \|f\|_2^2 \log \|f\|_1 \quad (3.2)$$

In particular, if  $\|f\|_1 \leq 1$  then (3.1) holds true.

**Proof :** If  $f \in L^1 \cap L^2$  with  $f \geq 0$  and  $\|f\|_1 = 1$  then  $d\nu = f \, d\mu$  is a probability measure. For every convex function  $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ , the Jensen inequality yields

$$\Psi \left( \int f \, d\nu \right) \leq \int \Psi(f) \, d\nu \quad (3.3)$$

We apply this to the convex function  $\Psi(x) = x \log x$ . Therefore,  $\int f \, d\nu = \|f\|_2^2$  and

$$2\|f\|_2^2 \log \|f\|_2 \leq \int f^2 \log f \, d\mu \quad (3.4)$$

We conclude (3.1) with  $\|f\|_1 = 1$ . To obtain the general case (3.2), we put  $f\|f\|_1^{-1}$  instead of  $f$  in (3.1) then the inequality follows. When  $\|f\|_1 \leq 1$ , (3.2) implies (3.1). The lemma is proved.

**Remark 3.2** *We can prove an analogue to the inequality (3.1) with  $\log_+$  instead of  $\log$ .*

The following lemma gives a universal bound for all the solutions of the differential inequality with parameter (3.6). We shall note that this bound doesn't depend on the initial condition. We also discuss the optimality of the result.

We need to introduce some notations and definitions. For any  $\eta, \lambda \in \mathbb{R}$  and for any continuous real-valued function  $b$  defined on  $(0, +\infty)$ , we define the following function:

$$H_{\eta, \lambda, b}(t) = \frac{2\lambda}{t^{\eta+1}} \int_0^t s^\eta b(s/\lambda) ds \quad (3.5)$$

(assuming this integral converges). This function plays the role of the function  $M_\eta$  of Section 1. We shall denote by  $H_{\eta, b}$  the function  $H_{\eta, (\eta+1)/2, b}$ .

**Lemma 3.3** *Let  $t_0 \in (0, +\infty]$ . Assume that  $\Phi \in C^1((0, +\infty), \mathbb{R})$  is a function satisfying the following differential inequality*

$$\Phi(s) \leq (-t/2)\Phi'(s) + b(t) \quad (3.6)$$

for all  $s > 0$  and all  $0 < t < t_0$ , with  $b$  a continuous real-valued function defined on  $(0, t_0)$ .

1. For any  $\eta > -1$ , let  $\lambda = \frac{\eta+1}{2}$ . Assume that  $H_{\eta, b}(t)$  converges for all  $t \in (0, \inf(t_0, s_0/\lambda))$ . Then for all  $t \in (0, \inf(t_0, s_0/\lambda))$ :

$$\Phi(t) \leq H_{\eta, b}(t). \quad (3.7)$$

2. Assume that  $\Phi$  is a non-negative function. For any  $\eta > -1$  and  $\lambda \geq \frac{\eta+1}{2}$ , we have for all  $0 < t < \inf(t_0, s_0/\lambda)$ :

$$\Phi(t) \leq H_{\eta, \lambda, b}(t) \quad (3.8)$$

Moreover, if  $b$  is non-increasing and non-negative, the function  $\lambda \rightarrow H_{\eta, \lambda, b}(t)$  is increasing for fixed  $t$  and fixed  $\eta$ . Hence, in that case, inequality (3.8) with  $\lambda = \frac{\eta+1}{2}$  implies the others.

3. For  $\eta > -1$  and  $\lambda = \frac{\eta+1}{2}$ . Then  $H_{\eta, \lambda, b}(s)$  satisfies (3.6) for all  $s \in (0, \inf(t_0, s_0/\lambda))$  with  $t = s/\lambda$ . In fact in that case, the inequality (3.6) is an equality.

**Proof of Lemma 3.3:** Let  $\eta > -1$  and  $\lambda > 0$ . For  $s > 0$  choose  $t = s/\lambda$  in (3.6). We multiply (3.6) by  $s^\eta$  and integrate over the interval  $(0, t]$ . Then

$$\int_0^t s^\eta \Phi(s) ds \leq (-1/2\lambda) \int_0^t s^{\eta+1} \Phi'(s) ds + \int_0^t s^\eta b(s/\lambda) ds. \quad (3.9)$$

The second integral is integrated by parts, so we get

$$\frac{1}{2\lambda} t^{\eta+1} \Phi(t) \leq \left[ \frac{\eta+1}{2\lambda} - 1 \right] \int_0^t s^\eta \Phi(s) ds + \int_0^t s^\eta b(s/\lambda) ds.$$

Let  $\lambda = (\eta + 1)/2$  if  $\Phi$  is real-valued and let  $\lambda \geq (\eta + 1)/2$  if  $\Phi \geq 0$ . The second term above is negative. Then

$$\frac{1}{2\lambda} t^{\eta+1} \Phi(t) \leq \int_0^t s^\eta b(s/\lambda) ds.$$

This proves 1 and 2 of the lemma. The statement 3 is easy to check by a direct computation.

Now, we can restate Theorem 2.1 with a slight modification of the expression of the bound (2.7). The proof depends upon Lemmas 3.3 and 3.1 and it is short. This proof is also simpler than the original proof of Theorem 2.1 because it doesn't use Theorem 2.3. But as already mentioned, an important disadvantage of Theorem 2.1 or Corollary 2.2.8 of [D] is that it doesn't enable us to treat the case  $\beta(\epsilon) = e^{\frac{\epsilon}{\alpha}}$ ,  $\alpha > 0$ . By a modification of Lemma 3.3 we shall provide in Section 9 an explicit bound of ultracontractivity property under the assumption  $\beta(\epsilon) = e^{\frac{\epsilon}{\alpha}}$  with  $0 < \alpha < 1$ .

**Theorem 3.4** *Suppose that the following logarithmic Sobolev inequality is valid for all  $\epsilon > 0$  and all  $0 \leq f \in \text{Quad}(A) \cap L^1 \cap L^\infty$ ,*

$$\int f^2 \log f d\mu \leq \epsilon Q(f) + \beta(\epsilon) \|f\|_2^2 + \|f\|_2^2 \log \|f\|_2 \quad (3.10)$$

*with  $\beta$  a continuous function. Then for all  $\eta > -1$*

$$\|e^{-At}\|_{\infty,2} \leq e^{M_\eta(t)} \quad (3.11)$$

*for all  $0 < t < \infty$  where  $M_\eta(t)$  is defined in (2.6).*

**Proof :** Assume that  $0 \leq f \in \text{Quad}(A) \cap L^1 \cap L^\infty$ . We set  $f_s = e^{-As} f = T_s f$  for  $s > 0$ . Suppose  $\|f\|_1 = 1$  then, by contraction property on  $L^1$  of the semigroup,  $\|f_s\|_1 \leq \|f\|_1 = 1$ . We check that  $0 \leq f_s \in \text{Quad}(A) \cap L^1 \cap L^\infty$ . We put  $f_s$  in (3.10),

$$\int f_s^2 \log f_s d\mu - \|f_s\|_2^2 \log \|f_s\|_2 \leq tQ(f_s) + \beta(t) \|f_s\|_2^2 \quad (3.12)$$

for all  $s, t > 0$ . We apply Lemma 3.1 with  $f_s$ , we deduce

$$\|f_s\|_2^2 \log \|f_s\|_2 \leq tQ(f_s) + \beta(t) \|f_s\|_2^2 \quad (3.13)$$

Let  $\Psi(s) = \|f_s\|_2^2$ . Then  $\Psi'(s) = -2Q(f_s)$ . Therefore,  $\Psi(s)$  satisfies

$$\frac{1}{2}\Psi(s) \log \Psi(s) \leq (-t/2)\Psi'(s) + \beta(t)\Psi(s) \quad (3.14)$$

Let  $\Phi(s) = \log \Psi(s)$  and changing  $t$  by  $t/2$ , then

$$\Phi(s) \leq (-t/2)\Phi'(s) + b(t) \quad (3.15)$$

for all  $s, t > 0$  with  $b(t) = 2\beta(t/2)$ . We apply Lemma 3.3 with  $t_0 = +\infty$ , for all  $\eta > -1$

$$\Phi(t) = \log \|T_t f\|_2^2 \leq H_{\eta, b}(t) = 2M_\eta(t) \quad (3.16)$$

Hence,

$$\|T_t f\|_2 \leq \exp(M_\eta(t)) \|f\|_1 \quad (3.17)$$

By duality,

$$\|T_t f\|_\infty \leq \exp(M_\eta(t)) \|f\|_2. \quad (3.18)$$

The proof of the theorem is completed.

We can localize this result in the sense that, if we assume (3.10) holds true for all  $0 < \varepsilon < \varepsilon_0$  then (3.11) holds true for  $t \in (0, 2\varepsilon_0)$ .

## 3.2 Another ultracontractive bound

Now, we prove another ultracontractive bound for the semigroup under the assumption that the generator satisfies LSIWP. We use a result due to T. Coulhon (see Prop. II.1 of [C]) and the lemma (3.1) of this article.

**Theorem 3.5** *Let  $A$  be a generator of a submarkovian semigroup and  $\beta$  be a function such that:*

$$\int f^2 \log(f/\|f\|_2) d\mu \leq t(Af, f) + \beta(t)\|f\|_2^2, t > 0. \quad (3.19)$$

*We define  $B(y) = \sup_{t>0}(ty/2 - t\beta(1/t))$ ,  $y \in \mathbb{R}$  and  $M(t)$  the inverse function of  $q(s) = \int_s^{+\infty} \frac{dy}{B(y)}$ ,  $s \in \mathbb{R}$  (We assume that  $\int^\infty \frac{dy}{B(y)} < \infty$ ). Then*

$$\|T_t f\|_\infty \leq e^{M(t)} \|f\|_1 \quad (3.20)$$

*for any  $t > 0$ .*

**Proof :** We assume that (3.19) is satisfied. The first step is to obtain a Nash type inequality. For that purpose, we apply lemma (3.1):

$$\|f\|_2^2 \ln \|f\|_2 \leq t(Af, f) + \beta(t)\|f\|_2^2, \forall t > 0, \quad \forall f \in \mathcal{D}(A), \|f\|_1 \leq 1. \quad (3.21)$$

Hence

$$\|f\|_2^2 \left[ \frac{1}{2t} \ln \|f\|_2^2 - \frac{1}{t} \beta(t) \|f\|_2^2 \right]^2 \leq (Af, f). \quad (3.22)$$

By optimisation over  $t > 0$  and by definition of  $B$ ,

$$\|f\|_2^2 B(\ln \|f\|_2^2) \leq (Af, f), \quad \forall f \in \mathcal{D}(A), \|f\|_1 \leq 1. \quad (3.23)$$

Since  $B$  is convex, it follows that  $B$  is continuous. Let's denote by  $\Theta(x) = xB(\ln x)$ ,  $x > 0$  (here we use notations of [C]). For the second step, we apply Prop.II.1 of [C] which says that

$$\|T_t f\|_2 \leq m(t) \|f\|_1$$

with  $m(t)$  the inverse function of  $p(t) = \int_t^{+\infty} \frac{dx}{\Theta(x)}$ ,  $t > 0$ . By a change of variable, we get

$$p(t) = \int_{\ln t}^{+\infty} \frac{dy}{B(y)}, t > 0.$$

Setting  $q(s) = \int_s^{+\infty} \frac{dy}{B(y)}$ ,  $s \in \mathbb{R}$ , we obtain  $p(t) = q(\ln t)$ ,  $t > 0$ . Thus  $m(t) = \exp(q^{-1}(t)) = e^{M(t)}$  with  $M(t)$  defined as in this theorem. This completes the proof.

## 4 Ultracontractivity implies LSIWP

In [D] and [DS], the authors show how we can deduce LSIWP from ultracontractivity using the complex interpolation of Stein. In this section, we give another way to deduce logarithmic Sobolev inequality with parameter from ultracontractivity. Nash type inequality is involved in the proof.

We first recall a result due to T.Coulhon which is one step to prove LSIWP from ultracontractivity property. We give a slightly different presentation of the statement of this theorem and we recall the proof for the convenience of the reader.

**Theorem 4.1** [C] *Let  $(T_t)$  be a symmetric semigroup on  $L^2$ . Suppose that*

$$\|T_t f\|_2^2 \leq m(t) \|f\|_1^2, \quad \forall t > 0. \quad (4.1)$$

*Then the following Nash type inequality is satisfied*

$$\|f\|_2^2 \Lambda(\ln \|f\|_2^2) \leq (Af, f), \quad \forall f \in \mathcal{D}(A), \|f\|_1 \leq 1, \quad (4.2)$$

*where  $\Lambda(s) = \sup_{t>0} (st - t \ln m(1/2t))$ ,  $s \in \mathbb{R}$ .*

The function  $\Lambda$  is the conjugate function (or so-called Legendre transform) of  $t \rightarrow t \ln m(1/2t)$ . The function  $x \rightarrow x\Lambda(\ln x)$  is nothing but the function  $\tilde{\Theta}(x) = \sup_{t>0} (x/2t) \log(x/m(t))$  of Proposition II.2 of [C]. The interest of the formulation (4.2) is that it expresses  $\tilde{\Theta}$  in terms of the Legendre transform  $\Lambda$  and is well-appropriate to deal with fractional powers of  $A$ . Indeed, in [BeMa] it is

proved that if  $A$  satisfies a Nash type inequality with  $\tilde{\Theta}(x) = x\Lambda(\ln x)$  then  $A^\alpha$  ( $0 < \alpha < 1$ ) satisfies a Nash type inequality with (roughly)  $\tilde{\Theta}_\alpha(x) = x\Lambda^\alpha(\ln x)$  where  $\Lambda^\alpha = \exp(\alpha \ln \Lambda)$ .

**Proof :** Nash type inequality (4.2) is proved by using a convexity argument and optimization over the time parameter. By Jensen's inequality,

$$e^{-2t(Af, f)/\|f\|_2^2} \leq \int_0^{+\infty} e^{-2t\lambda} d\mu(\lambda) = \|T_t f\|_2^2 / \|f\|_2^2$$

where  $d\mu(\lambda) = d(E_\lambda f, f) / \|f\|_2^2$  and  $(E_\lambda)$  is the spectral decomposition of  $A$ . By assumption (4.1) and  $\|f\|_1 \leq 1$ , we deduce

$$e^{-2t(Af, f)/\|f\|_2^2} \leq m(t) / \|f\|_2^2.$$

This can be written as

$$-2t(Af, f) \leq \|f\|_2^2 (\ln m(t) - \ln \|f\|_2^2)$$

or equivalently by changing  $t$  by  $1/2t$

$$\|f\|_2^2 (t \ln \|f\|_2^2 - t \ln m(1/2t)) \leq (Af, f).$$

We finishes the proof by optimizing over  $t > 0$ .

It is easily proved that Nash type inequality (4.2) is equivalent to what we shall call the relaxed Nash type inequality below

$$\|f\|_2^2 \log(\|f\|_2) \leq t(Af, f) + \log(\sqrt{m(t)}) \|f\|_2^2, \quad \forall t > 0, \|f\|_1 \leq 1. \quad (4.3)$$

We compare this inequality with the one we can deduce from Davies-Simon Theorem recalled in (2.4). Under the assumption of ultracontractivity property of Davies-Simon Theorem and with Lemma 3.1, we get the following relaxed Nash type inequality :

$$\|f\|_2^2 \log(\|f\|_2) \leq t(Af, f) + M(t) \|f\|_2^2, \quad \forall t > 0, \|f\|_1 \leq 1, \quad (4.4)$$

and the two functions  $M(t)$  and  $\log(\sqrt{m(t)})$  from (2.12) and (4.1) are the same i.e  $M(t) = \log(\sqrt{m(t)})$  (We assume that (2.12) and (4.1) are equalities). So the inequalities (4.3) and (4.4) are the same.

We now state the equivalence of relaxed Nash type inequality and LSIWP when  $Q(f) = (Af, f)$  is a Dirichlet form (see [FU]). We apply truncation method as developped in [BCLS]. This result is essentially contained in [BiMa]. We give the sketch of the proof for completeness.

**Theorem 4.2** 1. Assume that  $Q$  is a Dirichlet form and that the following inequality is satisfied

$$\|f\|_2^2 \log(\|f\|_2) \leq tQ(f) + M(t)\|f\|_2^2, \quad \forall t > 0, \|f\|_1 \leq 1. \quad (4.5)$$

Then

$$\int f^2 \log_+(f/\|f\|_2^2) d\mu \leq tQ(f) + \tilde{M}(t)\|f\|_2^2, \quad \forall t > 0 \quad (4.6)$$

with  $\tilde{M}(t) = \frac{1}{c_1}M(t/c_1) + c_2$  (The constants  $c_1, c_2 > 0$  do not depend on  $f$  and  $Q$ ).

2. Conversely (without any assumptions on the quadratic form  $Q$ ), if (4.6) is satisfied then (4.5) is also satisfied with  $M(t) = \tilde{M}(t)$  for all  $t > 0$ .

**Proof :**

1. The arguments are taken from [BiMa] and [BCLS]. We give the ingredients of the proof. Let  $f$  such that  $0 \leq f$  and  $\|f\|_2 = 1$ . Let  $k \in \mathbb{Z}$ , we define  $f_k = (f - 2^k) \wedge 2^k$ . Fix  $t > 0$ . By assumption,

$$\|f_k\|_2^2 \log(\|f_k\|_2/\|f_k\|_1) \leq tQ(f_k) + M(t)\|f_k\|_2^2.$$

We have  $2^{k-1} \leq \|f_k\|_2/\|f_k\|_1$  and  $2^{2k}\mu(2^k \leq f) \leq \|f_k\|_2^2$ . We set  $W(g) = tQ(g) + M(t)\|g\|_2^2$  (note that  $W$  is also a Dirichlet form). Then we deduce for any  $k \in \mathbb{Z}$ ,

$$2^{2k}\mu(2^k \leq f) \log 2^{k-1} \leq W(f_k).$$

By discretisation of the integral,

$$\int f^2 \log_+ f d\mu \leq \sum_{k=0}^{\infty} 2^{2(k+1)} \log 2^{k+1} \mu(2^k \leq f \leq 2^{k+1}).$$

Hence, altogether, we get for some  $c, c' > 0$ ,

$$\int f^2 \log_+ f d\mu \leq c \sum_{k=0}^{\infty} W(f_{k-1}) + c'\|f\|_2^2$$

We conclude by the fact that  $W$  is a Dirichlet form then

$$\sum_{k=0}^{\infty} W(f_{k-1}) \leq \sum_{k \in \mathbb{Z}} W(f_k) \leq W(f)$$

For a demonstration of the last statement see [BiMa]. This finishes the proof of the first statement.

2. We apply Lemma 3.1.



This finishes the proof of this theorem.

Combining Theorem 4.1 and Theorem 4.2, we obtain a new proof of LSIWP from ultracontractivity property. Indeed, we first apply Coulhon's result 4.1 then we get the so-called relaxed Nash type inequality (4.3). We now apply Theorem 4.2 to conclude. Note that in [D], they get  $\tilde{M}(t) = M(t)$  for all  $t > 0$ .

## 5 Relations between Nash type inequality and LSIWP

We now prove that Nash type inequality and LSIWP are (essentially) equivalent. We obtain this result by putting together arguments of Section 4 and using the fact that the function  $\Lambda$  in (5.1) is a Legendre transform. The natural assumption on  $\Lambda$  comes from the following remark: we have proved that ultracontractivity property (see (4.2)) or LSIWP (see (3.19) and (3.21)) implies a Nash type inequality of the form

$$\|f\|_2^2 \Lambda(\ln \|f\|_2^2) \leq (Af, f), \|f\|_1 \leq 1, f \in \mathcal{D}(A). \quad (5.1)$$

where  $\Lambda$  is given by the Legendre transformation (i.e the so-called conjugate function) of the function  $\psi$ ,

$$\Lambda(y) = \sup_{t>0} (ty/2 - \psi(t))$$

with  $\psi(t) = t\beta(1/t)$  (see (4.2)). It implies for any  $t > 0$  and any  $y \in \mathbb{R}$ ,

$$ty/2 - \Lambda(y) \leq \psi(t) \quad (5.2)$$

which says in particular that the function  $y \rightarrow ty/2 - \Lambda(y)$  is bounded for any  $t > 0$ . Note also, in the theory of Orlicz spaces, the functions  $\psi$  and  $\Lambda$  are  $N$ -functions in the sense of [A] and  $\psi$  is obtained by the duality formula, for any  $t > 0$ ,

$$\sup_{y \in \mathbb{R}} (ty/2 - \Lambda(y)) = \psi(t)$$

Here,  $\psi$  and  $\Lambda$  are not necessarily  $N$ -functions. In our case, it is not a problem because we only need the inequality (5.2) in our applications. So as we can see, Legendre transform (or more generally convexity) plays again an important role in our theory.

We are now in a position to give the natural condition on  $\Lambda$  to formulate our first statement: Nash type inequality  $\Rightarrow$  LSIWP.

**Theorem 5.1** *Assume that (5.1) holds true for some function  $\Lambda$  with  $(Af, f)$  a Dirichlet form and that  $\Lambda$  satisfies the following hypothesis: for any fixed  $t > 0$ , the function  $y \in \mathbb{R} \rightarrow ty/2 - \Lambda(y)$  is bounded above. Set*

$$N(t) = \sup_{y \in \mathbb{R}} (ty/2 - \Lambda(y)), \quad t > 0$$

and

$$\beta(t) = tN(1/t).$$

Then, there exists  $c_1, c_2 > 0$  such that, for all  $f \in \mathcal{D}(A)$ ,

$$\int f^2 \log_+(f/\|f\|_2) d\mu \leq t(Af, f) + \tilde{\beta}(t)\|f\|_2^2, \quad \forall t > 0 \quad (5.3)$$

with  $\tilde{\beta}(t) = \frac{1}{2c_1}\beta(\frac{2t}{c_1}) + c_2$ .

In fact, the main point of Theorem 5.1 is the existence of the function  $N$ . For instance, the second hypothesis of this theorem is satisfied if the following conditions (A1) and (A2) below holds true.

(A1): For any  $s > 0$  and any  $y_0 \in \mathbb{R}$ , the function  $y \rightarrow sy - \Lambda(y)$  is bounded above on the interval  $(-\infty, y_0)$ .

(A2):  $\lim_{y \rightarrow +\infty} \frac{\Lambda(y)}{y} = +\infty$ .

For example, the assumption (A1) is satisfied if  $\Lambda$  is non-negative or if there exists  $y_1 \in \mathbb{R}$  such that  $\Lambda(y) = 0$  for all  $y \leq y_1$  and  $\Lambda$  continuous or  $\lim_{y \rightarrow -\infty} \Lambda(y) = 0$  and  $\Lambda$  continuous.

As we have already mentioned, if  $\Lambda$  is given by the Legendre transformation of some function i.e  $\Lambda(y) = \sup_{t>0}(ty/2 - \psi(t))$  (finite at any point  $y \in \mathbb{R}$ ) then  $\Lambda$  satisfies immediately the hypothesis of our theorem. Also note that the transformation

$$t \rightarrow \beta(t) = tN(1/t).$$

is idempotent. So  $N(t) = t\beta(1/t)$ .

**Proof :** By Theorem 4.2, it is enough to prove (4.5). As a consequence of the definition of  $N(t)$ , we have for any  $t > 0$  and any  $y \in \mathbb{R}$ ,

$$yt/2 - \Lambda(y) \leq N(t)$$

or equivalently,

$$yt/2 - N(t) \leq \Lambda(y).$$

Let  $y = \ln \|f\|_2^2$  in the inequality just above and multiply it by  $\|f\|_2^2$ . Hence, by our assumption (5.1), we deduce

$$t\|f\|_2^2 \ln \|f\|_2^2 - N(t)\|f\|_2^2 \leq (Af, f).$$

We set  $t = 1/s, s > 0$ . This yields

$$\|f\|_2^2 \ln \|f\|_2^2 \leq s(Af, f) + sN(1/s)\|f\|_2^2$$

and by definition of  $\beta$ ,

$$\|f\|_2^2 \ln \|f\|_2^2 \leq s(Af, f) + \beta(s)\|f\|_2^2.$$

So, (4.5) is proved with  $M(t) = \frac{1}{2}\beta(2t)$ . Now, we apply Theorem 4.2. This finishes the proof.

We now state the converse of Theorem 5.1. Note that we do not need any assumption on the quadratic form  $(Af, f)$  for this converse.

**Theorem 5.2** Assume that, for all  $f \in \mathcal{D}(A)$ ,

$$\int f^2 \log_+(f/\|f\|_2) d\mu \leq t(Af, f) + \beta(t)\|f\|_2^2, \quad \forall t > 0 \quad (5.4)$$

is satisfied. Set

$$\Lambda(y) = \sup_{t>0} (ty/2 - N(t)), \quad y \in \mathbb{R}$$

with

$$N(t) = t\beta(1/t).$$

Then, for all  $f \in \mathcal{D}(A)$ ,

$$\|f\|_2^2 \Lambda(\ln \|f\|_2^2) \leq (Af, f), \|f\|_1 \leq 1. \quad (5.5)$$

The function  $\Lambda$  is automatically defined as can be seen in the course of the proof.

**Proof :** The assumption (5.4) implies obviously LSIWP. We now repeat the argument of the beginning of the proof of Theorem 3.5. We apply Lemma 3.1 to get (3.23) with  $B = \Lambda$ . This completes the proof.

We have shown that Nash type inequality are equivalent to LSIWP in the sense of Theorem 5.1 and Theorem 5.2.

## 6 Nash type inequality

In this section, we study some aspects of the relationship between ultracontractivity and Nash type inequality (see [C], [T] ) of the form :

$$B(\|f\|_2^2) \leq Q(f) \quad (6.1)$$

for all  $f \in \text{Quad}(A) \cap L^1$  with  $\|f\|_1 \leq 1$ . The classical Nash inequality corresponds to  $B(t) = ct^{1+2/n}$  (see [CKS]) i.e.

$$c\|f\|_2^{2+4/n} \leq Q(f)\|f\|_1^{4/n} \quad (6.2)$$

Let  $V, W$  be two continuous functions on  $[0, +\infty[$  and  $b_1$  a continuous function on  $]0, +\infty[$ . We begin by the following easy but important proposition :

**Proposition 6.1** *The two following inequalities are equivalent : for all  $t > 0$  and all  $f \in \text{Quad}(A) \cap L^1$  with  $\|f\|_1 \leq 1$ ,*

$$V(\|f\|_2^2) \leq tQ(f) + b_1(t)W(\|f\|_2^2) \quad (6.3)$$

and

$$B(\|f\|_2^2) \leq Q(f) \quad (6.4)$$

where is defined by  $B(x) = \sup_{s>0} (sV(x) - b(s)W(x)), x > 0$  with  $b(s) = sb_1(1/s)$ .

The inequality (6.3) will be called relaxed Nash type inequality. We shall apply this proposition in two important cases :

$$\text{case (A) :} \quad V(x) = x, \quad W(x) = 1. \quad (6.5)$$

$$\text{case (B) :} \quad V(x) = \frac{x}{2} \log x, \quad W(x) = x. \quad (6.6)$$

**Remark 6.2** *The assumptions on  $V, W, b_1$  and (6.3) implies that  $B(\|f\|_2^2)$  is finite. With  $V(x) = x$ ,  $W(x) = 1$  then  $B(x) = \sup_{s>0}(sx - b(s))$  is the complementary function (or Legendre transform) of  $b$ . In applications, we often recover  $b(s)$  by the same formula  $b(s) = \sup_{x>0}(xs - B(x))$  (see [A] p.229).*

*With the choice  $V(x) = \frac{x}{2} \ln x$  and  $W(x) = x$ ,  $B(x) = x \sup_{s>0}(\frac{s}{2} \ln x - b(s))$  is similar to the function  $\tilde{\Theta}$  introduced in section 4.*

**Proof :** The proof is easy and left to the reader.

By the remark just above, the problem of obtaining the Nash type inequality (6.4) is equivalent to show the inequality (6.3). But some difference may arise from the choice of the functions  $V$  and  $W$  as we shall see below. Under an ultracontractivity assumption on the semigroup, we apply the proposition 6.1 in case (A) and (B) respectively.

#### Case A

We start with this simple case.

**Theorem 6.3** *Let  $T_t$  be an ultracontractive semigroup such that*

$$\|T_t f\|_\infty \leq a(t) \|f\|_1 \quad (6.7)$$

*then*

$$\|f\|_2^2 \leq tQ(f) + a(t) \quad (6.8)$$

*for all  $t > 0$  and  $f \in \text{Quad}(A) \cap L^1$ , with  $\|f\|_1 \leq 1$ .*

*and*

$$B(\|f\|_2^2) \leq Q(f) \quad (6.9)$$

*for all  $t > 0$  and  $f \in \text{Quad}(A) \cap L^1$ , with  $\|f\|_1 \leq 1$ , where we set  $b(s) = sa(1/s)$  and  $B(x) = \sup_{s>0}(sx - b(s))$ .*

The inequality (6.8) is called super-Poincaré inequality in [W] (see (1.2) of [W]). See also Sec.5 of [W] for related results to the ultracontractivity property.

**Proof :** Let  $f \in L^1 \cap Quad(A)$  with  $\|f\|_1 \leq 1$ . Set  $f_s = T_s f$ , then for all  $t > 0$  :

$$f = f_t + \int_0^t A f_s ds$$

Hence,

$$\|f\|_2^2 = (f_t, f) + \int_0^t (A f_s, f) ds$$

and

$$\|f\|_2^2 \leq \|f_t\|_\infty \|f\|_1 + \int_0^t Q(f_{s/2}) ds$$

The function  $s \mapsto Q(f_s) = (A f_s, f_s)$  is non-increasing, thus

$$\|f\|_2^2 \leq a(t) + tQ(f)$$

We conclude the proof by applying Proposition 6.1 with  $V(x) = x, W(x) = 1$  and  $b_1(t) = a(t)$ .

In particular, if  $a(t) = ct^{-\frac{n}{2}}$  then we obtain the classical Nash inequality (6.2). This result is well-known (see [VSC]). In fact, (6.7) and (6.2) are equivalent (see [CKS]) and are also equivalent to Sobolev inequality (when  $n > 2$ ) :

$$\|f\|_{\frac{2n}{n-2}}^2 \leq cQ(f) \tag{6.10}$$

This last result is due to Varopoulos (see [V]).

For applications, an important family of functions is

$$a(t) = c_1 t^{-\alpha} \exp\left(\frac{c_2}{t^\gamma}\right), \quad \alpha \geq 0, \gamma \in \mathbb{R}. \tag{6.11}$$

This kind of function motivates our study. Of course, when  $t$  is small the main term is the exponential for which the computation of  $B$  just above is rather complicated. Indeed, with  $\gamma \neq 0$  and  $\alpha \neq 0$  the function  $B$  doesn't seem to be known explicitly. We can certainly estimate  $B(x)$  when  $x$  is large. But we shall see below that a better approach of Nash type inequality is to use the logarithmic Sobolev inequality. In that case  $a(t)$  is replaced by  $\ln a(t)$  with a change of the couple  $(V, W)$  and the corresponding function  $B(t)$  can be computed exactly (see case B).

Case B :

We have at least two possibilities to prove Nash type inequality from an ultracontractivity property for the case  $V(x) = \frac{x}{2} \ln x$  and  $W(x) = x$ . One way is to use the LSIWP and the convexity Lemma 3.1. An alternative proof is to use Coulhon's result recalled in Theorem 4.1.

**Theorem 6.4** *Let  $T_t$  be an ultracontractive semigroup such that*

$$\|T_t f\|_\infty \leq a(t) \|f\|_1 \tag{6.12}$$

or

$$\|T_t f\|_\infty \leq \sqrt{a(t)} \|f\|_2. \quad (6.13)$$

Then, for all  $t > 0$  and  $f \in \text{Quad}(A) \cap L^1$ , with  $\|f\|_1 \leq 1$  :

$$\|f\|_2^2 \log \|f\|_2 \leq tQ(f) + \frac{1}{2} \log a(t) \|f\|_2^2 \quad (6.14)$$

Or equivalently the following Nash type inequality

$$B(\|f\|_2^2) \leq Q(f) \quad (6.15)$$

with  $B(x) = x \sup_{s>0} (\frac{s}{2} \log x - b(s))$  and  $b(s) = \frac{s}{2} \log a(1/s)$ .

**Proof :** By applying Theorem 2.4 with  $M(t) = \frac{1}{2} \log a(t)$ , the following LSIWP is satisfied: for all  $t > 0$ ,

$$\int f^2 \log(f/\|f\|_2) d\mu \leq tQ(f) + M(t)\|f\|_2^2 \quad (6.16)$$

From the lemma 3.1, we deduce (6.14). We conclude by applying Proposition 6.1 with  $V(x) = \frac{x}{2} \log x$ ,  $W(x) = x$  and  $b_1(t) = \frac{1}{2} \log a(t)$ . This completes the proof.

The alternative proof using Coulhon's result is left to the reader (see Theorem 4.1).

We now give an example. When  $\alpha = 0$  in (6.11), we are able to explicit function  $B$  in Nash type inequality.

**Theorem 6.5** *If the following inequality is satisfied for some  $\gamma > 0$ ,*

$$\|T_t f\|_\infty \leq c_1 \exp\left(\frac{c_2}{t^\gamma}\right) \|f\|_1 \quad (6.17)$$

*Then there exist  $k, \beta > 0$  such that for all  $f \in \text{Quad}(A) \cap L^1$ , with  $\|f\|_1 \leq 1$ ,*

$$k \|f\|_2^2 \left[ \left( \log \frac{\|f\|_2}{\beta} \right)_+ \right]^{1+\frac{1}{\gamma}} \leq Q(f) \quad (6.18)$$

*Conversely (6.18) implies (6.17) with different constants  $c_1$  and  $c_2$ .*

**Proof :** We apply Theorem 6.4. Since  $B(t) = tD(\log \sqrt{t})$ , we have just to compute  $D(x) = \sup_{s>0} (sx - b(s))$  with  $b(s) = \frac{\log c_1}{2} s + \frac{c_2}{2} s^{1+\gamma}$ . For  $x \in \mathbb{R}$ , we easily study the extrema of  $h(s) = sx - b(s)$  and obtain  $D(x) = k [(x - k_1)_+]^{1+\frac{1}{\gamma}}$ ,  $x \in \mathbb{R}$  with  $k > 0$ . Let  $\beta = e^{k_1}$  then  $B(t) = kt \left[ \left( \log \frac{\sqrt{t}}{\beta} \right)_+ \right]^{1+\frac{1}{\gamma}}$ . The converse is proved by applying Proposition 6.1 and Theorem 2.1. The proof is completed.

**Remark 6.6** We denote that  $b(s)$  may not an  $N$ -function in the sense of [A] p.228 since  $\lim_{s \rightarrow 0} b(s)/s$  is not necessarily zero. But for all  $t, s > 0$ , the functions  $B$  and  $b$  satisfy

$$t(s \log \sqrt{t} - b(s)) \leq B(t)$$

Let  $v = \log \sqrt{t}$ , then

$$sv - H(v) \leq b(s)$$

with  $H(v) = e^{-2v}B(e^{2v})$ . Hence,

$$\tilde{H}(s) = \sup_{v \in \mathbb{R}} (sv - H(v)) \leq b(s)$$

It is easily seen that  $\tilde{H}(s) = b(s)$  when  $B(t) = kt \left[ \left( \log \frac{\sqrt{t}}{\beta} \right)_+ \right]^{1+\frac{1}{\gamma}}$ .

We now show that an ultracontractive bound on the semigroup implies a generalized Gross' inequality (see (6.20) below). Such an equality is easily deduced from LSIWP.

**Theorem 6.7** Let  $T_t$  be an ultracontractive semigroup such that

$$\|T_t f\|_\infty \leq a(t) \|f\|_1 \tag{6.19}$$

then

$$D \left( \int f^2 \log f \, d\mu \right) \leq Q(f) \tag{6.20}$$

for all  $0 \leq f \in \text{Quad}(A) \cap L^1 \cap L^\infty$  with  $\|f\|_2 = 1$  and where  $D(t) = \sup_{s>0} (st - b(s))$  with  $b(s) = \frac{s}{2} \log a(\frac{1}{s})$ . Equivalently,

$$\|f\|_2^2 D \left( \|f\|_2^{-2} \int f^2 \log (f/\|f\|_2) \, d\mu \right) \leq Q(f) \tag{6.21}$$

for all  $0 \leq f \in \text{Quad}(A) \cap L^1 \cap L^\infty$ .

**Proof :** By Davies-Simon Theorem 2.4, we have

$$\int f^2 \log f \, d\mu \leq tQ(f) + \frac{1}{2} \log a(t) \|f\|_2^2 + \|f\|_2^2 \log \|f\|_2 \tag{6.22}$$

for all  $t > 0$ . We now assume that  $\|f\|_2^2 = 1$  then changing  $t$  by  $1/s$ ,

$$s \left( \int f^2 \log f \, d\mu \right) - \frac{s}{2} \log a(1/s) \leq Q(f) \tag{6.23}$$

Thus, with  $D$  define as above, we conclude (6.20) and (6.21) by renormalisation. This proves the theorem.

We can see Theorem 6.4 as a corollary of Theorem 6.7 when  $D(t)$  is non decreasing in  $t$ . The proof is a simple application of the lemma 3.1 which can be also formulated as follows, for all  $0 \leq f \in L^1 \cap L^\infty$  with  $\|f\|_1 \leq 1$  :

$$\log \|f\|_2 \leq \|f\|_2^{-2} \int f^2 \log (f/\|f\|_2) d\mu \quad (6.24)$$

then from (6.21),

$$\|f\|_2^2 D(\log \|f\|_2) \leq Q(f) \quad (6.25)$$

But the first term of this inequality is  $B(\|f\|_2^2)$  in (6.15 ) because  $B(t) = tD(\log \sqrt{t})$ . We conclude Theorem 6.4.

## 7 Weak Sobolev inequalities and Nash Type inequalities.

We apply the results of the preceding section with  $a(t) = c_1 t^{-\frac{n}{2}}$  and recall explicitly some well-known results concerning semigroups with this polynomial ultracontractivity. Recall that, in our setting,

$$\|e^{-At}\|_{\infty,2} \leq c_1 t^{-\frac{n}{4}} \quad (7.1)$$

is equivalent to

$$\|e^{-At}\|_{\infty,1} \leq c'_1 t^{-\frac{n}{2}} \quad (7.2)$$

Let  $\mathcal{D}$  be the domain of  $A$ .

**Theorem 7.1** *The following inequalities are equivalent*

(i)

$$\|e^{-At} f\|_\infty \leq c_1 t^{-\frac{n}{4}} \|f\|_2 \quad (7.3)$$

for all  $f \in L^2$  and for all  $t > 0$ .

(ii)

$$\int f^2 \log f d\mu \leq \log \left( c_3 Q(f)^{\frac{n}{4}} \right) \quad (7.4)$$

for all  $f \in \mathcal{D} \cap L^1 \cap L^\infty$ ,  $f \geq 0$  and  $\|f\|_2 = 1$ .

(iii)

$$\|f\|_2^{2+\frac{4}{n}} \leq c_4 Q(f) \|f\|_1^{\frac{4}{n}} \quad (7.5)$$

for all  $f \in L^1 \cap \mathcal{D}$ .

(iv)

$$\|f\|_{\frac{2n}{n-2}}^2 \leq c_5 Q(f) \quad (7.6)$$

for all  $f \in \mathcal{D}$  (when  $n > 2$ ).



**Proof :** The equivalences between (7.3) and (7.5) and (7.6) (when  $n > 2$ ) are well-known, see [VSC],[D].

In [B], D.Bakry gives the arguments to prove that (7.4) is equivalent to (7.3) (see remark at the end of page 64 and Section 5 p.67 of [B]). Here, we focus on the inequality (7.4). We first prove that (7.3) implies (7.4) by applying Theorem 6.7 with  $a(t) = c_0 t^{-\frac{n}{2}}$ . A simple computation gives us  $D(y) = ce^{\frac{4y}{n}}$ ,  $y \in \mathbb{R}$ . Then, we deduce for all  $f \in \mathcal{D} \cap L^1 \cap L^\infty$ ,  $f \geq 0$  and  $\|f\|_2 = 1$  :

$$c \exp\left(\frac{4}{n} \int f^2 \log f d\mu\right) \leq Q(f) \quad (7.7)$$

because  $D$  is increasing and thus invertible, this inequality is equivalent to (7.4) i.e

$$\int f^2 \log f d\mu \leq \log\left(c_3 Q(f)^{\frac{n}{4}}\right) \quad (7.8)$$

with  $c_3 = c^{-\frac{n}{4}}$ .

We now prove that (7.4) implies (7.5) by applying Theorem 6.4. We get, for any  $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$ ,

$$\|f\|_2^{1+\frac{n}{2}} \leq c_3 Q(f)^{\frac{n}{4}} \quad (7.9)$$

with  $\|f\|_1 \leq 1$ . This proves (7.5).

The original proof of the implication (7.3) to (7.5) is obtained by using Theorem (6.3). But, as we have seen above, the applications of Theorem 6.3 have some limitations. So we shall prefer the approach with LSIWP given by Theorem 6.7.

The inequality (7.4) is called weak-Sobolev inequality of dimension  $n$  in [B]. In section 6, this inequality is generalized by inequality (6.20). Then (7.4) is a particular case of (6.20) with the function  $D(y) = ce^{4y/n}$ ,  $y \in \mathbb{R}$  which is invertible. More generally when  $D$  is invertible, the inequality (6.20) is equivalent to

$$\int f^2 \log f d\mu \leq D^{-1}(Q(f)) \quad (7.10)$$

for all  $f \in \mathcal{D} \cap L^1 \cap L^\infty$ ,  $f \geq 0$  and  $\|f\|_2 = 1$ .

## 8 Examples on the infinite Torus

In this section, we give families of examples of semigroups with one-exponential and double-exponential ultracontractivity property when the time is small (see definitions below). In fact, a natural setting for having such behaviors is the infinite torus  $\mathbb{T}^\infty$ . The reference for such examples is the paper by A.Bendikov [B2] (see also [B1]). This paper also contains much more examples of classes of ultracontractivity. In fact, much weaker behavior than polynomial ultracontractivity can be produced

(see[B2]).

Let  $\mathbb{T}^\infty$  be the infinite torus with its ordinary product structure. The group  $\mathbb{T}^\infty$  is a compact abelian group. The neutral element is denoted by 0 and  $dm$  the normalised Haar measure. This measure is the countable product of the normalised Haar measure of  $\mathbb{T}$  which is identified with  $[-\pi, \pi]$ . Let  $\mu_t$  the brownian semigroup on  $\mathbb{T}$  and  $\mathcal{A} = \{a_k\}_{k=1}^\infty$  a sequence of strictly positive numbers. For each  $t > 0$ , we define the product measure:

$$\mu_t^{\mathcal{A}} = \otimes_{k=1}^\infty \mu_{ta_k}$$

then  $(\mu_t^{\mathcal{A}})_{t>0}$  defines a symmetric convolution semigroup on  $\mathbb{T}^\infty$  denoted by  $(T_t^{\mathcal{A}})_{t \geq 0}$  and

$$\|T_t^{\mathcal{A}}\|_{1 \rightarrow \infty} = \mu_t^{\mathcal{A}}(0).$$

One important aspect of such semigroups for Harmonic Analysis theory in infinite dimensional spaces is that  $(\mu_t^{\mathcal{A}})$  is not necessarily absolutely continuous with respect to the Haar measure  $dm$ . We have to impose conditions on the sequence  $\mathcal{A}$  in order to get a continuous density. To this purpose, we set for  $x > 0$  :

$$\mathcal{N}^{\mathcal{A}}(x) = \#\{k \geq 1 : a_k \leq x\}.$$

It is proved in [B2] (Thm 3.6 p.51) that if

$$\log \mathcal{N}^{\mathcal{A}}(x) = o(x) \quad \text{as } x \rightarrow +\infty$$

then  $\mu_t^{\mathcal{A}}$  has a continuous density. The converse is also true. We also denote by  $\mu_t^{\mathcal{A}}$  the density when it exists. The infinitesimal generator is given formally as an infinite Laplacian  $A = \sum_{k=1}^\infty a_k \frac{\partial^2}{\partial^2 x_k}$ . In the two following subsections, we focus on two particular examples of ultracontractivity property.

## 8.1 One-exponential ultracontractivity

We shall say that a semigroup  $(T_t)$  satisfies a one-exponential ultracontractivity property at zero if there exists  $\alpha > 0$  and  $t_0 > 0$  such that

$$\|T_t\|_{1 \rightarrow +\infty} \leq c' \exp\left(\frac{c}{t^\alpha}\right), \quad 0 < t < t_0. \quad (8.1)$$

for some constants  $c, c' > 0$ . We shall say that a semigroup  $(T_t)$  has a strict one-exponential ultracontractivity property if (8.1) is satisfied and moreover

$$c' \exp\left(\frac{c}{t^\alpha}\right) \leq \|T_t\|_{1 \rightarrow +\infty}, \quad 0 < t < t_0. \quad (8.2)$$

with the same index  $\alpha$  but with possibly other constants  $c, c'$ .

Note that of (8.1) holds for some  $t < t_0$  then it holds for any  $t > 0$ . Indeed, the function  $\|T_t\|_{1 \rightarrow +\infty}$ , is non-increasing in  $t$ . We have the following family of examples satisfying a strict one-exponential ultracontractivity property:

**Theorem 8.1** (*[B2] Thm 3.18 p. 54*). Let  $\alpha > 0$ . Assume that  $\mathcal{N}^{\mathcal{A}}(x) \sim x^\alpha$  as  $x \rightarrow +\infty$ . Then we have

$$\log \mu_t^{\mathcal{A}}(0) \sim k(\alpha) t^{-\alpha} \text{ as } t \rightarrow 0 \quad (8.3)$$

In particular, there exists  $t_0, c_1(\alpha), c_2(\alpha)$  such that :  $\forall t \in ]0, t_0]$  ,

$$\exp\left(\frac{c_1(\alpha)}{t^\alpha}\right) \leq \|T_t\|_{1 \rightarrow +\infty} \leq \exp\left(\frac{c_2(\alpha)}{t^\alpha}\right) \quad (8.4)$$

Recall that  $\|T_t\|_{1 \rightarrow +\infty} = \mu_t^{\mathcal{A}}(0)$ . The condition on the sequence  $\mathcal{A}$  is satisfied, for instance with  $a_k = k^{1/\alpha}, k \geq 1$  with  $\alpha > 0$ .

Now, we apply Theorem 6.5 with  $X = \mathbb{T}^\infty$  and  $\mu$  the Haar measure on  $X$ . We have the following Nash type inequality.

**Theorem 8.2** *Under the assumptions of Theorem 8.1. There exists constants  $c_1, c_2 > 0$  such that :*

$$\|f\|_2^2 \left[ \log_+ \left( \frac{\|f\|_2}{c_1 \|f\|_1} \right) \right]^{1+\frac{1}{\alpha}} \leq c_2 Q(f) \quad (8.5)$$

for all  $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$ .

We deduce the following result by using cut-off method developed as in [BCLS] (see [BiMa] and also [W] for such inequalities).

**Theorem 8.3** *There exists  $c_3, c_4 > 0$  such that, for all  $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$ ,*

$$\int_{\mathbb{T}^\infty} f^2 \left[ \log \left( \frac{f}{c_3 \|f\|_2} + 1 \right) \right]^{1+\frac{1}{\alpha}} d\mu \leq c_4 (Q(f) + \|f\|_2^2). \quad (8.6)$$

Conversely, by a convexity argument similar to Lemma 3.1 (in the genral framework), (8.6) implies (8.5) with  $Q(f) + \|f\|_2^2$  instead of  $Q(f)$ .

## 8.2 Double-exponential ultracontractivity

We shall say that a semigroup  $(T_t)$  satisfies a double-exponential ultracontractivity property at zero if there exists  $\gamma > 0$  and  $t_0 > 0$  such that

$$\|T_t\|_{1 \rightarrow +\infty} \leq c' \exp\left(\exp\left(\frac{c}{t^\gamma}\right)\right), \quad 0 < t < t_0, \quad (8.7)$$

for some constants  $c, c' > 0$ . We shall say that a semigroup  $(T_t)$  has a strict double-exponential ultracontractivity property if (8.7) is satisfied and moreover

$$c' \exp\left(\exp\left(\frac{c}{t^\gamma}\right)\right) \leq \|T_t\|_{1 \rightarrow +\infty}, \quad 0 < t < t_0, \quad (8.8)$$

with the same index  $\gamma$  but with possibly other constants  $c, c'$ .

Note that if (8.7) holds for some  $t < t_0$  then it holds for any  $t > 0$ . We have the following family of examples satisfying a strict double-exponential ultracontractivity property:

**Theorem 8.4** (*[B2] Thm 3.27 p. 59*). *Let  $\gamma > 0$ . Assume that  $\log \mathcal{N}^{\mathcal{A}}(x) \sim x^{\gamma/\gamma+1}$  as  $x \rightarrow +\infty$ . Then we have*

$$\log \log \mu_t^{\mathcal{A}}(0) \sim c(\gamma) t^{-\gamma} \text{ as } t \rightarrow 0 \quad (8.9)$$

*In particular, there exists  $t_0, c_1(\gamma), c_2(\gamma) > 0$  such that :  $\forall t \in ]0, t_0]$  ,*

$$\exp(\exp\left(\frac{c_1(\gamma)}{t^\gamma}\right)) \leq \|T_t\|_{1 \rightarrow +\infty} \leq \exp(\exp\left(\frac{c_2(\gamma)}{t^\gamma}\right)) \quad (8.10)$$

The condition on the sequence  $\mathcal{A}$  is satisfied, for instance with  $a_k = [\ln(k+2)]^\delta$ ,  $k \geq 1$  with  $\delta = (\gamma+1)/\gamma$ .

Now, we apply Theorem 6.5 with  $X = \mathbb{T}^\infty$  and  $\mu$  the Haar measure on  $X$ . We have the following Nash type inequality,

**Theorem 8.5** *Under the assumptions of Theorem 8.4. There exists constants  $c_1, c_2 > 0$  such that :*

$$\|f\|_2^2 \log\left(\frac{\|f\|_2}{\|f\|_1}\right) \left[\log\left(c_1 \log_+ \frac{\|f\|_2}{\|f\|_1}\right)\right]_+^{\frac{1}{\gamma}} \leq c_2 Q(f) \quad (8.11)$$

for all  $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$ .

We deduce the following result. We set  $D_\gamma(y) = y_+ [\log_+ y_+]^{1/\gamma}$

**Theorem 8.6** *There exists  $c_3, c_4 > 0$  such that, for all  $0 \leq f \in \mathcal{D} \cap L^1 \cap L^\infty$ ,*

$$\int_{\mathbb{T}^\infty} f^2 D_\gamma\left(c_3 \log\left(\frac{f}{8\|f\|_2}\right)\right) d\mu \leq c_4 Q(f) \quad (8.12)$$

## 9 The double-exponential case

In this section, the assumptions are the same as in Section 2. In this general framework, we consider the relationship between the double-exponential ultracontractivity property and LSIWP. We recall that the semigroup satisfies the double-exponential ultracontractivity property if:

$$\|T_t f\|_\infty \leq a(t) \|f\|_1, \quad t > 0 \quad (9.1)$$

with  $a(t) = \exp(\exp(/t^\gamma))$  with  $\gamma > 0$  and  $c > 0$  (see Subsection 8.2). Applying Theorem 2.4, we get the following logarithmic Sobolev inequality with parameter : for all  $t > 0$  ,

$$\int f^2 \log f d\mu \leq tQ(f) + \beta(t)\|f\|_2^2 + \|f\|_2^2 \log \|f\|_2 \quad (9.2)$$

where  $\beta(t) = \frac{1}{2} \log a(t) = \frac{1}{2} \exp(c/t^\gamma)$ .

We now discuss about the converse result. Assume that (9.2) holds true. We remark that by formula (2.6) of Theorem 2.1 gives  $M_\eta(t) = \infty$  for all  $t > 0$  and all  $\eta > -1$ . So we get no information about ultracontractivity property of the semigroup with this formula. It is not completely surprising. Indeed, there exists a semigroup satisfying (9.2) with  $\gamma = 1$  but which is not ultracontractive hence (9.3) doesn't hold ([D]: example 2.3.5 p.73 and [DS]: (3) p.355 and p.359). So LSIWP (9.2) may not imply the ultracontractivity property of the semigroup (at least in that case when  $\gamma \geq 1$ ). In that section, we show that, if the condition (9.2) is satisfied with  $\gamma < 1$ , then the semigroup is ultracontractive of double-exponential type but with  $\gamma'$  different from  $\gamma$ . Moreover  $\gamma'$  tends to infinity as  $\gamma$  tends to 1.

This remark implies that Nash type inequality and ultracontractivity property are not equivalent in general (see [C] and also [BeMa]).

By the converse Theorem of Davies-Simon 2.3, if we suppose that the condition (9.2) is fulfilled with  $\gamma \in ]0, 1[$ , then the semigroup is ultracontractive. But Theorem 2.3 doesn't seem to give easily and explicitly a function  $b(t)$  such that :

$$\|T_t f\|_\infty \leq b(t)\|f\|_1 \quad (9.3)$$

In this paragraph, we modify the argument of proof of Lemma 3.3 to deal with the double-exponential case.

**Proposition 9.1** *If (9.2) is satisfied with  $\beta(t) = c_1 \exp(\frac{c_2}{t^\gamma})$ ,  $0 < \gamma < 1$  then (9.3) holds with  $b(t) = k_1 \exp\left(\exp(\frac{k_2}{t^\gamma})\right)$ ,  $\gamma' = \frac{\gamma}{1-\gamma}$  where  $k_1, k_2$  are some positive constants.*

**Remark 9.2** 1. *If we apply Theorem 2.4 and its (partial) converse proposition 9.1, we stay in the same class of functions of type double-exponential. But we lose the exponent  $\gamma$ . The question to know if the expression of the exponent  $\gamma' = \frac{\gamma}{1-\gamma}$  above is optimal is open.*

2. *We also note that  $\gamma'$  is singular when  $\gamma$  tends to 1. By a preceding remark  $\gamma = 1$  is really a critical index.*

**Proof :** The proof follows the same lines as the proof of Theorem 3.4 The only change we need is a modification of the lemma 3.3 when the function  $b(t) = c_1 \exp(\frac{c_2}{t^\gamma})$ . This is done with the following lemma :

**Lemma 9.3** Suppose that  $\Phi(s) \in C^1([0, +\infty[)$  satisfies ,

$$\Phi(s) \leq (-t/2)\Phi'(s) + c_1 \exp\left(\frac{c_2}{t^\gamma}\right) \quad (9.4)$$

for all  $s, t > 0$  and for a fixed  $\gamma$  such that  $0 < \gamma < 1$ . Then we have

$$\Phi(t) \leq k_1 \exp\left(\frac{k_2}{t^\alpha}\right) \quad (9.5)$$

for all  $t > 0$ , where  $\alpha = \frac{\gamma}{1-\gamma}$ ,  $k_1 = 2c_1$  and  $k_2 = c_2^{\frac{1}{1-\gamma}} \left(\frac{\gamma}{1-\gamma}\right)^{\frac{\gamma}{1-\gamma}}$ .

**Proof of the lemma :** Set  $t = \frac{s^{\beta+1}}{\lambda}$  with  $\lambda > 0$  and  $\beta > 0$  choosen later. We multiply (9.4) by  $\exp\left(\frac{-2c_3}{s^\beta}\right)s^{-\beta-1}$  with  $c_3 = c_2\lambda^\gamma$ . Then ,

$$\begin{aligned} s^{-\beta-1} \exp\left(\frac{-2c_3}{s^\beta}\right)\Phi(s) &\leq \frac{-1}{2\lambda} \exp\left(\frac{-2c_3}{s^\beta}\right)\Phi'(s) \\ &+ c_1 s^{-\beta-1} \exp\left(\frac{c_3}{s^{\gamma(\beta+1)}} - \frac{2c_3}{s^\beta}\right). \end{aligned} \quad (9.6)$$

We choose  $\beta$  such that  $\gamma(\beta + 1) = \beta$  then  $\beta = \frac{\gamma}{1-\gamma} > 0$ . We integrate (9.6) over the interval  $[0, t]$  for  $t > 0$ . Let  $I(t) = \int_0^t s^{-\beta-1} \exp\left(\frac{-2c_3}{s^\beta}\right)\Phi(s) ds$ . Thus

$$I(t) \leq \frac{-1}{2\lambda} A(t) + B(t) \quad (9.7)$$

with

$$A(t) = \int_0^t \exp\left(\frac{-2c_3}{s^\beta}\right)\Phi'(s) ds \quad (9.8)$$

and

$$B(t) = c_1 \int_0^t s^{-\beta-1} \exp\left(\frac{-c_3}{s^\beta}\right) ds. \quad (9.9)$$

These integrals converge because  $\beta > 0$  and  $c_3 > 0$ . The function  $B(t)$  can be explicetely computed

$$B(t) = \frac{c_1}{c_3\beta} \exp\left(\frac{-c_3}{t^\beta}\right). \quad (9.10)$$

$A(t)$  is computed by integration by parts

$$A(t) = \exp\left(\frac{-2c_3}{t^\beta}\right)\Phi(t) - 2\beta c_3 I(t). \quad (9.11)$$

From (9.6) we get

$$I(t) \leq \frac{-1}{2\lambda} \exp\left(\frac{-2c_3}{t^\beta}\right) \Phi(t) + \frac{\beta c_3}{\lambda} I(t) + \frac{c_1}{c_3 \beta} \exp\left(\frac{-c_3}{t^\beta}\right) \quad (9.12)$$

Now, we chose  $\lambda$  such that  $\lambda = \beta c_3$  then  $\lambda = (\beta c_2)^{\frac{1}{1-\gamma}}$ . Finally,

$$\frac{1}{2\lambda} \exp\left(\frac{-2c_3}{t^\beta}\right) \Phi(t) \leq \frac{c_1}{c_3 \beta} \exp\left(\frac{-c_3}{t^\beta}\right) \quad (9.13)$$

We conclude the lemma with  $k_1 = 2c_1$  and  $k_2 = c_2^{\frac{1}{1-\gamma}} \left(\frac{\gamma}{1-\gamma}\right)^{\frac{\gamma}{1-\gamma}}$  and then Proposition 9.1 is proved.

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