Smoothing effect and delocalization of interacting Bose-Einstein condensates in random potentials

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We theoretically investigate the physics of interacting Bose-Einstein condensates at equilibrium in a weak (possibly random) potential. We develop a perturbation approach to derive the condensate wave function for an amplitude of the potential smaller than the chemical potential of the condensate and for an arbitrary spatial variation scale of the potential. Applying this theory to disordered potentials, we find in particular that, if the healing length is smaller than the correlation length of the disorder, the condensate assumes a delocalized Thomas-Fermi profile. In the opposite situation where the correlation length is smaller than the healing length, we show that the random potential can be significantly smoothed and, in the mean-field regime, the condensate wave function can remain delocalized, even for very small correlation lengths of the disorder.

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I. INTRODUCTION

Ultracold atomic gases are currently attracting a lot of attention from both experimental and theoretical viewpoints. Taking advantage of the recent progress in cooling and trapping of neutral atoms [1], dilute atomic Bose-Einstein condensates (BECs) [2] and degenerate Fermi gases (DFGs) [3, 4, 5, 6] are now routinely produced at the laboratory. Using various techniques, space-dependent potentials can be designed almost on demand in these systems. For example, one can produce periodic [7, 8], quasi-periodic [9, 10, 11], or random potentials [12, 13, 14, 15, 16, 17] by using optical means. For these reasons and due to unique control and analysis possibilities, ultracold gases constitute a favorite playground for revisiting standard problems of condensed matter physics (CM) [18, 19, 20, 21].

Most current experiments with BECs lie in the mean-field regime where the Bose gas is described by a single wave function, ψ, governed by the (nonlinear) Gross-Pitaevskii equation [22]. Due to the interplay between the kinetic energy term and the interaction term, it is usually difficult to derive the exact solution of this equation. The importance of interactions can be characterized by the *healing length*, ξ, which defines the typical distance below which the spatial variations of ψ significantly contribute to the energy of the BEC, *via* the kinetic energy term [22]. In the Thomas-Fermi regime (TF), i.e. when ξ is significantly smaller than the typical variation scale, $\sigma_{\rm R}$, of the potential, $V(\mathbf{r})$, to which the BEC is subjected, the kinetic term is negligible and the BEC density simply follows the spatial variations of the potential[1]:

$$|\psi(\mathbf{r})|^2 \propto \mu - V(\mathbf{r}) \ . \qquad (1)$$

In the opposite situation ($\xi > \sigma_{\rm R}$), the kinetic term should be taken into account and the exact BEC wave function usually cannot be found analytically.

Besides a general interest, the question of determining the BEC wave function for an arbitrary ratio $\sigma_{\rm R}/\xi$ has direct applications to the case where $V(\mathbf{r})$ is a random potential. The physics of quantum systems in the presence of disorder is central in CM [23, 24, 25], owing to unavoidable defects in 'real-life systems'. One of the major paradigms of disordered quantum systems is due to Anderson who has shown that the eigenstates of single quantum particles in arbitrary weak random potentials can be localized, i.e. ψ shows an exponential decay at large distances[2] [26]. Recent experiments have studied the onset of strong or weak localization effects of light waves [27, 28] and microwaves [29, 30]. Ultracold matter waves are also widely considered as promising candidates to investigate Anderson localization in random [31, 32, 33] or quasi-random structures [10, 31, 34] and more generally to investigate the effects of disorder in various quantum systems (for a recent review, see Ref. [35] and references therein). It is expected that the dramatic versatility of ultracold gases would allow us for a direct comparison with theoretical studies of quantum disordered systems.

A key peculiarity of BECs is that interactions usually cannot be neglected and interaction-induced delocalization can compete with disorder-induced localization effects [15, 16, 17]. Generally, the interplay between the kinetic energy, the interactions and the disorder is still a open question that has motivated many works [36, 37, 38, 39, 40]. It is clear from Eq. (1) that, in the TF regime ($\sigma_{\rm R} \gg \xi$), where the interaction forces the wave function to adapt to the random potential, a BEC will not localize. Indeed, if $V(\mathbf{r})$ is a *homogeneous* random function[3] [41], so is the BEC wave function, ψ, which therefore, cannot decay at large distances. This has been con-

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[1] This is standard in the case of a harmonic confinement, $V(\mathbf{r}) = m\omega^2\mathbf{r}^2/2$. Although, there is no intrinsic typical variation scale, one can define $\sigma_{\rm R}$ as $m\omega^2\sigma_{\rm R}^2/2 = \mu$, i.e. $\sigma_{\rm R} = L_{\rm TF}$, the usual TF half size of the condensate and the validity of the TF regime reads $\xi \ll L_{\rm TF}$. For periodic, quasi-periodic or random potentials, $\sigma_{\rm R}$ is the spatial period or the correlation length (see section III for details).

[2] In 1D and 2D systems, all eigenstates are usually localized while in 3D, they are localized below the so-called *mobility edge*.

[3] In this context, the term 'homogeneous' means that all local statistical properties of the random potential are independent of the position.

firmed in recent experiments [15, 16, 17]. The question thus arises as to understand whether, as a naive transcription of the Ioffe-Regel criterion [42] would suggest, localization can happen when $\sigma_{\rm R} < \xi$.

In this paper, we show that this criterion is actually not sufficient for BECs at equilibrium if the interactions are non-negligible (i.e. if $\xi \ll L$, where L is the size of the system). We indeed show that interaction-induced delocalization still overcomes localization effects even when $\xi \gg \sigma_{\rm R}$. In fact, due to the *smoothing* of the random potential [43], the effect of disorder turns out to be reduced when $\xi/\sigma_{\rm R}$ increases.

In the following, we develop a general formalism based on perturbation theory (see section II) to determine the BEC wave function in any given weak potential, $V(\mathbf{r})$, for an arbitrary ratio $\sigma_{\rm R}/\xi$. We find that the BEC density, $|\psi|^2$, is still given by Eq. (1), except that the potential $V(\mathbf{r})$ has to be replaced by a *smoothed potential*, $\widetilde{V}(\mathbf{r})$. We derive an exact formula for the smoothed potential up to first order in the perturbation series. We then apply our results to the case where $V(\mathbf{r})$ is a 1D homogeneous random potential (see section III) and derive the statistical properties of the smoothed random potential, $\widetilde{V}(\mathbf{r})$. From this, we conclude that an interacting BEC remains delocalized, even for $\xi \gg \sigma_{\rm R}$ (if $\xi \ll L$).

II. SMOOTHING EFFECT IN INTERACTING BOSE-EINSTEIN CONDENSATES

Consider a low-temperature Bose gas in d dimensions with contact atom-atom interactions, $g_{d\rm D}\delta^{(d)}(\mathbf{r})$, where $g_{d\rm D}$ is the d-dimensional interaction parameter. In 3D geometries, $g_{\rm 3D} = 4\pi\hbar^2 a_{\rm sc}/m$, where $a_{\rm sc}$ is the scattering length [22], and m is the atomic mass. Low dimensional geometries (1D or 2D) can be realized in ultracold atomic samples using a tight radial confinement, so that the radial wave function is frozen to zero-point oscillations in the form $\varphi_\perp^0(\mathbf{r}_\perp)$, where $\mathbf{r}_\perp$ is the radial coordinate vector. In this case, $g_{d\rm D} = g_{\rm 3D}\int d\mathbf{r}_\perp\ |\varphi_\perp^0(\mathbf{r}_\perp)|^4$. For instance, one finds $g_{\rm 1D} = 2\hbar\omega_\perp a_{\rm sc}$, for a 2D harmonic radial confinement of frequency $\omega_\perp$. In addition, the Bose gas is assumed to be subjected to a given potential, $V(\mathbf{r})$, with a typical amplitude $V_{\rm R}$ and a typical variation scale $\sigma_{\rm R}$. Possibly, the potential, $V(\mathbf{r})$, may have various length scales. In this case, we assume that $\sigma_{\rm R}$ is the smallest. Assuming weak interactions, i.e. $\overline{n}^{2/d-1} \gg m g_{d\rm D}/\hbar^2$, where $\overline{n}$ the mean density [44, 45], we treat the BEC in the mean-field approach [22] and we use the Gross-Pitaevskii equation (GPE):

$$\mu\psi(\mathbf{r}) = \left[\frac{-\hbar^2\nabla^2}{2m} + V(\mathbf{r}) + g_{d\rm D}|\psi(\mathbf{r})|^2\right]\psi(\mathbf{r})\ , \quad (2)$$

where μ is the BEC chemical potential, and where the wave function, ψ, is normalized to the total number of condensed atoms, $\int d\mathbf{r}\ |\psi(\mathbf{r})|^2 = N$. Note that ψ minimizes the N-body energy functional so that ψ is necessary a real function (up to a non-physical uniform phase). In 1D and 2D geometries and in the absence of trapping, no true BEC can exist due to significant long-wavelength phase fluctuations [46]. In this case, no macroscopic wave function, ψ, can be defined. However, because density fluctuations are strongly suppressed in the presence of interactions, the Bose gas forms a *quasicondensate* [46] and the density, n, can be treated as a classical field. It turns out that $\sqrt{n}$ is governed by Eq. (2). Therefore, even though we only refer to BEC wave functions in the following, our formalism also applies to quasicondensates, after replacing ψ by $\sqrt{n}$.

A. The Thomas-Fermi regime

In the simplest situation, the healing length of the BEC is much smaller than the typical length scale of the potential ($\xi \ll \sigma_{\rm R}$). Therefore, the kinetic energy term in the GPE (2) is small and the BEC density, $|\psi|^2$, simply follows the spatial modulations of the potential:

$$\begin{aligned} &|\psi(\mathbf{r})|^2 = [\mu - V(\mathbf{r})]/g_{d\rm D} \quad &&\text{for } \mu > V(\mathbf{r}) \\ \text{and}\quad &|\psi(\mathbf{r})|^2 = 0 &&\text{otherwise.} \end{aligned} \quad (3)$$

This corresponds to the TF regime. Note that for $V_{\rm R} \ll \mu$, one has

$$\psi(z) \simeq \psi_0 - \frac{V(\mathbf{r})\psi_0}{2\mu}\ , \quad (4)$$

with $\psi_0 = \sqrt{\mu/g_{d\rm D}}$ being the BEC wave function at $V(\mathbf{r}) = 0$. Therefore, the BEC wave function itself follows the modulations of the potential $V(\mathbf{r})$.

B. Beyond the Thomas-Fermi regime: the smoothing effect

The situation changes when the healing length is of the order of, or larger than, the typical length scale of the potential ($\xi > \sigma_{\rm R}$). Indeed, the kinetic contribution limits the smallest variation length of the spatial modulations of a BEC wave function to a finite value of the order of the healing length [22]. Therefore, the BEC can only follow modulations of the potential on a length scale typically larger than ξ and Eq. (3) no longer holds.

For a weak amplitude of the potential[4], we can use perturbation theory techniques. We thus write the BEC wave function as $\psi(\mathbf{r}) = \psi_0 + \delta\psi(\mathbf{r})$ where we assume that $\delta\psi \ll \psi_0$, and ψ_0 is the zeroth-order solution of the GPE (2):

$$\mu\psi_0 = -\frac{\hbar^2}{2m}\nabla^2\psi_0 + g_{d\rm D}\psi_0^3\ . \quad (5)$$

Since the BEC is homogeneous at zeroth-order, one has $\psi_0 = \sqrt{\mu/g_{d\rm D}}$. Then, the first order term of the perturbation series is given by

$$-\frac{\hbar^2}{2m}\nabla^2(\delta\psi) - \left[\mu - 3g_{d\rm D}\psi_0^2\right]\delta\psi = -V(\mathbf{r})\psi_0\ . \quad (6)$$

[4] A precise condition for the validity of the perturbative approach will be given later [see Eq. (16)].

Since $\mu - 3g_{d\mathrm{D}}\psi_0^2 = -2\mu$, we are left with the equation

$$-\frac{\xi^2}{2}\nabla^2(\delta\psi) + \delta\psi = -\frac{V(\mathbf{r})\psi_0}{2\mu} \,, \tag{7}$$

where $\xi = \hbar/\sqrt{2m\mu}$ is the healing length of the BEC. We straightforwardly find the solution of Eq. (7), which reads

$$\delta\psi(\mathbf{r}) = -\int \mathrm{d}\mathbf{r}'\, G(\mathbf{r}-\mathbf{r}')\frac{V(\mathbf{r}')\psi_0}{2\mu} \,, \tag{8}$$

where $G(\mathbf{r})$ is the Green function of Eq. (7), defined as the solution of

$$\left[-\frac{\xi^2}{2}\nabla^2 + 1\right] G(\mathbf{r}) = \delta^{(d)}(\mathbf{r}) \,, \tag{9}$$

or equivalently, in Fourier space

$$\left[\frac{\xi^2}{2}|\mathbf{k}|^2 + 1\right] \widehat{G}(\mathbf{k}) = 1/(2\pi)^{d/2} \,, \tag{10}$$

where, $\widehat{G}(\mathbf{k}) = \frac{1}{(2\pi)^{d/2}} \int \mathrm{d}\mathbf{r}\; G(\mathbf{r})\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}$ is the Fourier transform of G. In contrast to the case of single particles, the Green function, $\widehat{G}(\mathbf{k})$, has no singularity point so that the perturbative approach can be safely applied for any wave vector $\mathbf{k}$.

The explicit formula for the Green function, G, depends on the dimension of the system. After some simple algebra, we find

$$\text{in 1D,} \qquad G(z) = \frac{1}{\sqrt{2}\xi}\exp\left(-\frac{|z|}{\xi/\sqrt{2}}\right) \tag{11}$$

$$\text{in 2D,} \qquad G(\rho) = \frac{1}{\pi\xi^2}\mathrm{K}_0\left(+\frac{|\rho|}{\xi/\sqrt{2}}\right) \tag{12}$$

$$\text{in 3D,} \qquad G(\mathbf{r}) = \frac{1}{2\pi\xi^2|\mathbf{r}|}\exp\left(-\frac{|\mathbf{r}|}{\xi/\sqrt{2}}\right) \,. \tag{13}$$

where K_0 is the modified Bessel function. Finally, up to first order in the perturbation series, the BEC wave function reads

$$\psi(\mathbf{r}) \simeq \psi_0 - \frac{\widetilde{V}(\mathbf{r})\psi_0}{2\mu} \tag{14}$$

with

$$\widetilde{V}(\mathbf{r}) = \int \mathrm{d}\mathbf{r}'\, G(\mathbf{r}')V(\mathbf{r}-\mathbf{r}') \,. \tag{15}$$

Interestingly enough, the Green function in any dimension shows a exponential decay, with a typical attenuation length, ξ, and is normalized to unity[5], $\int \mathrm{d}\mathbf{r}\; G(\mathbf{r}) = 1$. Therefore, $G(\mathbf{r})$ can be seen as a *smoothing function* with a typical width ξ. Indeed, it should be noted that Eq. (14) is similar to Eq. (4), except that the potential $V(\mathbf{r})$ which is relevant in the case $\xi \ll \sigma_{\mathrm{R}}$, changes to the potential $\widetilde{V}(\mathbf{r})$ for $\xi > \sigma_{\mathrm{R}}$. The potential $\widetilde{V}(\mathbf{r})$ is a convolution of $V(\mathbf{r})$ with a function which has a typical width ξ and thus corresponds to a *smoothed potential* with an amplitude smaller than V_{R}. In addition, if σ_{R} corresponds to the width of the correlation function of a random potential, V, the correlation length of the smoothed random potential, $\widetilde{V}$, is of the order of $\max(\sigma_{\mathrm{R}},\xi)$ [for details, see section III][6].

Note that, for $\xi \ll \sigma_{\mathrm{R}}$, $G(\mathbf{r})$ can be approximated by $\delta^{(d)}(\mathbf{r})$ in Eq. (15), and $\widetilde{V}(\mathbf{r}) \simeq V(\mathbf{r})$. We thus recover the results of section II A, valid for the TF regime.

The validity condition of the perturbation approach directly follows from Eq. (14):

$$\widetilde{V}(\mathbf{r}) \ll \mu \,. \tag{16}$$

Note that if $\xi \gg \sigma_{\mathrm{R}}$, the potential can be significantly smoothed so that the above condition can be less restrictive than the *a priori condition*, $V(\mathbf{r}) \ll \mu$.

The results of this section show that the potential, $V(\mathbf{r})$, can be significantly smoothed in interacting BECs. We stress that this applies to any kind of potentials provided that $\xi \ll L$ and $\widetilde{V}(\mathbf{r}) \ll \mu$. In the next section, we present an illustration of the smoothing effect in the case of a random potential.

III. APPLICATION TO A TRAPPED INTERACTING BOSE-EINSTEIN CONDENSATE IN A 1D RANDOM POTENTIAL

A. Trapped 1D Bose-Einstein condensate in a random potential

In this section, we consider a 1D Bose gas subjected to a weak homogeneous random potential, $V(z)$, with a vanishing average value ($\langle V\rangle = 0$), a standard deviation, V_{R}, and a spatial correlation length, σ_{R}, significantly smaller than the size of the system. In addition, we assume that the gas is trapped in a confining harmonic trap[7], $V_{\mathrm{h}}(z) = m\omega^2 z^2/2$ as in almost all current experiments on disordered BECs [14, 15, 16, 17]. We consider a situation such that $\hbar\omega \ll ng_{1\mathrm{D}} \ll \hbar^2n^2/m$, i.e. the Bose gas lies in the mean-field regime, and in the absence of disorder, the interactions dominate over the kinetic energy[8]. The situation mimics the experimental conditions of

[5] This property follows directly from the definition (10) of the Green function. Indeed, $\int \mathrm{d}\mathbf{r}\; G(\mathbf{r}) = (2\pi)^{d/2}\widehat{G}(\mathbf{k}=0) = 1$.

[6] In contrast, for example in the case of a deterministic periodic potential, $V(z) = V_{\mathrm{R}}\cos(kz)$, the variation scale, $\sigma_{\mathrm{R}} = 2\pi/k$, corresponds to the period of the potential, and we find $\widetilde{V}(z) = \frac{V_{\mathrm{R}}\cos(kz)}{1+k^2\xi^2}$. The potential is indeed smoothed as the amplitude of $\widetilde{V}$ is smaller than that of V. Nevertheless, the period of the smoothed potential, $\widetilde{V}$, is the same as that of the bare potential, V.

[7] All results also apply if there is no trapping. In this case, all zeroth-order terms simply do not depend on z.

[8] This corresponds to the usual TF regime for confined BECs in the absence of disorder [22]. However, no restriction is imposed for the ratio σ_{R}/ξ, so that the BEC can be out of the TF regime as defined in section I.

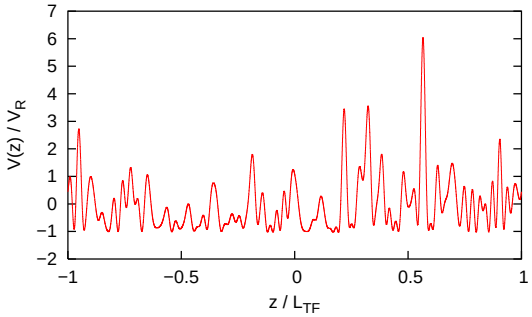

Figure 1: (color online) Example of the realization of a speckle random potential with $\sigma_{\rm R} \simeq 10^{-2} L_{\rm TF}$.

Ref. [15, 17]. The presence of the harmonic confinement introduces a low-momentum cut-off for the phase fluctuations so that the 1D Bose gas forms a true condensate at low temperatures [44, 45]. In this case, the BEC wave function is

$$\psi_0 = \sqrt{\mu_0(z)/g_{\rm 1D}}\,, \tag{17}$$

where $\mu_0(z) = \mu - m\omega^2 z^2/2$ is the local chemical potential. This corresponds to an inverted parabolic density profile with a half-size, $L_{\rm TF} = \sqrt{2\mu/m\omega^2}$, where the chemical potential is $\mu = \mu_{\rm TF} = \frac{\hbar\omega}{2}\left(\frac{3Nmg_{\rm 1D}l}{2\hbar^2}\right)^{2/3}$, with $l = \sqrt{\hbar/m\omega}$ being the extension of the ground state of the harmonic oscillator.

As $L_{\rm TF} \gg (\xi, \sigma_{\rm R})$, it is legitimate to use the local density approximation (LDA) [22], i.e. in a region significantly smaller than $L_{\rm TF}$, the quantities ψ_0 and μ_0 can be considered as uniform. We can thus directly apply the results of section II B. From Eqs. (14)-(17), we immediately find that

$$n(z) \simeq n_0(z) - \frac{\widetilde{V}(z)}{g_{\rm 1D}}\,, \tag{18}$$

where

$$\widetilde{V}(z) = \int {\rm d}z'\, \frac{\exp\left(\frac{-|z'|}{\xi_0(z)/\sqrt{2}}\right)}{\sqrt{2}\xi_0(z)} V(z-z')\,, \tag{19}$$

is the smoothed potential, with $\xi_0(z) = \hbar/\sqrt{2m\mu_0(z)}$ being the local healing length. The density profile of the BEC is thus expected to follow the modulations of a *smoothed random potential*.

Note that the total number of condensed atoms is $N = \int {\rm d}z\, |\sqrt{n_0(z)} + \delta\psi|^2 \simeq \int {\rm d}z\, \left(n_0(z) - \widetilde{V}(z)/g_{\rm 1D}\right)$ up to first order in $\widetilde{V}/\mu$. Since $\langle \widetilde{V} \rangle = 0$, one has $\langle N \rangle \simeq \int {\rm d}z\, n_0(z)$, owing to the assumed self-averaging property of the potential [41]. In addition, we have $\mu = \mu_{\rm TF}$.

We now compare our predictions to the exact solutions of the GPE (2) as obtained numerically. For the sake of concreteness, we consider a speckle random potential [47] similar to the one used in the recent experiments [14, 15, 16, 17] (see Fig. 1). Briefly, a speckle pattern consists in a random intensity distribution and is characterized by its statistical properties. First, the single-point amplitude distribution is a negative

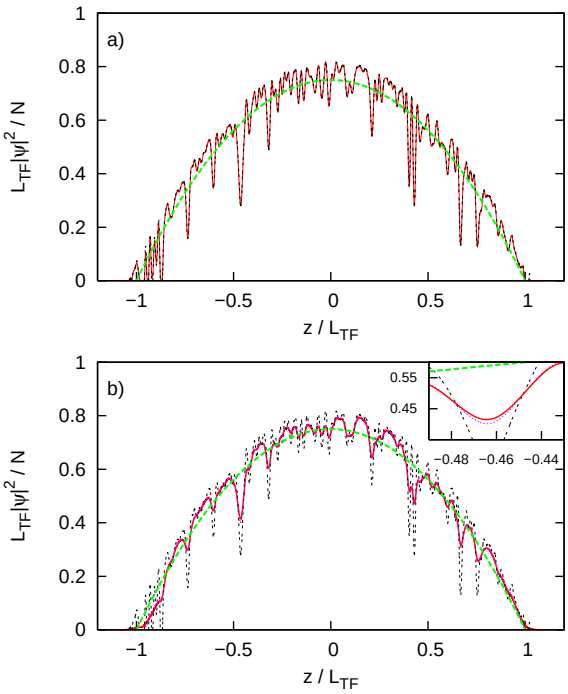

Figure 2: (color online) Density profiles of a BEC confined in a combined harmonic plus random potential ($V_{\rm R} = 0.1\mu$, $\sigma_{\rm R} = 7.5 \times 10^{-3} L_{\rm TF}$). The solid (red online) line corresponds to the numerically computed BEC wave function; the dashed (green online) line is the TF profile in the absence of disorder; and the black dotted line is a plot of the disordered TF profile [Eq. (3)]. a) Case where the healing length at the trap center, ξ, is smaller than the correlation length of the random potential: $\sigma_{\rm R}/\xi \simeq 10$. In this case, the density profile follows the modulations of the random potential according to Eq. (3). b) Opposite situation: $\sigma_{\rm R}/\xi \simeq 0.5$. In this case, the BEC density profile, obtained numerically, significantly differs from Eq. (3), but can hardly be distinguished from Eq. (18) [also plotted in Fig. 2b) as a dotted (purple online) line]. The inset shows a magnification of the plot in a very small region of the BEC.

exponential

$$\mathcal{P}[V(z)] = \frac{\exp[-(V(z)+V_{\rm R})/V_{\rm R}]}{|V_{\rm R}|} \quad \text{if } \frac{V(z)}{V_{\rm R}} \geq -1$$
$$\text{and} \quad \mathcal{P}[V(z)] = 0 \quad \text{otherwise}\,, \tag{20}$$

corresponding to the average value $\langle V \rangle = 0$ and the standard deviation $\Delta V = \sqrt{\langle [V(z) - \langle V \rangle]^2 \rangle} = |V_{\rm R}|$. Second, the spatial correlations are characterized by the autocorrelation function $C(\Delta z) = \langle V(\Delta z) V(0) \rangle$ which correlation length is denoted $\sigma_{\rm R}$ and can be chosen at will [17, 47]. For the numerical calculations, we numerically generate a 1D speckle pattern using a method similar to the one described in Ref. [48] in 1D and corresponding to the correlation function

$$C(\Delta z) = V_{\rm R}^2 \left|{\rm sinc}(\sqrt{3}\Delta z/\sqrt{2}\sigma_{\rm R})\right|^2\,, \tag{21}$$

where ${\rm sinc}(x) = \sin(x)/x$. For the sake of simplicity, it is useful to approximate $C(z)$ to a Gaussian function (see for

example section III B). Up to second order in $\Delta z/\sigma_{\rm R}$, we have $C(\Delta z) \simeq V_{\rm R}^2 \exp(-\Delta z^2/2\sigma_{\rm R}^2)$.

Numerical solutions of the GPE (2) are presented in Fig. 2 for two values of the ratio $\sigma_{\rm R}/\xi$, where ξ is the BEC healing length at the trap center. In the first case (Fig. 2a), we have $\xi \ll \sigma_{\rm R}$, and the density simply follows the modulations of the bare random potential, according to Eq. (3). In the second case (Fig. 2b), we have $\xi > \sigma_{\rm R}$, and as expected, the BEC wave function does not follow the modulations of the bare random potential $V(z)$ but actually follows smoother modulations of the smoothed potential $\widetilde{V}(z)$. Figure 2b (and the inset) shows that the numerically computed density can hardly be distinguished from Eq. (18). This supports the validity of our approach.

B. Statistical properties of the smoothed random potential

It is useful to compute the statistical properties of the smoothed random potential $\widetilde{V}(z)$ as they will be imprinted on the BEC density profile according to Eq. (18). From Eq. (15), we immediately find that, (i) $\widetilde{V}(z)$ is a random *homogeneous* potential, (ii) the average value of $\widetilde{V}$ vanishes,

$$\langle \widetilde{V} \rangle = \langle V \rangle = 0 \, , \tag{22}$$

and (iii) the correlation function of $\widetilde{V}$ is given by

$$\widetilde{C}_z(\Delta z) = \int {\rm d}u{\rm d}v \; C[\Delta z + (v-u)] G_z(u) G_{z+\Delta z}(v) \, , \tag{23}$$

where $C(\Delta z) = \langle V(\Delta z) V(0) \rangle$ is the correlation function of the bare potential V and $G_z(u)$ is given by Eq. (11) with ξ replaced by $\xi_0(z)$. In the following, we assume that $\Delta z \ll L_{\rm TF}$ so that $G_z \simeq G_{z+\Delta z}$ and we omit the subscripts. Assuming for simplicity a Gaussian correlation function for the bare random potential, $C(\Delta z) \simeq V_{\rm R}^2 \exp(-\Delta z^2/2\sigma_{\rm R}^2)$, we find after some algebra

$$\widetilde{C}(\Delta z) = V_{\rm R}^2 \, \Sigma \left(\frac{\sigma_{\rm R}}{\xi_0}, \frac{\Delta z}{\xi_0} \right) \, , \tag{24}$$

with

$$\begin{aligned} \Sigma(\overline{\sigma}_{\rm R}, \overline{\Delta z}) \; = \; & \overline{\sigma}_{\rm R}^2 \exp\left(-\frac{\overline{\Delta z}^2}{2\overline{\sigma}_{\rm R}^2} \right) \\ & + \frac{\sqrt{\pi}}{4} \overline{\sigma}_{\rm R} \left(1 - 2\overline{\sigma}_{\rm R}^2 - \sqrt{2}\, \overline{\Delta z} \right) \exp\left(\overline{\sigma}_{\rm R}^2 + \sqrt{2}\, \overline{\Delta z} \right) {\rm erfc}\left(\frac{2\overline{\sigma}_{\rm R}^2 + \sqrt{2}\, \overline{\Delta z}}{2\overline{\sigma}_{\rm R}} \right) \\ & + \frac{\sqrt{\pi}}{4} \overline{\sigma}_{\rm R} \left(1 - 2\overline{\sigma}_{\rm R}^2 + \sqrt{2}\, \overline{\Delta z} \right) \exp\left(\overline{\sigma}_{\rm R}^2 - \sqrt{2}\, \overline{\Delta z} \right) {\rm erfc}\left(\frac{2\overline{\sigma}_{\rm R}^2 - \sqrt{2}\, \overline{\Delta z}}{2\overline{\sigma}_{\rm R}} \right) \end{aligned} \tag{25}$$

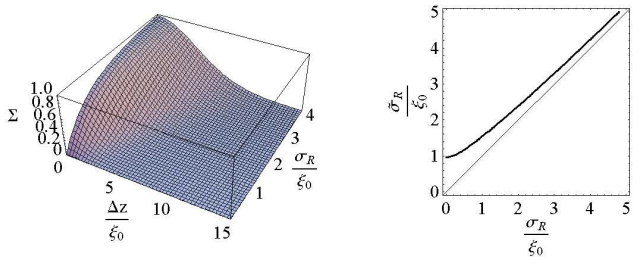

Figure 3: (color online) Left: Plot of the correlation function $\Sigma\left(\frac{\sigma_{\rm R}}{\xi_0}, \frac{\Delta z}{\xi_0}\right)$. Right: Width at $1/\sqrt{e}$ of the normalized correlation function $\Sigma\left(\frac{\sigma_{\rm R}}{\xi_0}, \frac{\Delta z}{\xi_0}\right) / \Sigma\left(\frac{\sigma_{\rm R}}{\xi_0}, 0\right)$.

where $\overline{\sigma}_{\rm R} = \sigma_{\rm R}/\xi_0$, $\overline{\Delta z} = \Delta z/\xi_0$ and ${\rm erfc}\,(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty {\rm d}t \; e^{-t^2}$ is the complementary error function. The correlation function $\Sigma\left(\frac{\sigma_{\rm R}}{\xi_0}, \frac{\Delta z}{\xi_0}\right)$ is plotted in Fig. 3.

This function Σ clearly decreases with $\sigma_{\rm R}/\xi_0$, indicating the onset of an increasing smoothing effect. At $\Delta z = 0$, we have a simple asymptotic expression for $\sigma_{\rm R} \gg \xi_0$:

$$\Sigma(\sigma_{\rm R}/\xi_0, 0) \simeq 1 - \left(\frac{\xi_0}{\sigma_{\rm R}} \right)^2 , \;\; \sigma_{\rm R} \gg \xi_0 \, . \tag{26}$$

So, as expected, $\Sigma(\sigma_{\rm R}/\xi_0, 0) \to 1$ as $\sigma_{\rm R}/\xi_0 \to \infty$, i.e. the random potential is hardly smoothed. For $\sigma_{\rm R} \ll \xi_0$:

$$\Sigma(\sigma_{\rm R}/\xi_0, 0) \simeq \frac{\sqrt{\pi}}{2} \frac{\sigma_{\rm R}}{\xi_0}, \;\; \sigma_{\rm R} \ll \xi_0 \, . \tag{27}$$

So, $\Sigma(\sigma_{\rm R}/\xi_0, 0) \to 0$ as $\sigma_{\rm R}/\xi_0 \to 0$, i.e. the amplitude of the smoothed random potential is significantly reduced compared to the amplitude of the bare random potential. Generally speaking, from Eq. (24), we have $\langle \widetilde{V}^2 \rangle = \widetilde{C}(0) = V_{\rm R}^2 \Sigma(\sigma_{\rm R}/\xi_0, 0)$. It follows that $\langle \widetilde{V}^2 \rangle$ is an increasing function of $\sigma_{\rm R}/\xi_0$ and that $\langle \widetilde{V}^2 \rangle \leq V_{\rm R}^2$. This is consistent with the idea of a *smoothing* of the random potential.

In addition, the correlation length, $\widetilde{\sigma}_{\rm R}$, of the smoothed random potential, $\widetilde{V}$, is given by the width at $1/\sqrt{e}$ of the function $\Delta z \to \Sigma(\overline{\sigma}_{\rm R}/\xi_0, \Delta z/\xi_0)$. At $\sigma_{\rm R} \gg \xi_0$, the smoothing is weak and $\widetilde{\sigma}_{\rm R} \simeq \sigma_{\rm R}$. For $\sigma_{\rm R} < \xi_0$, the smoothing is significant, so that $\widetilde{\sigma}_{\rm R}$ saturates at $\widetilde{\sigma}_{\rm R} \simeq \xi_0$, as expected. Roughly speaking, we have $\widetilde{\sigma}_{\rm R} \sim \max(\sigma_{\rm R}, \xi_0)$ [see Fig. 3].

C. Effect of disorder in interacting Bose-Einstein condensates

We finally discuss the properties of the BEC wave function in the presence of disorder. It follows from Eq. (18) that the BEC density follows the modulations of a random potential $\widetilde{V}$. In the TF regime ($\xi \ll \sigma_{\rm R}$), $\widetilde{V} \simeq V$, while when $\xi > \sigma_{\rm R}$, $\widetilde{V}$ is smoothed. Since $\widetilde{V}$ is a homogeneous random potential, there is no decay of the wave function. In particular, Anderson localization does not occur, even for $\xi \gg \sigma_{\rm R}$. In the case when $\xi > \sigma_{\rm R}$, it turns out that the BEC density is actually less affected by the random potential than in the TF regime ($\xi \ll \sigma_{\rm R}$). This is in striking contrast with the case of non-interacting particles where localization effects are usually stronger at low energy [41].

More quantitatively, using the statistical properties of the smoothed random potential, $\widetilde{V}$, one can easily compute the fluctuations $\Delta n(z) = \sqrt{\langle [n(z) - n_0(z)]^2 \rangle}$ of the BEC density around the average value $n_0(z) = [\mu - m\omega^2 z^2/2]/g_{\rm 1D}$. From Eq. (18), we find $\Delta n^2 \simeq \widetilde{C}(0)/g_{\rm 1D}^2$. Note that Δn^2 depends on the displacement from the trap center through the dependence of ξ_0 on z. At the trap center, we find

$$\Delta n_{\rm c} = \frac{V_{\rm R}}{g_{\rm 1D}} \sqrt{\Sigma(\sigma_{\rm R}/\xi, 0)} \, . \tag{28}$$

We recall that $\xi = \xi_0(0) = \hbar/\sqrt{2m\mu}$ is the BEC healing length in the trap center.

We have numerically extracted the fluctuations of the density in the trap center, according to the formula $\Delta n_{\rm c} \simeq \sqrt{\frac{1}{L_{\rm TF}/2} \int_{-L_{\rm TF}/4}^{+L_{\rm TF}/4} {\rm d}z \; [n(z) - n_0(z)]^2}$. This provides a good estimate of $\Delta n_{\rm c}$ as $\xi_0(z)$ changes by less than 3% in the range $[-L_{\rm TF}/4, +L_{\rm TF}/4]$. As shown in Fig. 4, the numerical results perfectly agree with Eq. (28) over a large range of the ratio $\xi/\sigma_{\rm R}$. The numerical calculations are performed for the speckle potential described in section III B and no fitting parameter has been used. In addition, note that we have used a single realization of the random potential for each point in Fig. 4. Averaging over disorder turned out to have little importance, since the random potential is almost self-averaging in the range $[-L_{\rm TF}/4, +L_{\rm TF}/4]$, if $\sigma_{\rm R} \ll L_{\rm TF}$.

Finally, we find from Eq. (16) that the perturbative approach that we have performed is valid whenever $\Delta n \ll n_0$, *i.e.* whenever

$$V_{\rm R} \sqrt{\Sigma(\sigma_{\rm R}/\xi_0, 0)} \ll \mu \, . \tag{29}$$

Note that this effect is more restrictive in the trap center where ξ_0 is minimum.

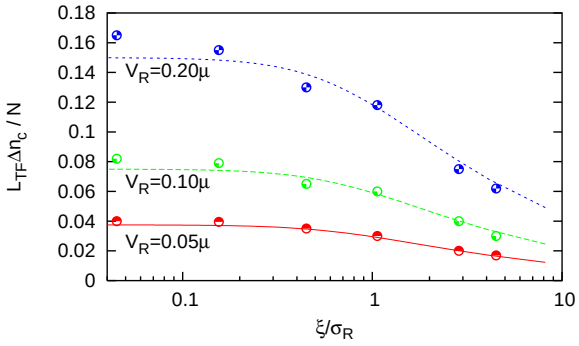

Figure 4: (color online) Amplitude of the fluctuations of the BEC density at the trap center, $\Delta n_{\rm c}$, in the combined harmonic plus random potential as a function of the ratio of the healing length to the correlation length of the disorder for several amplitudes of the random potential. The dots correspond to exact numerical results in the Gross-Pitaevskii approach [Eq. (2)] and the lines show the theoretical prediction [Eq. (28)].

IV. CONCLUSION

In summary, we have presented an analytical technique, based on the perturbation theory, to compute the static wave function of an interacting BEC subjected to a weak potential. This applies to the case where both the healing length of the BEC (ξ) and the spatial scale of the potential ($\sigma_{\rm R}$) are much smaller than the size of the system (L), but whithout restriction for the ratio $\xi/\sigma_{\rm R}$. In particular, we have shown that when the healing length is larger than the space scale of the potential, the BEC is sensitive to a *smoothed potential* which can be determined within our framework.

Applying these results to the case of a 1D random potential, we have shown that the wave function of a static interacting BEC is delocalized, similarly as in the TF regime [15]. This is confirmed by numerical calculations. The results of this analysis show that, for an interacting BEC at equilibrium, the larger the healing length, the smaller the perturbation induced by the disorder. It is worth noting that the conclusions of the present work hold for *static* BECs in the *mean-field regime* and when the interaction energy dominates over the kinetic energy in the absence of disorder, i.e. when the healing length is significantly smaller than the BEC half size ($\xi \ll L$). Going beyond the mean-field regime, it is interesting to study the interplay of interactions, disorder and kinetic energy in a Bose gas for interactions ranging from zero (where localization is expected) to the TF regime (where the BEC is delocalized as shown in this work). This question is addressed in Ref. [49].

Finally, we note that the transport properties of a BEC can show significantly different physics. For instance, localization has been studied in matter-wave beams [33] and in the expansion of an interacting BEC [15, 16, 17]. In the latter two cases, localization indeed does occur although non-negligible interactions can modify the usual picture of localization [15, 17, 33].

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