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# RESONANT EIGENSTATES FOR A QUANTIZED CHAOTIC SYSTEM

STÉPHANE NONNENMACHER AND MATHIEU RUBIN

ABSTRACT. We study the resonant eigenstates of the quantized “open baker’s map”, a simple model for quantum chaotic scattering, in the semiclassical limit. We first investigate the fractal dimension appearing in the Fractal Weyl law for the density of resonances, showing that it is not related with the “information dimension”. In a second step, we consider the *semiclassical measures* associated with sequences of resonant eigenstates. We show that these measures are conditionally invariant with respect to the classical dynamics, and generally exhibit interesting rich fractal structures.

## 1. INTRODUCTION

1.1. **Quantum cattering on  $\mathbb{R}^d$  and resonances.** In a typical scattering system, particles of positive energy come from infinity, interact with a localized potential  $V(q)$ , and then leave to infinity. The corresponding quantum Hamiltonian  $\hat{H} = -\hbar^2\Delta + V(q)$  has an absolutely continuous spectrum on the positive axis. However, the Green’s function  $G(z; q', q) = \langle q' | (\hat{H} - z)^{-1} | q \rangle$  admits a meromorphic continuation from the upper half plane  $\{\Im z > 0\}$  to (some part of) the lower half-plane  $\{\Im z < 0\}$ . This continuation generally has *poles*  $z_j = E_j - i\Gamma_j/2$ ,  $\Gamma_j > 0$ , which are called *resonances* of the scattering system.

The probability density of the corresponding “eigenfunction”  $\varphi_j(q)$  decays in time like  $e^{-t\Gamma_j/\hbar}$ , so physically  $\varphi_j$  represents a metastable state with decay rate  $\Gamma_j/\hbar$ , or lifetime  $\tau_j = \hbar/\Gamma_j$ . In the semiclassical limit  $\hbar \rightarrow 0$ , we will call “long-living” the resonances  $z_j$  such that  $\Gamma_j = \mathcal{O}(\hbar)$ , equivalently with lifetimes bounded away from zero.

The eigenfunction  $\varphi_j(q)$  is meaningful only near the interaction region, while its behaviour outside that region (exponentially increasing outgoing waves) is clearly unphysical. As a result, one practical method to compute resonances (at least approximately) consists in adding a *smooth absorbing potential*  $-iW(q)$  to the Hamiltonian  $\hat{H}$ , thereby obtaining a nonselfadjoint operator  $\hat{H}_W = \hat{H} - iW(q)$ . The potential  $W(q)$  is supposed to vanish in the interaction region, but is positive outside: its effect is to *absorb* outgoing waves, as opposed to a real positive potential which would reflect the waves back into the interaction region. Equivalently, the (nonunitary) propagator  $e^{-i\hat{H}_W/\hbar}$  kills wavepackets localized outside the interaction region.

The spectrum of  $\hat{H}_W$  in some neighbourhood of the positive axis is then made of discrete resonances  $\tilde{z}_j$  associated with square-integrable eigenfunctions  $\tilde{\varphi}_j$ . This method has been widely used in quantum chemistry in order to study reaction or dissociation dynamics [17, 33]; in those works it is implicitly assumed that eigenvalues  $\tilde{z}_j$  close to the real axis are

small perturbations of the resonances  $z_j$ , and that the corresponding eigenfunctions  $\varphi_j(q)$ ,  $\tilde{\varphi}_j(q)$  are close to one another in the interaction region. Very close to the real axis (namely, for  $|\Im \tilde{z}_j| = \mathcal{O}(\hbar^n)$  with  $n$  sufficiently large), Stefanov proved that the eigenvalues  $\tilde{z}_j$  are close to the resonances  $z_j$  of  $\hat{H}$  [36]. Such very long-living resonances are possible when the classical dynamics admits a trapped region of positive (Liouville) volume. In that case, resonances and the associated eigenfunctions can be approximated by quasimodes of an associated closed system [37].

**1.2. Resonances in chaotic scattering.** We will be interested in a different situation, where the set of trapped trajectories has volume zero, and is a fractal hyperbolic repeller. This case encompasses the famous 3-disk scatterer in 2 dimensions [8], or its smoothing, namely the 3-bump potential introduced in [34] and numerically studied in [16]. Resonances then lie deeper below the real line (typically,  $\Gamma_j \gtrsim \hbar$ ), and are not perturbations of a real spectrum. Previous studies have focussed in counting the number of resonances in small disks around the energy  $E$ , in the semiclassical régime, resulting in the following conjectured Weyl-type law:

$$(1.1) \quad \# \left\{ z_j \in \text{Res}(\hat{H}) : |E - z_j| \leq \gamma \hbar \right\} \sim C(\gamma) \hbar^{-d}.$$

Here the exponent  $d$  is related to the trapped set at energy  $E$ : the latter has (Minkowski) dimension  $2d+1$ . This asymptotics could be numerically checked for a variety of hyperbolic systems [16, 20, 10, 31, 23], but only upper bounds for the number of resonances could be rigorously proven [34, 42, 10, 35]. In §3.2 we numerically check the equivalent of this fractal law for an asymmetric version of the open baker’s map; apart from extending the results of [23], it allows to specify more precisely the dimension  $d$  appearing in the scaling law. To our knowledge, so far the only system for which the asymptotics corresponding to (1.1) could be explicitly computed is the “Walsh quantization” of the symmetric open baker’s map [23, 24], which is described in §5.

Beyond the Weyl law, the next step consists in studying the long-living resonant eigenstates  $\varphi_j$  or  $\tilde{\varphi}_j$ . Some rigorous results on this matter are announced in [25], while interesting numerics were performed by M. Lebental and coworkers for a model of open stadium billiard, relevant to describe an experimental micro-laser cavity [15]. To avoid the complications of “realistic” scattering systems, we focus on a simple model, namely the quantized open baker’s map studied in [24, 23, 13]. Such a model is meant to mimick the propagator of the nonselfadjoint Hamiltonian  $\hat{H}_W$ , in the case where the classical flow at energy  $E$  is chaotic in the interacting region. The classical open baker’s map lives on the 2-dimensional torus phase space (see §2.1), it is a highly chaotic (Bernoulli) system, and its *eigenmeasures* can be explicitly described (§2.3). The associated quantum map is a finite dimensional matrix, the eigenstates of which can be computed through a straightforward diagonalisation. The phase space distribution of the eigenstates will be characterized using Husimi measures (see §4.1). We will focus on the semiclassical limits of these Husimi measures, which are called semiclassical measures. We prove that (up to some subtleties due to discontinuities) any limit measure is necessarily an eigenmeasure of the classical

map, associated with a certain decay rate (Theorem 1). The characterization of semiclassical measures among all possible eigenmeasures remains an open problem. Inspired from the case of closed chaotic systems, in §4.3 we ask a few questions about the “unicity” of semiclassical measures; these questions remain open.

In a final attempt, we consider in §5 the Walsh quantization of the symmetric baker’s map: in that case, we can construct some explicit sequences of eigenstates, which converge to specific eigenmeasures. In that (nongeneric) framework, we can partially answer the above questions.

Let us mention that the eigenstates of the quantized open baker have been studied in parallel by J. Keating and coworkers, for both the “standard” [13] and “Walsh” quantization. Our theorem 1 provides a rigorous version of statements contained in their work.

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## 2. THE OPEN BAKER’S MAP

**2.1. Definition and symbolic dynamics.** Consider the phase space consisting in the 2-dimensional torus, or square  $\mathbb{T}^2 \simeq [0, 1) \times [0, 1)$ . A point on  $\mathbb{T}^2$  is described with the coordinates  $x = (q, p)$ , which we call respectively position (horizontal) and momentum (vertical), to insist on the symplectic structure  $dq \wedge dp$  on  $\mathbb{T}^2$ . We partition  $\mathbb{T}^2$  into three

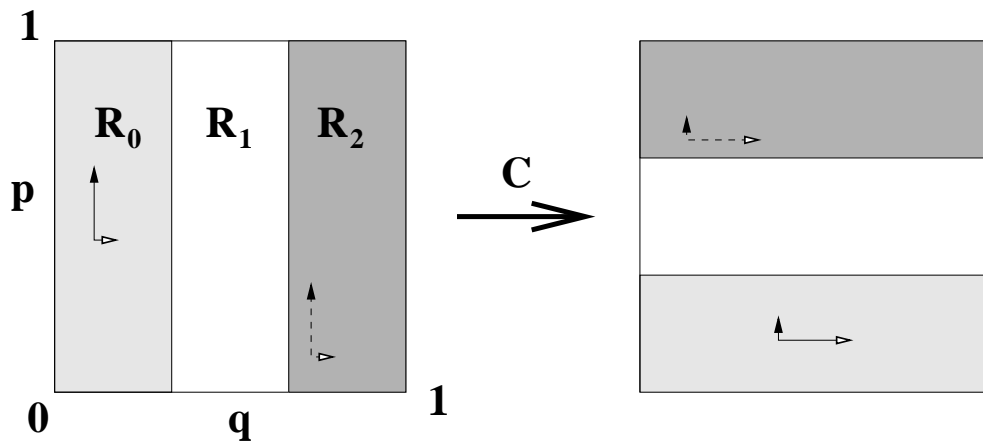


FIGURE 1. Sketch of the closed baker’s map  $A_{\mathbf{r}}$ , and its open counterpart  $B_{\mathbf{r}}$ , for the case  $\mathbf{r} = \mathbf{r}_{sym} = \{1/3, 1/3, 1/3\}$ . The three rectangles form a Markov partition.

vertical rectangles  $R_i$  with widths  $r_i$ , such that  $r_0 + r_1 + r_2 = 1$ , and first construct the following (closed) baker's map on  $\mathbb{T}^2$  (see Fig. 1):

$$(2.1) \quad (q, p) \mapsto A_{\mathbf{r}}(q, p) \stackrel{\text{def}}{=} (q', p') = \begin{cases} \left(\frac{q}{r_0}, p r_0\right) & \text{if } 0 \leq q < r_0 \\ \left(\frac{q-r_0}{r_1}, p r_1 + r_0\right) & \text{if } r_0 \leq q < r_0 + r_1 \\ \left(\frac{q-r_0-r_1}{r_2}, p r_2 + r_1 + r_0\right) & \text{if } r_0 + r_1 \leq q < 1 \end{cases} .$$

This baker's map is invertible and area-preserving on  $\mathbb{T}^2$ . It is discontinuous on the boundaries of the three rectangles  $R_i$ . This map is the archetype of a Bernoulli system: it can be put in correspondence with the right shift  $\sigma$  on the *symbolic space*  $\Sigma = \{0, 1, 2\}^{\mathbb{Z}}$ . Namely, to each bi-infinite sequence  $\epsilon = \dots \epsilon_{-2} \epsilon_{-1} \cdot \epsilon_0 \epsilon_1 \dots \in \Sigma$  corresponds the point  $x = J_{\mathbf{r}}(\epsilon) \in \mathbb{T}^2$ , which is defined as follows: the coordinates of  $x = (q, p)$  are given by

$$(2.2) \quad q(\epsilon) = \sum_{k=0}^{\infty} r_{\epsilon_0} r_{\epsilon_1} \cdots r_{\epsilon_{k-1}} \alpha_{\epsilon_k}, \quad p(\epsilon) = \sum_{k=1}^{\infty} r_{\epsilon_{-1}} r_{\epsilon_{-2}} \cdots r_{\epsilon_{-k+1}} \alpha_{\epsilon_{-k}},$$

where we have set  $\alpha_0 = 0$ ,  $\alpha_1 = r_0$ ,  $\alpha_2 = r_0 + r_1$ . The position coordinate (unstable direction) depends on symbols on the right of the dot, while the momentum coordinate (stable direction) depends on symbols on the left. In the symmetric case  $r_0 = r_1 = r_2 = 1/3$ , which we denote by  $\mathbf{r} = \mathbf{r}_{sym}$ , this mapping amounts to the ternary decomposition of  $q$  and  $p$ . The map  $J_{\mathbf{r}} : \Sigma \rightarrow \mathbb{T}^2$  described in (2.2) is surjective, “almost injective”, and conjugates  $A_{\mathbf{r}}$  with the shift:

$$(2.3) \quad A_{\mathbf{r}}(J_{\mathbf{r}}(\epsilon)) = J_{\mathbf{r}}(\sigma(\epsilon)) = J_{\mathbf{r}}(\dots \epsilon_{-1} \epsilon_0 \cdot \epsilon_1 \dots) .$$

For this reason,  $J_{\mathbf{r}}$  is often called a semiconjugacy between the two dynamical systems. This equation shows that for each time  $n \in \mathbb{Z}$ , the point  $A_{\mathbf{r}}^n(x)$  is in the rectangle  $R_{\epsilon_n}$ . A finite sequence  $\epsilon = \epsilon_{-m} \dots \epsilon_{-1} \cdot \epsilon_0 \dots \epsilon_{n-1}$  represents a *cylinder*, which is the set of sequences in  $\Sigma$  sharing the same symbols between indices  $-m$  and  $n-1$ . It is mapped through  $J_{\mathbf{r}}$  to a *rectangle* on the torus, which we denote by  $[\epsilon]$ . This rectangle has sides parallel to the two axes; it has width  $r_{\epsilon_0} r_{\epsilon_1} \cdots r_{\epsilon_{n-1}}$  and height  $r_{\epsilon_{-1}} \cdots r_{\epsilon_{-m}}$ .

To *open* this system, we now assume that the points in the middle rectangle  $R_1$  (“the hole”) escape to infinity, or equivalently, that the map is not defined on those points. We thus obtain an “open baker's map”  $B_{\mathbf{r}}$  defined on  $\mathcal{S} = R_0 \cup R_2$ :

$$(q, p) \mapsto B_{\mathbf{r}}(q, p) \stackrel{\text{def}}{=} (q', p') = \begin{cases} \left(\frac{q}{r_0}, p r_0\right) & \text{if } 0 \leq q < r_0 \\ \left(\frac{q-1+r_2}{r_2}, p r_2 + 1 - r_2\right) & \text{if } 1 - r_2 \leq q < 1 \end{cases} .$$

This open map is still easy to analyze through symbolic dynamics: the “hole”  $R_1$  corresponds to sequences  $\epsilon$  with  $\epsilon_0 = 1$ ; as a result,  $B_{\mathbf{r}}$  transforms the points  $x = J_{\mathbf{r}}(\epsilon)$  as follows:

$$(2.4) \quad \epsilon = \dots \epsilon_{-2} \epsilon_{-1} \cdot \epsilon_0 \epsilon_1 \dots \mapsto \begin{cases} \infty & \text{if } \epsilon_0 = 1 \\ \dots \epsilon_{-2} \epsilon_{-1} \epsilon_0 \cdot \epsilon_1 \dots = \sigma(\epsilon) & \text{if } \epsilon_0 \in \{0, 2\} \end{cases} .$$

The “inverse” of  $B_{\mathbf{r}}$  (which we will denote by  $B_{\mathbf{r}}^{-1}$ ) is defined on the set  $B_{\mathbf{r}}(\mathcal{S})$ , that is on the union of the bottom and top horizontal rectangles (see Fig. 1, right). It kills the sequences s.t.  $\epsilon_{-1} = 1$ , and if  $\epsilon_{-1} \neq 1$  it shifts the comma to the left.

**2.2. Trapped sets.** This symbolic dynamics allows to easily characterize the various trapped sets [24]: the set of points trapped in the future (forward trapped set)

$$\Gamma_- = \bigcap_{n=0}^{\infty} B_{\mathbf{r}}^{-n}(\mathcal{S}),$$

is made of the points  $x = J_{\mathbf{r}}(\epsilon)$  such that,  $\epsilon_n \in \{0, 2\}$  for all  $n \geq 0$ . This condition characterizes the positions of the points, while their momenta are unconstrained. Therefore, the set  $\Gamma_-$  takes the form of the direct product  $\Gamma_- = \mathcal{C}_{\mathbf{r}} \times [0, 1)$ , where  $\mathcal{C}_{\mathbf{r}}$  is a Cantor set of the interval. Similarly, the set  $\Gamma_+$  of points trapped in the past is made of all sequences  $\epsilon$  such that  $\epsilon_n \in \{0, 2\}$  for all  $n < 0$ . It is obviously given by the direct product  $\Gamma_+ = [0, 1) \times \mathcal{C}_{\mathbf{r}}$ . Finally, the trapped set  $K = \Gamma_- \cap \Gamma_+$  is given by  $K = \mathcal{C}_{\mathbf{r}} \times \mathcal{C}_{\mathbf{r}}$ , or sequences  $\epsilon$  with all  $\epsilon_n \in \{0, 2\}$ .

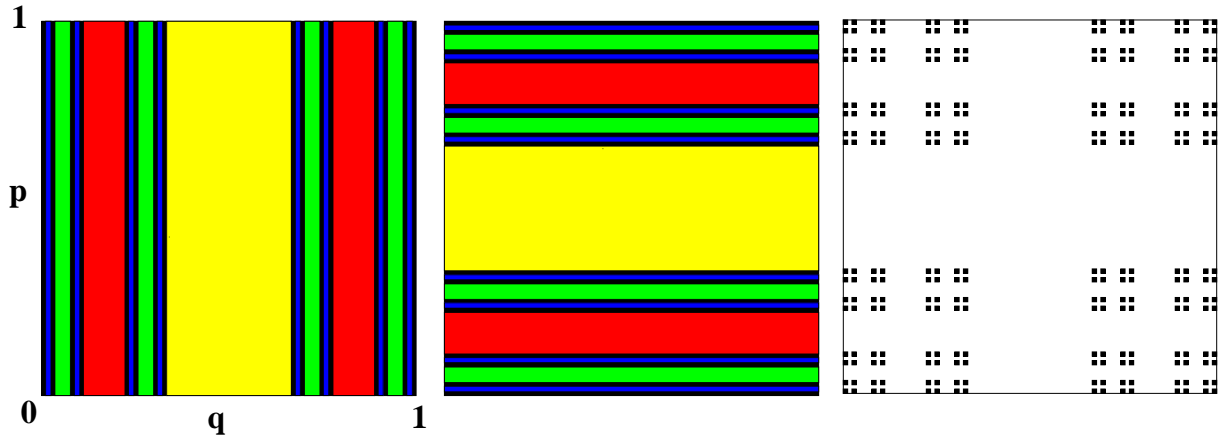


FIGURE 2. From left to right, approximations of the forward, backward trapped sets  $\Gamma_-$ ,  $\Gamma_+$  and  $K = \Gamma_- \cap \Gamma_+$ , for the open baker  $B_{\mathbf{r}_{sym}}$ . On the left and central plots, each color corresponds to points escaping at the same time in the future or past.

**2.3. Eigenmeasures of open maps.** Before giving the definition and some properties of the eigenmeasures of the open baker, we briefly recall how invariant measures emerge in the study of the quantized closed baker  $A_{\mathbf{r}, N}$ , and the associated quantum ergodicity theorem (as explained in §3,  $N = (2\pi\hbar)^{-1}$  is the semiclassical parameter in this framework).

**2.3.1. Quantum ergodicity for closed chaotic systems.** The quantum-classical correspondence between a closed symplectic map like  $A_{\mathbf{r}}$  and its quantization  $A_{\mathbf{r}, N}$  has one important consequence: in the semiclassical limit, stationary states of the quantum system (that is, eigenstates of  $A_{\mathbf{r}, N}$ ) should reflect the stationary properties of the classical map, namely

its invariant measures. To be more precise, for any sequence of eigenstates  $(\psi_N)_{N \rightarrow \infty}$  of the quantum map, one can associate a sequence of Husimi measures  $(H_{\psi_N})_{N \rightarrow \infty}$ , which are probability measures on  $\mathbb{T}^2$  (see the definition in §4.1). One can always extract a subsequence  $(H_{\psi_{N_k}})_{k \geq 1}$  weakly converging to some limit measure  $\mu$ , which one calls the **semiclassical measure** associated with the sequence  $(\psi_{N_k})_{k \geq 1}$ . If we omit the problems due to the discontinuities of  $A_{\mathbf{r}}$ , the quantum-classical correspondence implies that  $\mu$  must be invariant w.r.to the map  $A_{\mathbf{r}}$ , that is,

$$(2.5) \quad A_{\mathbf{r}}^* \mu = \mu \iff \text{for any Borel set } S \subset \mathbb{T}^2, \quad \mu(A_{\mathbf{r}}^{-1}(S)) = \mu(S).$$

$\mu$  is then also invariant w.r.to the inverse map  $A_{\mathbf{r}}^{-1}$ .

Furthermore, because the map  $A_{\mathbf{r}}$  is ergodic w.r.to the Lebesgue measure  $\mu_{Leb}$ , the Quantum Ergodicity theorem (or Shnirelman's theorem [30]) states that, for “almost any” sequence  $(\psi_N)_{N \rightarrow \infty}$ , the associated semiclassical measure is the Lebesgue measure  $\mu_{Leb}$ . Such a theorem was first proven for eigenstates of the Laplacian on compact surfaces of negative curvature [39, 5], then for more general Hamiltonians [12], billiards [9, 41] and maps [3, 40]. Quantum ergodicity for the baker's map was proven in [6].

It is generally unknown whether there exist “exceptional sequences” of eigenstates, converging to a different invariant measure. The absence of such sequences is expressed by the Quantum unique ergodicity conjecture [26], which has been proven only for few arithmetic systems [18, 14]; it has been disproved for systems enjoying very large spectral degeneracies at the quantum level, allowing for sufficient freedom to build up partially localized eigenstates [7, 1]. Some special eigenstates of the standard quantum baker with interesting multifractal properties have been recently identified by numerical means [22], but their persistence in the semiclassical limit remains unclear.

**2.3.2. Eigenmeasures for the open baker.** We now one opens the hole  $R_1$ , and consider the open map  $B_{\mathbf{r}}$ . An invariant measure for that map is necessarily supported on the trapped set  $K$ : any point outside  $\Gamma_-$  (resp. outside  $\Gamma_+$ ) will be expelled after a sufficient number of iterations of  $B_{\mathbf{r}}$  (resp. of  $B_{\mathbf{r}}^{-1}$ ).

In the following we will be lead to slightly relax the invariance condition, and consider measures which are invariant through  $B_{\mathbf{r}}^*$  *up to a multiplicative factor*. Such a measure  $\mu$  (called an “eigenmeasure”, or “conditionally invariant measure” of  $B_{\mathbf{r}}$ ) satisfies

$$(2.6) \quad B_{\mathbf{r}}^* \mu = \Lambda_{\mu} \mu \iff \text{for any Borel set } S \subset \mathbb{T}^2, \quad \mu(B_{\mathbf{r}}^{-1}(S)) = \Lambda_{\mu} \mu(S).$$

Here  $\Lambda_{\mu} \in [0, 1]$  (or rather  $\gamma_{\mu} = -\log \Lambda_{\mu}$ ) is called the “escape rate” or “decay rate” of the eigenmeasure  $\mu$ , and is given by  $\Lambda_{\mu} = \mu(\mathcal{S})$ . It corresponds to the fact that a fraction of the particles in the support of  $\mu$  escape at each step. We summarize some simple properties of eigenmeasures (the trapped sets  $\Gamma_+$  and  $K$  are sketched in Figure 2).

**Proposition 1.** *Let  $\mu$  be an eigenmeasure of  $B_{\mathbf{r}}^{-1}$  with decay rate  $\Lambda_{\mu}$ .*

*If  $\Lambda_{\mu} = 0$ ,  $\mu$  is supported in the hole  $R_1$ .*

*If  $\Lambda_{\mu} = 1$ ,  $\mu$  is supported in the trapped set  $K$  (invariant measure).*

*If  $0 < \Lambda_{\mu} < 1$ ,  $\mu$  is supported on the set  $\Gamma_+ \setminus K$ .*

To prove the last statement of the proposition, we notice that any subset  $S$  in the interior of  $\mathbb{T}^2 \setminus \Gamma_+$  is sent to infinity by a certain power of  $B_{\mathbf{r}}^{-1}$ ; by iterating (2.6) we see that if  $\Lambda_\mu > 0$ , then  $\mu(S) = 0$ . This shows that  $\mu$  is supported on  $\Gamma_+$ . On the other hand, (2.6) for  $S = K$  implies  $\mu(K)(1 - \Lambda_\mu) = 0$ .  $\square$

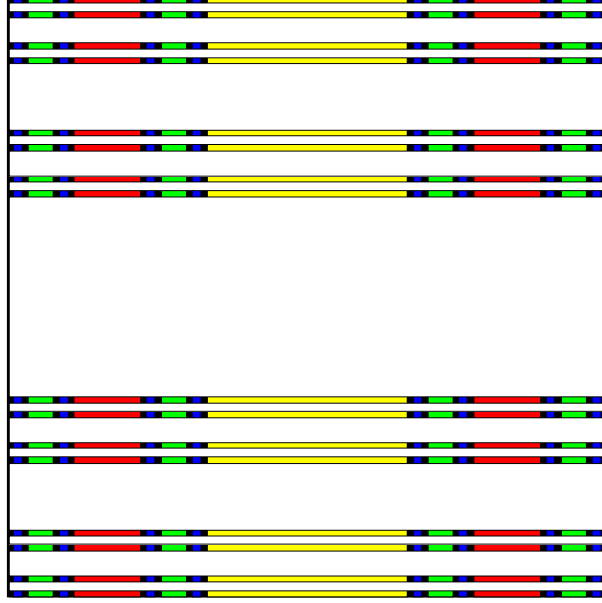


FIGURE 3. Various components  $\Gamma_+^{(n)}$  of the forward trapped set for the open baker  $B_{\mathbf{r}_{sym}}$ . The index  $n = 0, 1, 2, 3$  corresponds respectively to the color yellow, red, green, blue.

The set  $\Gamma_+ \setminus K$  can be naturally split into a disjoint union (see figure 3):

$$\Gamma_+ \setminus K = \bigsqcup_{n \geq 0} \Gamma_+^{(n)}, \quad \text{where} \quad \Gamma_+^{(0)} \stackrel{\text{def}}{=} \Gamma_+ \cap R_1, \quad \Gamma_+^{(n)} \stackrel{\text{def}}{=} B_{\mathbf{r}}^{-n} \Gamma_+^{(0)}, \quad n \geq 1.$$

Using this decomposition, we may construct any eigenmeasure as follows:

**Proposition 2.** *Take some  $\Lambda \in (0, 1)$  and  $\nu$  an arbitrary Borel probability measure on  $\Gamma_+^{(0)}$ . Define a probability measure  $\mu$  on  $\mathbb{T}^2$  as follows:*

$$(2.7) \quad \mu = (1 - \Lambda) \sum_{n \geq 0} \Lambda^n (A_{\mathbf{r}}^*)^n \nu = (1 - \Lambda) \sum_{n \geq 0} \Lambda^n (B_{\mathbf{r}}^*)^n \nu.$$

*Then  $B_{\mathbf{r}}^* \mu = \Lambda \mu$ . All  $\Lambda$ -eigenmeasures of  $B_{\mathbf{r}}$  are of this type.*

In §4 we will see that eigenmeasures of  $B_{\mathbf{r}}$  naturally appear as semiclassical limits of eigenstates of the quantized map  $B_{\mathbf{r}, N}$ . In the remainder of this section, we provide some examples of “remarkable” eigenmeasures of  $B_{\mathbf{r}}$ .



2.3.3. *Pure point eigenmeasures.* Applying the recipe of Proposition 2 to the Dirac measure  $\delta_{x_0}$  on an arbitrary point  $x_0 \in \Gamma_+^{(0)}$ , we obtain a simple, pure point  $\Lambda$ -eigenmeasure supported on the backwards trajectory  $(x_{-n} = B_{\mathbf{r}}^{-n}(x_0))_{n \geq 0}$ . For any  $\Lambda \in (0, 1)$ , we call this eigenmeasure

$$(2.8) \quad \mu_{x_0, \Lambda} \stackrel{\text{def}}{=} (1 - \Lambda) \sum_{n \geq 0} \Lambda^n \delta_{x_{-n}}.$$

We recall that pure point *invariant* measures of  $B_{\mathbf{r}}$  are localized on periodic orbits of  $B_{\mathbf{r}}$  on  $K$ , which form a countable set. On the opposite, for each  $\Lambda \in (0, 1)$  we have an uncountable family of eigenmeasures  $\mu_{x_0, \Lambda}$ , labelled by all  $x_0 \in \Gamma_+^{(0)}$ .

We also notice that the measure  $\mu$  constructed by iterating an arbitrary measure  $\nu$  on  $\Gamma_+^{(0)}$  can be expanded as

$$\mu = \int_{\Gamma_+^{(0)}} \mu_{x_0, \Lambda} d\nu(x_0).$$

2.3.4. *Natural eigenmeasure.* Hyperbolic systems with holes have been first studied by Markarian, Chernov and co-workers, who focussed on the existence and characterization of a “natural” eigenmeasure  $\mu_{nat}$ , absolutely continuous along the unstable direction [21, 4, 19]. This measure can be obtained by iterating any smooth measure  $\rho$  such that  $\rho(\Gamma_-) > 0$ :

$$(2.9) \quad \mu_{nat} = \lim_{n \rightarrow \infty} \frac{B_{\mathbf{r}}^n \rho}{\|B_{\mathbf{r}}^n \rho\|}, \quad \text{where } \|\mu\| \stackrel{\text{def}}{=} \mu(\mathbb{T}^2).$$

The eigenvalue  $\Lambda_{nat}$  associated with this measure is generally called “the decay rate of the system” by physicists. For the closed baker’s map  $A_{\mathbf{r}}$ , the natural measure is the Lebesgue measure  $\mu_{Leb}$ . As explained in §2.3.1, the quantum ergodicity theorem shows that this particular invariant measure is “favored” by quantum mechanics. One interesting question we will try to address is the relevance of  $\mu_{nat}$  with respect to the quantum open baker.

2.3.5. *Bernoulli eigenmeasures.* The first in (2.2), defines a mapping between the set  $\Sigma_+$  of one-sided sequences  $\cdot \epsilon_0 \epsilon_1 \dots$  and the position interval  $[0, 1] \ni q$ . By a slight abuse, we also call this mapping  $J_{\mathbf{r}}$ . Using this mapping, we may construct a certain family of measures on  $[0, 1]$  by pushing-forward Bernoulli measures on  $\Sigma_+$ .

Let us choose a *weight distribution*  $\mathbf{P} = \{P_0, P_1, P_2\}$ , where each  $P_\epsilon \in [0, 1]$  and  $P_0 + P_1 + P_2 = 1$ . Any finite sequence  $\epsilon = \cdot \epsilon_0 \epsilon_1 \dots \epsilon_{n-1} \subset \Sigma_+$  is mapped by  $J_{\mathbf{r}}$  onto a subinterval  $I_\epsilon \subset [0, 1]$ . By setting

$$\nu_{\mathbf{r}, \mathbf{P}}(I_\epsilon) = P_{\epsilon_0} \dots P_{\epsilon_{n-1}},$$

we define a Bernoulli measure  $\nu_{\mathbf{r}, \mathbf{P}}$  on  $[0, 1]$  (or equivalently on  $\Sigma_+$ ). Fractal properties of those measures are given in [11]. Below we list some of them.

If we take  $P_\epsilon = r_\epsilon$ , we recover  $\nu_{\mathbf{r}, \mathbf{P}} = \nu_{Leb}$  the Lebesgue measure on the interval. If for some  $\epsilon \in \{0, 1, 2\}$  we take  $P_\epsilon = 1$ , we get for  $\nu_{\mathbf{r}, \mathbf{P}}$  the Dirac measure at the point  $q(\cdot \epsilon \epsilon \dots)$ , which takes respectively the values  $0$ ,  $\frac{r_0}{1-r_1}$  and  $1$ . For any other distribution  $\mathbf{P}$ , the Bernoulli measure  $\nu_{\mathbf{r}, \mathbf{P}}$  is purely singular continuous, with obvious self-similar properties.

If the weight  $P_\epsilon$  vanishes,  $\nu_{\mathbf{r},\mathbf{P}}$  is supported on the Cantor set  $J_{\mathbf{r}}(\{\epsilon \in \Sigma_+, \epsilon_n \neq \epsilon\})$ . For any two distributions  $\mathbf{P} \neq \mathbf{P}'$ , the two measure  $\nu_{\mathbf{r},\mathbf{P}}, \nu_{\mathbf{r},\mathbf{P}'}$  are *mutually singular*, meaning that there exists disjoint subsets  $A, A'$  of  $[0, 1]$  such that  $\nu_{\mathbf{r},\mathbf{P}}(A) = \nu_{\mathbf{r},\mathbf{P}'}(A') = 1$ .

**Proposition 3.** *For any weight distribution  $\mathbf{P}$  such that  $P_1 < 1$ , there is a unique auxiliary distribution, namely  $\mathbf{P}^* = \left\{ \frac{P_0}{P_0+P_2}, 0, \frac{P_2}{P_0+P_2} \right\}$ , such that the product measure*

$$\mu_{\mathbf{r},\mathbf{P}}(dx) \stackrel{\text{def}}{=} \nu_{\mathbf{r},\mathbf{P}}(dq) \times \nu_{\mathbf{r},\mathbf{P}^*}(dp).$$

is an eigenmeasure of  $B_{\mathbf{r}}$ . The corresponding decay rate is

$$(2.10) \quad \Lambda_{\mathbf{P}} = 1 - P_1 = P_0 + P_2.$$

The measure of a rectangle  $[\epsilon]_{n,m}$  is simply

$$\mu_{\mathbf{r},\mathbf{P}}([\epsilon]_{n,m}) = P_{\epsilon_{-m}}^* \cdots P_{\epsilon_{-1}}^* P_{\epsilon_0} \cdots P_{\epsilon_{n-1}}.$$

If we take  $\mathbf{P} = \mathbf{r}$ , we obtain the ‘‘natural’’ eigenmeasure  $\mu_{\text{nat}} = \mu_{\mathbf{r},\mathbf{r}}$ , constructed by [21, 4] in a more general setting. This is the only eigenmeasure of  $B_{\mathbf{r}}$  which is absolutely continuous along the unstable (horizontal) direction. If  $\mathbf{P} \neq \mathbf{P}'$ , the Bernoulli eigenmeasures  $\mu_{\mathbf{r},\mathbf{P}}$  and  $\mu_{\mathbf{r},\mathbf{P}'}$  are mutually singular, eventhough they may share the same decay rate.

Some of these measures will appear in §5 as semiclassical measures for the Walsh-quantized open baker.

### 3. QUANTIZATION OF THE OPEN BAKER

We recall here the quantization of closed and open baker’s maps, following the original approach of Balasz-Voros, Saraceno, Saraceno-Vallejos [2, 27, 28]. The quantum Hilbert space corresponding to the torus phase space is  $N$ -dimensional, with  $N = (2\pi\hbar)^{-1}$ , so we denote it by  $\mathcal{H}_N$ . That space is spanned by the orthonormal *position basis*  $\{\mathbf{q}_j, j = 0, \dots, N - 1\}$ , localized at the discrete positions  $q_j = \frac{j}{N}$ . Through the discrete Fourier transform

$$(3.1) \quad (F_N)_{jk} = N^{-1/2} e^{-2i\pi jk/N}, \quad j, k = 0, \dots, N - 1,$$

this basis is transformed into the *momentum basis*  $\{\mathbf{p}_k, k = 0, \dots, N - 1\}$ :

$$\mathbf{p}_k = \sum_{j=0}^{N-1} (F_N^*)_{jk} \mathbf{q}_j.$$

The quantization of the closed baker’s map  $A_{\mathbf{r}}$  consists in a unitary operator acting on this Hilbert space. Strictly speaking, the quantization is well-defined only if the coefficients  $\mathbf{r}$  are such that

$$(3.2) \quad N r_i = N_i \in \mathbb{N}, \quad i = 0, 1, 2.$$

Yet, in the semiclassical regime ( $N \rightarrow \infty$ ) one can, if necessary, slightly modify the  $r_i$  by amounts  $\leq 1/N$  in order to satisfy this condition: such a modification is irrelevant for the

classical dynamics. Assuming (3.2), the quantization of  $A_{\mathbf{r}}$  on  $\mathcal{H}_N$  is given by the following unitary matrix in the position basis:

$$(3.3) \quad A_{\mathbf{r},N} = F_N^{-1} \begin{pmatrix} F_{N_0} & & \\ & F_{N_1} & \\ & & F_{N_2} \end{pmatrix}.$$

In §2.1, the open map  $B_{\mathbf{r}}$  was obtained by sending all points of the “hole”  $R_1$  to infinity, then applying  $A_{\mathbf{r}}$  to the remaining points. On the quantum side, this corresponds to kill all quantum states localized in  $R_1$ , and then apply  $A_{\mathbf{r},N}$ . Since the hole is defined in terms of positions, this “killing” is performed by the projector  $\Pi_{\mathcal{S},N} = I_N - \Pi_{hole,N}$ , where

$$\Pi_{hole,N} \stackrel{\text{def}}{=} \sum_{N_0 \leq j < N-N_2} |\mathbf{q}_j\rangle\langle \mathbf{q}_j|.$$

One ends up with the following matrix in the position basis:

$$(3.4) \quad B_{\mathbf{r},N} = F_N^{-1} \begin{pmatrix} F_{N_0} & & \\ & 0 & \\ & & F_{N_2} \end{pmatrix}.$$

This matrix is the “Weyl” quantization of  $B_{\mathbf{r}}$ . It is expected to share some semiclassical properties with the propagator  $\exp(-i\hat{H}_W/\hbar)$  of the “absorbing Hamiltonian”. In particular, the eigenvalues  $\{\lambda_j\}$  of  $B_{\mathbf{r},N}$  should be compared with  $\{e^{-i\tilde{z}_j/\hbar}\}$ , or with  $\{e^{-iz_j/\hbar}\}$ , where  $\{\tilde{z}_j\}$  (resp.  $\{z_j\}$ ) are the eigenvalues of  $\hat{H}_W$  (resp. the resonances of  $\hat{H}$ ).

**3.1. Fractal Weyl law for the quantized open baker.** The range of  $\Pi_{hole,N}$  is obviously in the kernel of  $B_{\mathbf{r},N}$ . We call “nontrivial” the spectrum of  $B_{\mathbf{r},N}$  on the complementary subspace. That spectrum is situated inside the unit disk. In the semiclassical limit  $N \rightarrow \infty$ , most of it accumulates near the origin, which corresponds to “short-lived states” [31]. We rather focus on “long-living” eigenvalues, situated away from the origin. In [24] we conjectured a fractal Weyl law for the semiclassical density of “long-living” eigenvalues.

**Conjecture 1** (Fractal Weyl Law). *Let  $B_{\mathbf{r}}$  be the open baker’s map described in §3. Then, for any radius  $0 < r < 1$ , there exists  $C_r \geq 0$  such that*

$$(3.5) \quad n(N,r) \stackrel{\text{def}}{=} \#\{\lambda \in \text{Spec}(B_{\mathbf{r},N}), : |\lambda| \geq r\} = C_r N^d + o(N^d), \quad N \rightarrow \infty, .$$

Here  $2d$  is the fractal dimension of the trapped set  $K$ .

So far, this conjecture could be proven only for the *Walsh* quantization  $\tilde{B}_k$  of the symmetric baker  $B_{\mathbf{r}_{sym}}$ , see section 5.2. This Weyl law was numerically checked for the sequence of quantum open bakers  $B_{\mathbf{r}_{sym},N}$ , especially for  $N$  along geometric sequences  $N = N_0 3^k$ ,  $k \geq 1$ . It was also checked for an open version of the kicked rotator [31]. Similar Weyl laws have been conjectured for Hamiltonian scattering systems [34], and upper bounds could be rigorously proven in various settings [42, 10, 35].

**3.2. Which dimension plays a role?** In the proofs for upper bounds, the exponent  $d$  is defined in terms of the *upper Minkowski dimension* of  $K$ . In the case of the open baker  $B_{\mathbf{r}}$ , we therefore expect that the exponent  $d$  appearing in the conjecture (resp.  $d + 1$ ) is given by the Minkowski dimension of the Cantor set  $\mathcal{C}_{\mathbf{r}}$  (resp.  $\Gamma_+$ ), which is equal to its box, Hausdorff and packing dimensions. We call this theoretical value  $d_0$ .

For the symmetric baker  $B_{\mathbf{r}_{sym}}$ , the Hausdorff dimension  $d_H(\Gamma_+) = d_0 + 1$  happens to be equal to the Hausdorff dimension of the natural measure  $\mu_{nat}$ , defined by

$$d_H(\mu_{nat}) = \inf_{A \subset \mathbb{T}^2, \mu_{nat}(A)=1} d_H(A).$$

For a more general map  $B_{\mathbf{r}}$ , that Hausdorff measure only satisfies the inequality  $d_H(\mu_{nat}) \leq d_H(\Gamma_+)$  (see the explicit expressions below). We want to investigate a possible ‘‘role’’ of the natural eigenmeasure  $\mu_{nat}$  regarding the structure of the resonance spectrum. It is therefore legitimate to ask the following

**Question 1.** *Is the correct exponent in the Weyl law (3.5) given by  $d_I = d_H(\mu_{nat}) - 1$  instead of  $d_0 = d_H(\Gamma_+) - 1$ ?*

Here the suffix  $I$  indicates that  $d_I$  is sometimes called the *information dimension* of the measure [11, 8]. As mentioned above, both dimensions are equal for the symmetric baker  $B_{\mathbf{r}_{sym}}$ , which has been numerically tested so far [23]. They are also equal in the case of a closed map on  $\mathbb{T}^2$ : in that case, the Weyl law has exponent 1 (the whole spectrum lies on the unit circle), and we have  $d_H(\mathbb{T}^2) = d_H(\mu_{Leb}) = 2$ .

However, for a nonsymmetric baker  $B_{\mathbf{r}}$ , the two dimensions take different values:

$$d_0 \text{ is the solution of } r_0^d + r_2^d = 1, \quad \text{while } d_I = \frac{r_0 \log p_0 + r_2 \log p_2}{r_0 \log r_0 + r_2 \log r_2}.$$

To answer this question, we considered a very asymmetric baker, namely taking  $\mathbf{r}_{asym}$  with  $r_0 = 1/32$ ,  $r_2 = 2/3$ . The two dimensions then take the values  $d_0 \approx 0.493$ ,  $d_I \approx 0.337$ . We computed the counting function  $n(N, r)$  for several radii  $0.1 \leq r \leq 1$  and several values of  $N$ . We then tried to fit the Weyl law (3.5) with an exponent  $d$  varying in a certain range, and computed the standard deviations (see Fig. 4). The numerical result is unambiguous: the best fit clearly occurs away from  $d_I$ , but it is close to  $d_0$ . This numerical test apparently rules out the possibility that  $d_I$  provides the correct exponent of the Weyl law, and suggests to take  $d = d_0$ .

To further illustrate the Weyl law for the asymmetric baker  $B_{\mathbf{r}_{asym}}$ , we plot in figure 5 (left) the counting functions  $n(N, r)$  as a function of  $r \in (0, 1)$ , for several values of  $N$ . On the right plot, we rescale  $n(N, r)$  by the power  $N^{-d_0}$ : the rescaled curves almost perfectly overlap, indicating that the scaling (3.5) is correct.

**Remark 1.** On figure 5 (right) the rescaled counting function seems to be ‘‘strictly decreasing’’ on an interval  $[\lambda_{\min}, \lambda_{\max}]$ , where  $\lambda_{\min} \approx 0.1$ ,  $\lambda_{\max} \approx 0.9$ . This implies that the spectrum of  $B_{\mathbf{r}_{asym}, N}$  becomes dense in the whole annulus  $\{\lambda_{\min} \leq |\lambda| \leq \lambda_{\max}\}$ , when  $N \rightarrow \infty$ . Therefore, for any  $\lambda \in [\lambda_{\min}, \lambda_{\max}]$ , one can consider sequences of eigenvalues  $(\lambda_N)_{N \geq 1}$  such that  $|\lambda_N| \xrightarrow{N \rightarrow \infty} \lambda$ .

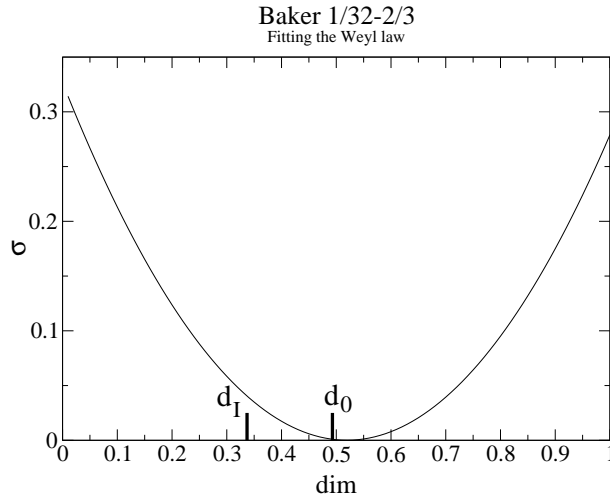


FIGURE 4. Standard deviations when fitting the Weyl law (3.5) to various dimensions, integrated on  $0.1 \leq r \leq 1$ . The two marks on the horizontal axis indicate the theoretical values  $d_I$  and  $d_0$ .

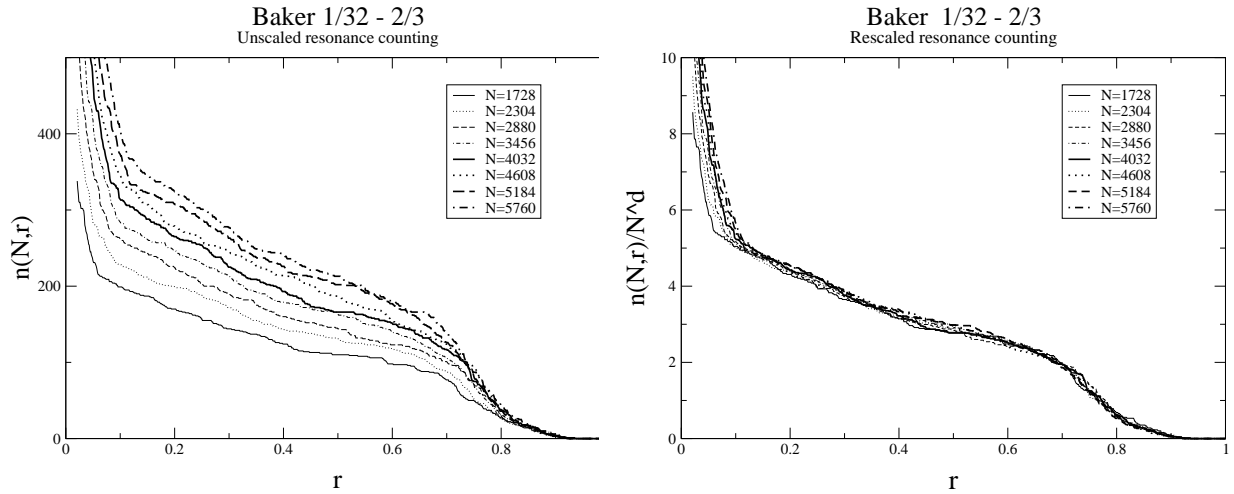


FIGURE 5. Left: number of resonances of the asymmetric baker  $B_{r,asy}$  outside the disks of radii  $r$ , for various values of Planck's constant  $N$ . Right: same curves vertically rescaled by  $N^{-d_0}$ .

#### 4. LOCALIZATION OF RESONANT EIGENSTATES

In this section, we study the phase space localization properties of the “long-living” right eigenstates of a given open baker  $B_{r,N}$ . That is, we will consider sequences  $(\psi_N)_{N \rightarrow \infty}$  such that, for any  $N$ ,

$$(4.1) \quad B_{r,N} \psi_N = \lambda_N \psi_N, \quad \|\psi_N\| = 1, \quad |\lambda_N| \geq \lambda_{min} > 0.$$

The lower bound  $\lambda_{min} > 0$  is arbitrary: its role is to ensure that all the eigenstates we consider are “long-living”. The phase space distribution of eigenstates is provided by the Husimi measures, which are defined in the next subsection.

**4.1. Phase space representation of quantum states.** We recall the definition and properties of *torus coherent states*, which we use to construct both the Husimi representation of quantum states, and the anti-Wick quantization of observables [3]. The Gaussian coherent state in  $L^2(\mathbb{R})$ , localized at the phase space point  $x = (q_0, p_0) \in \mathbb{R}^2$  and with squeezing  $\sigma > 0$ , is defined by the wavefunction

$$(4.2) \quad \Psi_{x,\sigma}(q) \stackrel{\text{def}}{=} \left(\frac{\sigma}{\pi\hbar}\right)^{1/4} e^{-i\frac{p_0 q_0}{2\hbar}} e^{i\frac{p_0 q}{\hbar}} e^{-\sigma\frac{(q-q_0)^2}{2\hbar}}$$

When  $\hbar = (2\pi N)^{-1}$ , that state can be periodized on the torus, to yield the torus coherent state  $\psi_{x,\sigma} \in \mathcal{H}_N$  with following components in the basis  $\{\mathbf{q}_j\}$ :

$$(4.3) \quad (\psi_{x,\sigma})_j = \frac{1}{\sqrt{N}} \sum_{\nu \in \mathbb{Z}} \Psi_{x,\sigma}(j/N + \nu), \quad j = 0, \dots, N-1.$$

This family of states is quasi-periodic w.r.to  $x$ : for any  $\mathbf{m} \in \mathbb{Z}^2$ ,  $\psi_{x+\mathbf{m},\sigma} = e^{i(\cdot)^*} \psi_{x,\sigma}$ . These states are asymptotically normalized in the semiclassical limit.

We use these coherent states to quantize classical observables  $f \in C^\infty(\mathbb{T}^2)$  into operators on  $\mathcal{H}_N$ . To any squeezing  $\sigma > 0$  and inverse Planck’s constant  $N > 0$  we associate the anti-Wick quantization

$$f \in C^\infty(\mathbb{T}^2) \mapsto \text{Op}_N^{\text{AW},\sigma}(f) \stackrel{\text{def}}{=} \int_{\mathbb{T}^2} |\psi_{x,\sigma}\rangle \langle \psi_{x,\sigma}| f(x) dx.$$

By duality, this quantization defines, for any state  $\psi \in \mathcal{H}_N$ , a Husimi distribution  $H_\psi^\sigma(x) dx$ :

$$\forall f \in C^\infty(\mathbb{T}^2), \quad \int f(x) H_\psi^\sigma(x) dx = \langle \psi | \text{Op}_N^{\text{AW},\sigma}(f) | \psi \rangle.$$

This distribution is actually a positive measure, with density given by the Husimi function

$$x \in \mathbb{T}^2 \mapsto H_\psi^\sigma(x) \stackrel{\text{def}}{=} N |\langle \psi_{x,\sigma} | \psi \rangle|^2.$$

For  $\|\psi\| = 1$ , this function defines a probability density on  $\mathbb{T}^2$ .

We will say that a sequence of states  $(\psi_N \in \mathcal{H}_N)_{N \rightarrow \infty}$  converges to the probability measure  $\mu$  if, for a given  $\sigma > 0$ , the Husimi measures  $H_{\psi_N}^\sigma$  weak-\* converge to the measure  $\mu$ . Equivalently,

$$(4.4) \quad \forall f \in C^\infty(\mathbb{T}^2), \quad \langle \psi_N | \text{Op}_N^{\text{AW},\sigma}(f) | \psi_N \rangle \xrightarrow{N \rightarrow \infty} \mu(f) = \int_{\mathbb{T}^2} f d\mu.$$

The measure  $\mu$  is then called the *semiclassical measure* associated with the sequence  $(\psi_N)$ .

Following [32], one can also consider quantizations (and dual Husimi representations) for which the squeezing parameter  $\sigma$  smoothly depends on the phase space point  $x$ , and is bounded away from 0. By adapting the proofs of [32] to the torus setting, one easily shows the following

**Proposition 4.** *Let  $\sigma_1, \sigma_2 \in C^\infty(\mathbb{T}^2, (0, \infty))$ . The two associated quantizations become close to one another when  $N \rightarrow \infty$ :*

$$\forall f \in C^\infty(\mathbb{T}^2), \quad \|\text{Op}_N^{\text{AW}, \sigma_1}(f) - \text{Op}_N^{\text{AW}, \sigma_2}(f)\| = \mathcal{O}(N^{-1}).$$

As a result, if the convergence property (4.4) holds for the squeezing  $\sigma_1$ , it also holds for  $\sigma_2$ .

**4.2. Semiclassical measures of resonant eigenstates.** We turn back to our sequence  $(\psi_N)_{N \rightarrow \infty}$  of long-living resonant eigenstates of  $B_{\mathbf{r}, N}$  (see (4.1)). As explained in §2.3.1, up to extracting a subsequence, we can assume that the Husimi measures  $(H_{\psi_N}^1)_{N \rightarrow \infty}$  converge to a certain probability measure  $\mu$ . The latter is called a *semiclassical measure* of the open baker  $B_{\mathbf{r}}$ . In Figure 6, we plot the Husimi functions  $H_{\psi_N}^1$  of some (right) eigenstates of  $B_{\mathbf{r}, N}$ , for the maps with respectively  $\mathbf{r} = \mathbf{r}_{\text{sym}}$  and  $\mathbf{r} = \mathbf{r}_2 = (1/9, 5/9, 1/3)$ . As a first remark, we notice that all Husimi functions are very inhomogeneous: they are very small in the horizontal strips which are in the complement of  $\Gamma_+$ . The proof of the proposition below shows that they are actually *exponentially small* outside of  $\Gamma_+$ , when  $N \rightarrow \infty$ .

Let  $D$  denote the interior of  $B_{\mathbf{r}}(R_0) \cup B_{\mathbf{r}}(R_2)$ , so that  $\partial D$  is the set of discontinuities for the map  $B_{\mathbf{r}}^{-1}$ . We also define  $\tilde{\Gamma}_+ \stackrel{\text{def}}{=} \Gamma_+ \cup \partial D$  (see Fig. 7). Taking the semiclassical limit, this nonhomogeneous distribution yields the following theorem, which is our central result.

**Theorem 1.** *Consider a semiclassical measure  $\mu$  for the open baker  $B_{\mathbf{r}}$ , limit of some sequence  $(\psi_N)_{N \rightarrow \infty}$  as in (4.1). Then:*

- i) the support of  $\mu$  is a subset of  $\tilde{\Gamma}_+$ .*
- ii) If  $\text{supp } \mu \cap D \neq \emptyset$ , there exists  $\Lambda \in [\lambda_{\min}^2, 1]$  such that the eigenvalues  $(\lambda_N)$  satisfy*

$$|\lambda_N|^2 \xrightarrow{N \rightarrow \infty} \Lambda.$$

*For any Borel subset  $S \subset \mathbb{T}^2$  such that  $S \cap \partial D = \emptyset$ , one has  $\mu(B_{\mathbf{r}}^{-1}(S)) = \Lambda \mu(S)$ .*

- iii) If  $\mu(\partial D) = \mu(B_{\mathbf{r}}^{-1}\partial D) = 0$ , then  $\mu$  is an eigenmeasure of  $B_{\mathbf{r}}$ , with decay rate  $\Lambda$ .*

*Proof.* In the proof we fix  $\mathbf{r}$ , and write  $B = B_{\mathbf{r}}$ . To prove *i)*, we will estimate the values of the Husimi functions away from  $\tilde{\Gamma}_+$ . For any small  $\delta > 0$ , we consider the set

$$\mathfrak{C}\tilde{\Gamma}_{+, \delta} \stackrel{\text{def}}{=} \left\{ x \in \mathbb{T}^2, \text{dist}(x, \tilde{\Gamma}_+) \geq \delta \right\},$$

which is composed of several connected components, each of them inside one of the rectangles  $B(R_i)$ . It is easy to see that each point  $x$  in this set escapes at certain time  $1 \leq n(x)$  through  $B^{-1}$ , where  $n(x)$  is constant on each connected component of  $\mathfrak{C}\tilde{\Gamma}_{+, \delta}$ .

Using the fact that  $\psi_N$  is an eigenstate of  $B_N$  and  $|\lambda_N| \geq \lambda_{\min}$ , for any  $x \in \mathfrak{C}\tilde{\Gamma}_{+, \delta}$  we rewrite  $H_{\psi_N}^\sigma(x)$  as follows:

$$(4.5) \quad H_{\psi_N}^\sigma(x) = N |\lambda_N|^{-2n(x)} |\langle \psi_{x, \sigma}, (B_N)^{n(x)} \psi_N \rangle|^2$$

$$(4.6) \quad = N |\lambda_N|^{-2n(x)} |\langle (B_N^\dagger)^{n(x)} \psi_{x, \sigma}, \psi_N \rangle|^2.$$

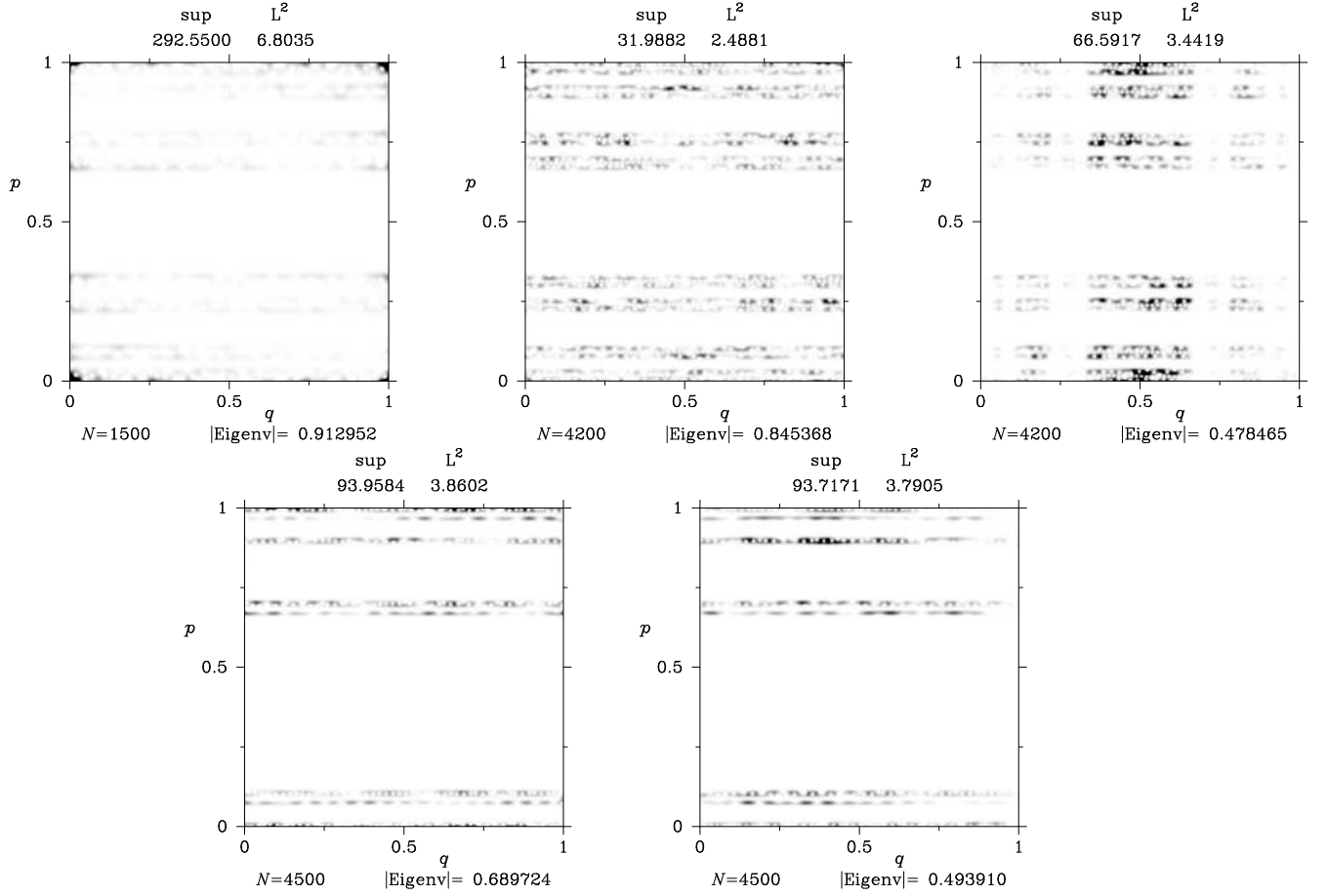


FIGURE 6. Top: Husimi functions of three eigenstates of  $B_{\mathbf{r}_{sym},N}$  (black=large values, white=0). Bottom: Husimi functions of 2 eigenstates of  $B_{\mathbf{r}_2,N}$ . We give the sup and  $L^2$  norms of the Husimi functions. The grey scale goes from white (zero) to black (maximum).

Slightly generalizing the estimates of [6], one can show that for any  $x \in \mathfrak{C}\tilde{\Gamma}_{+,\delta}$ , the coherent state  $\psi_{x,\sigma}$  propagates semiclassically through  $B_N^\dagger$ . Precisely, there exists  $C(\sigma, \delta) > 0$  such that, for any  $x \in \mathfrak{C}\tilde{\Gamma}_{+,\delta}$ ,

$$(4.7) \quad B_N^\dagger \psi_{x,\sigma} = \chi_{B(\mathcal{S})}(x) e^{i\theta} \psi_{x',\sigma'} + \mathcal{O}(e^{-NC(\sigma,\delta)}).$$

The parameters  $x', \sigma'$  are as follows:  $x' = B^{-1}(x)$ ,  $\sigma' = \sigma/r_\epsilon^2$  if  $x$  is in the rectangle  $B(R_\epsilon)$ .  $\chi_{B(\mathcal{S})}$  is the characteristic function on the set  $B(\mathcal{S}) = D$ . The center and squeezing of the coherent state are thus transported through the classical dynamics  $B^{-1}$ . The phase  $\theta$  can also be given explicitly. Notice that the new point  $x'$  is situated in a set  $\mathfrak{C}\tilde{\Gamma}_{+,\delta'}$  for some  $\delta' > 0$ .



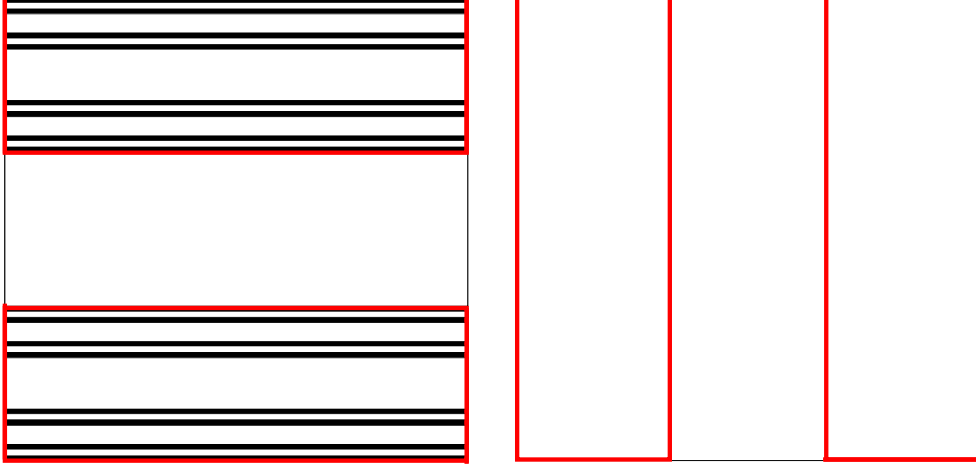


FIGURE 7. On the left, we show the set  $\tilde{\Gamma}_+$  ( $\partial D$  is indicated in thick red lines). On the right, we show  $B_r^{-1}(\partial D)$ .

Iterating this estimate exactly  $n(x)$  times, we obtain

$$\forall x \in \mathfrak{C}\tilde{\Gamma}_{+, \delta}, \quad (B_N^\dagger)^{n(x)} \psi_{x, \sigma} = \mathcal{O}(e^{-CN}).$$

Using the assumption  $|\lambda_N| \geq \lambda_{min}$ , we finally get the following exponential bound:

$$(4.8) \quad H_{\psi_N}^\sigma(x) = \mathcal{O}(e^{-CN}) \quad \text{uniformly for } x \in \mathfrak{C}\tilde{\Gamma}_{+, \delta}.$$

Passing to the limit  $N \rightarrow \infty$  this estimate shows that  $\mu(\mathfrak{C}\tilde{\Gamma}_{+, \delta}) = 0$ . Since  $\delta > 0$  was chosen arbitrary small, we have proven *i*).

The proof of the second statement uses the same methods. Take  $f \in C^\infty(\mathbb{T}^2)$  supported inside  $\mathfrak{C}\tilde{\Gamma}_{+, \delta}$ , and such that  $\mu(f) > 0$  (such a choice is possible from the assumption in *ii*). Inserting  $\lambda_N^{-1} B_N$  on both sides of  $\langle \psi_N, \text{Op}_N^{\text{AW}, \sigma}(f) \psi_N \rangle$ , we get

$$(4.9) \quad \langle \psi_N, \text{Op}_N^{\text{AW}, \sigma}(f) \psi_N \rangle = |\lambda_N|^{-2} \int_{\mathbb{T}^2} N dx f(x) |\langle (B_N^\dagger)^{n(x)} \psi_{x, \sigma}, \psi_N \rangle|^2$$

$$(4.10) \quad = |\lambda_N|^{-2} \int_{\mathbb{T}^2} N dx f(x) \chi_{B(\mathcal{S})}(x) |\langle \psi_{x', \sigma'}, \psi_N \rangle|^2 + \mathcal{O}(e^{-CN}).$$

Since the change of variables  $x \mapsto x'$  is symplectic on  $B(\mathcal{S})$ , the last line can be rewritten as

$$|\lambda_N|^{-2} \langle \psi_N, \text{Op}^{\text{AW}, \sigma'}(\chi_{\mathcal{S}} \times (f \circ B)) \psi_N \rangle + \mathcal{O}(e^{-CN}).$$

In the above expression, we can with no harm extend the definition of  $\sigma'$  outside  $\mathcal{S}$  into a smooth function  $\sigma'(x') > 0$ , therefore obtaining an “nice” anti-Wick quantization as in Prop. 4. In the semiclassical limit, the above matrix elements (without the prefactor  $|\lambda_N|^{-2}$ ) converge to  $\mu(\chi_{\mathcal{S}} \times f \circ B)$ , while the full expression should converge to  $\mu(f) > 0$ . This obviously implies that  $|\lambda_N|^2$  admits a limit  $\Lambda \in [\lambda_{min}^2, 1]$ , and that

$$\mu(f) = \Lambda^{-1} \mu(\chi_{\mathcal{S}} \times (f \circ B)).$$

The statement *ii*) then follows by standard arguments. To obtain the last statement, it suffices to decompose any measurable set  $A$  into  $A = (A \cap \partial D) \cup (A \setminus \partial D)$ . From *ii*), we have  $\mu(A \setminus \partial D) = \Lambda^{-1} \mu(B^{-1}(A \setminus \partial D))$ , while from the assumption,  $\mu(A \cap \partial D) = \mu(B^{-1}(A \cap \partial D)) = 0$ . We thus get  $\mu(A) = \Lambda^{-1} \mu(B^{-1}(A))$  for any set.  $\square$

**Remark 2.** We cannot rule out the possibility that  $\mu$  charges the discontinuity set  $\partial D \cup B^{-1}(\partial D)$ . Indeed, in some of the Husimi plots, we seem to see a “scar” (a strong peak of the Husimi function) at the point  $(0, 0)$ . On the other hand, all the Husimi functions we computed were small on  $\partial D \setminus \Gamma_+$ . We therefore conjecture that all semiclassical measures are supported on  $\Gamma_+$ , and are therefore eigenmeasures of  $B_{\mathbf{r}}$ .

**4.3. Abundance of semiclassical measures.** We now draw some consequences of the above theorem, and the remark which follows. Statement *ii*) of the theorem strongly constrains the converging sequences of eigenstates: the Husimi measures  $(H_{\psi_N}^1)$  can converge to some (eigen)measure  $\mu$  only if the corresponding eigenvalues  $(\lambda_N)$  asymptotically approach the circle of radius  $\sqrt{\Lambda}$ .

From the density argument in Remark 1, for any  $\Lambda \in [\lambda_{\min}^2, \lambda_{\max}^2]$  it is possible to construct sequences of eigenstates of  $B_{\mathbf{r}_{asym}, N}$ , such that  $|\lambda_N|^2 \xrightarrow{N \rightarrow \infty} \Lambda$ . Any semiclassical measure extracted from such a sequence will be an eigenmeasure with decay rate  $\Lambda$ . Therefore, two converging sequences  $(\psi_N)$ ,  $(\psi'_N)$  associated with limiting decays  $\Lambda \neq \Lambda'$  will necessarily converge to different eigenmeasures. This already shows that the semiclassical measures generated by all possible sequences in the annulus  $\{\lambda_{\min} \leq |\lambda| \leq \lambda_{\max}\}$  form an uncountable family. This should generalize to any open baker  $B_{\mathbf{r}}$ .

In section 2.3.2, we show that for each  $\Lambda \in (0, 1)$ , there are (uncountably) many possible eigenmeasures. A natural question thus concerns the variety of semiclassical measures, for a fixed  $\Lambda \in [\lambda_{\min}^2, \lambda_{\max}^2]$ :

**Question 2.**

- For a given  $\Lambda \in [\lambda_{\min}^2, \lambda_{\max}^2]$ , what are the semiclassical measures of  $B_{\mathbf{r}}$  of decay rate  $\Lambda$ ?
- is there a unique such measure? Otherwise, is some measure “favored”, in the sense that “almost all” sequences  $(\psi_N)$  with  $|\lambda_N|^2 \rightarrow \Lambda$  converge to  $\mu$ ?
- can the natural measure  $\mu_{nat}$  be obtained as a semiclassical measure?

The same type of questions appear in [13]. At present we are unable to answer them for the open baker  $B_{\mathbf{r}, N}$ . In the next section we try to address them for a solvable model, the Walsh-quantized open baker.

In the next section, we construct *quasimodes* for the quantum baker  $B_{\mathbf{r}, N}$ .

**4.4. Quasimodes.** From the explicit representation of eigenmeasures given in Proposition 2, it is tempting to construct approximate eigenstates by propagating backwards wavepackets localized on the set  $\Gamma_+^{(0)}$ . For  $\nu$  an arbitrary probability measure supported on  $\Gamma_+^{(0)}$ , assume that we can construct a family of states  $(\psi_{\nu, N})$  such that the Husimi measures  $H_{\psi_{\nu, N}}^1$  converge to the measure  $\nu$ . For any  $\lambda \in \mathbb{C}$ ,  $0 < |\lambda| < 1$ , we construct the following

state by mimicking Eq. (2.7):

$$(4.11) \quad \Psi_{\lambda,\nu,N} \stackrel{\text{def}}{=} \sqrt{1-|\lambda|^2} \sum_{n \geq 0} \lambda^n A_{\mathbf{r},N}^{-n} \psi_{\nu,N}.$$

Through Egorov's property, for  $n \in \mathbb{N}$  fixed and  $N \rightarrow \infty$ , the Husimi measures of the states  $A_{\mathbf{r},N}^{-n} \psi_{\nu,N}$  converge to the measures  $(A_{\mathbf{r}}^*)^n \nu$ , so the states  $(\Psi_{\lambda,\nu,N})$  obviously converge to the eigenmeasure  $\mu$  given in (2.7).

However, the state  $\Psi_{\lambda,\nu,N}$  is not an exact eigenstate of  $B_{\mathbf{r},N}$ , but only a quasimode. To check that statement, we need to apply  $B_{\mathbf{r},N} = A_{\mathbf{r},N} \Pi_{\mathcal{S},N}$  to that state:

$$B_{\mathbf{r},N} \Psi_{\lambda,\nu,N} = \sum_{n \geq 0} \lambda^n A_{\mathbf{r},N} \Pi_{\mathcal{S},N} A_{\mathbf{r},N}^{-n} \psi_{\nu,N}.$$

That state would be equal to  $\lambda \Psi_{\lambda,\nu,N}$  if we had simultaneously

$$(4.12) \quad \Pi_{\mathcal{S},N} \psi_{\nu,N} = 0 \quad \text{and} \quad \Pi_{\text{hole},N} A_{\mathbf{r},N}^{-n} \psi_{\nu,N} = 0 \quad \text{for all } n \geq 1.$$

These conditions can not be exactly fulfilled: if the first one is true (meaning that  $\psi_{\nu,N}$  is strictly localized in  $R_1$ ), then the ‘‘diffraction’’ effects of the quantum baker imply that the state  $A_{\mathbf{r},N}^{-1} \psi_{\nu,N}$  has nonzero components in  $R_1$  too.

However, the conditions can be satisfied up to a semiclassically small remainder, at least for  $n$  not too large. Instead of starting a general discussion, we will simply take for  $\psi_{\nu,N}$  a coherent state  $\psi_{x_o,\sigma}$  at some point  $x_o \in \Gamma_+^{(0)}$ . The ‘‘pure point quasimode’’  $\Psi_{\lambda,x_o}$  constructed by propagating this coherent state will converge to the pure point measure (2.8).

Let us take  $x_o$  away from  $\partial R_1$ , which ensures that  $\Pi_{\mathcal{S},N} \psi_{x_o,\sigma} = \mathcal{O}(e^{-CN})$ . We then evolve  $x_o$  backwards into  $x_{-n} = A_{\mathbf{r}}^{-n} x_o$ . The results of [6] show that, as long as  $x_{-n}$  stays away from  $\partial D$ , and the squeezing  $\sigma_{-n}$  is not too large, the backwards evolution of  $\psi_{x_o,\sigma}$  remains a coherent state:

$$(4.13) \quad A_{\mathbf{r},N}^{-n} \psi_{x_o,\sigma} = e^{i\theta_n} \psi_{x_{-n},\sigma_{-n}} + \mathcal{O}(e^{-CN}).$$

Because  $x_o$  was in  $\Gamma_+^{(0)}$ , the points  $x_{-n}$  are automatically in  $\Gamma_+^{(n)}$ , therefore far from the hole. As a consequence,  $\Pi_{\text{hole},N} \psi_{x_{-n},\sigma_{-n}} = \mathcal{O}(e^{-CN})$ , so that

$$(4.14) \quad B_{\mathbf{r},N} A_{\mathbf{r},N}^{-n} \psi_{x_o,\sigma} = A_{\mathbf{r},N}^{-n+1} \psi_{x_o,\sigma} + \mathcal{O}(e^{-CN}).$$

There remains to estimate the maximal  $n$  up to which the estimates (4.13,4.14) are correct. The estimate in [6, Prop.4.4] shows that (4.13) is valid as long as the distances between the momentum  $p(x_{-n})$  and the discontinuity points  $0, r_0, r_0 + r_1$  remain much larger than  $\sqrt{\sigma_{-n}/N}$ , the ‘‘height’’ of the coherent state in (4.13). A favorable choice for  $x_o = J_{\mathbf{r}}(\epsilon)$  consists in a sequence  $\epsilon$  such that the indices  $\epsilon_{-i}$  regularly jump between 0 and 2 when  $i \rightarrow -\infty$ . For simplicity we assume that both indices asymptotically occur with the same frequency. This ensures that the momenta  $p(x_{-n})$  stay at finite distance from  $0, r_0, r_0 + r_1$ . Yet, the squeezing parameter  $\sigma_{-n} = \sigma(r_{\epsilon_{-1}} \cdots r_{\epsilon_{-n}})^{-2}$  exponentially increases with  $n$ , more or less like  $\sigma(r_0 r_2)^{-n}$ . If we take an initial state  $\psi_{x_o,\sigma}$  very elongated along the horizontal, by selecting  $\sigma = N^{-1+\delta}$ , the estimates (4.13,4.14) are valid until the time  $n_{\text{break}} = (2 -$

$\delta) \frac{\log N}{|\log r_0 r_2|}$ , which is a sort of Ehrenfest time. Summing (4.14) over these times, we get the quasimode estimate

$$(4.15) \quad B_{\mathbf{r},N} \Psi_{\lambda,x_o} = \lambda \Psi_{\lambda,x_o} + \mathcal{O}(\lambda^{n_{break}}).$$

The remainder is of order  $\hbar^\alpha$  with  $\alpha = (2 - \delta) \frac{\log |\lambda|}{\log r_0 r_2}$ . We thus have a better quasimode if the quasi-eigenvalue  $\lambda$  is small.

In our numerical texts, we never found an eigenstate which would seem close to such a “pure point quasimode”.

## 5. A SOLVABLE TOY MODEL: WALSH QUANTIZATION OF THE OPEN BAKER

We now present an alternative (Walsh) quantization of the open symmetric baker  $B_{\mathbf{r}_{sym}}$ , introduced in [23, 24]. A similar quantization of the standard (closed) baker was proposed and studied in [29, 38]. We will drop the index  $\mathbf{r}_{sym}$  from our notations, and call  $B = B_{\mathbf{r}_{sym}}$ . We first recall the “Walsh quantum kinematics”.

**5.1. Walsh quantization of the torus.** Walsh quantization is possible for Planck’s constant taking values  $N = 3^k$ ,  $k \in \mathbb{N}$ . It uses the tensor product decomposition of the Hilbert space  $\mathcal{H}_N = (\mathbb{C}^3)^{\otimes k}$ . As notice in §2.1, each discrete position  $q_j = j/N$  can be represented by its ternary sequence  $q_j \equiv \cdot \epsilon_0 \epsilon_2 \dots \epsilon_{k-1}$ , with symbols  $\epsilon_i \in \{0, 1, 2\}$ . Accordingly, each position eigenstate  $\mathbf{q}_j$  can be represented as a tensor product:

$$\mathbf{q}_j = e_{\epsilon_0} \otimes e_{\epsilon_2} \otimes \dots \otimes e_{\epsilon_{k-1}} = |\cdot \epsilon_0 \epsilon_2 \dots \epsilon_{k-1}\rangle,$$

where  $\{e_0, e_1, e_2\}$  is the canonical (orthonormal) basis of  $\mathbb{C}^3$ . Walsh quantization consists in replacing the discrete Fourier transform (3.1) by the Walsh(-Fourier) transform  $W_N$ , which is a unitary operator preserving the tensor product structure of  $\mathcal{H}_N$ . We define it through its inverse  $W_N^*$ , which maps the position basis to the orthonormal basis of “Walsh momentum states”:

$$\tilde{\mathbf{p}}_j = W_N^* \mathbf{q}_j \stackrel{\text{def}}{=} F_3^* e_{\epsilon_{k-1}} \otimes F_3^* e_{\epsilon_{k-2}} \otimes \dots \otimes F_3^* e_{\epsilon_1} \otimes F_3^* e_{\epsilon_0}$$

(here  $F_3^*$  is the inverse Fourier transform on  $\mathbb{C}^3$ ). To agree with our notations of §2.1, we will index the symbols relative to the momentum coordinate by *negative* integers, so the Walsh momentum states will be denoted by

$$\tilde{\mathbf{p}}_j = |\epsilon_{-k} \dots \epsilon_{-1} \cdot\rangle, \quad \text{where } p_j = j/N \equiv \cdot \epsilon_{-1} \dots \epsilon_{-k}.$$

Each position (resp. momentum) state is microlocalized in a “quantum rectangle” of width (resp. height)  $3^{-k} = 1/N$  and height (resp. width) unity. More generally, for any  $\ell \in \{0, \dots, k\}$ , one can construct an orthonormal basis of  $\ell$ -(Walsh-)coherent states, localized in rectangles  $[\boldsymbol{\epsilon}] = [\boldsymbol{\epsilon}]_\ell$ , where  $\boldsymbol{\epsilon} = \epsilon_{-k+\ell} \dots \epsilon_{-1} \cdot \epsilon_0 \dots \epsilon_{\ell-1}$ . Such rectangles also have “quantum area”  $1/N$ . The  $\ell$ -coherent state supported by the rectangle  $[\boldsymbol{\epsilon}]_\ell$  takes the form

$$|[\boldsymbol{\epsilon}]_\ell\rangle \stackrel{\text{def}}{=} e_{\epsilon_0} \otimes \dots \otimes e_{\epsilon_{\ell-1}} \otimes F_3^* e_{\epsilon_{-k+\ell}} \otimes \dots \otimes F_3^* e_{\epsilon_{-1}}.$$

Like the squeezing parameter  $\sigma$  in the Gaussian case (see section 3), the index  $\ell$  describes the “aspect ratio” of the coherent state. One important difference between Gaussian and

Walsh coherent states lies in the fact that the latter is *strictly* localized both in momentum and position. Another difference is that, for each  $\ell$ , the  $\ell$ -coherent states make up an orthonormal basis, instead of an overcomplete family.

Each  $\ell$ -basis of coherent states can be used to define a Walsh-anti-Wick quantization and Walsh-Husimi functions. The latter are constant in each  $\ell$ -rectangle:

$$(5.1) \quad WH_{\psi_N}^\ell(x) = 3^k |\langle \psi_N | [\epsilon]_\ell \rangle|^2, \quad x \in [\epsilon]_\ell.$$

In the semiclassical limit, we will consider Walsh-Husimi representations such that  $\ell \sim k/2$  (which corresponds to  $\sigma \sim 1$ ). Semiclassical measures are defined as the weak limits of sequences  $(WH_{\psi_N}^\ell)$  of Walsh-Husimi measures, where  $N = 3^k \rightarrow \infty$ , and  $\ell = \ell(k) \sim k/2$ . It is shown in [1] that the precise choice of  $\ell = \ell(k)$  does not affect the limit measure, as long as both

$$(5.2) \quad \ell(k) \rightarrow \infty \quad \text{and} \quad k - \ell(k) \rightarrow \infty \quad \text{as} \quad k \rightarrow \infty.$$

**5.2. Walsh quantization of the open baker.** Here we recall the Walsh quantization of the open baker  $B = B_{\mathbf{r}_{sym}}$  [24, 1]. Mimicking the standard quantization (3.4), we simply replace the Fourier transforms  $F_N^*$ ,  $F_{N/3}$  by their Walsh analogues  $W_N$ ,  $W_{N/3}$  (with  $N = 3^k$ ,  $k \geq 0$ ), so that the Walsh-quantized open baker is given by the following matrix in the position basis:

$$(5.3) \quad \tilde{B}_N \stackrel{\text{def}}{=} W_N^* \begin{pmatrix} W_{N/3} & & \\ & 0 & \\ & & W_{N/3} \end{pmatrix}.$$

For any set of vectors  $v_0, \dots, v_{k-1} \in \mathbb{C}^3$ , this operator acts as follows on the tensor product state  $v_0 \otimes \dots \otimes v_{k-1}$ :

$$(5.4) \quad \tilde{B}_N(v_0 \otimes v_1 \dots \otimes v_{k-1}) = v_1 \otimes \dots \otimes v_{k-1} \otimes \tilde{F}_3^* v_0.$$

Here  $\tilde{F}_3^* = F_3^* \pi_{02}$ , where  $\pi_{02}$  is the orthogonal projector on  $\mathbb{C}e_0 \oplus \mathbb{C}e_2$  in  $\mathbb{C}^3$ .

This simple expression allows one to completely analyze the spectrum of  $\tilde{B}_N$  (see [24, Prop. 5.5]). It is determined by the two nontrivial eigenvalues  $\lambda_-$ ,  $\lambda_+$  of the matrix  $\tilde{F}_3^*$ . These eigenvalues have moduli  $|\lambda_+| \approx 0.8443$ ,  $|\lambda_-| \approx 0.6838$ . The spectrum of  $\tilde{B}_N$  has a gap: the long-living eigenvalues are contained in the annulus  $\{|\lambda_-| \leq |\lambda| \leq |\lambda_+|\}$ , while the rest of the spectrum lies at the origin. Most of the eigenvalues are degenerate. If we count multiplicities, the long-living spectrum satisfies the following asymptotics when  $k \rightarrow \infty$ :

$$(5.5) \quad \forall r > 0, \quad \# \left\{ \lambda_j \in \text{Spec}(\tilde{B}_N), |\lambda_j| \geq r \right\} = C_r 2^k + o(2^k),$$

$$C_r = \begin{cases} 1, & r < r_0 \\ 0, & r > r_0 \end{cases}, \quad r_0 \stackrel{\text{def}}{=} |\lambda_- \lambda_+|^{1/2} = 3^{-1/4}.$$

This nontrivial spectrum is carried by a subspace  $\mathcal{H}_{N, long}$  of exact dimension  $2^k$ . Since the trapped set for  $B = B_{\mathbf{r}_{sym}}$  has dimension  $2d = 2 \frac{\log 2}{\log 3}$ , the above asymptotics agrees with the Fractal Weyl law (3.5), when  $N$  is restricted to the values  $3^k$ . Although the density

of resonances is peaked on the circle  $\{|\lambda| = r_0\}$ , the spectrum densely fills the annulus  $\{|\lambda_-| \leq |\lambda| \leq |\lambda_+|\}$  as  $k \rightarrow \infty$  (see Figure 8). In the next section we construct some long-

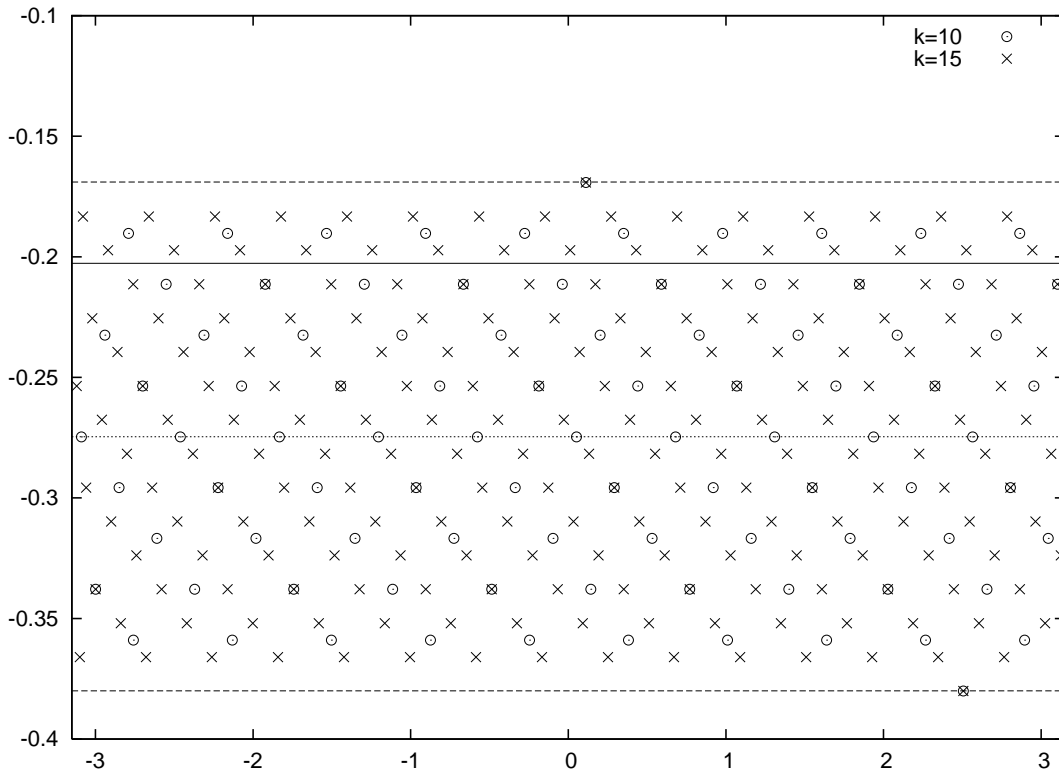


FIGURE 8. Nontrivial spectrum of the Walsh open baker  $\tilde{B}_N$  for  $N = 3^{10}$  (circles) and  $3^{15}$  (crosses), using a logarithmic representation (horizontal= $\arg \lambda_j$ , vertical= $\log |\lambda_j|$ ). We plot horizontal lines at the extremal radii  $|\lambda_{\pm}|$  of the spectrum (dashed), at the radius  $r_0 = |\lambda_- \lambda_+|^{1/2}$  of highest degeneracies (dotted) and at the radius corresponding to the natural measure (full).

living eigenstates of  $\tilde{B}_N$  and analyze their Walsh-Husimi measures. Notice that, due to the degeneracies of the spectrum, there is generally a large freedom to select eigenstates ( $\psi_N$ ) associated with a sequence of eigenvalues ( $\lambda_N$ ). Intuitively, that freedom should provide more possibilities for semiclassical measures.

We mention that J. Keating and coworkers have recently studied the eigenstates of a slightly different version of the Walsh-baker, namely a matrix  $\tilde{B}'_N$  obtained by replacing  $F_3$  by the “half-integer Fourier transform”  $(G_3)_{jj'} = N^{-1} e^{-2i\pi(j+1/2)(j'+1/2)/3}$  (private communication).

**5.3. Resonance eigenstates of the Walsh open baker.** We first provide the analogue of Theorem 1 for the Walsh-baker. We remind that a semiclassical measure  $\mu$  is now a

weak limit of some sequence of Walsh-Husimi measures  $(WH_{\psi_N}^\ell)_{N=3^k \rightarrow \infty}$ , where  $\psi_N$  is an eigenstate of  $B_N$ , and the index  $\ell \approx k/2$ .

**Theorem 2.** *Let  $\mu$  be a semiclassical measure for a sequence of long-living eigenstates of the Walsh-baker  $\tilde{B}_N$ . Then  $\mu$  is an eigenmeasure for  $B = B_{\mathbf{r}_{sym}}$ , and the corresponding decay rate  $\Lambda$  satisfies  $\Lambda = \lim_{N \rightarrow \infty} |\lambda_N|^2$ .*

Notice that all problems due to the discontinuities of  $B$  have disappeared.

*Sketch of Proof.* The proof is very similar with that of Theorem 1. The main ingredient is the (now exact) propagation of Walsh coherent states through  $\tilde{B}_N$ . For any  $0 \leq \ell \leq k-1$ , consider an  $\ell$ -rectangle  $[\epsilon]_\ell$ . Classically, this rectangle is either killed by  $B^{-1}$  (if  $\epsilon_{-1} = 1$ ), or it is mapped to the rectangle  $[\epsilon]_{\ell+1} = [\epsilon_{-k+\ell} \dots \epsilon_{-2} \cdot \epsilon_{-1} \epsilon_0 \dots \epsilon_{\ell-1}]$ . From the expression (5.4), one easily checks that the coherent state  $|\![\epsilon]_\ell\rangle$  is transformed accordingly by  $\tilde{B}_N^\dagger$ :

$$\tilde{B}_N^\dagger |[\epsilon]_\ell\rangle = (1 - \delta_{\epsilon_{-1}, 1}) |[\epsilon]_{\ell+1}\rangle.$$

This exact expression, which is the quantum counterpart of the classical shift (2.4), should be compared with the approximate expression (4.7). The rest of the proof is the same as for Theorem 1, using the family of Walsh-anti-Wick quantizations introduced in [1].  $\square$

From the structure of the spectrum, we can construct sequences of eigenvalues  $(\lambda_N)$  converging to any circle of radius  $\lambda \in [|\lambda_-|, |\lambda_+|]$ . We also know that any semiclassical measure is an eigenmeasure of  $B$ , so we can ask Questions 2 in the present framework (setting  $\lambda_{\max/\min} = |\lambda_\pm|$ ). In the next section we give partial answers to these questions.

Concerning the last point in Question 2, we notice that the decay rate of the natural measure for  $B = B_{\mathbf{r}_{sym}}$  is  $\Lambda_{nat} = 2/3$ , which is contained in the interval  $[|\lambda_-|^2, |\lambda_+|^2]$  (see figure 8). It is therefore relevant to ask whether  $\mu_{nat}$  can be a semiclassical measure. At present we are not able to answer that question.

**5.4. Construction of some eigenstates of  $\tilde{B}_N$ .** In this section we construct one particular eigenbasis of  $\tilde{B}_N$ , restricted to the subspace  $\mathcal{H}_{N, long}$  of long-living eigenstates (i.e. the subspace carrying the nontrivial spectrum).

The construction starts from the (right) eigenvectors  $v_-$ ,  $v_+ \in \mathbb{C}^3$  of  $\tilde{F}_3^*$  associated with  $\lambda_\pm$ . Notice that these two vectors (which we assume normalized) are not orthogonal to each other. For any sequence  $\boldsymbol{\eta} = \eta_0 \dots \eta_{k-1}$ ,  $\eta_i \in \{\pm\}$ , we form the tensor product state

$$|\boldsymbol{\eta}\rangle \stackrel{\text{def}}{=} v_{\eta_0} \otimes v_{\eta_1} \dots \otimes v_{\eta_{k-1}}.$$

The action (5.4) of  $\tilde{B}_N$  implies that

$$\tilde{B}_N |\boldsymbol{\eta}\rangle = \lambda_{\eta_0} |\tau(\boldsymbol{\eta})\rangle,$$

where  $\tau$  acts as a cyclic shift on the sequence:  $\tau(\eta_0 \dots \eta_{k-1}) = \eta_1 \dots \eta_{k-1} \eta_0$ . The orbit  $\{\tau^j(\boldsymbol{\eta}), j \in \mathbb{Z}\}$  contains  $\ell_\boldsymbol{\eta}$  elements, where the *period*  $\ell_\boldsymbol{\eta}$  of the sequence  $\boldsymbol{\eta}$  necessarily divides  $k$ . The states  $\{|\tau^j(\boldsymbol{\eta})\rangle, j = 0, \dots, \ell_\boldsymbol{\eta} - 1\}$  are not orthogonal to each other, but form a linearly independent family, which generates the  $\tilde{B}_N$ -invariant subspace  $\mathcal{H}_\boldsymbol{\eta} \subset \mathcal{H}_{N, long}$ .

The eigenvalues of  $\tilde{B}_N$  restricted to  $\mathcal{H}_\eta$  are of the form  $\lambda_{\eta,r} = e^{2i\pi r/\ell_\eta} \left( \prod_{j=0}^{\ell_\eta-1} \lambda_{\eta_j} \right)^{1/\ell_\eta}$  (with indices  $r = 1, \dots, \ell_\eta$ ), and the corresponding (non-normalized) eigenvectors read

$$(5.6) \quad |\psi_{\eta,r}\rangle = \sum_{j=0}^{\ell_\eta-1} c_{\eta,r,j} |\tau^j(\eta)\rangle, \quad c_{\eta,r,j} = \prod_{m=0}^{j-1} \frac{\lambda_{\eta_m}}{\lambda_{\eta,r}}.$$

Up to a normalization factor, this state is unchanged if  $\eta$  is replaced by  $\tau(\eta)$ . In the next subsections we investigate the localization properties of some of these eigenstates, by computing their Walsh-Husimi functions.

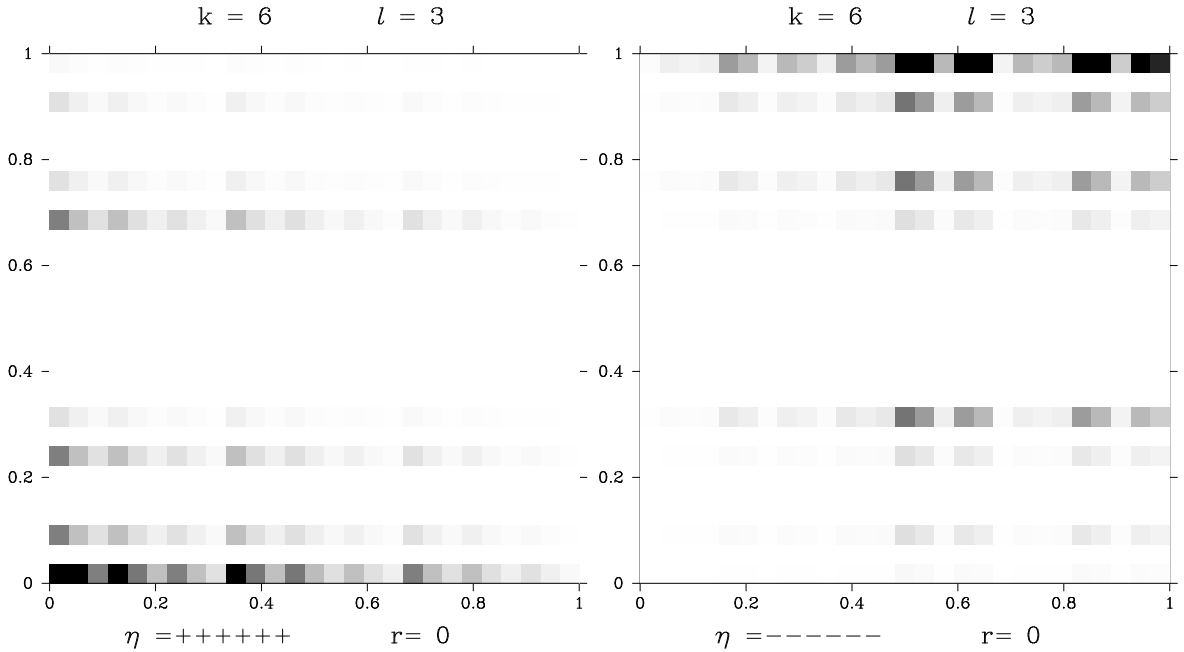


FIGURE 9. Walsh-Husimi functions of the extremal eigenstates  $\psi_{+,N}$ ,  $\psi_{-,N}$  of  $\tilde{B}_N$ , for  $N = 3^6$ ,  $\ell = 3$ . These are coarse-grained versions of the Bernoulli measures  $\rho_{\mathbf{P}_+}$  and  $\rho_{\mathbf{P}_-}$ .

5.4.1. *Extremal eigenstates.* The simplest case is provided by the sequence  $\eta = + + \dots +$ , which has period 1, so that  $|\psi_{+,N}\rangle = |\eta\rangle = v_+^{\otimes k}$  is the (unique) eigenstate associated with the largest eigenvalue  $\lambda_+$  (the longest-living resonance, see fig. 8).  $|\psi_{+,N}\rangle$  is normalized since  $v_+$  is so. For any choice of index  $0 \leq \ell \leq k$ , the Walsh-Husimi function of  $|\psi_{+,N}\rangle$  factorizes:

$$\forall x \in [\epsilon]_\ell, \quad WH_{\psi_{+,N}}^\ell(x) = 3^k \prod_{j=0}^{\ell-1} |\langle v_+, e_{\epsilon_j} \rangle|^2 \prod_{j=-1}^{-k+\ell} |\langle v_+, F_3^* e_{\epsilon_j} \rangle|^2.$$



The second product involves the vector  $w_+ \stackrel{\text{def}}{=} F_3 v_+$ , with components  $w_{+,\epsilon} = (1 - \delta_{\epsilon,1})v_{+,\epsilon}/\lambda_+$ . Following the formalism of §2.3.5, let  $\rho_{\mathbf{P}_+} = \rho_{\mathbf{r}_{sym}, \mathbf{P}_+}$  be the Bernoulli eigenmeasure of  $B$ , with weights  $P_{+,\epsilon} = |v_{+,\epsilon}|^2$ ,  $P_{+,\epsilon}^* = |w_{+,\epsilon}|^2$ . The above expression shows that the Husimi measure  $WH_{\psi_{+,N}}^\ell$  is equal to the measure  $\rho_{\mathbf{P}_+}$ , *conditioned on the grid formed by the  $\ell$ -rectangles*. Since the diameters of the rectangles decrease to zero as  $k \rightarrow \infty$  (assuming  $\ell(k) \sim k/2$ ), the Husimi measures  $(H_{\psi_{+,N}}^\ell)$  therefore converge to  $\rho_{\mathbf{P}_+}$ .

One can similarly show that the Husimi functions of the eigenstates  $\psi_- = v_-^{\otimes k}$ , associated with the eigenvalue  $\lambda_-$ , converge to the Bernoulli eigenmeasure  $\rho_{\mathbf{P}_-}$ , with weights  $P_{-,\epsilon} = |v_{-,\epsilon}|^2$ ,  $P_{-,\epsilon}^* = |w_{-,\epsilon}|^2$ , where  $w_{-,\epsilon} = (1 - \delta_{\epsilon,1})v_{-,\epsilon}/\lambda_-$ .

In figure 9 we plot the Husimi functions for  $\psi_{+,N}$  and  $\psi_{-,N}$  using the “isotropic squeezing”  $\ell = k/2$ .

Considering the fact that the eigenvalues  $\lambda_N$  close to the circles of radii  $|\lambda_+|$  and  $|\lambda_-|$  have small degeneracies, we propose the following

**Conjecture 2.** *Any sequence of eigenstates  $(\psi_N)$  with eigenvalues such that  $|\lambda_N| \rightarrow |\lambda_+|$  (resp.  $|\lambda_N| \rightarrow |\lambda_-|$ ) converges to the semiclassical measure  $\rho_{\mathbf{P}_+}$  (resp.  $\rho_{\mathbf{P}_-}$ ).*

This conjecture has been partially proven for the quantum baker  $\tilde{B}'_N$  studied by Keating *et al.*: in that case the two eigenvectors of  $\tilde{G}_3^*$  replacing  $v_\pm$  are orthogonal to each other, which greatly simplifies the computations. The limit measure  $\rho_{\mathbf{P}'_+}$  is then the “uniform” measure on the trapped set  $K$ , with  $\mathbf{P}'_+ = \mathbf{P}'_+^* = (1/2, 0, 1/2)$ .

5.4.2. *Non-unique semiclassical measures in the “bulk”.* In this section, we choose a rational number  $t = \frac{m}{n}$  with  $n > 0$ ,  $1 \leq m \leq n - 1$  (nonnecessarily coprime), and construct some families of eigenstates with eigenvalues of modulus  $\lambda_t = |\lambda_-|^{1-t} |\lambda_+|^t$ . In particular, we show that there exist *several* semiclassical measures of decay rate  $\Lambda = \lambda_t^2$ . This partially answers (in the case of the Walsh baker) the second point in Question 2.

For simplicity we only consider subsequences along the values  $k = nk'$ , with  $k' \in \mathbb{N}$ . To construct our sequence  $\boldsymbol{\eta}$ , we select an  $n$ -sequence  $\boldsymbol{\eta}_n$  containing  $m$  (+) and  $n - m$  (−), and repeat it  $k'$  times: we then obtain  $\boldsymbol{\eta} = (\boldsymbol{\eta}_n)^{k'}$ , which has the same period  $\ell_{\boldsymbol{\eta}}$  as the sequence  $\boldsymbol{\eta}_n$  (so that  $\ell_{\boldsymbol{\eta}}|n$ ). For this sequence  $\boldsymbol{\eta}$  we can construct  $\ell_{\boldsymbol{\eta}}$  eigenstates  $\psi_{\boldsymbol{\eta},r}$ , using the formula (5.6). The corresponding eigenvalues  $\lambda_{\boldsymbol{\eta},r}$  all have modulus  $\lambda_t$ .

**Proposition 5.** *i) Select  $\boldsymbol{\eta}_n$  as above. All sequences of eigenstates  $\{\psi_{\boldsymbol{\eta},r}\}_{k=nk' \rightarrow \infty}$ , with  $\boldsymbol{\eta} = (\boldsymbol{\eta}_n)^{k'}$  and  $r = r(k')$  arbitrary, converge to the same semiclassical measure  $\rho_{\boldsymbol{\eta}_n}$ , which is a linear combination of Bernoulli measures for  $B^n$  (see (5.11)).*

*ii) If  $\boldsymbol{\eta}_n$  and  $\boldsymbol{\eta}'_n$  are two  $n$ -sequences with  $m$  (+), which are not related by a cyclic permutation, then the semiclassical measures  $\rho_{\boldsymbol{\eta}_n}$ ,  $\rho_{\boldsymbol{\eta}'_n}$  (which share the same decay rate) are mutually singular.*

In Figure 10 we plot the Husimi functions of two states  $\psi_{\boldsymbol{\eta},0}$  constructed by periodizing two 4-sequences not cyclically related. The two measures visibly do not “charge” the same parts of the phase space.

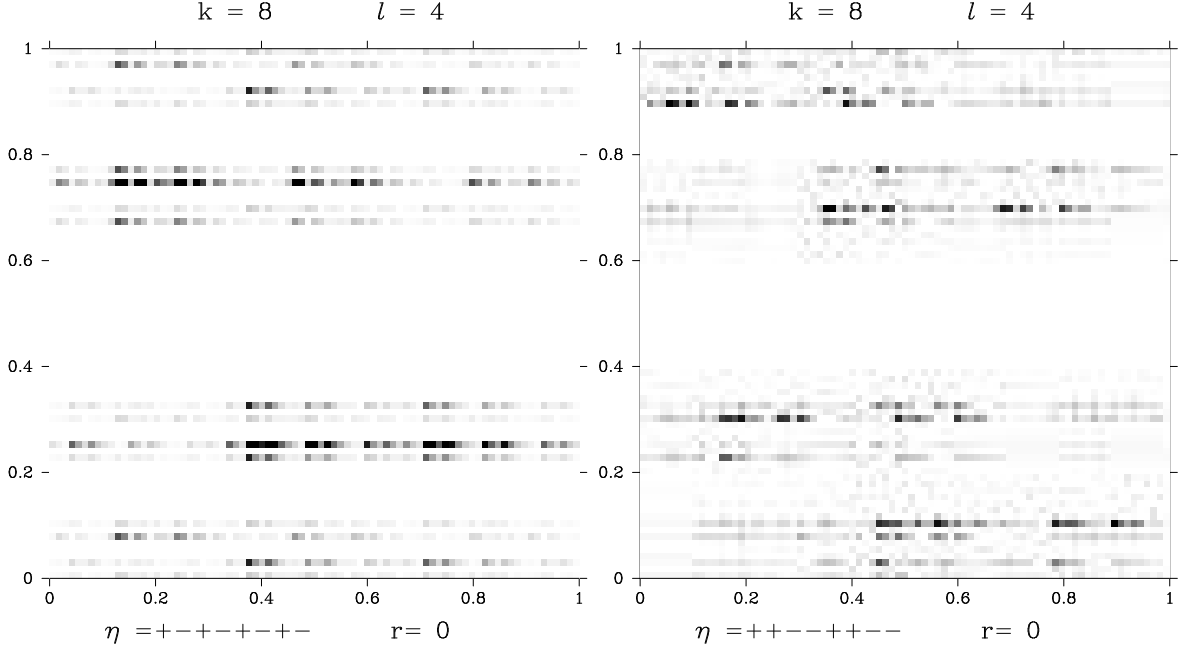


FIGURE 10. Walsh-Husimi functions of two eigenstates  $\psi_{\boldsymbol{\eta},r}$  constructed by periodizing  $\boldsymbol{\eta}_4 = + - + -$  (left) and  $\boldsymbol{\eta}_4 = + + - -$  (right), taking  $k' = 2$  and  $r = 0$  in both cases.

*Proof.* From (5.6), each state  $\psi_{\boldsymbol{\eta},r}$  is a combination of  $\ell_{\boldsymbol{\eta}}$  states  $|\tau^j(\boldsymbol{\eta})\rangle$ . When  $k' \rightarrow \infty$ , these  $\ell_{\boldsymbol{\eta}}$  states are asymptotically orthogonal to each other. Indeed, their overlaps can be decomposed as

$$\langle \boldsymbol{\eta} | \tau^j(\boldsymbol{\eta}) \rangle = (\langle \boldsymbol{\eta}_n | \tau^j(\boldsymbol{\eta}_n) \rangle)^{k'} \quad j = 0, \dots, \ell_{\boldsymbol{\eta}} - 1,$$

and for any  $j \neq 0 \bmod \ell_{\boldsymbol{\eta}}$  we have  $\boldsymbol{\eta}_n \neq \tau^j(\boldsymbol{\eta}_n)$ , which implies  $|\langle \boldsymbol{\eta}_n | \tau^j(\boldsymbol{\eta}_n) \rangle| \leq |\langle v_+, v_- \rangle|^2 < 1$ . As a result, the asymptotic normalization of  $\psi_{\boldsymbol{\eta},r}$  is

$$\|\psi_{\boldsymbol{\eta},r}\|^2 = \sum_{j=0}^{\ell_{\boldsymbol{\eta}}-1} |c_{\boldsymbol{\eta},r,j}|^2 + o(1), \quad k' \rightarrow \infty.$$

To study the phase space distribution of  $\psi_{\boldsymbol{\eta},r}$ , it is sufficient [1] to fix some rectangle  $[\boldsymbol{\alpha}] = [\alpha_{-l'} \dots \alpha_{-1} \cdot \alpha_0 \dots \alpha_{l-1}]$  and compute the weight of  $\psi_{\boldsymbol{\eta},r}$  on that rectangle:

$$(5.7) \quad \int_{[\boldsymbol{\alpha}]} H_{\psi_{\boldsymbol{\eta},r}}^{\ell}(x) dx = \langle \psi_{\boldsymbol{\eta},r} | \Pi_{[\boldsymbol{\alpha}]} | \psi_{\boldsymbol{\eta},r} \rangle = \sum_{j,j'=0}^{n-1} c_{\boldsymbol{\eta},r,j} \bar{c}_{\boldsymbol{\eta},r,j'} \langle \tau^{j'}(\boldsymbol{\eta}) | \Pi_{[\boldsymbol{\alpha}]} | \tau^j(\boldsymbol{\eta}) \rangle.$$

Above we assume that  $\ell > l$  and  $k - \ell > l'$ . The projector on  $[\boldsymbol{\alpha}]$  is given by:

$$\Pi_{[\boldsymbol{\alpha}]} = \pi_{\alpha_0} \otimes \pi_{\alpha_1} \dots \pi_{\alpha_{l-1}} \otimes (I)^{\otimes k-l-l'} \otimes F_3^* \pi_{\alpha_{-l'}} F_3 \dots F_3^* \pi_{\alpha_{-1}} F_3.$$

The tensor factor  $(I)^{k-l-l'-1}$  implies that each matrix element  $\langle \tau^{j'}(\boldsymbol{\eta}) | \Pi_{[\boldsymbol{\alpha}]} | \tau^j(\boldsymbol{\eta}) \rangle$  contains  $k' - \mathcal{O}(1)$  factors  $\langle \tau^{j'}(\boldsymbol{\eta}_n) | \tau^j(\boldsymbol{\eta}_n) \rangle = \langle \boldsymbol{\eta}_n | \tau^{j-j'}(\boldsymbol{\eta}_n) \rangle$ ; for the same reasons as above, this

element is small if  $j \neq j'$  and  $k' \gg 1$ . We are then lead to consider only the diagonal elements  $j = j'$ , which are easily computed:

$$(5.8) \quad \langle \tau^j(\boldsymbol{\eta}) | \Pi_{[\boldsymbol{\alpha}]} | \tau^j(\boldsymbol{\eta}) \rangle = \prod_{i=0}^{l-1} P_{\eta_{j+i}, \alpha_i} \prod_{i'=1}^{l'} P_{\eta_{j-i}, \alpha_{-i'}}^*.$$

Here we used as above the weights  $P_{\pm, \epsilon} = |v_{\pm, \epsilon}|^2$ ,  $P_{\pm, \epsilon}^* = |w_{\pm, \epsilon}|^2$ , and extended the definition of  $\eta_i$  to  $i \in \mathbb{Z}$  by periodicity. The right hand side exactly corresponds to  $\tilde{\rho}_{\tau^j(\boldsymbol{\eta}_n)}([\boldsymbol{\alpha}])$ , where  $\tilde{\rho}_{\tau^j(\boldsymbol{\eta}_n)}$  is a Bernoulli eigenmeasure for the map  $B^n$ , which we now describe. The symbolic dynamics for  $B^n$  uses  $3^n$  symbols, which are in one-to-one correspondence with  $n$ -sequences  $\{\epsilon_0 \dots \epsilon_{n-1}\}$ . Adapting the formalism of §2.3.5 to this new symbol space, one constructs the Bernoulli measure  $\tilde{\rho}_{\tau^j(\boldsymbol{\eta}_n)}$  using the following weight distributions  $\tilde{\mathbf{P}}, \tilde{\mathbf{P}}^*$ :

$$(5.9) \quad \tilde{P}_{\epsilon_0 \dots \epsilon_{n-1}} = \prod_{i=0}^{n-1} P_{\eta_{j+i}, \epsilon_i} \quad \text{and} \quad \tilde{P}_{\epsilon_{-1} \dots \epsilon_{-n}}^* = \prod_{i=1}^n P_{\eta_{j-i}, \epsilon_{-i}}^*.$$

One easily checks that the weight (5.8) is equal to  $\tilde{\rho}_{\tau^j(\boldsymbol{\eta}_n)}([\boldsymbol{\alpha}])$ .

The  $\ell_{\boldsymbol{\eta}}$  measures  $\tilde{\rho}_{\tau^j(\boldsymbol{\eta}_n)}$  are related to one another through the map  $B$ :

$$(5.10) \quad B^* \tilde{\rho}_{\tau^j(\boldsymbol{\eta}_n)} = |\lambda_{\eta_j}|^2 \tilde{\rho}_{\tau^{j+1}(\boldsymbol{\eta}_n)}.$$

Finally, the (non-normalized) semiclassical measure associated with the sequence  $\{\psi_{\boldsymbol{\eta}, r}\}_{k' \rightarrow \infty}$  (with  $r = r(k')$  arbitrary) is the weighted sum

$$(5.11) \quad \rho_{\boldsymbol{\eta}_n} = \sum_{j=0}^{\ell_{\boldsymbol{\eta}}-1} C_{\boldsymbol{\eta}_n, j} \tilde{\rho}_{\tau^j(\boldsymbol{\eta}_n)}, \quad \text{where} \quad C_{\boldsymbol{\eta}_n, j} = \left| \prod_{m=0}^{j-1} \frac{\lambda_{\eta_m}}{\lambda_t} \right|^2.$$

This is an eigenmeasure of  $B$ , with decay rate  $\Lambda = \lambda_t^2$  and total mass  $\sum_{j=0}^{n-1} C_{\boldsymbol{\eta}_n, j}$ .

The proof of the second statement uses the following fact: for any  $j, j' \in \mathbb{Z}$ , the weight distributions  $\tilde{\mathbf{P}}$  and  $\tilde{\mathbf{P}}'$  defining respectively the Bernoulli measures  $\tilde{\rho}_{\tau^j(\boldsymbol{\eta}_n)}$  and  $\tilde{\rho}_{\tau^{j'}(\boldsymbol{\eta}'_n)}$  (see (5.9)) are *different*. As a result, these two measures are mutually singular (see the end of §2.3.5), and so are the two linear combinations  $\rho_{\boldsymbol{\eta}_n}, \rho_{\boldsymbol{\eta}'_n}$ .  $\square$

## 6. CONCLUDING REMARKS

Our understanding of semiclassical measures associated with long-living resonant eigenstates remains very partial. The only “robust” result we obtain is the fact that these measures are eigenmeasures of the classical dynamics, and their decay rate is directly related with those of the corresponding resonant eigenstates. This result, which derives from Egorov’s theorem, can be proven as well for Hamiltonian scattering systems [25, Theorem 3].

Like for “closed chaotic systems”, there remains to understand which eigenmeasures can be obtained as semiclassical limits of resonant eigenstates (Question 2). The nongeneric example of the Walsh-quantized baker provides some answers (*e.g.* the existence of Bernoulli semiclassical measures, their nonunicity of semiclassical measures for decay rates in the

“bulk” of the spectrum), but we have no idea whether these results apply to more general systems. The “natural eigenmeasure” does not seem to play a particular role at the quantum level.

A tempting way of constraining the possible semiclassical measures would be to adapt the “entropic” methods of [1] to the present nonunitary framework. A desirable output of these methods would be, for instance, to forbid semiclassical measures from being of the pure point type described in §2.3.3. We leave such an approach to further study.

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