



HAL
open science

Capacitive flows on a 2D random net

Olivier Garet

► **To cite this version:**

| Olivier Garet. Capacitive flows on a 2D random net. 2006. hal-00090053v2

HAL Id: hal-00090053

<https://hal.science/hal-00090053v2>

Preprint submitted on 28 Aug 2006 (v2), last revised 14 May 2009 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

CAPACITIVE FLOWS ON A 2D RANDOM NET

OLIVIER GARET

ABSTRACT. This paper concerns maximal flows on \mathbb{Z}^2 traveling from a convex set to infinity, the flows being restricted by a random capacity. For every compact convex set A , we prove that the maximal flow $\Phi(nA)$ between nA and infinity is such that $\Phi(nA)/n$ almost surely converges to the integral of a deterministic function over the boundary of A . The limit can also be interpreted as the optimum of a deterministic continuous max-flow problem. We derive some properties of the infinite cluster in supercritical Bernoulli percolation.

1. INTRODUCTION

The aim of this paper is the study of maximal flows on \mathbb{Z}^2 traveling from a convex set A to infinity, the flows being restricted by a random capacity.

Formally, let $\vec{\mathbb{E}}^2 = \{(x, y) \in \mathbb{Z}^2 \times \mathbb{Z}^2 : \|x - y\|_1 = 1\}$ and $\mathbb{E}^2 = \{\{x, y\} \in \mathbb{Z}^2 \times \mathbb{Z}^2 : \|x - y\|_1 = 1\}$, where $\|\cdot\|_1$ is the ℓ^1 -norm: $\|(a, b)\|_1 = |a| + |b|$.

As usually, we denote by $\mathbb{L}^2 = (\mathbb{Z}^2, \mathbb{E}^2)$ the unoriented square lattice.

We say that a map $f : \vec{\mathbb{E}}^2 \rightarrow \mathbb{R}$ is a flow if $f(x, y) = -f(y, x)$ holds for each edge $(x, y) \in \vec{\mathbb{E}}^2$.

Let $(t_e)_{e \in \mathbb{E}^2}$ be a family of positive numbers.

We say that f is a capacitive flow from A to infinity if it satisfies

$$(1) \quad \begin{cases} |f(x, y)| \leq t_{\{x, y\}} & \text{for each bond } (x, y) \in \vec{\mathbb{E}}^2 \\ \text{Div } f(x) = 0 & \text{for } x \in \mathbb{Z}^2 \setminus A, \end{cases}$$

where $\text{Div } j(x) = \sum_{y \in \mathbb{Z}^2; \|x-y\|_1=1} j(x, y)$.

We denote by $\text{Capflow}(A, \infty)$ the set of capacitive flows from A to infinity. The aim is to study the maximal flow from a convex set A to infinity, that is

$$(2) \quad \max \left\{ \sum_{x \in A \cap \mathbb{Z}^2} \text{Div } j(x); j \in \text{Capflow}(A, \infty) \right\},$$

when the $(t_e)_{e \in \mathbb{E}^2}$ are given by some collection of independent identically distributed random variables.

Links with first passage percolation. The efficiency of methods coming from first passage percolation in studying the maximum flow through a randomly capacitated network was initially pointed out by Grimmett and Kesten [11].

2000 *Mathematics Subject Classification.* 60K35, 82B43.

Key words and phrases. first-passage percolation, maximal flows.

The point is that max-flow problem can be often translated into a min-cut problem thanks to the max-flow min-cut theorem of Ford and Fulkerson [7]. In the current setting, we can prove that

$$(3) \quad \max \left\{ \sum_{x \in A} \text{Div } j(x); j \in \text{Capflow}(A, \infty) \right\} = \text{Mincut}(A, \infty) \text{ a.s.},$$

where

$$(4) \quad \text{Mincut}(A, \infty) = \min \left\{ \sum_{e \in C} t_e; C \subset \mathbb{E}^2 \text{ and every infinite path in } \mathbb{L}^2 \text{ starting from } A \text{ meets } C \right\}$$

Such sets C are called cutsets (relatively to A). On \mathbb{L}^2 , cutsets can be characterized as follows: Let $\mathbb{Z}_*^2 = \mathbb{Z}^2 + (1/2, 1/2)$, $\mathbb{E}_*^2 = \{\{a, b\}; a, b \in \mathbb{Z}_*^2 \text{ and } \|a - b\|_1 = 1\}$ and $\mathbb{L}_*^2 = (\mathbb{Z}_*^2, \mathbb{E}_*^2)$. It is easy to see that \mathbb{L}_*^2 is isomorphic to \mathbb{L}^2 .

For each bond $e = \{a, b\}$ of \mathbb{L}^2 (*resp.* \mathbb{L}_*^2), let us denote by $s(e)$ the only subset $\{i, j\}$ of \mathbb{Z}_*^2 (*resp.* \mathbb{Z}^2) such that the quadrangle $aibj$ is a square in \mathbb{R}^2 . s is clearly an involution, and it is not difficult to see that s is a one-to-one correspondence between the cutsets in \mathbb{L}^2 and the sets in \mathbb{E}_*^2 that contains a closed path surrounding A . If C is minimal for inclusion, $s(C)$ is just a path surrounding A , then the quantity $\sum_{e \in C} t_e$ can be interpreted as the length of the path in a first-passage percolation scheme on \mathbb{Z}_*^2 . Using this duality, Grimmett and Kesten [11] have been able to study the maximal flow between parallel sides of a rectangle whose bases follow the orientation of the lattice.

This leads us to recall a fundamental result in first-passage percolation:

Assume that m is a probability measure on $[0, +\infty)$, such that

$$(5) \quad m(0) < 1/2 \text{ and } \exists c > 0, \int_{[0, \infty)} \exp(cx) dm(x) < \infty.$$

On $\Omega = [0, +\infty)^{\mathbb{E}^2}$, consider the probability measure $\mathbb{P} = m^{\otimes \mathbb{E}^2}$. For $e \in \mathbb{E}^2$, we define $t_e(\omega) = \omega_e$, thus the variables $(t_e)_{e \in \mathbb{E}^2}$ are independent identically distributed random variables with common law m .

For $\gamma \subset \mathbb{E}^2$, we define $l(\gamma) = \sum_{e \in \gamma} t_e$. We denote by $d(a, b)$ the length of the shortest path from a to b , that is

$$d(a, b) = \inf \{l(\gamma); \gamma \text{ contains a path from } a \text{ to } b\}.$$

Then, by the Cox–Durrett Shape Theorem [5], there exists a norm μ on \mathbb{R}^2 such that

$$(6) \quad \lim_{\|x\|_1 \rightarrow +\infty} \frac{d(0, x)}{\mu(x)} = 1 \text{ a.s.}$$

We can also define l^* by $l^*(A) = l(s(A))$ and a (random) distance d^* on $\mathbb{Z}_*^2 \times \mathbb{Z}_*^2$ by

$$d^*(a, b) = \inf \{l^*(\gamma); \gamma \text{ contains a path from } a \text{ to } b\}.$$

Since \mathbb{L}_*^2 is isomorphic to \mathbb{L}^2 , it is easy to see that $d^*(\cdot, \cdot)$ enjoys the same asymptotic properties as $d(\cdot, \cdot)$ does.

Main results. We first recall some common notations: \mathcal{H}^1 is the 1-dimensional normalized Hausdorff measure, λ^2 is the 2-dimensional Lebesgue measure, div is the usual divergence operator, and $C_c^1(\mathbb{R}^2, E)$ is the set of compactly supported C^1 vector functions from \mathbb{R}^2 to E . For each set A with finite perimeter, we define δ^*A to be the subset formed by the points $x \in \delta A$ where δA admits an unique outer normal, which is denoted by $\nu_A(x)$.

The main goal of the paper is the following theorem:

Theorem 1.1. *We suppose that $m(0) < 1/2$ and that m has an exponential moment. Then, for each convex set A with 0 in the interior, we have*

$$\lim_{n \rightarrow +\infty} \frac{\operatorname{Mincut}(nA, \infty)}{n} = \int_{\delta^*A} \mu(\nu_A(x)) d\mathcal{H}^1(x).$$

Equivalently,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \max \left\{ \sum_{x \in nA \cap \mathbb{Z}^2} \operatorname{Div} j(x); j \in \operatorname{Capflow}(nA, \infty) \right\} = \sup \left\{ \int_A \operatorname{div} f d\lambda^2(x); f \in C_c^1(\mathbb{R}^2, \mathcal{W}_\mu) \right\},$$

where

$$\mathcal{W}_\mu = \{x \in \mathbb{R}^2 : \langle x, w \rangle \leq \mu(w) \text{ for all } w\}.$$

Note that \mathcal{W}_μ is sometimes called the Wulff crystal associated to μ .

If we observe the last equality, we can see that the optimal value of a discrete random max-flow problem converges (after a suitable renormalization) to the optimum of a deterministic continuous max-flow problem.

In fact, we even have exponential bounds for the fluctuations around

$$(7) \quad \mathcal{I}(A) = \sup \left\{ \int_A \operatorname{div} f d\lambda^2(x); f \in C_c^1(\mathbb{R}^2, \mathcal{W}_\mu) \right\}.$$

Indeed, we prove that

Theorem 1.2. *For each $\varepsilon > 0$, there exist constants $C_1, C_2 > 0$, such that*

$$(8) \quad \forall n \geq 0 \quad \mathbb{P} \left(\frac{\operatorname{Mincut}(nA, \infty)}{n\mathcal{I}(A)} \notin (1 - \varepsilon, 1 + \varepsilon) \right) \leq C_1 \exp(-C_2 n).$$

With the help of Menger's Theorem, we get the following corollaries:

Corollary 1.3. *We consider supercritical Bernoulli percolation on the square lattice, where the edges are open with probability $p > p_c(2) = 1/2$. Then, for each convex set A with 0 in the interior, the maximal number $\operatorname{dis}(A)$ for a collection of disjoint open paths from A to infinity satisfies*

$$(9) \quad \exists C_1, C_2 > 0 \quad \forall n \geq 0 \quad \mathbb{P} \left(\frac{\operatorname{dis}(nA)}{n\mathcal{I}(A)} \notin (1 - \varepsilon, 1 + \varepsilon) \right) \leq C_1 \exp(-C_2 n),$$

where $\mathcal{I}(A)$ is the quantity defined in (7) in the case where the law m of passage times is the Bernoulli law $(1-p)\delta_0 + p\delta_1$.

This corollary has itself an easy and pleasant consequence:

Corollary 1.4. *We consider supercritical Bernoulli percolation on the square lattice. For each integer k , there almost surely exist k disjoint open bi-infinite paths.*

Note however that this amazing corollary is not really new, since it can be obtained as a consequence of Grimmett and Marstrand [9] – see also Grimmett [10], page 148, Th. 7.2.(a).

The paper is organized as follows: in Section 2, we remind some basic properties in first-passage percolation and prove some useful properties of the functional \mathcal{I} . Next, the proof of Theorem 1.2 falls naturally into two parts: Section 3 deals with the upper large deviations appearing in the Theorem, whereas Section 4 is about the lower ones. We complete the proof of Theorem 1.1 and establish the corollaries in Section 5. In the final section, we discuss the possibility of an extension to higher dimensions.

2. PRELIMINARY RESULTS

Notation. We denote by $\langle \cdot, \cdot \rangle$ the natural scalar product on \mathbb{R}^2 and by $\|\cdot\|_2$ the associated norm.

We also denote by \mathcal{S} the Euclidean unit sphere: $\mathcal{S} = \{x \in \mathbb{R}^2 : \|x\|_2 = 1\}$.

2.1. First-passage percolation. Let us introduce some notation and results related to first passage percolation. As previously, we suppose that (5) is satisfied and denote by μ the norm given by (6).

It will be useful to use

$$\mu_{\max} = \sup\{\mu(x); \|x\|_1 = 1\} \text{ and } \mu_{\min} = \inf\{\mu(x); \|x\|_1 = 1\}.$$

Of course, $0 < \mu_{\min} \leq \mu_{\max} < +\infty$ and we have

$$\forall x \in \mathbb{R}^2 \quad \mu_{\min}\|x\|_1 \leq \mu(x) \leq \mu_{\max}\|x\|_1.$$

The speed of convergence in Equation (6) can be precised:

Proposition 2.1 (Large deviations, Grimmett-Kesten [11]). *Then for any $\varepsilon > 0$, there exist $C_3, C_4 > 0$ such that*

$$\forall x \in \mathbb{Z}^2 \quad \mathbb{P}(d(0, x) \in [(1 - \varepsilon)\mu(x), (1 + \varepsilon)\mu(x)]) \geq 1 - C_3 \exp(-C_4\|x\|_1).$$

Note that in [11], the proof of this result is only written in the direction of the first axis, *i.e.* for $x = ne_1$. Nevertheless, it applies in any direction and computations can be followed in order to preserve a uniform control, whatever direction one considers. See for instance Garet and Marchand [8] for a detailed proof in an analogous situation. The control of $\mathbb{P}(d(0, x) > (1 + \varepsilon)\mu(x))$ is also a byproduct of the foregoing lemma 3.2.

2.2. Properties of \mathcal{I} . Since μ is a norm, it is obviously a convex function that does not vanish on the Euclidean sphere \mathcal{S} . So, it follows from Proposition 14.3 in Cerf [3] that

$$(10) \quad \int_{\partial^* A} \mu(\nu_A(x)) \, d\mathcal{H}^1(x) = \mathcal{I}(A)$$

holds for every Borel set of finite perimeter, which includes compact convex sets and polygons.

From Equation (7), it is easy to see that

$$(11) \quad \mathcal{I}(\lambda A) = \lambda \mathcal{I}(A)$$

holds for each Borel set A and each $\lambda > 0$.

Lemma 2.2. $\mathcal{I}(A) > 0$ for each convex set A with non-empty interior.

Proof. For each $x \in \partial^* A$, $\mu(\nu_A(x)) \geq \mu_{\min} \|\nu_A(x)\|_1 \geq \mu_{\min} \|\nu_A(x)\|_2 = \mu_{\min}$, so it follows from (10) that $\mathcal{I}(A) \geq \mu_{\min} \mathcal{H}^1(\partial A)$. \square

The next lemma clarifies the link between $\mathcal{I}(A)$ and μ when A is a polygon. Loosely speaking, $\mathcal{I}(A)$ is simply the μ -length of the polygon.

Lemma 2.3. Let A be a polygon whose sides are $[s_0, s_1], [s_1, s_2], \dots, [s_{n_e-1}, s_{n_e}]$, with $s_{n_e} = s_0$. We have

$$\mathcal{I}(A) = \sum_{i=0}^{n_e-1} \mu(s_i - s_{i+1}).$$

Proof. For $x = (a, b) \in \mathbb{Z}^2$, define $x^\perp = (-b, a)$. The map $\mathbb{R}^{\mathbb{Z}^2} \rightarrow \mathbb{R}^{\mathbb{Z}^2}$ that maps $(t_{x,y})_{\{x,y\} \in \mathbb{E}^2}$ to $(t_{-y,x})_{\{x,y\} \in \mathbb{E}^2}$ leaves $m^{\otimes \mathbb{E}^2}$ invariant, so it follows from (6) that $\mu(z) = \mu(z^\perp)$ holds for each $z \in \mathbb{Z}^2$. Since μ is homogeneous and continuous, the formula $\mu(z) = \mu(z^\perp)$ also holds for each $z \in \mathbb{R}^2$. We have

$$\begin{aligned} \int_{\partial^* A} \mu(\nu_A(x)) d\mathcal{H}^1(x) &= \sum_{i=0}^{n_e-1} \|s_i - s_{i+1}\|_2 \mu \left(\left(\frac{s_i - s_{i+1}}{\|s_i - s_{i+1}\|_2} \right)^\perp \right) \\ &= \sum_{i=0}^{n_e-1} \|s_i - s_{i+1}\|_2 \mu \left(\frac{s_i - s_{i+1}}{\|s_i - s_{i+1}\|_2} \right) \\ &= \sum_{i=0}^{n_e-1} \mu(s_i - s_{i+1}). \end{aligned}$$

\square

The next property of \mathcal{I} is fundamental and will play an important role in the proof of lower large deviations. Basically, it says that the shortest path that allows to surround a convex polygon is the frontier of the polygon itself.

Lemma 2.4. Let A, B be two polygons with $B \subset A$. We suppose that B is convex. Then $\mathcal{I}(B) \leq \mathcal{I}(A)$.

Proof. We proceed by induction on the number $n(A, B)$ of vertices of B which do not belong to ∂A . When $n = 0$, we just apply the triangle inequality. When $n > 1$, we build a polygon A' with $B \subset A' \subset A$, $\mathcal{I}(A') \leq \mathcal{I}(A)$ and $n(A', B) < n(A, B)$ as follows: let z be a vertex of B which is not in ∂A . Since B is convex, there exists an affine map φ with $\varphi(z) = 0$ and $\varphi(x) < 0$ for x in $B \setminus \{z\}$. Let D be the connected component of z in $A \cap \{x \in \mathbb{R}^2 : \varphi(x) \geq 0\}$. D is a polygon which has a side F in $\{x \in \mathbb{R}^2 : \varphi(x) \geq 0\}$. Note $A' = A \setminus D$. Denote by s_a and s_b the ends of F and define $\mu(F) = \mu(s_b - s_a)$. We have $\mathcal{I}(A) = (\mathcal{I}(A') - \mu(F)) + (\mathcal{I}(D) - \mu(F))$. By the triangle inequality $\mu(F) \leq \mathcal{I}(D)/2$, so $\mathcal{I}(A') \leq \mathcal{I}(A)$. \square

We will also need convenient approximations of convex sets by convex polygons. This is the goal of the next lemma.

Lemma 2.5. Let A be a bounded convex set with 0 in the interior of A . For each $\varepsilon > 0$, there exist convex polygons P and Q such that

$$0 \in P \subset A \subset Q \text{ and } \mathcal{I}(Q) - \varepsilon \leq \mathcal{I}(A) \leq \mathcal{I}(P) + \varepsilon.$$

Proof. A proof of the existence of Q can be found in Lachand-Robert and Oudet [13] in a more general setting. The existence of P is simpler: let $(A_p)_{p \geq 1}$ such that

- for each $p \geq 1$, A_p is a convex polygon
- for each $p \geq 1$, $A_p \subset A$
- $0 \in A_p$ for large p .
- $\lim_{p \rightarrow +\infty} \lambda^2(A \setminus A_p) = 0$.

(For example, take A_p as the convex hull of x_1, \dots, x_p , where $(x_p)_{p \geq 1}$ is dense in ∂A : this ensures that $\bigcup_{p \geq 1} A_p \supset A \setminus \partial A$.) For fixed $f \in C_c^1(\mathbb{R}^2, \mathcal{W}_\mu)$, $A \mapsto \int_A \operatorname{div} f \, d\lambda^2(x)$ is continuous with respect to the L^1 convergence of Borel sets, so $A \mapsto \mathcal{I}(A)$ is lower semicontinuous. Then, $\mathcal{I}(A) \leq \varliminf_{p \rightarrow +\infty} \mathcal{I}(A_p)$, so there exists $p \geq 1$ with $\mathcal{I}(A) \leq \mathcal{I}(A_p) + \varepsilon$ and $0 \in A_p$. \square

3. UPPER LARGE DEVIATIONS

Theorem 3.1. *For each $\varepsilon > 0$ there exist constants $C_5, C_6 > 0$, such that*

$$(12) \quad \mathbb{P}(\operatorname{Mincut}(nA, \infty) \geq n\mathcal{I}(A)(1 + \varepsilon)) \leq C_5 \exp(-C_6 n),$$

The proof falls naturally into three parts

- (1) Approximate nA by a polygon
- (2) Parallel outside nA (but close to nA) the boundary of the polygon: it creates a new polygon
- (3) Hope that successive vertices of the newly created polygon can be joined by a path which is short enough and does not enter in nA

Therefore, we need a lemma which roughly says that one can find a path from x to y that has length smaller than $(1 + \varepsilon)\mu(x - y)$ and is not far from a straight line. To this aim, we must introduce some definitions:

Let $y, z \in \mathbb{R}^2$, $\hat{x} \in \mathcal{S}$, and $R, h > 0$. We define

$$d(y, \mathbb{R}\hat{x}) = \|y - \langle y, \hat{x} \rangle \hat{x}\|_2, \text{ (the Euclidean distance from } y \text{ to the line } \mathbb{R}\hat{x}\text{)}$$

$$\operatorname{Cyl}_z(\hat{x}, R, h) = \{y \in \mathbb{Z}^2 : d(y - z, \mathbb{R}\hat{x}) \leq R \text{ and } 0 \leq \langle y - z, \hat{x} \rangle \leq h\},$$

For $R > 0$ and $z, z' \in \mathbb{R}^2$ with $z \neq z'$, we also define

$$\widetilde{\operatorname{Cyl}}(z, z', R) = \operatorname{Cyl}_z\left(\frac{z' - z}{\|z' - z\|_2}, R, \|z - z'\|_2\right).$$

Lemma 3.2. *For $z \in \mathbb{R}^2$, $\hat{x} \in \mathcal{S}$, and sufficiently large h and r , we can define s_0 (resp. s_f) to be the integer point in $\operatorname{Cyl}_z(\hat{x}, r, h)$ which is the closest to z (resp. $z + h\hat{x}$). We also define the longitudinal crossing time $t_{\operatorname{long}}(\operatorname{Cyl}_z(\hat{x}, h, r))$ of the cylinder $\operatorname{Cyl}_z(\hat{x}, r, h)$ as the minimal time needed to cross it from s_0 to s_f , using only edges in the cylinder.*

Then for any $\varepsilon > 0$, and any function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\lim_{r \rightarrow +\infty} f(r) = +\infty$, there exist two strictly positive constants C_7 and C_8 such that

$$\begin{aligned} & \forall z \in \mathbb{R}^2 \quad \forall \hat{x} \in \mathcal{S} \quad \forall h > 0 \quad \mathbb{P}(t_{\operatorname{long}}(\operatorname{Cyl}_z(\hat{x}, f(h), h)) \geq \mu(\hat{x})(1 + \varepsilon)h) \\ & \leq C_7 \exp(-C_8 h). \end{aligned}$$

Proof. For $x \in \mathbb{Z}^2$ and $t \geq 0$, let

$$\mathcal{B}_x(t) = \{y \in \mathbb{Z}^2 : \mu(x - y) \leq t\}.$$

For $x, y \in \mathbb{Z}^2$ denote by $I_{x,y}$ the length of the shortest path from x to y which is inside $\mathcal{B}_x(1, 25\mu(x - y)) \cap \mathcal{B}_y(1, 25\mu(x - y))$. Of course $I_{x,y}$ has the same law than $I_{0,x-y}$, and we simply define $I_x = I_{0,x}$. We begin with an intermediary lemma.

Lemma 3.3. *For each $\varepsilon \in (0, 1]$, there exists $M_0 = M_0(\varepsilon)$ such that for each $M \geq M_0$ there exist $c = c(\varepsilon, M) < 1$ and $t = t(\varepsilon, M) > 0$ with*

$$\|x\| \in [M/2, 2M] \implies \mathbb{E} \exp(t(I_x - (1 + \varepsilon)\mu(x))) \leq c.$$

Proof. Let Y be a random variable with law m and let $\gamma > 0$ be such that $\mathbb{E} e^{2\gamma Y} < +\infty$. First, Equation (6) easily implies the following almost sure convergence:

$$\lim_{\|x\| \rightarrow +\infty} \frac{I_x}{\mu(x)} = 1.$$

By considering a deterministic path from 0 to x with length $\|x\|$, we see that I_x is dominated by a sum of $\|x\|$ independent copies of Y denoted by $Y_1, \dots, Y_{\|x\|}$, and thus $I_x/\|x\|$ is dominated by

$$\frac{1}{\|x\|} \sum_{k=1}^{\|x\|} Y_k.$$

This family is equi-integrable by the law of large numbers. So $(I_x/\|x\|)_{x \in \mathbb{Z}^2 \setminus \{0\}}$ and then $(I_x/\mu(x))_{x \in \mathbb{Z}^2 \setminus \{0\}}$ are also equi-integrable families, which implies that

$$(13) \quad \lim_{\|x\| \rightarrow +\infty} \frac{\mathbb{E} I_x}{\mu(x)} = 1.$$

Note that for every $y \in \mathbb{R}$ and $t \in (0, \gamma]$,

$$e^{ty} \leq 1 + ty + \frac{t^2}{2} y^2 e^{t|y|} \leq 1 + ty + \frac{t^2}{\gamma^2} e^{2\gamma|y|}.$$

Let $\tilde{I}_x = I_x - (1 + \varepsilon)\mu(x)$ and suppose that $t \in (0, \gamma]$. Then, since $|\tilde{I}_x| \leq I_x + 2\mu(x)$, the previous inequality implies that

$$e^{t\tilde{I}_x} \leq 1 + t\tilde{I}_x + \frac{t^2}{\gamma^2} e^{4\gamma\mu(x)} e^{2\gamma I_x}.$$

Since $\mu(x) \leq \|x\|\mu_{\max}$ and $I_x \leq Y_1 + \dots + Y_{\|x\|}$, we can define $\rho = e^{4\gamma\mu_{\max}} \mathbb{E} e^{2\gamma Y}$ and thus obtain

$$\mathbb{E} e^{t\tilde{I}_x} \leq 1 + t[\mathbb{E} \tilde{I}_x + \frac{t}{\gamma^2} \rho^{\|x\|}]$$

Considering Equation (13), let M_0 be such that $\|x\| \geq M_0/2$ implies $\frac{\mathbb{E} I_x}{\mu(x)} \leq 1 + \varepsilon/3$.

For x such that $\|x\| \geq M_0$, we have $\mathbb{E} \tilde{I}_x \leq -\frac{2}{3}\varepsilon\mu(x)$, so

$$\mathbb{E} e^{t\tilde{I}_x} \leq 1 + t[-\frac{2}{3}\varepsilon\mu(x) + \frac{t}{\gamma^2} \rho^{\|x\|}] \leq 1 + t[-\frac{2}{3}\varepsilon\mu_{\max} + \frac{t}{\gamma^2} \rho^{\|x\|}].$$

Therefore, we can take $t = t(\varepsilon, M) = \min(\gamma, \gamma^2\mu_{\max}\frac{\varepsilon}{3}\rho^{-2M})$ and $c = c(\varepsilon, M) = 1 - \frac{1}{3}\varepsilon\mu_{\max}t(\varepsilon, M)$. \square

Let us come back now to the proof of lemma 3.2. Let $\varepsilon \in (0, 1)$ and consider the integer $M_0 = M_0(\varepsilon/3)$ given by the previous lemma. Let $M_1 = M_1(\varepsilon)$ be an integer greater than M_0 and such that

$$(14) \quad (1 + \varepsilon/3) \left(1 + \frac{\mu_{\max}}{\mu_{\min}} \frac{2}{M_1} \right) \leq 1 + \varepsilon/2.$$

Let $N = 1 + \text{ip}(h/M_1) - \text{ip}(x)$ denotes the integer part of x – and, for each $i \in \{0, \dots, N\}$ denote by x_i the integer point in the cylinder which is the closest to $z + \frac{ih\hat{x}}{N}$. Note that

$$\left(1 - \frac{1}{N} \right) M_1 \leq \frac{h}{N} \leq M_1.$$

1. Let i_0 be an integer with $i_0 \geq \max\left(\frac{1, 25(2+M_1)\sqrt{2}}{\mu_{\min}}, 2\right)$. There exists a deterministic path inside the cylinder from x_0 to x_{i_0} (*resp.* x_{N-i_0} to x_N) which uses less than $2i_0h/N$ edges: we denote by L_{start} (*resp.* L_{end}) the random length of this path. Markov's inequality easily gives

$$(15) \quad \mathbb{P}(L_{start} > \frac{\varepsilon}{4}\mu(\hat{x})h) + \mathbb{P}(L_{end} > \frac{\varepsilon}{4}\mu(\hat{x})h)$$

$$(16) \quad \leq 2(\mathbb{E} e^{2\gamma Y})^{2i_0M_1} \exp(-\frac{\gamma\varepsilon}{2}h\mu_{\min}) \leq C'e^{-C_5h}.$$

2. For each $i, j \in \{0, \dots, N-1\}$, we have $|\mu(x_i - x_j) - |j - i|h\mu(\hat{x})/N| \leq 2\mu_{\max}$, thus, if h is larger than some h_0 , $\mathcal{B}_{x_i}(1, 25\mu(x_i - x_{i+1})) \cap \mathcal{B}_{x_j}(1, 25\mu(x_j - x_{j+1})) = \emptyset$ as soon as $|j - i| \geq 2$.

Let $h_1 = h_1(\varepsilon, f) \geq h_0$ be such that $\forall h \geq h_0$, $f(h) \geq i_0$. If we take h larger than h_1 , then the set

$$\bigcup_{i=i_0}^{N-i_0-1} \mathcal{B}_{x_i}(1, 25\mu(x_i - x_{i+1}))$$

is completely inside the cylinder. So, provided that $h \geq h_1$, we have, inside the cylinder, a path from x_0 to x_N with length

$$L_{start} + \sum_{i=i_0}^{N-i_0-1} I_{x_i, x_{i+1}} + L_{end}$$

$$\text{Let } S_{odd} = \sum_{\substack{2 \leq i \leq N-3 \\ i \text{ odd}}} I_{x_i, x_{i+1}} \text{ and } S_{even} = \sum_{\substack{2 \leq i \leq N-3 \\ i \text{ even}}} I_{x_i, x_{i+1}}.$$

By the definition of $(x_i)_{1 \leq i \leq N}$, we have

$$\begin{aligned} \sum_{\substack{2 \leq i \leq N-3 \\ i \text{ odd}}} \mu(x_{i+1} - x_i) &\leq \sum_{\substack{2 \leq i \leq N-3 \\ i \text{ odd}}} \frac{h\mu(\hat{x})}{N} + 2\mu_{\max} \\ &\leq \frac{N-3}{2} \left(\frac{h\mu(\hat{x})}{N} + 2\mu_{\max} \right) \\ &\leq \frac{N}{2} \frac{h\mu(\hat{x})}{N} + (N-1) \frac{\mu_{\max}}{\mu_{\min}} \mu(\hat{x}) \\ &\leq \frac{h\mu(\hat{x})}{2} \left(1 + 2 \frac{\mu_{\max}}{\mu_{\min}} \frac{1}{M_1} \right) \end{aligned}$$

Then, using (14), we can write for each $t \geq 0$:

$$\begin{aligned}
\mathbb{P}(S_{\text{odd}} \geq \frac{h\mu(\hat{x})}{2}(1 + \varepsilon/2)) &\leq \mathbb{P}(S_{\text{odd}} \geq (1 + \varepsilon/3) \sum_{\substack{2 \leq i \leq N-3 \\ i \text{ odd}}} \mu(x_{i+1} - x_i)) \\
&\leq \mathbb{P}(\sum_{\substack{2 \leq i \leq N-3 \\ i \text{ odd}}} I_{x_i, x_{i+1}} - (1 + \varepsilon/3)\mu(x_{i+1} - x_i) \geq 0) \\
&\leq \mathbb{P}(\exp(t \sum_{\substack{2 \leq i \leq N-3 \\ i \text{ odd}}} I_{x_i, x_{i+1}} - (1 + \varepsilon/3)\mu(x_{i+1} - x_i)) \geq 1) \\
&\leq \mathbb{E} \exp(t \sum_{\substack{2 \leq i \leq N-3 \\ i \text{ odd}}} I_{x_i, x_{i+1}} - (1 + \varepsilon/3)\mu(x_{i+1} - x_i)) \\
&\leq \prod_{\substack{2 \leq i \leq N-3 \\ i \text{ odd}}} \mathbb{E} \exp(t I_{x_i, x_{i+1}} - (1 + \varepsilon/3)\mu(x_{i+1} - x_i))
\end{aligned}$$

We take now $t = t(\varepsilon/3, M_1(\varepsilon))$ and $\rho = \rho(\varepsilon/3, M_1(\varepsilon))$. For each i , we have $\mu(x_i - x_{i+1}) \in [M_1/2, 2M_1]$, thus we can apply the previous lemma and get

$$\begin{aligned}
\mathbb{P}(S_{\text{odd}} \geq \frac{h\mu(\hat{x})}{2}(1 + \varepsilon/2)) &\leq \rho^{(N-5)/2} \\
&\leq \rho^{hM_1(\varepsilon)/2-3/2} = A \exp(-Bh)
\end{aligned}$$

with $A = \rho^{-3/2}$ and $B = -\frac{1}{2M_1(\varepsilon)} \ln \rho$.

Similarly, $\mathbb{P}(S_{\text{even}} \geq \frac{h\mu(\hat{x})}{2}(1 + \varepsilon/2)) \leq A \exp(-Bh)$, so it suffices to put the pieces together to conclude the proof. \square

Proof of Theorem 3.1. We first deal with the case where A is a convex polygon. Let us denote by s_0, s_1, \dots, s_{n_e} the vertices of A , with $s_{n_e} = s_0$. We suppose that the vertices are in trigonometric order. For each $i \in \{0, n_e - 1\}$, let v_i be such that $\langle v_i, s_{i+1} - s_i \rangle = 0$ and $\langle v_i, s_i \rangle = 1$. For $x \in \mathbb{R}^2$, define $\varphi_i(x) = \langle v_i, x \rangle$. With our conventions

$$nA = \bigcap_{i=0}^{n_e-1} \{x \in \mathbb{R}^2 : \varphi_i(x) \leq n\}.$$

For $z \in \mathbb{R}^2$, we define $\text{Int}(z)$ as the only $x \in \mathbb{Z}_*^2$ such that $z \in x + [-1/2, 1/2) \times [-1/2, 1/2)$. Let $\varepsilon > 0$. For $i \in \{0, \dots, n_e\}$, let $y_i = \text{Int}(n(1 + \varepsilon)s_i)$.

Our goal is to build for each i a path from y_i to y_{i+1} which does not enter nA and is short enough. Define $M = \max\{\|v_i\|_2; 0 \leq i \leq n_e - 1\}$ and $S = \max\{\mu(s_i - s_{i+1}); 0 \leq i \leq n_e - 1\}$.

It is easy to see that

$$\forall r \geq 0 \quad \varphi_i \geq n(1 + \varepsilon) - M\left(\frac{\sqrt{2}}{2} + r\right) \text{ on } \widetilde{\text{Cyl}}(y_i, y_{i+1}, r)$$

Moreover, for each $i \in \{0, \dots, n_e - 1\}$, we have

$$M\left(\frac{\sqrt{2}}{2} + \frac{\varepsilon}{4MS}\|y_i - y_{i+1}\|_2\right) \leq M\left(\frac{\sqrt{2}}{2} + \frac{\varepsilon}{4MS}(nS + \sqrt{2})\right) \leq \frac{n\varepsilon}{2},$$

provided that n is large enough. Therefore, it follows that $\varphi_i \geq (1 + \varepsilon/2)n$ on $\widetilde{\text{Cyl}}(y_i, y_{i+1}, \frac{\varepsilon}{4MS} \|y_i - y_{i+1}\|_2)$, which means that this set is off nA .

Since $\mu(y_i - (1 + \varepsilon)ns_i) \leq \mu_{\max}$, we know that

$$\sum_{i \in \{0, \dots, n_e\}} \mu(y_i - y_{i+1}) \leq n(1 + \varepsilon)\mathcal{I}(A) + 2n_e\mu_{\max} \leq n(1 + \varepsilon)^2\mathcal{I}(A)$$

provided that n is large enough.

Then, one can see that for n greater than some (deterministic) integer n_0 , the event

$$A_n = \bigcap_{i \in \{0, \dots, n_e-1\}} \left\{ t_{\text{long}} \left(\widetilde{\text{Cyl}}(y_i, y_{i+1}, \frac{\varepsilon}{4MS} \|y_i - y_{i+1}\|_2) \right) < (1 + \varepsilon)\mu(y_{i+1} - y_i) \right\}$$

satisfies

$$A_n \subset \{\text{Cut}(nA) \leq n(1 + \varepsilon)^3\mathcal{I}(A)\}$$

We are now ready to apply lemma 3.2 with $f(h) = \frac{\varepsilon}{4MS}h$. It comes that

$$\begin{aligned} P(\text{Cut}(nA) > n(1 + \varepsilon)^3\mathcal{I}(A)) &\leq \mathbb{P}(A_n^c) \\ &\leq \sum_{i=0}^{n_e-1} C_7 \exp(-C_8 \|y_i - y_{i+1}\|_2) \\ &\leq \sum_{i=0}^{n_e-1} C_7 e^{C_8\sqrt{2}} \exp(-C_8 \|s_i - s_{i+1}\|_2 n), \\ &\leq c_1 \exp(-c_2 n), \end{aligned}$$

with $c_1 = n_e C_7 e^{C_8\sqrt{2}}$ and $c_2 = C_8 \min_i \|s_i - s_{i+1}\|_2$.

Since ε is arbitrary, the theorem follows when A is a polygon.

Let us go to the general case: by lemma 2.5, there exists a convex polygon Q with $Q \supset A$ and $(1 + \varepsilon)\mathcal{I}(A) \leq (1 + \varepsilon/2)\mathcal{I}(Q)$

By its very definition, $\text{Mincut}(nA, \infty) \leq \text{Mincut}(nQ, \infty)$. Then

$$\begin{aligned} \mathbb{P}(\text{Mincut}(nA, \infty) \geq n\mathcal{I}(A)(1 + \varepsilon)) &\leq \mathbb{P}(\text{Mincut}(nQ, \infty) \geq n\mathcal{I}(A)(1 + \varepsilon)) \\ &\leq \mathbb{P}(\text{Mincut}(nQ, \infty) \leq n\mathcal{I}(Q)(1 + \varepsilon/2)). \end{aligned}$$

Hence the result follows from the polygonal case. \square

4. LOWER LARGE DEVIATIONS

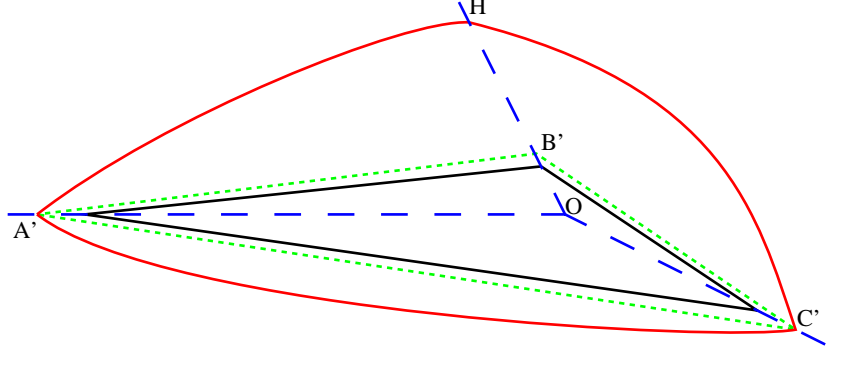
Theorem 4.1. *For each $\varepsilon > 0$, there exist constants $C_9, C_{10} > 0$, such that*

$$(17) \quad \forall n \geq 1 \quad \mathbb{P}(\text{Mincut}(nA, \infty) \leq n\mathcal{I}(A)(1 - \varepsilon)) \leq C_9 \exp(-C_{10}n),$$

The choice of a strategy for the proof of lower large deviations is more difficult than for the upper ones. An important point is that it is hopeless to consider the sides of the polygon separately.

Indeed, consider the following picture: the red curve and the green one surround the black triangle. Of course, it is expected that the minimal cutset looks like the green triangle rather than like the red ones. However, the red path from A' to H is shorter than the green one from A' to B' . But this advantage is lost on the next side, because the red path from H to C' is much longer than the green one

from B' to C' . So it appears that we must think globally, using the perimeter of surrounding curves. To this aim, Lemma 2.4 will be particularly useful.



Proof. Again, we first deal with the case where A is a convex polygon whose sides are $[s_0, s_1], [s_1, s_2], \dots, [s_{n_e-1}, s_{n_e}]$, with $s_{n_e} = s_0$. We denote by $L_{n,i}$ the points $x \in \mathbb{Z}_*^2$ that touch a bond which intersects $[1, +\infty)n s_i$.

Lemma 4.2. *For each $\varepsilon > 0$, there exist $C_{11} = C_{11}(\varepsilon), C_{12} = C_{12}(\varepsilon)$, such that*

$$\mathbb{P}(\exists i \in \{0, n_e - 1\} \exists (x, y) \in L_{n,i} \times L_{n,i+1} \quad d(x, y) \leq (1 - \varepsilon)\mu(x - y)) \leq C_{11} \exp(-C_{12}n).$$

Proof. Since $\{0, \dots, n_e - 1\}$ is finite, it is sufficient to prove that for each i, j with $0 \leq i < j < n_e$, there exists $C_{11}(i, j) > 0$ and $C_{12}(i, j) > 0$ with

$$\mathbb{P}(\exists (x, y) \in L_{n,i} \times L_{n,j} \quad d(x, y) \leq (1 - \varepsilon)\mu(x - y)) \leq C_{11}(i, j) \exp(-C_{12}(i, j)n)$$

Thanks to Proposition 2.1, we can write

$$\begin{aligned} & \mathbb{P}(\exists (x, y) \in L_{n,i} \times L_{n,j} \quad d(x, y) \leq (1 - \varepsilon)\mu(x - y)) \\ & \leq \sum_{(x, y) \in L_{n,i} \times L_{n,j}} \mathbb{P}(d(x, y) \leq (1 - \varepsilon)\mu(x - y)) \\ & \leq \sum_{(x, y) \in L_{n,i} \times L_{n,j}} C_3 \exp(-C_4 \|x - y\|_2) \\ & \leq C_3 \sum_{p=0}^{+\infty} |A_p| \exp(-C_4 p), \end{aligned}$$

where

$$A_p = \{(x, y) \in L_{n,i} \times L_{n,j}; \|x - y\|_2 \in [p, p + 1)\}.$$

Let $\alpha = d_2([1, +\infty)s_i, [1, +\infty)s_j)$ and $\theta = \arccos \frac{\langle s_i, s_j \rangle}{\|s_i\|_2 \|s_j\|_2}$. We can see that

- $|A_p| = 0$ for $p \leq n\alpha - 3$.
- $|A_p| \leq \frac{2000}{\sin^2 \theta} (1 + p)^2$ for each $p \geq 0$.

The first point is clear. Let us prove the second point: for each $k \in \{i, j\}$, let $s'_k = s_k / \|s_k\|_2$. Obviously, $A_p \subset B_p \times B'_p$, where $B_p = \{x \in L_{n,i}; d_2(x, \mathbb{R}s'_i) \leq p + 3\}$ and $B'_p = \{y \in L_{n,j}; d_2(y, \mathbb{R}s'_j) \leq p + 3\}$.

For $r \in \mathbb{R}$, define

$$f(r) = \sum_{x \in B_p} \mathbb{1}_{\{|rs'_i - x| \leq \sqrt{2}\}}.$$

Since $d_2(x, \mathbb{R}s_i) \leq 1$ for each $x \in L_{n,i}$, it follows that

$$\int_{\mathbb{R}} f(r) dr \geq 2|B_p|.$$

For a given r , the sum defining $f(r)$ has at most 9 non vanishing terms, thus we have

$$f(r) \leq 9 \mathbb{1}_{\{d_2(rs'_i, \mathbb{R}s_j) \leq p+3+\sqrt{2}\}}.$$

Then

$$|B_p| \leq \frac{1}{2} \int_{\mathbb{R}} f(r) dr \leq 9 \times \frac{1}{\beta_{i,j}} (p+3+\sqrt{2}) \leq \frac{9(3+\sqrt{2})}{\beta_{i,j}} (p+1),$$

where $\beta_{i,j} = |s'_i - \langle s'_i, s'_j \rangle s'_j| = \sqrt{1 - \langle s'_i, s'_j \rangle^2}$.

Similarly, $|B'_p| \leq \frac{9(3+\sqrt{2})}{\beta_{i,j}} (p+1)$. Finally, $|A_p| \leq \frac{2000}{\sin^2 \theta} (1+p)^2$.

Let K' be such that $\frac{2000}{\sin^2 \theta} C_3 (1+p)^2 \leq K' \exp(\frac{C_4}{2} p)$ holds for each $p \geq 0$: we have

$$\begin{aligned} & \mathbb{P}(\exists (x, y) \in L_{n,i} \times L_{n,j} : d(x, y) \leq (1-\varepsilon)\mu(x-y)) \\ & \leq \sum_{p=\text{Int}(n\alpha-3)}^{+\infty} K' \exp(-\frac{C_4}{2} p) \leq \frac{K' e^{2C_4}}{1 - \exp(-\frac{C_4}{2})} \exp(-\frac{C_4 \alpha}{2} n), \end{aligned}$$

which completes the proof of the lemma. \square

We go back to the proof of Theorem 4.1.

Suppose that $\text{Mincut}(nA, \infty) < (1-\varepsilon)n\mathcal{I}(A)$. Then, there exists a closed path in the dual lattice that surrounds nA and has a length $l(\gamma)$ smaller than $(1-\varepsilon)n\mathcal{I}(A)$. This path γ necessarily cuts the half-lines $([1, +\infty)ns_i)_{0 \leq i \leq n_e-1}$ in some points $y_0, y_1, \dots, y_{n_e-1}$. We also define $y_e = y_0$. We can choose these points in such a way that γ visits the $(y_i)_{0 \leq i \leq n_e}$ in the natural order. Let x_i the point in $L_{n,i}$ which is such that $\|y_i - x_i\|_1 \leq 1/2$. Obviously,

$$(18) \quad \sum_{i=0}^{n_e-1} d(x_i, x_{i+1}) \leq l(\gamma) \leq (1-\varepsilon)n\mathcal{I}(A).$$

Let B be the polygon determined by the y_i : we have

$$\mathcal{I}(B) = \sum_{i=0}^{n_e-1} \mu(y_i - y_{i+1}) \leq \sum_{i=0}^{n_e-1} (\mu(x_i - x_{i+1}) + \mu_{\max})$$

nA is convex and contained in B , so by lemma 2.4, $\mathcal{I}(B) \geq \mathcal{I}(nA)$. It follows that

$$\begin{aligned} \sum_{i=0}^{n_e-1} d(x_i, x_{i+1}) &\leq (1-\varepsilon)n\mathcal{I}(A) \leq (1-\varepsilon)\mathcal{I}(B) \\ &\leq \sum_{i=0}^{n_e-1} (1-\varepsilon)(\mu(x_i - x_{i+1}) + \mu_{\max}) \\ &\leq \sum_{i=0}^{n_e-1} (1-\varepsilon/2)\mu(x_i - x_{i+1}), \end{aligned}$$

provided that $n \geq \frac{1}{\alpha}(1 + \frac{2\mu_{\max}}{\varepsilon\mu_{\min}})$.

So, for large n , the event $\{\text{Mincut}(nA, \infty) < (1-\varepsilon)\}$ implies the existence of $i \in \{0, \dots, n_e-1\}$, $x_i \in L_{n,i}$ and $x_{i+1} \in L_{n,i+1}$ with

$$d(x_i, x_{i+1}) \leq (1-\varepsilon/2)\mu(x_i - x_{i+1}).$$

Then, we have

$$\begin{aligned} &\mathbb{P}(\text{Mincut}(nA, \infty) < (1-\varepsilon)n\mathcal{I}(A)) \\ &\leq \mathbb{P}(\exists i \in \{0, \dots, n_e-1\} \exists (x, y) \in L_{n,i} \times L_{n,i+1} d(x, y) \leq (1-\varepsilon/2)\mu(x - y)) \\ &\leq C_{11}(\varepsilon/2) \exp(-C_{12}(\varepsilon/2)n), \end{aligned}$$

thanks to lemma 4.2. This ends the proof for the case where A is a polygon.

Let us go to the general case: by lemma 2.5, there exists a convex polygon P with $0 \in P$, $P \subset A$ and $(1-\varepsilon)\mathcal{I}(A) \leq (1-\varepsilon/2)\mathcal{I}(P)$

By its very definition, $\text{Mincut}(nA, \infty) \geq \text{Mincut}(nP)$ Then

$$\begin{aligned} \mathbb{P}(\text{Mincut}(nA, \infty) \leq n\mathcal{I}(A)(1-\varepsilon)) &\leq \mathbb{P}(\text{Mincut}(nP) \leq n\mathcal{I}(A)(1-\varepsilon)) \\ &\leq \mathbb{P}(\text{Mincut}(nP) \leq n\mathcal{I}(P)(1-\varepsilon/2)), \end{aligned}$$

which has just been proved to decrease exponentially fast with n . \square

5. FINAL PROOFS

5.1. Proof of the theorems. Obviously, Theorem 1.2 is just Theorem 3.1 and Theorem 4.1 together. Since $\mathcal{I}(A) = \int_{\partial^* A} \mu(\nu_A(x)) d\mathcal{H}^1(x)$, the first equality in Theorem 1.1 directly follows from Theorem 1.2 with the help of the Borel–Cantelli lemma.

It is worth saying a word about Equation (3), because the Ford–Fulkerson theorem is initially concerned with finite graphs. Let us recall a version of this theorem.

Proposition 5.1 (Ford–Fulkerson). *For each finite graph $G = (V, E)$ and every disjoint subsets A and B of V , we have*

$$(19) \quad \max\left\{\sum_{x \in A} \text{Div } j(x); j \in \text{Capflow}(A, B)\right\} = \text{Mincut}(A, B),$$

where

$$(20) \quad \text{Mincut}(A, B) = \min\left\{\sum_{x \in C} t_x; \text{ every path in } G \text{ from } A \text{ to } B \text{ meets } C\right\}$$

and $\text{Capflow}(A, B)$ is the set of flows j that satisfy $|j(x, y)| \leq t_{\{x, y\}}$ for each $\{x, y\} \in E$ and $\text{Div } j(x) = 0$ for $x \in V \setminus (A \cup B)$.

In fact, in the initial paper [7] and in most books, A and B are just singletons. The reduction to this case is easy: because of the antisymmetry property, the contribution of edges inside A to $\sum_{x \in A} \text{Div } j(x)$ is null, so we do not change the max-flow (nor the min-cut) if we identify the points that are in A . Obviously, the max-flow and the min-cut are not changed either when we identify the points that are in B .

Now let $G_n = (V_n, E_n)$ be the restriction of \mathbb{L}^2 to $V_n = \{x \in \mathbb{Z}^2; \|x\|_1 \leq n\}$ and denote by B_n the boundary of V_n .

Let f be a flow from A to infinity: f is a flow from A to B_n , so $\sum_{x \in A} \text{Div } j(x) \leq \text{Mincut}(A, B_n)$. By the definition of a cutset, a minimal cutset from A to infinity is the external boundary of a finite connected set containing A . In particular, a minimal cutset is finite. It follows that $\inf_{n \geq 1} \text{Mincut}(A, B_n) = \text{Mincut}(A, \infty)$. Then, $\sup\{\sum_{x \in A} \text{Div } j(x); j \in \text{Capflow}(A, \infty)\} \leq \text{Mincut}(A, \infty)$. Conversely, let j_n be a flow that realizes $\max\{\sum_{x \in A} \text{Div } j(x); j \in \text{Capflow}(A, B_n)\}$. We can extend j_n to $\overrightarrow{\mathbb{E}^d}$ by putting $j_n(e) = 0$ outside E_n . Obviously $j_n \in \prod_{e \in \overrightarrow{\mathbb{E}^d}} [-t_e, +t_e]$, thus the sequence $(j_n)_{n \geq 1}$ admits a subsequence $(j_{n_k})_{k \geq 1}$ converging to some $j' \in \prod_{e \in \overrightarrow{\mathbb{E}^d}} [-t_e, +t_e]$ in the product topology. Easily, j' is antisymmetric.

$$\begin{aligned} \sum_{x \in A} \text{Div } j'(x) &= \lim_{k \rightarrow +\infty} \sum_{x \in A} \text{Div } j_{n_k}(x) \\ &= \lim_{k \rightarrow +\infty} \max\left\{ \sum_{x \in A} \text{Div } j(x); j \in \text{Capflow}(A, B_k) \right\} \\ &= \lim_{k \rightarrow +\infty} \text{Mincut}(A, B_{n_k}) \\ &= \inf_{n \geq 1} \text{Mincut}(A, B_n) = \text{Mincut}(A, \infty). \end{aligned}$$

For $x \in \mathbb{Z}^2 \setminus A$, there exists k_0 such that $x \in V_n \setminus (B_{n_k} \cup A)$ for $k \geq k_0$: then $\text{Div } j_{n_k}(x) = 0$ for $k \geq k_0$ which ensures that $\text{Div } j'(x) = 0$. It is now easy to see that j' is a capacitive flow from A to infinity, which completes the proof of Equation (3) and, therefore, the proof of Theorem 1.1.

5.2. Proof of the corollaries. Let us now recall Menger's Theorem (see for example Diestel [6] for a proof.)

Proposition 5.2 ("Menger's Theorem"). *Let $G = (V, E)$ be a finite graph and $A, B \subset V$. Then the minimum number of vertices separating A from B is equal to the maximum number of disjoint paths from A to B .*

We can now prove Corollary 1.3.

Proof. Consider the probability space $(\Omega, \mathcal{B}, \mathbb{P})$, with $\Omega = \{0, 1\}^{\mathbb{E}^2}$ and $\mathbb{P} = \text{Ber}(p)^{\otimes \mathbb{E}^2}$. As usually, e is said to be open if $\omega_e = 1$ and closed otherwise. Let $R = \{e \in \mathbb{E}^2 : \omega_e = 1\}$ and define V_n and E_n as previously. Let $H_n = (V_n, E_n \cap R)$. It is easy to see that the minimum number of vertices separating A from B_n is equal to $\text{Mincut}(A, B_n)$, where the capacity flow is defined by $t_e = 1 - \omega_e$. The variables $(t_e)_{e \in \mathbb{E}^2}$ are independent identically distributed with $m = \text{Ber}(1 - p)$ as common law. Then, by Menger's theorem, the maximum number of disjoint paths from A to B_n is $\text{Mincut}(A, B_n)$. By a classical compactness argument, the maximum number of disjoint paths from A to infinity is the limit of the maximum number of disjoint paths from A to B_n . Therefore, $\text{dis}(A) = \lim_{n \rightarrow +\infty} \text{Mincut}(A, B_n) = \text{Mincut}(A, \infty)$.

Since $p > p_c(2) = 1/2$, $m(0) = 1 - p < 1/2$, and we can apply Theorem 1.2 to complete the proof of Corollary 1.3. \square

We prove finally Corollary 1.4

Proof. Let us denote by I_k the event : “there exist k disjoint open bi-infinite paths.”. I_k is obviously translation-invariant, so by the ergodic theorem, its probability is null or full. Let $A = [-1, 1]^2$ and $S_n = \{\text{dis}(An) \geq n\mathcal{I}(A)/2\}$. For large n enough, we have $n\mathcal{I}(A)/2 > 2k$ and $\mathbb{P}(S_n) > 1/2$. Denote by T_n the event “all edges inside nA are open”. It is not difficult to see that $T_n \cap S_n \subset I_k$. But T_n and S_n are independent, so $P(I_k) \geq P(T_n \cap S_n) = P(T_n)P(S_n) > 0$. Finally, $P(I_k) = 1$. \square

6. PERSPECTIVES

It is to be expected that these results still holds in higher dimensions. In fact, we make the following conjecture:

Conjecture 6.1. *We suppose that $m(0) < 1 - p_c(\mathbb{Z}^d)$ and that m has an exponential moment. Then, there exists a map μ on the unit sphere such that for each convex set A with 0 in the interior, we have*

$$\lim_{n \rightarrow +\infty} \frac{\text{Mincut}(nA, \infty)}{n^{d-1}} = \int_{\partial^* A} \mu(\nu_A(x)) d\mathcal{H}^{d-1}(x).$$

Equivalently,

$$\lim_{n \rightarrow +\infty} \frac{1}{n^{d-1}} \max \left\{ \sum_{x \in nA \cap \mathbb{Z}^d} \text{Div } j(x); j \in \text{Capflow}(nA, \infty) \right\} = \sup \left\{ \int_A \text{div } f d\lambda^d(x); f \in C_c^1(\mathbb{R}^d, \mathcal{W}_\mu) \right\},$$

where

$$\mathcal{W}_\mu = \{x \in \mathbb{R}^d : \langle x, w \rangle \leq \mu(w) \text{ for all } w\}.$$

Of course, the situation is more complicated when $d \geq 3$ because cutsets are not paths, therefore their capacity can not be interpreted in term of first-passage percolation. In a seminal paper [12], Kesten put the basis of a generalization of first-passage percolation which seems to be the appropriate tool for the problem considered here. Basically, he studies the minimal cut between opposite sides of a parallelepiped with (e_1, e_2, e_3) as axes. This allows to define a quantity ν which is a good candidate for $\mu(e_1)$. Later, Boivin [2] extended some of Kesten’s results. Particularly, he defined a function on the unit sphere of \mathbb{R}^3 which may be convenient for our purpose. The condition $m(0) < 1 - p_c$ is coherent with some previous results: indeed, Zhang [15] proved that $\nu = 0$ for $m(0) \geq 1 - p_c$ whereas Chayes and Chayes [4] had proved (at least in the Bernoulli case) that $\nu > 0$ for $m(0) \geq 1 - p_c$ using a result of Aizenman, Chayes, Chayes, Fröhlich and Russo [1]. Note that Th  ret [14] recently proved some results that give an independent proof of this fact. So, $m(0) < 1 - p_c$ seems to be a natural assumption for the conjecture. This is also coherent with the expected domain of validity for the d -dimensional version of corollary 1.3. Of course, this conjecture is at present far from being solved, because some of the quantities that are used in the present proof do not have an obvious equivalent in higher dimensions. However, we think that the conjecture presented here is a good motivation to continue the study initiated in Kesten [12].

REFERENCES

- [1] M. Aizenman, J. T. Chayes, L. Chayes, J. Fröhlich, and L. Russo. On a sharp transition from area law to perimeter law in a system of random surfaces. *Comm. Math. Phys.*, 92(1):19–69, 1983.
- [2] Daniel Boivin. Ergodic theorems for surfaces with minimal random weights. *Ann. Inst. H. Poincaré Probab. Statist.*, 34(5):567–599, 1998.
- [3] Raphaël Cerf. The Wulff crystal in Ising and percolation models. In *École d'été de probabilités de Saint-Flour, XXIV—2004*, volume 1878 of *Lecture Notes in Math.*, page 268 p. Springer, Berlin, 2006.
- [4] J. T. Chayes and L. Chayes. Bulk transport properties and exponent inequalities for random resistor and flow networks. *Comm. Math. Phys.*, 105(1):133–152, 1986.
- [5] J. Theodore Cox and Richard Durrett. Some limit theorems for percolation processes with necessary and sufficient conditions. *Ann. Probab.*, 9(4):583–603, 1981.
- [6] Reinhard Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, 2005.
- [7] L. R. Ford, Jr. and D. R. Fulkerson. Maximal flow through a network. *Canad. J. Math.*, 8:399–404, 1956.
- [8] Olivier Garet and Régine Marchand. Large deviations for the chemical distance in supercritical Bernoulli percolation. *Ann. Probab.*, To appear, Preprint available at <http://arxiv.org/abs/math.PR/0409317>.
- [9] G. R. Grimmett and J. M. Marstrand. The supercritical phase of percolation is well behaved. *Proc. Roy. Soc. London Ser. A*, 430(1879):439–457, 1990.
- [10] Geoffrey Grimmett. *Percolation*. Springer-Verlag, Berlin, second edition, 1999.
- [11] Geoffrey Grimmett and Harry Kesten. First-passage percolation, network flows and electrical resistances. *Z. Wahrsch. Verw. Gebiete*, 66(3):335–366, 1984.
- [12] Harry Kesten. Surfaces with minimal random weights and maximal flows: a higher-dimensional version of first-passage percolation. *Illinois J. Math.*, 31(1):99–166, 1987.
- [13] Thomas Lachand-Robert and Édouard Oudet. Minimizing within convex bodies using a convex hull method. *SIAM J. Optim.*, 16(2):368–379 (electronic), 2005.
- [14] Marie Thérêt. On the small maximal flows in first passage percolation. *Preprint, available at <http://arxiv.org/abs/math.PR/0607252>*, 2006.
- [15] Yu Zhang. Critical behavior for maximal flows on the cubic lattice. *J. Statist. Phys.*, 98(3-4):799–811, 2000.

LABORATOIRE DE MATHÉMATIQUES, APPLICATIONS ET PHYSIQUE MATHÉMATIQUE D'ORLÉANS
 UMR 6628, UNIVERSITÉ D'ORLÉANS, B.P. 6759, 45067 ORLÉANS CEDEX 2 FRANCE
E-mail address: `Olivier.Garet@univ-orleans.fr`