

Robust \mathcal{H}_∞ reduced order filtering for uncertain bilinear systems

H. Souley Ali, M. Zasadzinski*, H. Rafaralahy and M. Darouach

CRAN – CNRS

IUT de Longwy, Université Henri Poincaré – Nancy I
186, rue de Lorraine, 54400 Cosnes et Romain, FRANCE

Tel : +33 +3 82 39 62 22 – Fax : +33 +3 82 39 62 91

E-mail : {souley,mzasad,hrafa,darouach}@iut-longwy.uhp-nancy.fr

Abstract

This paper investigates both the \mathcal{H}_∞ and robust \mathcal{H}_∞ reduced order unbiased filtering problems for respectively a nominal bilinear system and a bilinear system affected by norm-bounded structured uncertainties in all the system matrices. First, an algebraic framework is used to solve the unbiasedness condition and second, a change of variable is introduced on the inputs of the system to reduce the conservatism inherent to the requirement of exponential convergence of the filter. Then the reduced order filtering solution is obtained through LMI with an equality constraint by transforming the problem into a robust state feedback in the nominal case and a robust static output feedback in the presence of uncertainties. In the last case, an additional bilinear matrix equality must also be solved.

Keywords : Bilinear systems, Structured uncertainties, Robust reduced order filtering, Unbiasedness, \mathcal{H}_∞ optimization, LMI.

1 Introduction

In the last decades, due to the fact that many physical processes may be appropriately modeled as bilinear systems when linear models are inadequate, great interest has been accorded to the state estimation of bilinear systems. Most work on state estimation of bilinear systems has been done on observers design (Funahashi, 1979; Tibken and Hofer, 1989; Derese *et al.*, 1979; Wang and Kao, 1991). The \mathcal{H}_∞ filtering for linear systems has been deeply visited in the full order case (Nagpal and Khargonekar, 1991; Shaked and Theodor, 1992b; Shaked and Theodor, 1992a) and even in the reduced order one (Grigoriadis and Watson, 1997; Watson and Grigoriadis, 1998). The reduced order filtering problem presented by (Watson and Grigoriadis, 1998) is an unbiased one and considers only the nominal case. The full order \mathcal{H}_∞ filtering for nonlinear systems has been studied by (de Souza *et al.*, 1993; Reif *et al.*, 1999) and is also shown to be equivalent to an \mathcal{L}_2 gain attenuation problem for the mapping between the disturbances and the estimation error. The robust full order filtering for uncertain linear systems has been treated in (Fu *et al.*, 1992; Li and Fu, 1997; Pálhares and Peres, 2000; Sayed, 2001). The robust filtering gets its importance from the necessity to still keep good performances even if parameter uncertainties affect the system.

In this paper, a reduced order \mathcal{H}_∞ functional filtering method is proposed to reconstruct a linear combination of the states of a bilinear system by exploiting the nonlinearities in the nominal and the robust cases as ‘structured uncertainties’. This is achieved through the design of a filter whose dynamics has the same dimension as this linear combination. In addition to the exponential convergence and \mathcal{L}_2 gain attenuation requirements, the filter must also be unbiased, i.e. the estimation error does not depend explicitly on the states of the system. The proposed approach is based on the resolution of algebraic

*Corresponding author : E-mail: mzasad@iut-longwy.uhp-nancy.fr – Fax : +33 +3 82 39 62 91

Sylvester equations to find conditions for the existence of the unbiased reduced order filter. Then the exponential convergence and \mathcal{L}_2 gain attenuation problems are reduced to a robust state feedback in the nominal case. It is shown that the robust functional unbiased filtering problem for uncertain bilinear systems subjected to time-varying norm-bounded uncertainties can be seen as a particular case of a static output feedback one under some conditions. This problem requires to solve **Linear Matrix Inequalities** (LMI) with an additional non convex **Bilinear Matrix Equality** (BME) constraint.

The paper is organized as follows. The conditions for the unbiasedness, exponential convergence and \mathcal{L}_2 gain attenuation of a reduced order \mathcal{H}_∞ functional filter for continuous-time nominal bilinear systems are studied in section 2. It is shown through section 3 that the robust filtering problem for bilinear systems affected by structured norm-bounded time-varying uncertainties can be solved as a static output feedback problem. An illustrative example is given in section 4. Then, some conclusions are presented in section 5.

Notations. $\|x\| = \sqrt{x^T x}$ and $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$ are the Euclidean vector norm and the spectral matrix norm respectively where $\lambda_{\max}(A^T A)$ is the maximal eigenvalue of the symmetric matrix $A^T A$. A^\dagger is a generalized inverse of matrix A satisfying $A = AA^\dagger A$. $\text{bdiag}(A_1, \dots, A_k)$ denotes a block-diagonal matrix with A_1, \dots, A_k as block-diagonal ‘elements’ and $\text{herm}(A) = A + A^T$. $\mathcal{L}_2[0, \infty)$ is the space of signals with bounded energy.

2 Reduced order unbiased \mathcal{H}_∞ filtering in the nominal case

2.1 Problem Formulation

Consider the nominal bilinear system described by

$$\dot{x} = A_0 x + \sum_{i=1}^m A_i u_i x + B w \quad (1a)$$

$$y = C x + D w \quad (1b)$$

$$z = L x \quad (1c)$$

where $u(t) = [u_1(t) \dots u_m(t)]^T \in \mathbb{R}^m$ is the known control input vector, $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^p$ is the measured output and $z(t) \in \mathbb{R}^r$ is the vector to be estimated where $r \leq n$. The vector $w(t) \in \mathbb{R}^q$ represents the disturbance vector. A_0, A_i, B, C, D and L are known constant matrices. Without loss of generality, it is assumed that $\text{rank } L = r$. The problem is to estimate the vector $z(t)$ from the measurements $y(t)$ and the inputs $u(t)$. For many physical processes, disturbances are continuous and inputs are continuous and bounded, which is reflected in the following assumption for bilinear system (1).

Assumption 1. *The disturbance signal $w(t)$ is continuous. The control inputs, $u(t)$, are continuous and bounded, i.e. $u(t) \in \Omega \subset \mathbb{R}^m$, where*

$$\Omega := \{u(t) \in \mathbb{R}^m \mid u_{i,\min} \leq u_i(t) \leq u_{i,\max}, i = 1, \dots, m\}. \quad (2)$$

The proposed reduced order functional filter is given by

$$\dot{\eta} = H_0 \eta + \sum_{i=1}^m H_i u_i \eta + J_0 y + \sum_{i=1}^m J_i u_i y \quad (3a)$$

$$\hat{z} = \eta + E y \quad (3b)$$

where $\hat{z}(t) \in \mathbb{R}^r$ is the estimate of $z(t)$.

The estimation error is given by

$$e = z - \hat{z} = L x - \hat{z} = \bar{e} - E D w \quad (4)$$

$$\bar{e} = \Psi x - \eta \quad (5)$$

$$\Psi = L - E C. \quad (6)$$

The generalization of the well-known \mathcal{H}_∞ attenuation in the linear case to the nonlinear systems is the \mathcal{L}_2 gain attenuation defined as follows (van der Schaft, 1992).

Definition 1. Let $\gamma > 0$, the mapping from $w(t)$ to $e(t)$ is said to have \mathcal{L}_2 gain less than or equal to γ if

$$\int_0^\infty \|e(t)\|^2 dt \leq \gamma^2 \int_0^\infty \|w(t)\|^2 dt \quad (7)$$

$\forall w(t) \in \mathcal{L}_2[0, \infty)$, and with zero initial conditions.

In this paper, the problem of the filter design is to determine H_0, H_i, J_0, J_i and E such that :

- (i) the filter (3) is unbiased if $w(t) = 0$ (see (Seron *et al.*, 1997), p. 176), i.e. the estimation error is independent of x ,
- (ii) the filter (3) is exponentially convergent for $u(t) \in \Omega$ and $w(t) = 0$,
- (iii) the mapping from the disturbance input $w(t)$ to the estimation error $e(t)$ has \mathcal{L}_2 gain less than a given scalar γ for $u(t) \in \Omega$.

2.2 Unbiasedness condition fulfillment

From (4), notice that the time derivative of the error e is function of the time derivative of the disturbances w . To avoid the use of \dot{w} in the dynamics of the error e , consider \bar{e} as a new ‘state vector’. Then the \mathcal{L}_2 gain from w to e has the following state space realization

$$\dot{\bar{e}} = (H_0 + \sum_{i=1}^m u_i H_i) \bar{e} + (\Psi A_0 - H_0 \Psi - J_0 C) x \quad (8a)$$

$$+ \sum_{i=1}^m (\Psi A_i - H_i \Psi - J_i C) u_i x + (\Psi B - J_0 D - \sum_{i=1}^m J_i D u_i) w \quad (8b)$$

$$e = \bar{e} - EDw \quad (8c)$$

and the unbiasedness of the filter is achieved if and only if the following Sylvester equations

$$\Psi A_i - H_i \Psi - J_i C = 0 \quad i = 0, \dots, m \quad (9)$$

hold. As L is of full row rank, (9) is equivalent to

$$(\Psi A_i - H_i \Psi - J_i C) \begin{bmatrix} L^\dagger & I_n - L^\dagger L \end{bmatrix} = 0 \quad i = 0, \dots, m \quad (10)$$

and since $\text{rank } L = r$, one has $LL^\dagger = I_r$. Using the definition of Ψ , (10) is equivalent to

$$0 = \Psi A_i L^\dagger - H_i \Psi L^\dagger - J_i C L^\dagger \quad i = 0, \dots, m \quad (11a)$$

$$0 = \Psi \bar{A}_i + H_i E \bar{C} - J_i \bar{C} \quad i = 0, \dots, m \quad (11b)$$

where

$$\bar{A}_i = A_i (I_n - L^\dagger L), \quad i = 0, \dots, m \quad (12a)$$

$$\bar{C} = C (I_n - L^\dagger L). \quad (12b)$$

Using (6), relation (11a) can be rewritten as

$$H_i = \bar{\bar{A}}_i - \mathcal{K}_i \bar{\bar{C}}_i \quad i = 0, \dots, m \quad (13)$$

where, for $i = 0, \dots, m$,

$$\bar{\bar{A}}_i = LA_i L^\dagger, \quad \bar{\bar{C}}_i = \begin{bmatrix} CA_i L^\dagger \\ CL^\dagger \end{bmatrix}, \quad (14a)$$

$$\mathcal{K}_i = \begin{bmatrix} E & K_i \end{bmatrix} \text{ with } K_i = J_i - H_i E \quad (14b)$$

Then relation (11b) can be expressed in the following compact form

$$\mathcal{K}\Sigma = L\bar{A} \quad (15)$$

where

$$\bar{A} = \begin{bmatrix} \bar{A}_0 & \dots & \bar{A}_m \end{bmatrix} \quad (16a)$$

$$\Sigma = \begin{bmatrix} C\bar{A} \\ \text{bdiag}(\bar{C}, \dots, \bar{C}) \end{bmatrix} \quad (16b)$$

$$\mathcal{K} = \begin{bmatrix} E & K_0 & \dots & K_m \end{bmatrix}, \quad (16c)$$

and a general solution to (15), if it exists, is given by

$$\mathcal{K} = \begin{bmatrix} E & K_0 & \dots & K_m \end{bmatrix} = L\bar{A}\Sigma^\dagger + Z(I_{(m+2)p} - \Sigma\Sigma^\dagger) \quad (17)$$

where $Z = [Z_E \ Z_0 \ \dots \ Z_m]$ is an arbitrary matrix.

Lemma 1. *The unbiasedness of the filter (3) is achieved if and only if the following rank condition holds*

$$\text{rank} \begin{bmatrix} LA \\ CA \\ \text{bdiag}(C, \dots, C) \\ \text{bdiag}(L, \dots, L) \end{bmatrix} = \text{rank} \begin{bmatrix} CA \\ \text{bdiag}(C, \dots, C) \\ \text{bdiag}(L, \dots, L) \end{bmatrix} \quad (18)$$

where

$$A = \begin{bmatrix} A_0 & \dots & A_m \end{bmatrix}. \quad (19)$$

Proof. Using the previous developments, filter (3) is unbiased, i.e. relation (9) holds, if and only if there exists a solution \mathcal{K} to (15), that is if and only if

$$L\bar{A}(I_n - \Sigma^\dagger\Sigma) = 0, \quad (20)$$

which is equivalent to

$$\text{rank} \begin{bmatrix} L\bar{A} \\ \Sigma \end{bmatrix} = \text{rank } \Sigma. \quad (21)$$

The rest of the proof can be obtained from (Darouach *et al.*, 2001). \square

2.3 Unbiasedness condition under $ED = 0$

The mapping from w to e , given by (8a), becomes

$$\dot{\bar{e}} = \left(H_0 + \sum_{i=1}^m u_i H_i \right) \bar{e} + \left(\Psi B - K_0 D - H_0 E D - \sum_{i=1}^m u_i (K_i D + H_i E D) \right) w \quad (22a)$$

$$e = \bar{e} - EDw \quad (22b)$$

or equivalently

$$\begin{aligned} \dot{\bar{e}} = & \left(\bar{A}_0 + \sum_{i=1}^m u_i \bar{A}_i - \left(L \bar{A} \Sigma^\dagger - Z(I_{(m+2)p} - \Sigma \Sigma^\dagger) \right) \Lambda_1(u) \right) \bar{e} \\ & + \left(L B - \left(L \bar{A} \Sigma^\dagger - Z(I_{(m+2)p} - \Sigma \Sigma^\dagger) \right) \Lambda_2(u) \right) w \\ & + \left(\bar{A}_0 + \sum_{i=1}^m u_i \bar{A}_i - \left(L \bar{A} \Sigma^\dagger - Z(I_{(m+2)p} - \Sigma \Sigma^\dagger) \right) \Lambda_1(u) \right) E D w \end{aligned} \quad (23a)$$

$$e = \bar{e} - E D w \quad (23b)$$

where

$$\Lambda_1(u) = \begin{bmatrix} C A_0 L^\dagger + \sum_{i=1}^m u_i C A_i L^\dagger \\ \psi_C(u) \end{bmatrix}, \quad \Lambda_2(u) = \begin{bmatrix} C B \\ \psi_D(u) \end{bmatrix} \quad (24)$$

with $\psi_C(u) = [L^\dagger{}^T C^T \quad u_1 L^\dagger{}^T C^T \quad \dots \quad u_m L^\dagger{}^T C^T]^T$ and $\psi_D(u) = [D^T \quad u_1 D^T \quad \dots \quad u_m D^T]^T$.

Due to the term $Z(I_{(m+2)p} - \Sigma \Sigma^\dagger) \Lambda_1(u) E$, the error is bilinear in the gain parameter Z in system (23). This bilinearity is intrinsically linked to the unbiasedness condition (9). Indeed, the ‘bilinearity’ $H_i \Psi$ in (9) yields a gain K_i (see (14b)) containing the term $H_i E$. In order to avoid this bilinearity, we consider $E D = 0$ in the sequel; this allows to have an LMI tractable formulation for the problem instead of a bilinear matrix inequality intractable one. Adding the constraint $E D = 0$, relations (15), (16) and (17) become

$$\mathcal{K} \bar{\Sigma} = \begin{bmatrix} 0 & L \bar{A} \end{bmatrix} \quad (25)$$

where

$$\bar{\Sigma} = \begin{bmatrix} D & C \bar{A} \\ 0 & \text{bdiag}(\bar{C}, \dots, \bar{C}) \end{bmatrix} \quad (26)$$

and a general solution to (25), if it exists, is given by

$$\mathcal{K} = \begin{bmatrix} 0 & L \bar{A} \end{bmatrix} \bar{\Sigma}^\dagger + Z(I_{(m+2)p} - \bar{\Sigma} \bar{\Sigma}^\dagger). \quad (27)$$

Then, the following lemma is derived from lemma 1.

Lemma 2. *The unbiasedness of the filter (3) is achieved under $E D = 0$ if and only if*

$$\text{rank} \begin{bmatrix} 0 & L A \\ D & C A \\ 0 & \text{bdiag}(C, \dots, C) \\ 0 & \text{bdiag}(L, \dots, L) \end{bmatrix} = \text{rank} \begin{bmatrix} D & C A \\ 0 & \text{bdiag}(C, \dots, C) \\ 0 & \text{bdiag}(L, \dots, L) \end{bmatrix}. \quad (28)$$

Now, assume that condition (28) in lemma 2 holds. Then relation (9) is verified with \mathcal{K} given by (27) and, hence, $\bar{e}(t) = e(t)$ in (23), i.e.

$$\begin{aligned} \dot{e} = & \left(\bar{A}_0 + \sum_{i=1}^m u_i \bar{A}_i - \left(\begin{bmatrix} 0 & L \bar{A} \end{bmatrix} \bar{\Sigma}^\dagger - Z(I_{(m+2)p} - \bar{\Sigma} \bar{\Sigma}^\dagger) \right) \Lambda_1(u) \right) e \\ & + \left(L B - \left(\begin{bmatrix} 0 & L \bar{A} \end{bmatrix} \bar{\Sigma}^\dagger - Z(I_{(m+2)p} - \bar{\Sigma} \bar{\Sigma}^\dagger) \right) \Lambda_2(u) \right) w. \end{aligned} \quad (29)$$

As the item (i) of the design objectives has been solved, it remains to treat the points (ii) and (iii) of these objectives.

2.4 Exponential convergence and \mathcal{L}_2 gain attenuation

Here, a change of variables is introduced by considering each $u_i(t)$ in equation (29) as a ‘structured uncertainty’. Notice that the definition of the ‘uncertainty set’ Ω in relation (2) can lead to some conservatism (see (Boyd *et al.*, 1994)) since, in the general case, $|u_{i,\min}| \neq |u_{i,\max}|$ with $|u_{i,\min}| \neq 1$ and $|u_{i,\max}| \neq 1$. To reduce this conservatism, each $u_i(t)$ can be rewritten as follows

$$u_i(t) = \alpha_i + \sigma_i \varepsilon_i(t) \quad (30)$$

where $\alpha_i \in \mathbb{R}$ and $\sigma_i \in \mathbb{R}$ are given by

$$\alpha_i = \frac{u_{i,\min} + u_{i,\max}}{2}, \quad \sigma_i = \frac{u_{i,\max} - u_{i,\min}}{2} \quad i = 1, \dots, m, \quad (31)$$

$\alpha_0 = 1$ and $\sigma_0 = 0$. The new ‘uncertain’ variable is $\varepsilon(t) \in \bar{\Omega} \subset \mathbb{R}^m$ where the polytope $\bar{\Omega}$ is defined as

$$\bar{\Omega} := \{\varepsilon(t) \in \mathbb{R}^m \mid \varepsilon_{i,\min} = -1 \leq \varepsilon_i(t) \leq \varepsilon_{i,\max} = 1 \quad \text{for } i = 1, \dots, m\}. \quad (32)$$

By using relations (30)-(32), the dynamics of the error $e(t)$ in (29) can be rewritten as follows

$$\dot{e} = (\mathbb{A} - Z\mathbb{C} + (\tilde{\mathbb{A}} - Z\tilde{\mathbb{C}}) \Delta_e(\varepsilon) \bar{H}_e) e + (\mathbb{B} - Z\mathbb{G} + (\tilde{\mathbb{B}} - Z\tilde{\mathbb{G}}) \Delta_w(\varepsilon) \bar{H}_w) w \quad (33)$$

with

$$\mathbb{A} = \sum_{i=0}^m \alpha_i \bar{A}_i - \begin{bmatrix} 0 & L\bar{A} \end{bmatrix} \bar{\Sigma}^\dagger \begin{bmatrix} \sum_{i=0}^m \alpha_i C A_i L^\dagger \\ \alpha_C \end{bmatrix}, \quad \mathbb{C} = (I_{(m+2)p} - \bar{\Sigma} \bar{\Sigma}^\dagger) \begin{bmatrix} \sum_{i=0}^m \alpha_i C A_i L^\dagger \\ \alpha_C \end{bmatrix}, \quad (34a)$$

$$\tilde{\mathbb{A}} = \begin{bmatrix} \sigma_1 \bar{A}_1 & \dots & \sigma_m \bar{A}_m \end{bmatrix} - \begin{bmatrix} 0 & L\bar{A} \end{bmatrix} \bar{\Sigma}^\dagger \Gamma, \quad \tilde{\mathbb{B}} = - \begin{bmatrix} 0 & L\bar{A} \end{bmatrix} \bar{\Sigma}^\dagger \mathbb{D}, \quad \tilde{\mathbb{G}} = (I_{(m+2)p} - \bar{\Sigma} \bar{\Sigma}^\dagger) \mathbb{D}, \quad (34b)$$

$$\tilde{\mathbb{C}} = (I_{(m+2)p} - \bar{\Sigma} \bar{\Sigma}^\dagger) \Gamma, \quad \mathbb{B} = LB - \begin{bmatrix} 0 & L\bar{A} \end{bmatrix} \bar{\Sigma}^\dagger \begin{bmatrix} \alpha_0 CB \\ \alpha_D \end{bmatrix}, \quad \mathbb{G} = (I_{(m+2)p} - \bar{\Sigma} \bar{\Sigma}^\dagger) \begin{bmatrix} \alpha_0 CB \\ \alpha_D \end{bmatrix}. \quad (34c)$$

and

$$\Gamma = \begin{bmatrix} \sigma_1 C A_1 L^\dagger & \dots & \sigma_m C A_m L^\dagger \\ 0 & \dots & 0 \\ \text{bdiag}(\sigma_1 C L^\dagger, \dots, \sigma_m C L^\dagger) \end{bmatrix}, \quad \mathbb{D} = \begin{bmatrix} 0 \\ \text{bdiag}(\sigma_1 D, \dots, \sigma_m D) \end{bmatrix}, \quad (35a)$$

$$\alpha_C = \begin{bmatrix} \alpha_0 L^{\dagger T} C^T & \dots & \alpha_m L^{\dagger T} C^T \end{bmatrix}^T, \quad \alpha_D = \begin{bmatrix} \alpha_0 D^T & \dots & \alpha_m D^T \end{bmatrix}^T. \quad (35b)$$

From (32), the ‘uncertain’ matrices $\Delta_e(\varepsilon)$ and $\Delta_w(\varepsilon)$ are bounded as

$$\|\Delta_e(\varepsilon)\| \leq 1 \quad \text{and} \quad \|\Delta_w(\varepsilon)\| \leq 1 \quad (36)$$

where $\Delta_e(\varepsilon) = \text{bdiag}(\varepsilon_1 I_r, \dots, \varepsilon_m I_r) \in \mathbb{R}^{mr \times mr}$, $\Delta_w(\varepsilon) = \text{bdiag}(\varepsilon_1 I_q, \dots, \varepsilon_m I_q) \in \mathbb{R}^{mq \times mq}$, $\bar{H}_e = [I_r \dots I_r]^T \in \mathbb{R}^{mr \times r}$ and $\bar{H}_w = [I_q \dots I_q]^T \in \mathbb{R}^{mq \times q}$.

According to the previous developments, the error dynamics (33) can be rewritten as the following system

$$\dot{e} = (\mathbb{A} - Z\mathbb{C}) e + \left(\begin{bmatrix} \tilde{\mathbb{A}} & \tilde{\mathbb{B}} & \mathbb{B} \end{bmatrix} - Z \begin{bmatrix} \tilde{\mathbb{C}} & \tilde{\mathbb{G}} & \mathbb{G} \end{bmatrix} \right) \begin{bmatrix} p_e \\ p_w \\ w \end{bmatrix} \quad (37a)$$

$$\begin{bmatrix} q_e \\ q_w \\ e \end{bmatrix} = \begin{bmatrix} \bar{H}_e \\ 0 \\ I_r \end{bmatrix} e + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{H}_w \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_e \\ p_w \\ w \end{bmatrix} \quad (37b)$$

connected with

$$\underbrace{\begin{bmatrix} p_e \\ p_w \end{bmatrix}}_{\tilde{p}} = \begin{bmatrix} \Delta_e(\varepsilon) & 0 \\ 0 & \Delta_w(\varepsilon) \end{bmatrix} \underbrace{\begin{bmatrix} q_e \\ q_w \end{bmatrix}}_{\tilde{q}}. \quad (38)$$

At this step, the \mathcal{H}_∞ reduced order unbiased filtering can be solved as a particular case of a dual robust state feedback problem with structured uncertainties. The following theorem ensures the exponential convergence of the filter (3) and the \mathcal{L}_2 gain attenuation from $w(t)$ to $e(t)$.

Theorem 1. *Suppose that assumption 1 and condition (28) hold. If there exist $P = P^T > 0$, $S_e = \text{bdiag}(\mu_{1,e}I_r, \dots, \mu_{m,e}I_r) > 0$, $S_w = \text{bdiag}(\mu_{1,w}I_q, \dots, \mu_{m,w}I_q) > 0$, Y and a scalar $\rho > 0$ such that (\bullet ' is the transpose of the off-diagonal part)*

$$\begin{bmatrix} \Upsilon & \bullet & \bullet & \bullet & \bullet & 0 \\ \tilde{\mathbb{A}}^T P - \tilde{\mathbb{C}}^T Y & -S_e & 0 & 0 & 0 & 0 \\ \tilde{\mathbb{B}}^T P - \tilde{\mathbb{G}}^T Y & 0 & -S_w & 0 & 0 & 0 \\ \mathbb{B}^T P - \mathbb{G}^T Y & 0 & 0 & -\gamma^2 I_q & 0 & \bullet \\ S_e \bar{H}_e & 0 & 0 & 0 & -S_e & 0 \\ 0 & 0 & 0 & S_w \bar{H}_w & 0 & -S_w \end{bmatrix} < 0 \quad (39)$$

holds with $Y = Z^T P$ and $\Upsilon = \text{herm}(PA - Y^T C) + (1 + \rho)I_r$, then the reduced order unbiased filter (3) for the bilinear system (1) is exponentially convergent and has a \mathcal{L}_2 gain from $w(t)$ to $e(t)$ less than or equal to γ .

Proof. By considering system (37)-(38) as a diagonal norm-bounded linear differential inclusion, the following auxiliary system is derived from (37) (Boyd *et al.*, 1994)

$$\dot{e} = (\mathbb{A} - ZC)e + \left(\left[\tilde{\mathbb{A}}S_e^{-1/2} \quad \tilde{\mathbb{B}}S_w^{-1/2} \mid \gamma^{-1}\mathbb{B} \right] - Z \left[\tilde{\mathbb{C}}S_e^{-1/2} \quad \tilde{\mathbb{G}}S_w^{-1/2} \mid \gamma^{-1}\mathbb{G} \right] \right) \begin{bmatrix} p_e \\ p_w \\ w \end{bmatrix} \quad (40a)$$

$$\begin{bmatrix} q_e \\ q_w \\ e \end{bmatrix} = \begin{bmatrix} S_e^{1/2} \bar{H}_e \\ 0 \\ I_r \end{bmatrix} e + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \gamma^{-1} S_w^{1/2} \bar{H}_w \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_e \\ p_w \\ w \end{bmatrix} \quad (40b)$$

where matrices $S_e = \text{bdiag}(\mu_{1,e}I_r, \dots, \mu_{m,e}I_r) > 0$ and $S_w = \text{bdiag}(\mu_{1,w}I_q, \dots, \mu_{m,w}I_q) > 0$ satisfy $\Delta_e(\varepsilon)S_e = S_e\Delta_e(\varepsilon)$ and $\Delta_w(\varepsilon)S_w = S_w\Delta_w(\varepsilon)$ in order to take the structure of $\Delta_e(\varepsilon)$ and $\Delta_w(\varepsilon)$ into account (Boyd *et al.*, 1994) ($\mu_{i,e}$ and $\mu_{i,w}$, $i = 1, \dots, m$, are positive scalars).

Let $Y = Z^T P$, then by using the bounded real lemma (Boyd *et al.*, 1994), system (37)-(38) is exponentially convergent and has a \mathcal{L}_2 gain from w to e less than or equal to γ if there exist $P = P^T > 0$, $S_e > 0$, $S_w > 0$, Y and a scalar $\rho > 0$ such that matrices in system (40) satisfy the following inequality

$$\begin{bmatrix} \Upsilon & \bullet & \bullet & \bullet & \bullet & 0 & \bullet \\ S_e^{-1/2}(\tilde{\mathbb{A}}^T P - \tilde{\mathbb{C}}^T Y) - I & 0 & 0 & 0 & 0 & 0 & 0 \\ S_w^{-1/2}(\tilde{\mathbb{B}}^T P - \tilde{\mathbb{G}}^T Y) & 0 & -I & 0 & 0 & 0 & 0 \\ \gamma^{-1}(\mathbb{B}^T P - \mathbb{G}^T Y) & 0 & 0 & -I & 0 & \bullet & 0 \\ S_e^{1/2} \bar{H}_e & 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & \gamma^{-1} S_w^{1/2} \bar{H}_w & 0 & -I & 0 \\ I & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0.$$

Pre- and post-multiplying this inequality by

$\text{bdiag}(I_n, S_e^{1/2}, S_w^{1/2}, \gamma I_q, S_e^{1/2}, S_w^{1/2}, I_r)$ and using the Schur lemma lead to the LMI (39).

Under condition (28), by using $Z = P^{-1}Y^T$ and (27), the matrices of filter (3) are given by (13)-(14). \square

3 Robust reduced order unbiased \mathcal{H}_∞ filtering

In this section, the following uncertain bilinear system is considered

$$\dot{x} = (A_0 + \Delta_{A_0}(t))x + \sum_{i=1}^m (A_i + \Delta_{A_i}(t))u_i x + (B + \Delta_B(t))w \quad (41a)$$

$$y = (C + \Delta_C(t))x + (D + \Delta_D(t))w \quad (41b)$$

$$z = Lx \quad (41c)$$

where $x(t)$, $y(t)$, $z(t)$, $w(t)$ and $u(t)$ have been defined in section 2. The uncertain matrices $\Delta_{A_0}(t)$, $\Delta_B(t)$, $\Delta_C(t)$, $\Delta_D(t)$ and $\Delta_{A_i}(t)$ are given by

$$\begin{bmatrix} \Delta_{A_0}(t) & \Delta_B(t) \\ \Delta_C(t) & \Delta_D(t) \end{bmatrix} = \begin{bmatrix} M_{0,x} \\ M_{0,y} \end{bmatrix} \Delta_0(t) \begin{bmatrix} E_{0,x} & E_{0,w} \end{bmatrix} \quad (42a)$$

$$\begin{bmatrix} \Delta_{A_1}(t) & \dots & \Delta_{A_m}(t) \end{bmatrix} = \begin{bmatrix} M_{1,x} & \dots & M_{m,x} \end{bmatrix} \underbrace{\text{bdiag}(\Delta_1(t), \dots, \Delta_m(t))}_{\Delta(t)} \begin{bmatrix} E_{1,x}^T & \dots & E_{m,x}^T \end{bmatrix}^T \quad (42b)$$

where $M_{i,x} \in \mathbb{R}^{n \times \bar{\ell}_i}$, $M_{0,y} \in \mathbb{R}^{p \times \bar{\ell}_0}$, $E_{i,x} \in \mathbb{R}^{\bar{\ell}_i \times n}$ and $E_{0,w} \in \mathbb{R}^{\bar{\ell}_0 \times q}$ ($i = 0, \dots, m$) are known constant matrices which specify how the elements of the matrices of the nominal system are affected by the uncertain parameters in $\Delta_i(t)$, $i = 0, \dots, m$. The continuous time-varying uncertainties in (42) are assumed to satisfy :

Assumption 2. *There exist $\Delta_{i,j}(t) \in \mathbb{R}^{\ell_{i,j} \times \ell_{i,j}}$, $j = 1, \dots, s_i$ and $i = 0, \dots, m$, such that*

$$\begin{cases} \Delta_i(t) = \text{bdiag}(\Delta_{i,1}(t), \dots, \Delta_{i,s_i}(t)), i = 0, \dots, m \\ \|\Delta_i(t)\| \leq I_{\bar{\ell}_i}, \forall t \geq 0, i = 0, \dots, m \end{cases} \quad (43)$$

where $\bar{\ell}_i = \sum_{j=1}^{s_i} \ell_{i,j}$.

The constraint $ED = 0$ in section 2.3 is replaced by

$$E \begin{bmatrix} M_{0,y} & D \end{bmatrix} = 0 \quad (44)$$

in the sequel of section 3. Then equations (25), (26), (27) and (28) must be replaced by

$$\mathcal{K}\hat{\Sigma} = \begin{bmatrix} 0 & 0 & L\bar{A} \end{bmatrix}, \quad (45a)$$

$$\hat{\Sigma} = \begin{bmatrix} M_{0,y} & D & C\bar{A} \\ 0 & 0 & \text{bdiag}(\bar{C}, \dots, \bar{C}) \end{bmatrix}, \quad (45b)$$

$$\mathcal{K} = \begin{bmatrix} 0 & 0 & L\bar{A} \end{bmatrix} \hat{\Sigma}^\dagger + Z(I_{(m+2)p} - \hat{\Sigma}\hat{\Sigma}^\dagger), \quad (45c)$$

$$\text{rank} \begin{bmatrix} 0 & 0 & LA \\ M_{0,y} & D & CA \\ 0 & 0 & \text{bdiag}(C, \dots, C) \\ 0 & 0 & \text{bdiag}(L, \dots, L) \end{bmatrix} = \text{rank} \begin{bmatrix} M_{0,y} & D & CA \\ 0 & 0 & \text{bdiag}(C, \dots, C) \\ 0 & 0 & \text{bdiag}(L, \dots, L) \end{bmatrix}. \quad (45d)$$

Introducing the following augmented state vector $\xi = [x^T \ e^T]^T$, using (29) and the change of variable

in (30)-(32) enable one to express the system obtained by the concatenation of (3) and (41)-(42) as

$$\begin{aligned} \dot{\xi}(t) = & \left(\left[\begin{array}{c} M_{0,x} \\ \Psi M_{0,x} - \sum_{i=0}^m \alpha_i J_i M_{0,y} \end{array} \right] \Delta_0(t) \begin{bmatrix} E_{0,x} & 0 \end{bmatrix} + \sum_{i=0}^m \alpha_i \left(\begin{bmatrix} A_i & 0 \\ 0 & H_i \end{bmatrix} + \begin{bmatrix} M_{i,x} \\ \Psi M_{i,x} \end{bmatrix} \Delta_i(t) \begin{bmatrix} E_{i,x} & 0 \end{bmatrix} \right) \right. \\ & \left. + \sum_{i=1}^m \sigma_i \varepsilon_i \left(\begin{bmatrix} A_i & 0 \\ 0 & H_i \end{bmatrix} + \begin{bmatrix} M_{i,x} \\ \Psi M_{i,x} \end{bmatrix} \Delta_i(t) \begin{bmatrix} E_{i,x} & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -J_i M_{0,y} \end{bmatrix} \Delta_0(t) \begin{bmatrix} E_{0,x} & 0 \end{bmatrix} \right) \right) \xi \\ & + \left(\begin{bmatrix} B \\ \Psi B - \sum_{i=0}^m \alpha_i J_i D \end{bmatrix} + \begin{bmatrix} M_{0,x} \\ \Psi M_{0,x} - \sum_{i=0}^m \alpha_i J_i M_{0,y} \end{bmatrix} \Delta_0(t) E_{0,w} \right. \\ & \left. + \sum_{i=1}^m \sigma_i \varepsilon_i \left(\begin{bmatrix} 0 \\ -J_i D \end{bmatrix} + \begin{bmatrix} 0 \\ -J_i M_{0,y} \end{bmatrix} \Delta_0(t) E_{0,w} \right) \right) w \end{aligned} \quad (46a)$$

$$e = \begin{bmatrix} 0 & I_r \end{bmatrix} \xi \quad (46b)$$

where Ψ satisfies the unbiasedness relation (9). The uncertain system (46) is equivalent to the following one

$$\dot{\xi}(t) = \begin{bmatrix} \sum_{i=0}^m \alpha_i A_i & 0 \\ 0 & \mathbb{A} - Z\mathbb{C} \end{bmatrix} \xi + \begin{bmatrix} B \\ \mathbb{B} - Z\mathbb{G} \end{bmatrix} w + \begin{bmatrix} \bar{A}_\sigma & 0 & 0 \\ 0 & \tilde{\mathbb{A}} - Z\tilde{\mathbb{C}} & \tilde{\mathbb{B}} - Z\tilde{\mathbb{G}} \end{bmatrix} \begin{bmatrix} p_x \\ p_e \\ p_w \end{bmatrix} \quad (47a)$$

$$+ \begin{bmatrix} M_{0,x} & \bar{M}_\alpha \\ \mathbb{B}_M - Z\mathbb{G}_M & \mathbb{B}_{\bar{M}_\alpha} - Z\mathbb{G}_{\bar{M}_\alpha} \end{bmatrix} \begin{bmatrix} p_0 \\ p \end{bmatrix} + \begin{bmatrix} 0 & \bar{M}_\sigma \\ \tilde{\mathbb{B}}_{\bar{M}_\sigma} - Z\tilde{\mathbb{G}}_{\bar{M}_\sigma} & \mathbb{B}_{\bar{M}_\sigma} - Z\mathbb{G}_{\bar{M}_\sigma} \end{bmatrix} \begin{bmatrix} \bar{p}_0 \\ \bar{p} \end{bmatrix} \quad (47b)$$

$$\hat{q} = \mathbb{C}_q \xi + \mathbb{D}_{qw} w \quad (47c)$$

$$e = \begin{bmatrix} 0 & I_r \end{bmatrix} \xi \quad (47d)$$

connected with

$$\underbrace{\begin{bmatrix} p_x^T & p_e^T & p_w^T & p_0^T & p^T & \bar{p}_0^T & \bar{p}^T \end{bmatrix}^T}_{\hat{p}} = \underbrace{\text{bdiag}(\Delta_x(\varepsilon), \Delta_e(\varepsilon), \Delta_w(\varepsilon), \Delta_0(t), \Delta(t), \bar{\Delta}_0(\varepsilon, t), \bar{\Delta}(\varepsilon, t))}_{\hat{\Delta}(\varepsilon, t)} \underbrace{\begin{bmatrix} q_x^T & q_e^T & q_w^T & q_0^T & q^T & \bar{q}_0^T & \bar{q}^T \end{bmatrix}^T}_{\hat{q}} \quad (48)$$

where

$$\begin{aligned} \bar{A}_\sigma &= [\sigma_1 A_1 \quad \cdots \quad \sigma_m A_m], \bar{M}_\alpha = [\alpha_1 M_{1,x} \quad \cdots \quad \alpha_m M_{m,x}], \bar{M}_\sigma = [\sigma_1 M_{1,x} \quad \cdots \quad \sigma_m M_{m,x}], \\ \bar{M}_\alpha &= [\alpha_0 M_{0,y}^T \quad \cdots \quad \alpha_m M_{m,y}^T]^T, \mathbb{B}_M = LM_{0,x} - [0 \quad 0 \quad L\bar{A}] \hat{\Sigma}^\dagger \begin{bmatrix} CM_{0,x} \\ \bar{M}_{\alpha,y} \end{bmatrix}, \\ \mathbb{G}_M &= (I_{(m+2)p} - \hat{\Sigma} \hat{\Sigma}^\dagger) \begin{bmatrix} CM_{0,x} \\ \bar{M}_{\alpha,y} \end{bmatrix}, \mathbb{B}_{\bar{M}_\alpha} = L\bar{M}_\alpha - [0 \quad 0 \quad L\bar{A}] \hat{\Sigma}^\dagger \begin{bmatrix} C\bar{M}_\alpha \\ 0 \end{bmatrix}, \\ \mathbb{G}_{\bar{M}_\alpha} &= (I_{(m+2)p} - \hat{\Sigma} \hat{\Sigma}^\dagger) \begin{bmatrix} C\bar{M}_\alpha \\ 0 \end{bmatrix}, \mathbb{B}_{\bar{M}_\sigma} = L\bar{M}_\sigma - [0 \quad 0 \quad L\bar{A}] \hat{\Sigma}^\dagger \begin{bmatrix} C\bar{M}_\sigma \\ 0 \end{bmatrix}, \\ \mathbb{G}_{\bar{M}_\sigma} &= (I_{(m+2)p} - \hat{\Sigma} \hat{\Sigma}^\dagger) \begin{bmatrix} C\bar{M}_\sigma \\ 0 \end{bmatrix}, \tilde{\mathbb{B}}_{\bar{M}_\sigma} = -[0 \quad 0 \quad L\bar{A}] \hat{\Sigma}^\dagger \mathbb{M}, \tilde{\mathbb{G}}_{\bar{M}_\sigma} = (I_{(m+2)p} - \hat{\Sigma} \hat{\Sigma}^\dagger) \mathbb{M}, \end{aligned}$$

$$\mathbb{M} = \begin{bmatrix} 0 \\ \text{bdiag}(\sigma_1 M_{0,y}, \dots, \sigma_m M_{0,y}) \end{bmatrix}, \mathbb{C}_q = \begin{bmatrix} \bar{H}_x & 0 \\ 0 & \bar{H}_e \\ 0 & 0 \\ E_{0,x} & 0 \\ \bar{E}_x & 0 \\ \bar{E}_{0,x} & 0 \\ \bar{E}_x & 0 \end{bmatrix}, \mathbb{D}_{qw} = \begin{bmatrix} 0 \\ 0 \\ \bar{H}_w \\ E_{0,w} \\ 0 \\ \bar{E}_{0,w} \\ 0 \end{bmatrix},$$

$$\begin{aligned} \bar{H}_x &= [I_n \ \dots \ I_n]^T \in \mathbb{R}^{mn \times n}, \bar{E}_{0,x} = [E_{0,x}^T \ \dots \ E_{0,x}^T]^T \in \mathbb{R}^{m\bar{\ell}_0 \times n}, \\ \bar{E}_x &= [E_{1,x}^T \ \dots \ E_{m,x}^T]^T \in \mathbb{R}^{(\bar{\ell}_1 + \dots + \bar{\ell}_m) \times n}, \bar{E}_{0,w} = [E_{0,w}^T \ \dots \ E_{0,w}^T]^T \in \mathbb{R}^{m\bar{\ell}_0 \times q}, \\ \Delta_x(\varepsilon) &= \text{bdiag}(\varepsilon_1 I_n, \dots, \varepsilon_m I_n), \bar{\Delta}_0(\varepsilon, t) = \text{bdiag}(\varepsilon_1 \Delta_0(t), \dots, \varepsilon_m \Delta_0(t)), \\ \bar{\Delta}(\varepsilon, t) &= \text{bdiag}(\varepsilon_1 \Delta_1(t), \dots, \varepsilon_m \Delta_m(t)). \end{aligned}$$

The following inequalities hold from (32) and the definition of $\Delta_i(t)$

$$\|\Delta_x(\varepsilon)\| \leq 1, \|\bar{\Delta}_0(\varepsilon, t)\| \leq 1 \text{ and } \|\bar{\Delta}(\varepsilon, t)\| \leq 1. \quad (49)$$

Note that the unbiasedness condition (9) for the filter (3) is verified for the nominal case. In the system (47), the determination of gain matrix Z can be transformed into a robust static output feedback control problem by introducing the following auxiliary system

$$\begin{cases} \dot{\xi} = \mathbf{A}\xi + \bar{\mathbf{B}}_w \bar{w} + \mathbf{B}_u \bar{u} \\ \bar{z} = \bar{\mathbf{C}}_z \xi + \bar{\mathbf{D}}_{zw} \bar{w} \\ \bar{y} = \mathbf{C}_y \xi + \bar{\mathbf{D}}_{yw} \bar{w} \end{cases} \quad (50a)$$

$$\bar{u} = -Z \bar{y} \quad (50b)$$

where Z is the static output feedback controller to be designed in order to achieve stability and attenuation from the ‘augmented perturbation’ $\bar{w}^T(t) = [\hat{p}^T(t) \ w^T(t)]^T$ to the ‘augmented controlled output’ $\bar{z}^T(t) = [\hat{q}^T(t) \ e^T(t)]$. $\bar{u}(t)$ and $\bar{y}(t)$ play the role of ‘control input’ and ‘measured output’, respectively. The matrices of system (50) are given by

$$\begin{aligned} \bar{\mathbf{B}}_w &= [\mathbf{B}_p \mathbf{S}^{-1/2} \ \gamma^{-1} \mathbf{B}_w], \bar{\mathbf{D}}_{yw} = [\mathbf{D}_{yp} \mathbf{S}^{-1/2} \ \gamma^{-1} \mathbf{D}_{yw}], \bar{\mathbf{C}}_z = \begin{bmatrix} \mathbf{S}^{1/2} \mathbf{C}_q \\ \mathbf{C}_e \end{bmatrix}, \mathbf{A} = \begin{bmatrix} \sum_{i=0}^m \alpha_i A_i & 0 \\ 0 & \mathbf{A} \end{bmatrix}, \\ \bar{\mathbf{D}}_{zw} &= \begin{bmatrix} \mathbf{S}^{1/2} \mathbf{D}_{qp} \mathbf{S}^{-1/2} & \gamma^{-1} \mathbf{S}^{1/2} \mathbf{D}_{qw} \\ \mathbf{D}_{ep} \mathbf{S}^{-1/2} & \gamma^{-1} \mathbf{D}_{ew} \end{bmatrix}, \mathbf{B}_u = \begin{bmatrix} 0 \\ I_r \end{bmatrix}, \mathbf{C}_y = \begin{bmatrix} 0 & \mathbf{C} \end{bmatrix}, \\ [\mathbf{B}_p \ \mathbf{B}_w] &= \begin{bmatrix} \bar{A}_\sigma & 0 & 0 & M_{0,x} & \bar{M}_\alpha & 0 & \bar{M}_\sigma & B \\ 0 & \tilde{\mathbf{A}} & \tilde{\mathbf{B}} & \mathbb{B}_M & \mathbb{B}_{\bar{M}_\alpha} & \tilde{\mathbf{B}}_{\bar{M}_\sigma} & \mathbb{B}_{\bar{M}_\sigma} & \mathbb{B} \end{bmatrix}, [\mathbf{D}_{yp} \ \mathbf{D}_{yw}] = \begin{bmatrix} 0 & \tilde{\mathbf{C}} & \tilde{\mathbf{G}} & \mathbb{G}_M & \mathbb{G}_{\bar{M}_\alpha} & \tilde{\mathbf{G}}_{\bar{M}_\sigma} & \mathbb{G}_{\bar{M}_\sigma} & \mathbb{G} \end{bmatrix}, \\ \begin{bmatrix} \mathbf{C}_q \\ \mathbf{C}_e \end{bmatrix} &= \begin{bmatrix} \mathbf{C}_q \\ 0 \ I_r \end{bmatrix}, \begin{bmatrix} \mathbf{D}_{qp} & \mathbf{D}_{qw} \\ \mathbf{D}_{ep} & \mathbf{D}_{ew} \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{D}_{qw} \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

where \mathbf{S} is a block-diagonal matrix with the same ‘structure’ as the uncertain matrix $\hat{\Delta}(\varepsilon, t)$ given in (48), i.e.

$$\mathbf{S} = \text{bdiag}(S_x, S_e, S_w, S_\Delta^0, S_\Delta, \bar{S}_\Delta^0, \bar{S}_\Delta) > 0 \quad (51)$$

with

$$\left\{ \begin{array}{l} S_x = \text{bdiag}(\mu_{1,x}I_n, \dots, \mu_{m,x}I_n) \\ S_e = \text{bdiag}(\mu_{1,e}I_r, \dots, \mu_{m,e}I_r) \\ S_w = \text{bdiag}(\mu_{1,w}I_q, \dots, \mu_{m,w}I_q) \\ S_\Delta^0 = \text{bdiag}(\mu_{0,1}I_{\ell_{0,1}}, \dots, \mu_{0,s_0}I_{\ell_{0,s_0}}) \\ S_\Delta = \text{bdiag}(\text{bdiag}(\mu_{1,1}I_{\ell_{1,1}}, \dots, \mu_{1,s_1}I_{\ell_{1,s_1}}), \dots, \text{bdiag}(\mu_{m,1}I_{\ell_{m,1}}, \dots, \mu_{m,s_m}I_{\ell_{m,s_m}})) \\ \bar{S}_\Delta^0 = \text{bdiag}(\text{bdiag}(\tilde{\mu}_{1,1}I_{\ell_{0,1}}, \dots, \tilde{\mu}_{1,s_0}I_{\ell_{0,s_0}}), \dots, \text{bdiag}(\tilde{\mu}_{m,1}I_{\ell_{0,1}}, \dots, \tilde{\mu}_{m,s_0}I_{\ell_{0,s_0}})) \\ \bar{S}_\Delta = \text{bdiag}(\text{bdiag}(\bar{\mu}_{1,1}I_{\ell_{1,1}}, \dots, \bar{\mu}_{1,s_1}I_{\ell_{1,s_1}}), \dots, \text{bdiag}(\bar{\mu}_{m,1}I_{\ell_{m,1}}, \dots, \bar{\mu}_{m,s_m}I_{\ell_{m,s_m}})) \end{array} \right.$$

where $\mu_{i,x}$, $\mu_{i,e}$, $\mu_{i,w}$, $\mu_{0,h}$, $\mu_{i,j}$, $\tilde{\mu}_{i,h}$ and $\bar{\mu}_{i,j}$ are scalars to be chosen ($i = 1, \dots, m$, $h = 1, \dots, s_0$ and $j = 1, \dots, s_i$).

Notice that the following relation holds

$$\widehat{\Delta}(\varepsilon, t)\mathbf{S} = \mathbf{S}\widehat{\Delta}(\varepsilon, t). \quad (52)$$

Then, the gain Z is designed in the following theorem.

Theorem 2. *Assume that assumptions 1 and 2 and relation (45d) hold, there exists a robust functional unbiased reduced order filter (3) for the uncertain system (41) if there exist matrices $\mathbf{P} = \mathbf{P}^T > 0$, $\mathbf{Q} = \mathbf{Q}^T > 0$, $\mathbf{S} > 0$ and $\bar{\mathbf{S}} > 0$ such that (with $\bar{\ell} = \bar{\ell}_1 + \dots + \bar{\ell}_m$, $\bar{s} = m(n + r + q) + \bar{\ell}_0(m + 1) + 2\bar{\ell}$)*

$$\left[\begin{array}{c|c} \bar{\mathbf{K}}_y & 0 \\ \hline 0 & I_{r+\bar{s}} \end{array} \right]^T \left[\begin{array}{ccc|cc} \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} & \mathbf{P}\mathbf{B}_w & \mathbf{P}\mathbf{B}_p & \mathbf{C}_e^T & \mathbf{C}_q^T\mathbf{S} \\ \mathbf{B}_w^T\mathbf{P} & -\gamma^2 I_q & 0 & \mathbf{D}_{ew}^T & \mathbf{D}_{qw}^T\mathbf{S} \\ \mathbf{B}_p^T\mathbf{P} & 0 & -\mathbf{S} & \mathbf{D}_{ep}^T & \mathbf{D}_{qp}^T\mathbf{S} \\ \hline \mathbf{C}_e & \mathbf{D}_{ew} & \mathbf{D}_{ep} & -I_r & 0 \\ \mathbf{S}\mathbf{C}_q & \mathbf{S}\mathbf{D}_{qw} & \mathbf{S}\mathbf{D}_{qp} & 0 & -\mathbf{S} \end{array} \right] \left[\begin{array}{c|c} \bar{\mathbf{K}}_y & 0 \\ \hline 0 & I_{r+\bar{s}} \end{array} \right] < 0, \quad (53a)$$

$$\left[\begin{array}{c|c} \bar{\mathbf{K}}_u & 0 \\ \hline 0 & I_{q+\bar{s}} \end{array} \right]^T \left[\begin{array}{ccc|cc} \mathbf{Q}\mathbf{A}^T + \mathbf{A}\mathbf{Q} & \mathbf{Q}\mathbf{C}_e^T & \mathbf{Q}\mathbf{C}_q^T & \mathbf{B}_w & \mathbf{B}_p\bar{\mathbf{S}} \\ \mathbf{C}_e\mathbf{Q} & -I_r & 0 & \mathbf{D}_{ew} & \mathbf{D}_{ep}\bar{\mathbf{S}} \\ \mathbf{C}_q\mathbf{Q} & 0 & -\bar{\mathbf{S}} & \mathbf{D}_{qw} & \mathbf{D}_{qp}\bar{\mathbf{S}} \\ \hline \mathbf{B}_w^T & \mathbf{D}_{ew}^T & \mathbf{D}_{qw}^T & -\gamma^2 I_q & 0 \\ \bar{\mathbf{S}}\mathbf{B}_p^T & \bar{\mathbf{S}}\mathbf{D}_{ep}^T & \bar{\mathbf{S}}\mathbf{D}_{qp}^T & 0 & -\bar{\mathbf{S}} \end{array} \right] \left[\begin{array}{c|c} \bar{\mathbf{K}}_u & 0 \\ \hline 0 & I_{q+\bar{s}} \end{array} \right] < 0, \quad (53b)$$

$$I_{n+r} = \mathbf{P}\mathbf{Q}, \quad (53c)$$

where $\mathbf{S} = \bar{\mathbf{S}}^{-1}$ satisfy (51), and where $\bar{\mathbf{K}}_y$ and $\bar{\mathbf{K}}_u$ are two matrices whose columns span the null spaces of $[\mathbf{C}_y \ \mathbf{D}_{yp} \ \mathbf{D}_{yw}]$ and $[\mathbf{B}_u^T \ 0]$, respectively. All gains Z are given by

$$Z = \bar{\mathbf{B}}_R^\dagger \bar{\mathbf{K}} \bar{\mathbf{C}}_L^\dagger + \bar{Z} - \bar{\mathbf{B}}_R^\dagger \bar{\mathbf{B}}_R \bar{Z} \bar{\mathbf{C}}_L \bar{\mathbf{C}}_L^\dagger \quad (54)$$

with

$$\mathbf{K} = -\mathbf{R}_1^{-1} \bar{\mathbf{B}}_L^T \mathbf{V}_1 \bar{\mathbf{C}}_R^T \left(\bar{\mathbf{C}}_R \mathbf{V}_1 \bar{\mathbf{C}}_R^T \right)^{-1} + \mathbf{R}_1^{-1} \mathbf{V}_2^{1/2} \mathbf{R}_2 \left(\bar{\mathbf{C}}_R \mathbf{V}_1 \bar{\mathbf{C}}_R^T \right)^{-1/2}$$

$$\mathbf{V}_1 = \left(\bar{\mathbf{B}}_L \mathbf{R}_1^{-1} \bar{\mathbf{B}}_L^\dagger - \bar{\mathbf{Q}} \right)^{-1} > 0$$

$$\mathbf{V}_2 = \mathbf{R}_1 - \bar{\mathbf{B}}_L^T \left(\mathbf{V}_1 - \mathbf{V}_1 \bar{\mathbf{C}}_R^T \left(\bar{\mathbf{C}}_R \mathbf{V}_1 \bar{\mathbf{C}}_R^T \right)^{-1} \bar{\mathbf{C}}_R \mathbf{V}_1 \right) \bar{\mathbf{B}}_L$$

$$\left[\begin{array}{c|c} \bar{\mathbf{B}} & \bar{\mathbf{Q}} \\ \hline \bullet & \bar{\mathbf{C}} \end{array} \right] = \left[\begin{array}{ccc|cc} -\bar{\mathbf{B}}_u & \mathbf{Q}\mathbf{A}^T + \mathbf{A}\mathbf{Q} & \mathbf{Q}\mathbf{C}_e^T & \mathbf{Q}\mathbf{C}_q^T & \mathbf{B}_w & \mathbf{B}_p\bar{\mathbf{S}} \\ 0 & \mathbf{C}_e\mathbf{Q} & -I_r & 0 & \mathbf{D}_{ew} & \mathbf{D}_{ep}\bar{\mathbf{S}} \\ 0 & \mathbf{C}_q\mathbf{Q} & 0 & -\bar{\mathbf{S}} & \mathbf{D}_{qw} & \mathbf{D}_{qp}\bar{\mathbf{S}} \\ 0 & \mathbf{B}_w^T & \mathbf{D}_{ew}^T & \mathbf{D}_{qw}^T & -\gamma^2 I_q & 0 \\ 0 & \bar{\mathbf{S}}\mathbf{B}_p^T & \bar{\mathbf{S}}\mathbf{D}_{ep}^T & \bar{\mathbf{S}}\mathbf{D}_{qp}^T & 0 & -\bar{\mathbf{S}} \\ \hline \bullet & \mathbf{C}_y\mathbf{Q} & \bar{\mathbf{S}}\mathbf{D}_{yp} & \mathbf{D}_{yw} & 0 & 0 \end{array} \right]$$

and \mathbf{R}_1 , \mathbf{R}_2 and \bar{Z} are arbitrary matrices satisfying $\mathbf{R}_1 = \mathbf{R}_1^T > 0$ and $\|\mathbf{R}_2\| < 1$. $\bar{\mathbf{B}}_L$, $\bar{\mathbf{B}}_R$, $\bar{\mathbf{C}}_L$ and $\bar{\mathbf{C}}_R$ are any full rank factors such that $\bar{\mathbf{B}} = \bar{\mathbf{B}}_L \bar{\mathbf{B}}_R$ and $\bar{\mathbf{C}} = \bar{\mathbf{C}}_L \bar{\mathbf{C}}_R$.

Proof. By using the bounded real lemma (Boyd *et al.*, 1994), the problem to be solved has a solution if there exist $\mathbf{S} > 0$ (given by (51)) and $\mathbf{P} = \mathbf{P}^T > 0$ such that

$$\begin{bmatrix} \text{herm}(\mathbf{P}(\mathbf{A} - \mathbf{B}_u \mathbf{Z} \mathbf{C}_y)) & \mathbf{P}(\bar{\mathbf{B}}_w - \mathbf{B}_u \mathbf{Z} \bar{\mathbf{D}}_{yw}) & \bar{\mathbf{C}}_z^T \\ (\bar{\mathbf{B}}_w - \mathbf{B}_u \mathbf{Z} \bar{\mathbf{D}}_{yw})^T \mathbf{P} & -I_{q+\bar{s}} & \bar{\mathbf{D}}_{zw}^T \\ \bar{\mathbf{C}}_z & \bar{\mathbf{D}}_{zw} & -I_{r+\bar{s}} \end{bmatrix} < 0.$$

or, from the projection lemma (Iwasaki and Skelton, 1994), if there exist matrices $\mathbf{P} = \mathbf{P}^T > 0$, $\mathbf{Q} = \mathbf{Q}^T > 0$, $\mathbf{S} > 0$ and $\bar{\mathbf{S}} > 0$ with $\mathbf{P} = \mathbf{Q}^{-1}$ and $\bar{\mathbf{S}} = \mathbf{S}^{-1}$ such that relations (53a), (53b) and (53c) hold. Pre- and post-multiplying the above inequality by $\text{bdiag}(\mathbf{Q}, I_{q+\bar{s}}, I_{r+\bar{s}})$ give the following inequality

$$\bar{\mathbf{Q}} + \bar{\mathbf{B}} \mathbf{Z} \bar{\mathbf{C}} + \bar{\mathbf{C}}^T \mathbf{Z}^T \bar{\mathbf{B}}^T < 0, \quad (55)$$

and the relation (54) can be deduced from (55) by using formulas in (Iwasaki and Skelton, 1994). Since matrix \mathbf{S} given by (51) satisfies (52), the structure of $\hat{\Delta}(\varepsilon, t)$ is taken into account (Boyd *et al.*, 1994). The robust reduced order filter is finally obtained by using relations (13), (14), (45c) and (54). \square

4 Illustrative example

Consider the following uncertain bilinear system

$$\dot{x} = \left(\begin{bmatrix} -0.146 & 0 \\ -0.1763 & -1.197 \end{bmatrix} + \Delta_{A_0}(t) \right) x + \left(\begin{bmatrix} -0.097 & 0.09 \\ 0.08 & 0.05 \end{bmatrix} + \Delta_{A_1}(t) \right) u x + \left(\begin{bmatrix} 0.8 \\ 0.1 \end{bmatrix} + \Delta_B(t) \right) w \quad (56a)$$

$$y = \left(\begin{bmatrix} 0.7 & 0 \\ 0 & 0.01 \end{bmatrix} + \Delta_C(t) \right) x + \left(\begin{bmatrix} 0.09 \\ 0.09 \end{bmatrix} + \Delta_D(t) \right) w \quad (56b)$$

$$z = \begin{bmatrix} -0.3 & 0.2 \end{bmatrix} x \quad (56c)$$

with $u_{\min} = -0.1 \leq u(t) \leq u_{\max} = 0.1$ and

$$M_{0,x} = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.32 \end{bmatrix}, M_{0,y} = \begin{bmatrix} 0.04 & 0.1 \\ 0.4 & 0.8 \end{bmatrix}, E_{0,x} = \begin{bmatrix} 0.1 & 0.3 \\ 0.1 & 0.2 \end{bmatrix}, E_{0,w} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad (57a)$$

$$M_{1,x} = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.21 \end{bmatrix}, E_{1,x} = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.21 \end{bmatrix}. \quad (57b)$$

By choosing $S_w = 10^{-2}$, $S_e = 10^{-4}$ and $\rho = 0.1$ in the nominal case, the theorem 1 gives $P = 36.539413$, $Z = [-4.7728522 \ 0 \ 36.710613 \ 0 \ 0 \ -0.5195422]$ and $\gamma = 0.9$. The nominal filter matrices are given by

$$H_0 = -1.14558, H_1 = 0.00122, J_0 = [-0.28542 \ -1.01853], \quad (58a)$$

$$J_1 = [0.04578 \ -0.51832], E = [-0.19293 \ 0.19293]. \quad (58b)$$

Then, apply theorem 2 to obtain a robust reduced order unbiased \mathcal{H}_∞ filter for the system (56). First of all, note that the rank condition (45d) is verified. The different matrices of the static output feedback problem (50a)-(50b) can be computed easily.

The next step is the resolution of relations (53a), (53b) and (53c). Notice that each of relations (53a) and (53b) are linear in \mathbf{S} and \mathbf{S}^{-1} , separately. But these inequalities are not jointly linear in \mathbf{S} . To overcome this, choose $\mathbf{S} = \text{bdiag}(10^{-4}I_4, 5 \times 10^{-4}I_4, 10.1 \times 10^{-4}I_4)$.

The resolution of LMI (53a) and (53b) with $\mathbf{P} = \mathbf{P}^T > 0$ and $\mathbf{Q} = \mathbf{Q}^T > 0$ gives $\gamma = 25$ and \mathbf{P}, \mathbf{Q} such that

$$\mathbf{PQ} = \begin{bmatrix} 205.03434 & 447.93402 & -1.527 \times 10^{11} \\ 67.712912 & 2413.4772 & -8.341 \times 10^{11} \\ -159.08618 & -5637.0533 & 3.861 \times 10^{12} \end{bmatrix}.$$

This value is obviously far from the equality (53c) of theorem 2. After running the cone complementary linearization algorithm (El Ghaoui *et al.*, 1997) three times, the solutions $\mathbf{P} = \mathbf{P}^T > 0$ and $\mathbf{Q} = \mathbf{Q}^T > 0$ verify

$$\mathbf{PQ} = \begin{bmatrix} 1 & -3.033 \times 10^{-09} & 1.96 \times 10^{-12} \\ -3.596 \times 10^{-10} & 1 & 3.632 \times 10^{-13} \\ 3.052 \times 10^{-9} & 5.095 \times 10^{-09} & 1 \end{bmatrix} \approx I.$$

Matrices $\bar{\mathbf{B}}_L, \bar{\mathbf{B}}_R, \bar{\mathbf{C}}_L, \bar{\mathbf{C}}_R, \bar{\mathbf{Q}}, \mathbf{R}_1, \mathbf{R}_2, \mathbf{V}_1, \mathbf{V}_2$ and \mathbf{K} are then computed using relations in theorem 2. Finally, choosing $\bar{\mathbf{Z}} = [1 \ 0 \ 0 \ 0 \ 0 \ 0]$, the gain \mathbf{Z} is given by $\mathbf{Z} = [1 \ 0 \ -0.002892 \ 0.134976 \ 0.001680 \ -0.078405]$ (see (54)), and the filter matrices by

$$H_0 = -1.20321, H_1 = -0.08111, E = [0 \ 0], \quad (59a)$$

$$J_0 = [-0.50346 \ 0.12425], J_1 = [0.02967 \ -0.07780]. \quad (59b)$$

Since matrix $[M_{0,y} \ D]$ is of full row rank, then from (44), $E = 0$. The nominal filter (58) obtained from theorem 1 and the robust filter (59) designed using theorem 2 are applied to the uncertain bilinear system defined by (56) to (57).

The perturbation $w(t)$ and the control input $u(t)$ are given by figure 1. In figures 2 to 5, the uncertainties affect the nominal system. Three cases are considered : the case 1 corresponds to the nominal one, the case 2 to the robust one with $\Delta_0(t)$ and $\Delta_1(t)$ given in figures 2 and 3, and the case 3 to the robust one with $\Delta_0(t)$ and $\Delta_1(t)$ given in figures 4 and 5. The estimation error $e(t) = z(t) - \hat{z}(t)$ is given in figures 6, 7 and 8.

Figures 6, 7 and 8 show that the robust filter is sensitive to the disturbance $w(t)$, and less sensitive to parametric uncertainties $\Delta_0(t)$ and $\Delta_1(t)$. But, on the same figures, it can be noted that the nominal filter is sensitive to these uncertainties. Moreover, if the uncertainties are time-varying (figure 8), then the nominal filter is more sensitive. This motivates the importance of taking time-varying uncertainties in the synthesis of a robust filter into account. For the nominal filter, the tighter optimization on γ may explain the lack of robustness with respect to parametric uncertainties. Figures 7 and 8 also prove that the time response of the nominal filter to the initial conditions is smaller than its time response to the disturbance $w(t)$. On the other hand these time responses are small and appreciably the same for the robust filter in cases 1, 2 and 3. Note too that the time response of the nominal filter is smaller in case 1 (without uncertainty, figure 6) than that in cases 2 and 3 (with uncertainties, figures 7 and 8) before the occurrence of $w(t)$.

5 Conclusion

This paper has presented a simple solution to the \mathcal{H}_∞ unbiased functional reduced order filtering problem via LMI methods for bilinear systems. The results of the nominal case are extended to the robust filtering for bilinear systems subject to structured norm-bounded time-varying uncertainties in all system matrices. After giving conditions for the existence of the unbiased filter in the nominal case, the filter design is reduced to a robust state feedback problem in the nominal case and to a robust static output feedback one when the bilinear system is affected by the uncertainties. There is an additional non convex relation to solve in the robust case.

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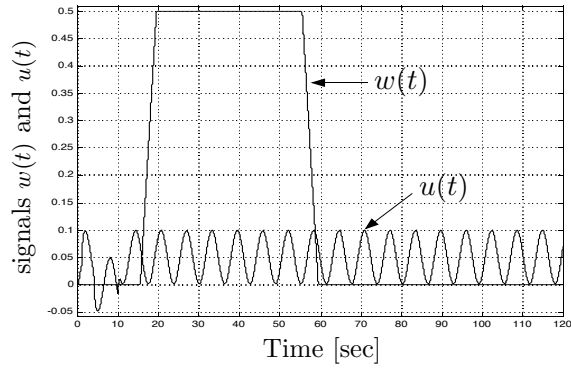


Figure 1: Perturbation $w(t)$ and control input $u(t)$.

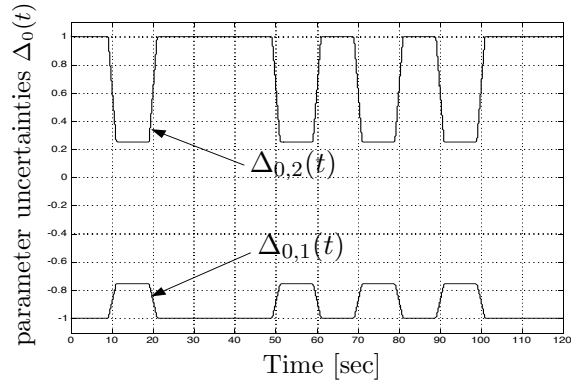


Figure 2: Parameter uncertainties $\Delta_0(t)$ in case 2.

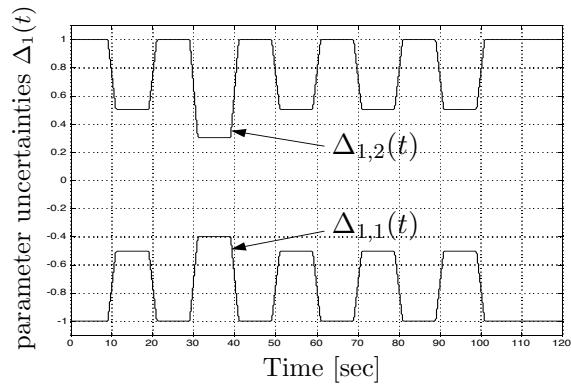


Figure 3: Parameter uncertainties $\Delta_1(t)$ in case 2.

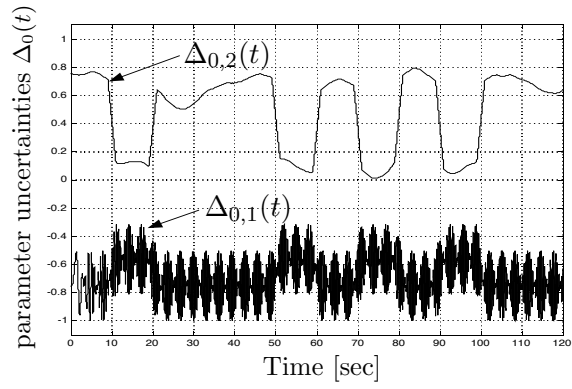


Figure 4: Parameter uncertainties $\Delta_0(t)$ in case 3.

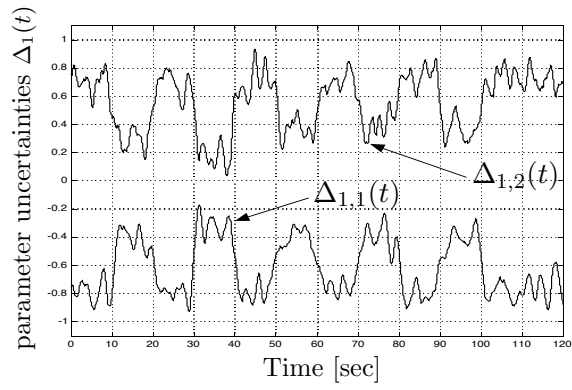


Figure 5: Parameter uncertainties $\Delta_1(t)$ in case 3.

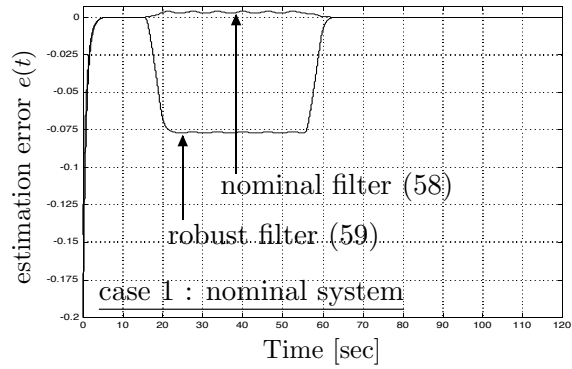


Figure 6: Robust and nominal filters : estimation error $e(t)$.

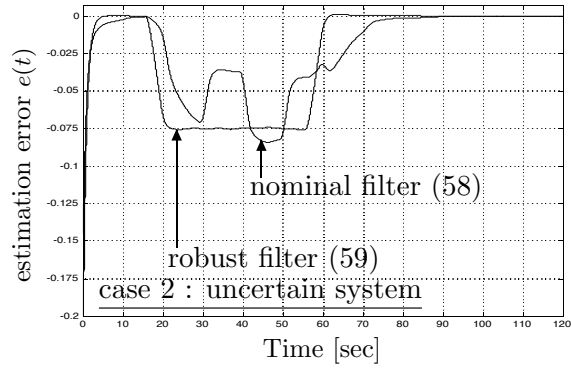


Figure 7: Robust and nominal filters : estimation error $e(t)$.

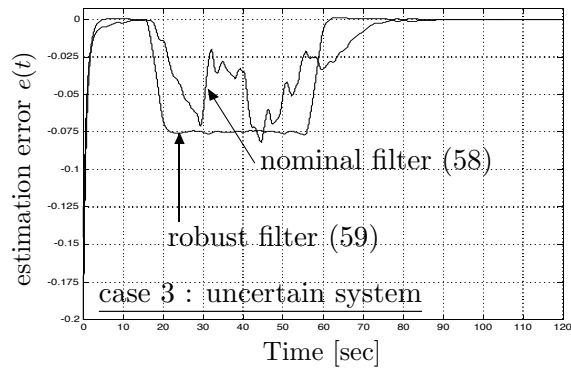


Figure 8: Robust and nominal filters : estimation error $e(t)$.