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# Density of periodic points, invariant measures and almost equicontinuous points of Cellular automata 

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#### Abstract

Revisiting the notion of $\mu$-almost equicontinuous cellular automata introduced by R. Gilman, we show that the sequence of image measures of a shift ergodic measure $\mu$ by iterations of such automata converges in Cesàro mean to an invariant measure $\mu_{c}$. Moreover the dynamical system (cellular automaton $F$, invariant measure $\mu_{c}$ ) has still the $\mu_{c}$-almost equicontinuity property and the set of periodic points is dense in the topological support of the measure $\mu_{c}$. We also show that the density of periodic points in the topological support of a measure $\mu$ occurs for each $\mu$-almost equicontinuous cellular automaton when $\mu$ is an invariant and shift ergodic measure. Finally using most of these results we give a non trivial example of a couple ( $\mu$-equicontinuous cellular automaton $F$, shift and $F$-invariant measure $\mu$ ) such that the restriction of $F$ to the topological support of $\mu$ has no equicontinuous points.


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## 1 Introduction

A one-dimensional cellular automaton (CA) is a discrete mathematical idealization of a space-time physical system. The space, called configuration space, is the set of doubly infinite sequences of elements of a finite set $A$. The discrete time is represented by the action of a CA on this space. Using extensive computer simulations, Wolfram in [9] has proposed a first empirical (visual) classification of one dimensional CA. In [5] Gilman propose a formal and measurable classification by roughly dividing the set of all CA in two parts, those with almost equicontinuous points or equicontinuous points and those with almost expansive points (partition in order and disorder). The Gilman's classification is defined thanks to Bernoulli measures and corresponds to the Wolfram's classification based on simulations that use random entries. The measure does not need to be invariant, so the Gilman's classification can be apply to any CA. In [7], Kůrka, introduce a topological classification based on the equicontinuity, sensitiveness and expansiveness properties. If a CA has equicontinuous points, then there exist finite configurations that stop the propagation of the perturbations on the one dimensional lattice. If a CA has $\mu$-almost equicontinuous points then the probability that a perturbation move to infinity is equal to zero (see [5]). Remark that the class of CA with almost equicontinuous points contains the topological class of CA with equicontinuous points. In this paper, we consider the definitions of Gilman ( $\mu$-expansiveness and $\mu$-equicontinuity) in the more general case of probability measure on the configuration space $A^{\mathbb{Z}}$. In this case we call $\mu$-equicontinuous CA, any CA which has a set of measure one of $\mu$ equicontinuous points. Our main goal is to study the $\mu$-equicontinuous CA when $\mu$ is an invariant measure and show the existence of such a measure. Here we prove (see Theorem 1) that if $\mu$ is a shift ergodic measure and $F$ a CA which has $\mu$-equicontinuous points then the sequence $\left(\mu \circ F^{-n}\right)$ converges in Cesàro mean to an $F$-invariant measure $\mu_{c}$. We also show that this automaton $F$ is still a $\mu_{c}$-equicontinuous CA. Then, we describe properties of $\mu$-equicontinuous CA when $\mu$ is an $F$-invariant measure. Here we show that (see Proposition 4) under these assumptions, the measure entropy is equal to zero. If the measure is shift ergodic or is the measure $\mu_{c}$ given by Theorem 1 (see Proposition 5 and 6), the set of $F$-periodic points is dense in the topological support of this invariant measure. This result extends a previous result on the density of periodic points of surjective CA with equicontinuous points acting on a mixing subshift of finite type (see [1]). Remark that in
[3] Boyle and Kitchen have shown that closing CA always have a dense set of periodic points. The expansive CA and some other CA with equicontinuous points belong to this large class. In [5] Gilman gives an example of a $\mu$-equicontinuous CA that has no equicontinuous points. The invariant measure $\mu_{c}$, (limit by Cesàro mean of $\left(\mu \circ F^{n}\right)$ ) that we can construct (using our results) for this particular automaton still has $\mu_{c}$-equicontinuous points and no equicontinuous points, but the restriction of this CA to the topological support of $\mu_{c}$ always has equicontinuous points. Using most of our results, we describe a particular CA called $F_{e}$ with a non trivial dynamic which keep the sensitiveness property (no equicontinuous points) if we restrict its action to the topological support $S\left(\mu_{c}\right)$ of the invariant measure $\mu_{c}$. The proofs of sensitiveness and $\mu_{c}$-equicontinuity of the automaton $\left(S\left(\mu_{c}\right), F_{e}\right)$ use non classical arguments.

## 2 Definitions and preliminary results

### 2.1 Symbolics systems and cellular automata

Let $A$ be a finite set or alphabet. Denote by $A^{*}$ the set of all concatenations of letters in $A$. These concatenations are called words. The length of a word $u \in A^{*}$ is denoted by $|u|$. The set of bi-infinite sequences $x=\left(x_{i}\right)_{i \in \mathbb{Z}}$ is denoted by $A^{\mathbb{Z}}$. A point $x \in A^{\mathbb{Z}}$ is called a configuration. For integers $i, j$ with $i \leq j$ we denote by $x(i, j)$ the word $x_{i} \ldots x_{j}$ and by $x(i, \infty)$ the infinite sequence $\left(v_{n}\right)_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$ one has $v_{n}=x_{i+n}$. For each integer $t$ and each word $u=u_{1} \ldots u_{|u|}$, we call cylinder the set $[u]_{t}=\{x \in$ $\left.A^{\mathbb{Z}}: x_{t}=u_{1} \ldots ; x_{t+|u|}=u_{|u|}\right\}$. We endow $A^{\mathbb{Z}}$ with the product topology of the discrete topologies on the sets $A$. For this topology $A^{\mathbb{Z}}$ is a compact metric space. A metric compatible with this topology can be defined by the distance $d(x, y)=2^{-i}$ where $i=\min \left\{|j|\right.$ such that $\left.x_{j} \neq y_{j}\right\}$. The shift $\sigma: A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ is defined by: $\sigma(x)=\left(x_{i+1}\right)_{i \in \mathbb{Z}}$. The dynamical system $\left(A^{\mathbb{Z}}, \sigma\right)$ is called the full shift. A subshift $X$ is a closed shift-invariant subset $X$ of $A^{\mathbb{Z}}$. Consider a probability measure $\mu$ on the Borel sigma-algebra $\mathcal{B}$ of $A^{\mathbb{Z}}$. If $\mu$ is $\sigma$-invariant then the topological support of $\mu$ (which is the smallest closed subset of measure 1) is a subshift denoted by $S(\mu)$. If $\alpha=\left\{A_{1}, \ldots, A_{n}\right\}$ and $\beta=\left\{B_{1}, \ldots, B_{m}\right\}$ are two partitions of a compact space $X$, denote by $\alpha \vee \beta$ the partition $\left\{A_{i} \cap A_{j} ; 1 \leq i \leq n ; 1 \leq j \leq m\right\}$. The metric entropy $h_{\mu}(T)$ of a transformation $T$ is an isomorphism invariant between two $\mu$-preserving
transformations. Put $H_{\mu}(\alpha)=\sum_{A \in \alpha} \mu(A) \log \mu(A)$. The entropy of the partition $\alpha$ is defined as $h_{\mu}(\alpha)=\lim _{n \rightarrow \infty} \frac{1}{n} H_{\mu}\left(\vee_{i=0}^{n-1} T^{-i} \alpha\right)$ and the entropy of $(X, T, \mu)$ as $\sup _{\alpha} h_{\mu}(\alpha)$. A cellular automaton is a continuous self-map $F$ on $A^{\mathbb{Z}}$ commuting with the shift. The Curtis-Hedlund-Lyndon theorem states that for every $F$ there exist an integer $r$ and a block map $f$ from $A^{2 r+1}$ to $A$ such that: $F(x)_{i}=f\left(x_{i-r}, \ldots, x_{i}, \ldots, x_{i+r}\right)$. The integer $r$ is called the radius of the CA. If $X$ is a subshift of $A^{\mathbb{Z}}$ and one has $F(X) \subset X$, the restriction of $F$ to $X$ determines a dynamical system $(X, F)$; it is called a CA on $X$. For example, given any shift invariant measure we can consider the restriction of $F$ to $S(\mu)$. A closed subset of $Y \subset A^{\mathbb{Z}}$ (not necessarily shift-invariant) such that $F(Y) \subset Y$ is said $F$-invariant. The omega limit set of any set $S \subset A^{\mathbb{Z}}$ under the action of $F$ is denoted by $w(S, F)=\lim _{n \rightarrow \infty} \cap_{j=0}^{n} \cup_{i=j}^{\infty}\left\{F^{i}(S)\right\}$.

### 2.2 Almost equicontinuous points of cellular automata

In [5] Gilman shows that considering any Bernoulli measure $\mu$, it is possible to divide the set of all CA in the three following classes: The class of CA where there exist equicontinuous points, the class of CA with $\mu$-almost equicontinuous points but without equicontinuous point and the class of $\mu$-almost expansive CA. In this section we recall the topological and classical definitions for the expansive and equicontinuous classes of CA acting on $A^{\mathbb{Z}}$ and we extend the Gilman's measurable definitions to any probability measure $\mu$.

For any integer $n \geq 0$ and point $x \in A^{\mathbb{Z}}$, we denote by $B_{n}(x)$ the set of points $y$ such that for all $i \in \mathbb{N}$ one has $d\left(F^{i}(x), F^{i}(y)\right) \leq 2^{-(n+1)}$ and by $C_{n}(x)$ the set of points $y$ such that $y_{j}=x_{j}$ with $-n \leq j \leq n$.

## Definition 1 Equicontinuity

-A point $x \in A^{\mathbb{Z}}$ is called an equicontinuous point if for all positive integer $n$ there exists another positive integer $m$ such that $B_{n}(x) \supset C_{m}(x)$.
-A CA is called almost equicontinuous when there exists equicontinuous points.
$-A C A$ is equicontinuous when all points in $A^{\mathbb{Z}}$ are equicontinuous.
-A point $x$ is $\mu$-almost equicontinuous if for all $m \in \mathbb{N}$ one has

$$
\lim _{n \rightarrow \infty} \frac{\mu\left(C_{n}(x) \cap B_{m}(x)\right)}{\mu\left(C_{n}(x)\right)}=1 .
$$

- A CA is $\mu$-almost equicontinuous if there exists a set of full measure of $\mu$-almost equicontinuous points.

The last definition extends the notion of $\mu$-equicontinuity to all measurable CA $\left(A^{\mathbb{Z}}, F, \mu\right)$. If $x$ is a $\mu$-equicontinuous point which belongs to the topological support $S(\mu)$ of some measure $\mu$ then $x$ is also a $\mu$-equicontinuous point. Remark that for CA, to have no equicontinuous points is equivalent to sensitiveness (see [7]).

## Definition 2 Expansiveness

-A Cellular automaton is positively expansive if there exists a positive integer $n$ such that for all $x \in A^{\mathbb{Z}}$ one has $B_{n}(x)=\{x\}$.
-A cellular automaton $F$ is almost expansive if there exists a positive integer $n$ such that for all $x \in A^{\mathbb{Z}}, \mu\left(B_{n}(x)\right)=0$.

In [5] Gilman proves the following proposition that allows to establish a classification based on $\mu$-equicontinuity. Since in [5], the proof of the original Proposition 1 requires only the ergodicity of any Bernoulli measure, it can be extended to any shift ergodic measure.

Proposition 1 [5] Let $\mu$ be a shift-ergodic measure and $F$ a cellular automaton with radius $r$. The following properties are equivalent:
(i) $F$ has a $\mu$-equicontinuous point.
(ii) There exists a point $x \in A^{\mathbb{Z}}$ such that $\mu\left(B_{m}(x)\right)>0$ for all $m \geq 0$.
(iii) There exists a point $x \in A^{\mathbb{Z}}$ such that $\mu\left(B_{r}(x)\right)>0$.
(iv) The set of $\mu$-equicontinuous points has measure 1 for $F$.

When $\mu$ is a shift ergodic measure and $F$ has no $\mu$-equicontinuous points then $F$ is an an almost expansive CA. A point $x$ is an equicontinuous point when the interior of $B_{n}(x)$ is non empty for all $n \in \mathbb{N}$.

In [5], Gilman introduce a measure-theoretic analogue of the interior of a set. For any measurable set $E$, define $\rho_{E}(x)=\lambda$ if $\lim _{n \rightarrow \infty} \frac{\mu\left(C_{n}(x) \cap E\right)}{\mu\left(C_{n}(x)\right)}=\lambda$ and call $E^{\mu}$ the set $\left\{x \in E: \rho_{E}(x)=1\right\}$.

Lemma 1 (Lebesgue) If $\mu$ is a Borel probability measure and $E$ any measurable set, we have $\mu\left(E^{\mu}\right)=\mu(E)$.

A point $x$ is a $\mu$-equicontinuous point if for all $n \in \mathbb{N}$ one has $x \in B_{n}^{\mu}(x)$.
The next topological result is due to Gilman (see [6]). We give here a detailed proof of this key result.

Proposition 2 [6] If there exist a point $x$ and an integer $m \neq 0$ such that $B_{n}(x) \cap \sigma^{-m} B_{n}(x) \neq \emptyset$ with $n \geq r$ (the radius of the automaton $F$ ) then the common sequence $\left(F^{i}(y)(-n, n)\right)_{i \in \mathbb{N}}$ of all points $y \in B_{n}(x)$ is ultimately periodic.

## Proof:

First remark that for each shift periodic point $\bar{w}$ of period $P$, the cardinal of the set $\left\{F^{i}(\bar{w}) \mid i \in \mathbb{N}\right\}$ is finite and less or equal to $(\# A)^{P}$ (for all integer $k \geq 0$ one has $\left.\sigma^{P} \circ F^{k}(\bar{w})=F^{k} \circ \sigma^{P}(\bar{w})=F^{k}(\bar{w})\right)$. This implies that the sequence $\left(F^{i}(\bar{w})\right)_{i \in \mathbb{N}}$ is ultimately periodic. Since all the elements of $B_{n}(x)$ share the same ultimately periodic sequence $\left(F^{i}(x)(-n, n)\right)_{i \in \mathbb{N}}$, we only need to show that $B_{n}(x)$ contains a shift periodic element $\bar{w}$ to finish the proof. Now suppose without loosing generalities that $m>0$, pick a point $y_{1} \in B_{n}(x) \cap \sigma^{-m}\left(B_{n}(x)\right)$ and put $w:=y_{1}(-n,-n+m-1)$.

We claim that for all points $x$, integers $n \geq r$ and $y \in B_{n}(x)$, all points $z, z^{\prime}$ that verify $z(-\infty,-n)=y(-\infty,-n), z(-n+1,+\infty)=x(-n+1,+\infty)$, $z^{\prime}(-\infty,-n)=x(-\infty,-n)$ and $z^{\prime}(-n+1,+\infty)=y(-n+1,+\infty)$ belong to $B_{n}(x)$.
We prove the claim only for $z$ using a recurrence proof. To simplify, we indifferently denote by $f$, the local rule of $F$ which is a block map from $A^{2 r+1}$ to $A$ and also all the finite extensions of $f$ which are block maps from $A^{2 r+1+k}$ to $A^{k+1}$ with $k \in \mathbb{N}$. Since $y \in B_{n}(x)$ we have $z(-n, n)=y(-n, n)=$ $x(-n, n)$. Suppose that for $i>0$ one has $F^{t}(z)(-n, n)=F^{t}(x)(-n, n)$ for $0 \leq t \leq i$. In this case we have $F^{t}(z)(-\infty,-n)=F^{t}(y)(-\infty,-n)$ for $0 \leq t \leq i$ and $F^{i+1}(z)(-n, 0)=f\left(F^{i}(z)(-n-r, r)\right)=f\left[F^{i}(z)(-r-n,-n-\right.$ 1) $\left.F^{i}(x)(-n, r)\right]=f\left[F^{i}(y)(-r-n,-n-1) F^{i}(x)(-n, r)\right]=F^{i+1}(x)(-n, 0)$. Since $F^{i+1}(z)(1, n)=f\left(F^{i}(x)(-r+1, n+r)\right)=F^{i+1}(x)(1, n)$ it follows that $F^{i+1}(z)(-n, n)=F^{i+1}(x)(-n, n)$. We can conclude saying that $\left(F^{k}(z)(-n\right.$ , $n))_{k \in \mathbb{N}}=\left(F^{k}(x)(-n, n)\right)_{k \in \mathbb{N}}$ which implies that $z \in B_{n}(x)$.

Now we apply the claim to a point $\sigma^{m}\left(y_{1}\right) \in \sigma^{m} B_{n}(x) \cap B_{n}(x)$. The points $y_{1}$ and $\sigma^{m}\left(y_{1}\right)$ belong to $B_{n}(x)$, so the point $y_{2}$ such that $y_{2}(-\infty,-n)=$ $\sigma^{m}\left(y_{1}\right)(-\infty,-n)$ and $y_{2}(-n+1,+\infty)=y_{1}(-n+1,+\infty)$ belongs to $B_{n}(x)$. We can see that $y_{2}(-n-m,-n+m-1)=w w$ and $y_{2} \in \sigma^{m} B_{n}(x) \cap$ $B_{n}(x) \cap \sigma^{-m} B_{n}(x)$. Next we construct $y_{3}$ by applying the claim to $\sigma^{m}\left(y_{2}\right)$ and $\sigma^{-m}\left(y_{2}\right)$ (remark that $\left.\sigma^{m}\left(y_{2}\right) \in B_{n}\left(\sigma^{-m}\left(y_{2}\right)\right)=B_{n}(x)\right)$. The point $y_{3}$ is such that $y_{3}(-\infty,-n)=\sigma^{m}\left(y_{2}\right)(-\infty,-n)$ and $y_{3}(-n+1,+\infty)=$ $\sigma^{-m}\left(y_{2}\right)(-n+1,+\infty)$. Since we have $y_{3}(-n-2 m, 2 m-1)=w w w w$, we can repeat the same process to $\sigma^{2 m}\left(y_{3}\right)$ and $\sigma^{-2 m}\left(y_{3}\right)$ and construct a point $y_{4}$
such that $y_{4}(-n-4 m, 4 m-1)=w^{8}$. Finally, the sequence $\left(y_{n}\right)_{n \in \mathbb{N}}$ of points of $B_{n}(x)$ that we construct by this algorithm converges to the shift periodic point $\bar{w}=^{\infty} w^{\infty}=\ldots w w w w \ldots$. Since $B_{n}(x)$ is closed and compact, $\bar{w}$ is in $B_{n}(x)$.

In [6] Gilman state the following result using the ergodic properties of any Bernoulli measure $\mu$.

Proposition 3 [6] Let $\mu$ be a shift ergodic measure. If a cellular automaton $F$ has a $\mu$-equicontinuous point, then for all $\epsilon>0$ there exists a $F$-invariant closed set $Y$ such that $\mu(Y)>1-\epsilon$ and the restriction of $F$ to $Y$ is equicontinuous.

## Proof:

Let $x$ be a $\mu$-equicontinuous point and $m$ and integer greater than $r$, the radius of $F$. Since $\mu$ is a shift ergodic measure and $\mu\left(B_{m}(x)\right)>0$, for each integer $k \geq 0$ and $\mu$-almost all points $y$, there exist positive integers $i, j$ greater than $k$ such that $y \in \sigma^{-i} B_{m}(x) \cap \sigma^{j} B_{m}(x)$. Then from Proposition 2, the sequences $\left(u_{n}\right)_{n \in \mathbb{N}}=\left(F^{n}(y)(i-m, i+m)\right)_{n \in \mathbb{N}}$ and $\left(v_{n}\right)_{n \in \mathbb{N}}=\left(F^{n}(y)(-j-\right.$ $m,-j+m))_{n \in \mathbb{N}}$ are ultimately periodic with respective preperiods $p p_{u}$ and $p p_{v}$. For all $n \in \mathbb{N}$, denote by $w_{n}$ the word $F^{n}(y)(-j+m+1, i-m)$ and remark that for each integer $n \geq 1$, one has $w_{n}=f\left(u_{n-1} w_{n-1} v_{n-1}\right)$ where $f$ is a block map. It follows that if a word $w=w_{t}$ with $t \geq p p_{u}, p p_{v}$ appears infinitely often in $\left(w_{n}\right)_{n \in \mathbb{N}}$, the sequence $\left(w_{n+t}\right)_{n \in \mathbb{N}}$ is periodic. This implies that $\left(w_{n}\right)_{n \in \mathbb{N}}$ and $\left(F^{n}(y)(-k, k)\right)_{n \in \mathbb{N}}$ are ultimately periodic sequences. Let $P$ be a map from $\mathbb{N}$ to $\mathbb{N}^{2}$ and $Y_{P(k)}$ be the set of points $y$ such that each sequence $\left(F^{n}(y)(-k, k)\right)_{n \in \mathbb{N}}$ are periodic of period $p(k)$ and preperiod $p p(k)$ where $(p(k), p p(k))=P(k)$. As $\mu$ is shift ergodic measure, for each real $\epsilon>0$ there exists a map $P_{\epsilon}: \mathbb{N} \rightarrow \mathbb{N}^{2}$ such that for all $n \in \mathbb{N}$ we have $\mu\left(Y_{P_{\epsilon}(n)}\right)>1-\epsilon \times 2^{-n}$. Since each $Y_{P_{\epsilon}(k)}$ is closed and $F$-invariant, the set $Y_{\epsilon}=\lim _{n \rightarrow \infty} \cap_{k=1}^{n} Y_{P_{\epsilon}(k)}$ is closed and $F$-invariant too. Clearly $\mu\left(Y_{\epsilon}\right)>1-\epsilon$ and each point $y \in Y_{\epsilon}$ is an equicontinuous point since for each integer $k \geq 0$ one has $C_{l \times r}(y) \subset B_{k}(y)$ with $l=p(k)+p p(k)$ and $(p(k), p p(k))=P_{\epsilon}(k)$.

## 3 Results on invariant measures

### 3.1 Measure entropy and density of the set of periodic points

Proposition 4 The measure entropy $h_{\mu}(F)$ of a $\mu$-equicontinuous and $\mu$ invariant cellular automaton $F$ is equal to zero.

## Proof:

Let $\alpha_{p}$ be the partition of $A^{\mathbb{Z}}$ by the $2 p+1$ central coordinates. Two points $x$ and $y$ belong to the same element of $\alpha_{p}$ if and only if $x(-p, p)=y(-p, p)$. Let $\alpha_{p}^{n}(x)$ be the element of the partition $\alpha_{p} \cap F^{-1} \alpha_{p} \ldots F^{-n+1} \alpha_{p}$ which contains $x$. Clearly for all $n \in \mathbb{N}$, we have $\alpha_{p}^{n}(x) \supset B_{p}(x)$. Since almost all points are $\mu$-equicontinuous points, there exist a set $Z$ with measure 1 such that if $y \in Z$ then $\mu\left(B_{m}(y)\right)>0$ for all integer $m \geq 0$. This implies that for almost all $y$ and all positive integer $p$, we have $\lim _{n \rightarrow \infty} \frac{-\log \mu\left(\alpha_{p}^{n}(y)\right)}{n} \leq$ $\lim _{n \rightarrow \infty} \frac{-\log \mu\left(B_{p}(x)\right)}{n}=0$. Using the Shannon-Breiman-McMillan theorem which tell that

$$
h_{\mu}\left(F, \alpha_{p}\right)=\int_{A^{\mathbb{Z}}} \lim _{n \rightarrow \infty} \frac{-\log \mu\left(\alpha_{p}^{n}(y)\right)}{n} d \mu(y)=0,
$$

we can conclude that $h_{\mu}(F)=\lim _{p \rightarrow \infty} h_{\mu}\left(F, \alpha_{p}\right)=0$.
In Proposition 4, the measure $\mu$ does not need to be shift invariant. From Proposition 4, all cellular automaton $F$ which has equicontinuous points in the topological support of a shift ergodic and $F$-invariant measure $\mu$ verifies $h_{\mu}(F)=0$. This result about CA with equicontinuous points first appears in [8].

In [2] it is shown that if a measure is shift and $F$-invariant, the entropy of the CA $\left(A^{\mathbb{Z}}, F, \mu\right)$ is equal to zero when some discrete analogues of Lyapunov exponents are null. Remark that these Lyapunov exponents can be equal to zero for almost expansive CA (see [2]) but even if they are always null when there exists a set of full measure of equicontinuous points (see [8]), they are not in general equal to zero for $\mu$-equicontinuous CA (it can be easily seen in examples of section 3.3).

Proposition 5 If a cellular automaton $F$ has some $\mu$-equicontinuous points where $\mu$ is a $F$-invariant and shift ergodic measure then the set of $F$-periodic points is dense in the topological support of $\mu$.

## Proof:

It is sufficient to show that for each point $z$ in $S(\mu)$ (the topological support of $\mu)$ and positive integer $p$ we can construct a $\sigma$ and $F$ periodic point $\bar{w}={ }^{\infty} w^{\infty}$ (bi-infinite repetition of the word $w$ ) such that $\bar{w}(-p, p)=z(-p, p)$. Remark that if $x$ is a $\mu$-equicontinuous point then $\mu\left(B_{r}(x)\right)>0$ where $r$ is the radius of $F$. Since $z$ is in $S(\mu)$ and $\mu$ is a shift ergodic measure then $\mu\left(C_{p}(z)\right)>0$ and there exist $(i, j) \in \mathbb{N}^{2}$ such that $\mu\left(C_{p}(z) \cap \sigma^{-(i+p)} B_{r}(x) \cap \sigma^{j+p} B_{r}(x)\right)>0$. To simplify, write $S=C_{p}(z) \cap \sigma^{-(i+p)} B_{r}(x) \cap \sigma^{j+p} B_{r}(x)$. Clearly there exists a point $y \in S$ such that $\mu\left([y(-r-i-p, j+p-r-1)]_{-r-i-p} \cap S\right)>0$. Denote by $S^{\prime}$ the set $[y(-r-i-p, j+p-r-1)]_{-r-i-p} \cap S$ and remark that using the Poincaré recurrence theorem, we obtain that there exists an integer $m>0$ such that $\mu\left(S^{\prime} \cap F^{-m} S^{\prime}\right)>0$. Remark that all points $y^{\prime} \in S^{\prime}$ share the same sequence $\left(F^{n}\left(y^{\prime}\right)(-r-i-p, j+p-r-1)\right)_{n \in \mathbb{N}}$ since it result of the same combination of block maps on the word $y(-i-p+r+1, j+p-r-1)$ and on the two sequences $\left(F^{n}(x)(-i-p-r,-i-p+r)\right)_{n \in \mathbb{N}}$ and $\left(F^{n}(x)(j+p-r, j+p+r)\right)_{n \in \mathbb{N}}$. It follows that $\forall y^{\prime} \in S^{\prime}$ one has $F^{m}\left(y^{\prime}\right)(-r-i-p, j+p-r-1)=y^{\prime}(-r-i-p, j+p-r-1)$. From Proposition 2 and its proof, since $\sigma^{-i} B_{r}(x) \cap \sigma^{j} B_{r}(x) \neq \emptyset$, the shift periodic point $\bar{w}={ }^{\infty} w^{\infty}$ such that $w=\bar{w}(-r-i-p, j+p-r-1)=$ $y(-r-i-p, j+p-r-1)$ belongs to the set $S^{\prime}$. Finally, we obtain that $F^{m}(\bar{w})(-r-i-p, j+p-r-1)=\bar{w}(-r-i-p, j+p-r-1)$ and since the common $\sigma$-period of $\bar{w}$ and $F^{m}(\bar{w})$ is less or equal to $|w|=2 p+2 r+i+j$, we get that $F^{m}(\bar{w})=\bar{w}$ which finish the proof.

### 3.2 Invariant measures as limit of Cesàro means

Proposition 2 also allows us to prove a Cesàro mean convergence result.
Theorem 1 Let $\mu$ be a shift-ergodic measure. If a cellular automaton $F$ has some $\mu$-almost equicontinuous points then the sequence $\left(\mu \circ F^{-n}\right)_{n \in \mathbb{N}}$ converges vaguely in Cesàro mean to an invariant measure $\mu_{c}$.

## Proof

To show that the sequence of measure $\left(\frac{1}{n} \sum_{i=0}^{n-1} \mu \circ F^{-i}\right)_{n \in \mathbb{N}}=\left(\mu_{n}\right)_{n \in \mathbb{N}}$ converges vaguely in measure, we need to show that for all $x \in S(\mu)$ and $m \in \mathbb{N}$ the sequence $\left(\mu_{n}\left(C_{m}(x)\right)\right)_{n \in \mathbb{N}}$ converges. From Proposition 3 there exists a set $Y_{\epsilon}$ of measure greater than $1-\epsilon$ such that for all points $y \in Y_{\epsilon}$ and positive integer $k$ the sequences $\left(F^{n}(y)(-k, k)\right)_{n \in \mathbb{N}}$ are eventually periodic with
preperiod $p p_{\epsilon}(k)$ and period $p_{\epsilon}(k)$. Hence for all $x \in A^{\mathbb{Z}}$ and integer $k \geq m$

$$
\begin{aligned}
\mu_{n}\left(C_{m}(x) \cap Y_{\epsilon}\right)= & \frac{1}{n} \sum_{i=0}^{p p_{\epsilon}(k)-1} \mu\left(F^{-i}\left(C_{m}(x)\right) \cap Y_{\epsilon}\right) \\
& +\frac{1}{n} \sum_{i=p p_{\epsilon}(k)}^{n-1} \mu\left(F^{-i}\left(C_{m}(x)\right) \cap Y_{\epsilon}\right) .
\end{aligned}
$$

The first term tends to 0 ; using periodicity one gets

$$
\lim _{n \rightarrow \infty} \mu_{n}\left(C_{m}(x) \cap Y_{\epsilon}\right)=\frac{1}{p_{\epsilon}(k)} \sum_{i=0}^{p_{\epsilon}(k)-1} \mu\left(F^{-\left(i+p p_{\epsilon}(k)\right)}\left(C_{m}(x) \cap Y_{\epsilon}\right) .\right.
$$

Clearly we have $\lim _{\epsilon \rightarrow 0} \mu_{n}\left(C_{m}(x) \cap Y_{\epsilon}\right)=\mu_{n}\left(C_{m}(x)\right)$. The convergence is uniform with respect to $\epsilon$ since for all $x$ and $m \in \mathbb{N}$

$$
\left|\mu_{n}\left(C_{m}(x) \cap Y_{\epsilon}\right)-\mu_{n}\left(C_{m}(x)\right)\right| \leq \frac{n \epsilon}{n}=\epsilon .
$$

Consequently, letting $\epsilon$ going to 0 , we get the result by inverting the limits

$$
\begin{gathered}
\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu \circ F^{-i}\left(C_{m}(x)\right)=\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \lim _{\epsilon \rightarrow 0} \mu \circ F^{-i}\left(C_{m}(x) \cap Y_{\epsilon}\right) \\
=\lim _{\epsilon \rightarrow 0} \lim _{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu \circ F^{-i}\left(C_{m}(x) \cap Y_{\epsilon}\right) \\
=\lim _{\epsilon \rightarrow 0} \frac{1}{p_{\epsilon}(k)} \sum_{i=0}^{p_{\epsilon}(k)-1} \mu\left(F^{-\left(i+p p_{\epsilon}(k)\right)}\left(C_{m}(x)\right) \cap Y_{\epsilon}\right)=\mu_{c}\left(C_{m}(x)\right) .
\end{gathered}
$$

We denote by $\mu_{c}$ the Cesàro mean limit of $\left(\mu \circ F^{n}\right)_{n \in \mathbb{N}}$.
In the following of this section, we suppose that $\mu_{c}$ is the Probability measure which came from the Cesàro mean of the sequence $\left(\mu \circ F^{-n}\right)_{n \in \mathbb{N}}$ when $F$ is a $\mu$-equicontinuous CA and $\mu$ is a shift ergodic measure.

Theorem 2 If $\mu$ is a shift ergodic measure and $F$ has a $\mu$-equicontinuous point then $F$ is also a $\mu_{c}$-equicontinuous $C A$.

Proof
We need to show that there exists a set of measure one (for the measure $\mu_{c}$ ) of $\mu_{c}$-equicontinuous points. We will prove this by showing that $\mu_{c}$-almost
all points $y$ belong $B_{m}^{\mu_{c}}(y)$ for all $m \in \mathbb{N}$. Since $F$ is a $\mu$-equicontinuous CA then there exists a point $x$ such that $x \in B_{m}^{\mu}(x)$ for all $m \in \mathbb{N}$. Moreover, for all positive integer $k$ one has $\mu\left(B_{k}(x)\right)>0$.

For all $m \in \mathbb{N}$, define $Y_{m}=\cup_{i, j \in \mathbb{N}^{2}}\left(\sigma^{-i-m} B_{r}(x) \cap \sigma^{j+m} B_{r}(x)\right)$ where $r$ is the radius of the automaton $F$. Since $\mu$ is a shift ergodic measure, for all $m \in \mathbb{N}$ we get $\mu\left(Y_{m}\right)=1$. Then consider the omega limit sets of $Y_{m}$ called $\Omega_{m}:=w\left(Y_{m}, F\right)=\lim _{n \rightarrow \infty} \cap_{j=0}^{n} \cup_{i=j}^{\infty} F^{i}\left(Y_{m}\right)$. Clearly for all $m \in \mathbb{N}$, we have $\mu_{c}\left(\Omega_{m}\right)=1$. Let $\Lambda(F)=w\left(A^{\mathbb{Z}}, F\right)$, the omega limit set of the $F$. Since $\mu$ is a shift ergodic measure then there exists an integer $k$ such that $B_{r}(x) \cap \sigma^{k} B_{r}(x) \neq \emptyset$ and using Proposition 2 we obtain that the sequence $\left(F^{n}(x)(-r, r)\right)_{n \in \mathbb{N}}$ is ultimately periodic of period $p$. It follows that there exist $p$ points $z_{0}, \ldots z_{p-1}$ such that we have $w\left(B_{r}(x), F\right)=\left\{B_{r}\left(z_{l}\right) \cap \Lambda(F) \mid 0 \leq l \leq\right.$ $p-1\}$. This implies that $\Omega_{m}=\cup_{z \in\left[z 0 \ldots z_{p-1}\right]} \cup_{i, j \in \mathbb{N}^{2}}\left(\sigma^{-i-m} B_{r}(z) \cap \sigma^{j+m} B_{r}(z)\right) \cap$ $\Lambda(F)$. Define the set $\Omega_{m}^{\prime}=\cup_{z \in\left[z_{0} \ldots z_{p-1}\right]} \cup_{i, j \in \mathbb{N}^{2}}\left(\sigma^{-i-m} B_{r}(z) \cap \sigma^{j+m} B_{r}(z)\right)^{\mu_{c}}$ $\cap \Lambda(F)$. Since for any measurable set $E$ one has $\mu(E)=\mu\left(E^{\mu_{c}}\right)$ (see Lemma 1) and that we need to take off a countable number of sets of measure zero from $\Omega_{m}$ to obtain $\Omega_{m}^{\prime}$, we get that $\mu_{c}\left(\Omega_{m}^{\prime}\right)=1$ for all $m \in \mathbb{N}$.

Define the set $\Omega:=\cap_{m \in \mathbb{N}} \Omega_{m}^{\prime}$ and remark that $\mu_{c}(\Omega)=1$. Clearly, for all $y \in \Omega$ and all $m \in \mathbb{N}$, there exist $i, j \geq m$ and $z \in w(x, F)$ such that $y \in\left(\sigma^{-i} B_{r}(z) \cap \sigma^{j} B_{r}(z)\right)^{\mu_{c}}$. Since for all $y^{\prime} \in\left(\sigma^{-i} B_{r}(z) \cap \sigma^{j} B_{r}(z)\right)^{\mu_{c}}$, the sequence $\left(F^{n}\left(y^{\prime}\right)(-m, m)\right)_{n \in \mathbb{N}}$ depends only on $y^{\prime}(-m, m)$ and the common sequences $\left(F^{n}\left(y^{\prime \prime}\right)(-r+j, j+r)\right)_{n \in \mathbb{N}}$ and $\left(F^{n}\left(y^{\prime \prime}\right)(-r-i,-i+r)\right)_{n \in \mathbb{N}}$ where $y^{\prime \prime}$ is any point in $\left(\sigma^{-i} B_{r}(z) \cap \sigma^{j} B_{r}(z)\right)^{\mu_{c}}$, for all $k \geq m$, we have $C_{k}(y) \cap$ $\left(\sigma^{-i} B_{r}(z) \cap \sigma^{j} B_{r}(z)\right)^{\mu_{c}} \subset B_{m}(y)$. Since $y \in C_{k}(y) \cap\left(\sigma^{-i} B_{r}(z) \cap \sigma^{j} B_{r}(z)\right)^{\mu_{c}}$, we get that $y \in B_{m}(y)^{\mu_{c}}$ which finish the proof.

Proposition 6 If $\mu_{c}$ and $F$ are respectively a measure and a cellular automaton that verify the assumptions of Theorem 2 then the set of F-periodic points is dense in $S\left(\mu_{c}\right)$ (the topological support of $\mu_{c}$ ).

## Proof

Let $x$ be a $\mu$-equicontinuous point. From the proof and definitions of Theorem 2 , there exists a finite number of points $z_{0}, \ldots z_{p-1} \in w(\{x\}, F)$ such that $\forall m \in \mathbb{N}$, the sets $\Omega_{m}=\cup_{z \in\left[z_{0} \ldots z_{p-1}\right]} \cup_{i, j \in \mathbb{N}^{2}}\left(\sigma^{-i-m} B_{r}(z) \cap \sigma^{j+m} B_{r}(z)\right) \cap \Lambda(F)$ have full measure with respect to the invariant measure $\mu_{c}$. It follows that for each point $y \in S\left(\mu_{c}\right)$ and $k \in \mathbb{N}$, one has $\mu_{c}\left(C_{k}(y) \cap \Omega_{m}\right)>0(\forall m \in \mathbb{N})$. Since for each $y \in S\left(\mu_{c}\right)$ and $k \in \mathbb{N}$ one has $\mu_{c}\left(C_{k}(y) \cap \Omega_{k}\right)>0$, using the $\Sigma$-additivity of $\mu_{c}$, we can see that there always exist integers $i, j \geq k$
and point $z \in w(\{x\}, F)$ such that $\mu_{c}\left(C_{k}(y) \cap \sigma^{-i} B_{r}(z) \cap \sigma^{j} B_{r}(z)\right)>0(\mathrm{I})$. Finally, using the final part of the proof of Proposition 5 that only use the $F$-invariance of $\mu_{c}$ and inequality (I), we get that there exists a dense set of $F$-periodic points in $S\left(\mu_{c}\right)$.

We remark that the measure $\mu_{c}$ is not necessarily shift ergodic.

### 3.3 Examples of $\mu$-equicontinuous CA without equicontinuous points

In [5] Gilman gives an example of a $\mu$-equicontinuous CA $F_{s}$ that has no equicontinuous points. The automaton $F_{s}$ act on $\{0,1,2\}^{\mathbb{Z}}$ and is defined thank to the following block map of radius 1 .

$$
\begin{array}{|c|c|c|c|c|c|c|c|c}
* 00 & * 01 & * 02 & * 10 & * 11 & * 12 & * 20 & * 21 & * 22 \\
0 & 1 & 0 & 0 & 1 & 0 & 2 & 0 & 2
\end{array}
$$

where $*$ stands for any letter in $\{0,1,2\}$. Considering 0 as a background element, the 2's move straight down, 1's move to the left and 1 and 2 collide annihilate each other. In this case the measure $\mu$ is a Bernoulli measure on $\{0,1,2\}^{\mathbb{Z}}$ and the existence of $\mu$-equicontinuous points depends on the parameters $p(0), p(1), p(2)$ of this measure. In [5] it is shown that if $p(2)>p(1)$ then the probability that a 2 is never annihilated is positive and this implies that there exist $\mu$-equicontinuous points. Nevertheless for this interesting example, there is no Bernoulli measures $\mu$ which is preserving by this automaton $F$. In this case a "natural" way to obtain an invariant measure $\mu^{\prime}$ such that $F_{s}$ is a $\mu^{\prime}$-equicontinuous CA is to use Cesàro mean of image by $F_{s}$ of an appropriate Bernoulli measure using Theorem 1. The dynamical system $\left(\{0,1,2\}^{\mathbb{Z}}, F_{s}, \mu^{\prime}\right)$ we obtain is again a $\mu^{\prime}$-equicontinuous CA which is sensitive (without equicontinuous points) but the CA $\left(S\left(\mu^{\prime}\right), F_{s}, \mu^{\prime}\right)$ where $S\left(\mu^{\prime}\right)$ is the topological support of $\mu^{\prime}$ has always equicontinuous points since $S\left(\mu^{\prime}\right)=\{0,2\}^{\mathbb{Z}}$ and the restriction of $F_{s}$ to $\{0,2\}^{\mathbb{Z}}$ is the identity map. Since it has more interest to consider a dynamical system $(S(\mu), F, \mu)$ instead of the system $\left(A^{\mathbb{Z}}, F, \mu\right)$ when we mix topological and measurable conditions, we will describe here a CA called $F_{e}$ and a measure $\mu_{c}$ such that $\left(S\left(\mu_{c}\right), F_{e}\right)$ is sensitive and $\mu_{c}$-equicontinuous.

Roughly, to have $\mu$-equicontinuous points but no equicontinuous points requires that there exist some 'perturbations' that can move to infinity but
the probability that these perturbations move to infinity is equal to zero. One way to get theses properties for an automaton $F$ and an invariant measure $\mu_{c}$ (obtained thanks to the results of theorem 1 ), is that $\left(A^{\mathbb{Z}}, F\right)$ generates permanently 'propagating structures" of all type of sizes. The "length of life" of the "propagating structures" depends on their sizes. This is roughly the dynamic of the following cellular automaton $F_{e}$.

### 3.3.1 Definition of the cellular automaton $F_{e}$

The automaton $F_{e}$ we consider act on $X=X^{1} \times X^{2}$ where $X^{1}=\left\{E_{0}, E_{1}, E_{2}\right.$, $\left.E_{3}, 0, R, L\right\}^{\mathbb{Z}}$ and $X^{2}=\{0,1\}^{\mathbb{Z}}$. We define $F_{e}$ as the composition of 3 other CA : $F_{e}=F_{3} \circ F_{2} \circ F_{1}$. To simplify, we write $\hat{E}=\left\{E_{0}, E_{1}, E_{2}, E_{3}\right\}$ and $\bar{E}=\{0, L, R\}$. We denote by $x=\left(x^{1}, x^{2}\right)$ any point $x \in X$ and by $x_{i}^{j}$ the letter in position $i$ of $x^{j}(1 \leq j \leq 2)$. Next we call $\mathbf{1}_{S}(x)$ the map which is equal to one if $x \in S$ and zero otherwise.

The automaton $F_{1}$ is the identity on $X^{1}$ and its restriction to $X^{2}$ came from the following block map $f_{1}$ of radius 3
$f_{1}\left(x_{i-3}^{2}, x_{i-2}^{2}, x_{i-1}^{2}, x_{i}^{2}, x_{i+1}^{2}, x_{i+2}^{2}, x_{i+3}^{2}\right)=\mathbf{1}_{\{1\}}\left(x_{i-3}^{2}\right) \times \mathbf{1}_{\{1\}}\left(x_{i-2}^{2}\right) \times \mathbf{1}_{\{1\}}\left(x_{i-1}^{2}\right)$.
The automaton $F_{2}$ is still the identity on $X^{0} \times X^{1}$ but its action on $X^{2}$ depends on $X^{1}$. The block map $f_{2}$ is defined by:

$$
f_{2}\binom{x_{i-2}^{1}, x_{i-1}^{1}, x_{i}^{1}, x_{i+1}^{1}, x_{i+2}^{1}}{x_{i-2}^{2}, x_{i-1}^{2}, x_{i}^{2}, x_{i+1}^{2}, x_{i+2}^{2}}=\binom{x_{i}^{1}}{\vee_{j=0}^{2} \mathbf{1}_{\left\{E_{0}\right\}}\left(x_{i-j}^{1}\right)}
$$

where $\vee_{i=0}^{2} \mathbf{1}_{\left\{E_{0}\right\}} x_{i-j}^{1}$ is equal to 1 if at least one $x_{i-j}^{1}$ is equal to 1 and equal to 0 in all the other cases.

The automaton $F_{3}$ is the identity map on $X^{2}$ and is defined thanks to a local rule $f_{3}$ on $X^{1}$.

When central coordinate $x_{i}$ is an element of $\bar{E}$, the block map $f_{3}$ is defined by the following rules:

$$
\begin{aligned}
f_{3}\left(x_{i-11}^{1}, \ldots x_{i}^{1}, \ldots x_{i+11}^{1}\right) & =R \text { if } x_{i-10}^{1}, \ldots x_{i}^{1}, \ldots x_{i+m}^{1}=R 0^{11+m} \\
& \text { where } m=\min \left\{10, \min \left\{k-1 \mid x_{i+k} \in \hat{E}\right\}\right\} \\
& =L \text { if } x_{i-m}^{1}, \ldots x_{i}^{1}, \ldots x_{i+10}^{1}=0^{11+m} L \\
& \text { where } m=\min \left\{10, \min \left\{k-1 \mid x_{i-k} \in \hat{E}\right\}\right\} \\
& =R \text { if } \exists 0 \leq k, j \leq 9 \operatorname{such} \text { that } \\
& x_{i-j-k}^{1}, \ldots, x_{i}^{1}, \ldots, x_{i+10}^{1}=E^{*} 0^{k} L 0^{j+10} \\
& \text { with } j+2 k+1=10 \text { and } E^{*} \in \hat{E} . \\
& =L \text { if } \exists 0 \leq k, j \leq 9 \text { such that } \\
& x_{i-10}^{1}, \ldots, x_{i}^{1}, \ldots, x_{i+j+k}^{1}=0^{10+j} R 0^{k} E^{*} \\
& \text { with } j+2 k+1=10 \text { and } E^{*} \in \hat{E} .
\end{aligned}
$$

If the central coordinate $x_{i}$ is an element of $\hat{E}$, for each $i \in\{0,1,2,3\}$, one has

$$
f_{3}\left(x_{i-10}^{1}, \ldots, E_{i}, \ldots x_{i+10}^{1}\right)=E_{i+1} \text { if } x_{i-k}^{1}, \ldots, x_{i}^{1}=R 0^{k-1} E_{i} \text { with } 0 \leq k \leq 9
$$ where the addition ' $i+1$ ' is modulo 4 .

For all the other cases where the central coordinate $x_{i}$ is an element $E^{*}$ in $\hat{E}$, we have $f_{3}\left(x_{i-10}^{1}, \ldots, E^{*}, \ldots x_{i+10}^{1}\right)=E^{*}$. In all other cases not described above we have, $f_{3}\left(x_{i-10}^{1}, \ldots, x_{i}^{1}, \ldots x_{i+10}^{1}\right)=0$.

### 3.3.2 The invariant measure $\mu_{c}$

Let $S^{1}$ be a subshift of finite type defined by the following list of accepted words: words of type $L E_{j} 0^{k}$ with ( $1 \leq j \leq 3$ and $k=170$ ); words of type $0 E_{l} 0^{k}$ with ( $0 \leq l \leq 3$ and $k=170$ ); words of type $0^{l}$ with $l \in \mathbb{N}$. A typical configuration in $S^{1}$ is
$\ldots 00 E_{2} 000 \ldots 00 L E_{0} 00 \ldots 00 L E_{1} 00 \ldots 00 E_{1} 00 \ldots 00 E_{3} 00 \ldots 00 L E_{0} 00 \ldots$
We denote by $\mu_{1}$ the Parry measure on $S^{1}$ (see for example [4]), by $\delta_{\infty 0^{\infty}}$ the Dirac measure on the point ${ }^{\infty} 0^{\infty}=\ldots 00000 \ldots$ and we call $\mu_{I}$ the measure $\mu_{1} \times \delta_{\infty_{0} \infty}$ on $X$. Since $\mu_{I}$ is a shift ergodic measure, using Proposition 1, Theorem 1 and $2, F_{e}$ is a $\mu_{c}$-equicontinuous CA (with $\left.\mu_{c}=\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu_{I} \circ F_{e}^{i}\right)$ if there exists a point $x$ such that $\mu_{I}\left(B_{r}(x)\right)>0$ . Remark that $S^{1} \times{ }^{\infty} 0^{\infty}$ is not an invariant set for $F_{e}$. In Section 3.3.3 we will characterize the topological support of $\mu_{c}$ by describing the action of $F_{3}$ on the non-invariant set $S^{1}$.

### 3.3.3 The dynamic of $F_{e}$

In this subsection, we describe the global dynamic of $F_{e}$ by showing in first place the contribution of each of the 3 cellular automata $F_{1}, F_{2}$ and $F_{3}$.

The dynamic of $F_{1}$
The action of $F_{1}$ on $X^{2}$ is only the shift of consecutive sequences of letters " 1 " (that we call "trains of 1 ") of one coordinate to the right and the destruction of the two last letters " 1 " at the left side of this train.

Action of $F_{1}$ on a configuration of $X^{2}$ :

$$
\stackrel{F_{1}}{\stackrel{F_{1}}{\mapsto} \ldots 00001111000 \ldots} \stackrel{F_{1}}{\mapsto} \ldots 00000001100 \ldots{ }_{\mapsto}^{\mapsto} \ldots 0000000000 \ldots
$$

Remark that a train of 1 with a length $2 k+1$ will move of $k$ coordinates to the right before collapsing.

The dynamic of $F_{2}$
The action of the cellular automaton $F_{2}$ is to 'create' a sequence of three letters ' 1 ' in $X^{2}$ when there is a letter $E_{0}$ in $x_{i}^{1}$.

Example:

$$
x=\begin{array}{llll}
x^{1} \ldots E_{0} * * \ldots \\
x^{2} \ldots 00000 \ldots & F_{2}(x) & \ldots E_{0} * * \ldots & F_{2}(x) \\
\mapsto & \ldots 11100 \ldots & \ldots E_{0} * * \ldots \\
\ldots 11100 \ldots
\end{array}
$$

The symbol * replace any letter in $\{0, L, R\}$.
Action of $F_{2} \circ F_{1}$
If $F_{e}^{i}\left(x^{1}\right)=E_{0}$ for $0 \leq i \leq n-1$, then there is at least a train of 1 of length $n+3$ moving to the left of at least $\left\lceil\frac{n+3}{2}\right\rceil$ coordinates.

Example of the action of $F_{2} \circ F_{1}$ (on $x=x_{x^{1}}^{2}$ where $*$ replace any letter in $\hat{E} \cup \bar{E}$ )

$$
\begin{array}{ll}
x^{1}={ }^{\infty} 0 E_{0} * * * * * * * \ldots \\
x^{2}={ }^{\infty} 0000000000 \cdots & \left(F_{2} \circ F_{1}\right)^{n}
\end{array}{ }^{\infty} \underbrace{\infty}_{n+3 \text { times }} 0 E_{0} * * * * * * * \ldots \quad=F^{n}(x)^{1} .
$$

The oscillator dynamic of $F_{3}$ and the subshifts $S^{1}$ and $S\left(\mu_{c}\right)$
First remark that $S\left(\mu_{c}\right) \subset w\left(S^{1} \times{ }^{\infty} 0^{\infty}, F_{e}\right)$ where $w\left(S^{1} \times{ }^{\infty} 0^{\infty}, F_{e}\right)=$ $\lim _{n \rightarrow \infty} \cap_{j=0}^{n} \cup_{i=j}^{\infty} F_{e}^{i}\left(S^{1} \times{ }^{\infty} 0^{\infty}\right)$.

When we apply $F_{3}$ to a finite configuration of the subshift $S^{1}$, it is easy to see that only finite configurations of the type $0^{k} L E_{j} 0^{l}$ will be affected by the first iteration of the automaton $F_{3}$ where $k, l \in \mathbb{N}^{2}$ and $j \in[0 . .3]$. We have $F_{3}\left(\ldots 0^{k} L E_{j} 0^{l+m} \ldots\right)=\ldots L 0^{10} E_{j} 0^{l} \ldots$ and $F_{3}^{2}\left(\ldots 0^{k} L E_{j} 0^{l} \ldots\right)=$ $\ldots L 0^{20} E_{j} 0^{l} \ldots$ where $m, k \geq 170, l \in \mathbb{N}$ and $0 \leq j \leq 3$. Remark that the $L$ moves to the left indefinitely unless it would approach a letter $E^{*}$ in $\hat{E}$. If there is less than nine 0 between the $L$ and the $E^{*}$, the $L$ disappears and a $R$ appears at $i$ coordinates to the right of $E^{*}$ if the $L$ was at $10-i$ coordinates to the right of the $E^{*}$. An similar process occurs when after some iterations, the $R^{\prime} s$ appears and move to the right until they encounter a letter $E^{*} \in \hat{E}$. The letter $R$ disappears, it appears a $L$ which return to the left and the letter $E^{*}=E_{j}$ in the neighborhood become $E_{j+1} \bmod 4$.

Let's see a typical evolution of an oscillator.
$\cdots \underbrace{E^{*} 0 \ldots 0 R 0}_{\text {oscillator size } 50 n} \ldots 0 E_{i} \ldots \Rightarrow \underbrace{E_{3}^{*} 0 \ldots R 0^{k} E_{i}}_{\text {same oscillator }} \ldots \stackrel{F_{3}}{\Rightarrow \Rightarrow \ldots \underbrace{E^{*} 0 \ldots 0 L 0^{l} E_{i+1} \bmod 4}_{\text {same oscillator }}} \cdots$
where $k \leq 9, n \geq 4$ and $l+k=10$.
We will describe only the projection on $X^{1}$ of the subshift $S\left(\mu_{c}\right)=$ $w\left(S^{1}, F_{e}\right) \times w\left({ }^{\infty} 0^{\infty}, F_{e}\right)$. In $w\left(S^{1}, F_{e}\right)$ : (i) there is at least 170 letters in $\bar{E}$ between two $E^{*}$. (ii) There is at most one letter $M \in\{L, R\}$ between two $E^{*}$. (iii) At the right side of the last $E^{*}$ to the right (if it exists) there are only letters 0 . (iv) (iii) At the left side of the last $E^{*}$ (if it exists ), there are only letters 0 . (v) There is no configurations of the type $E^{*} 0^{m} E_{0} *^{l} E^{*}$ where $*$ is in place of any letter in $\{0, R, L\}$ and $l \geq 170$. We remark that the configurations of $S\left(\mu_{c}\right)$ can be generated by a finite automata which means that $S\left(\mu_{c}\right)$ is a sofic subshift.

Action of $F_{e}$
Since the action of $F_{e}$ on letters $E^{*} \in \hat{E}$ is the identity or a permutation in $\hat{E}$, the set of configurations that contains an infinite number of letters in $\hat{E}$ has measure one in $S\left(\mu_{c}\right)$.

Under the action of $F_{2} \circ F_{1}$, a configuration of the type $\ldots E_{0} \ldots \times^{\infty} 0^{\infty}$ will generate in $X^{2}$ (at the same coordinate that $E_{0}$ ) a train of 1 until the $E_{0}$ will change in $E_{i}($ with $i \geq 1)$. Under the action of $F_{e}$ (see action of $F_{2} \circ F_{1}$ ) a configuration $\ldots E^{*} 0^{l} M 0^{k} E_{0} \ldots \times^{\infty} 0^{\infty}\left(E^{*} \in \hat{E}, M \in\{R, L\}\right)$ will produce a train of 1 of length $\mathcal{L}$ such that $\left\lfloor\frac{m}{5}\right\rfloor+2 \leq \mathcal{L} \leq\left\lfloor\frac{m}{5}\right\rfloor+4$ with $m=l+k+1$. In the following we will choose $\left\lfloor\frac{m}{5}\right\rfloor+2$ or $\left\lfloor\frac{m}{5}\right\rfloor+4$ for the length $\mathcal{L}$ of this
train of 1 according to the context.
Let see a typical dynamic of an oscillator transmitter

We call oscillator transmitter of size $l+m+1$, any pattern of the form $E^{*} 0^{l} M 0^{m} E^{*}$ and void oscillator any pattern of the form $E^{*} 0^{k} E^{* *}$ where $k, l+$ $m+1 \geq 170$, each $E^{*}$ belongs to $\hat{E}$, each $E^{* *} \in\left\{E_{1}, E_{2}, E_{3}\right\}$ and $M \in\{R, L\}$. Remark that oscillators of the type $E^{*} 0^{k} E_{0}$ does not belong to the language of $S\left(\mu_{c}\right)$.

For each $l \in \mathbb{N}$, denote by $\mathfrak{C}_{l}[i]$ the union of all the sets $[U]_{i} \times X^{2} \subset X$ where $U=\left[E^{*} 0^{j} M 0^{k} E^{*}\right]_{i}$ is a cylinder in $X^{1}, M$ replaces one letter in $\{L, R\}$, each $E^{*}$ are any letters in $\hat{E}$ and $j, k, l$ verify $j+k+1=l$. Let $\overline{\mathcal{C}_{k}[i]}$ be the union of sets $\left(\left[E^{*} 0^{k} E^{* *}\right]_{i} \times X^{2}\right)$ where $E^{*}$ replace any letter in $\hat{E}, k \geq 170$ and $E^{* *} \in\left\{E_{1}, E_{2}, E_{3}\right\}$. We call respectively oscillators transmitter in position $i$ and void oscillators in positions $i$ the sets $\mathcal{C}_{l}[i]$ and $\overline{\mathfrak{C}_{l}[i]}$. Remark that $\mu_{I}\left(\mathcal{C}_{l}[i]\right)=\mu_{I}\left(\left[E^{*} 0^{i-1} R E^{*}\right] \times{ }^{\infty} 0^{\infty}\right)$. Without taking into consideration the position of the oscillators, we will call respectively $\mathfrak{C}_{l}$ and $\overline{\mathcal{C}_{l}}$ the oscillators transmitter and void oscillators of size $l$.

Propagation of trains of 1 generated by oscillator transMITTER

Since an oscillator transmitter $\mathcal{C}_{l}$ generate a train of 1 with a length $\mathcal{L}$ at most equal to $\left\lfloor\frac{l}{5}\right\rfloor+4$ and that train looses 2 elements when it moves of 1 coordinate, it can influences some patterns situated at $\left(\left\lfloor\frac{l}{5}\right\rfloor+4\right)+\left(\left\lfloor\frac{l}{5}\right\rfloor+4\right) / 2=$ $\left\lfloor\frac{3 l}{10}\right\rfloor+6$ coordinates to the right of the right extremity of the oscillator if there is no concatenation process with another train created by other oscillators transmitter. Since the proofs of Proposition 7 and 8 only require the understanding of the propagation of trains of 1 when the initial configuration is in $S^{1} \times{ }^{\infty} 0^{\infty}$, we will only consider concatenation process with trains generated by oscillators transmitter situated to the right side. For two consecutive oscillators $\mathcal{C}_{l} \mathcal{C}_{m}$, a trains of 1 generated by the first one $\mathcal{C}_{l}$ will reach the coordinates situated under the beginning of $\mathcal{C}_{m}$ only if $m \leq\left\lfloor\frac{3 l}{10}\right\rfloor+6$


Figure 1: An illustration of the dynamic of $F_{e}$ on 5 oscillators $\mathfrak{C}_{4200}, \mathfrak{C}_{600}$ and three $\mathcal{C}_{300}$ and the resulting dynamic of train of 1 in $X^{2}$. For simplification, we do not specify the states of the oscillators and their evolution because the interesting part of their dynamic can be deduce from the evolution of the train of 1 in $X^{2}$. Each line represents the sequence of images of $x^{2}$ after every 60 iterations of $F_{e}$. The black horizontal lines represent the trains of 1 and blank ones, the sequences of 0 . The extremity of the oscillators are delimited by arrows. The first oscillator of size 300 is a void oscillator which never generates any trains of 1 in $X^{2}$. To the left side, there is one large oscillator transmitter of size 6000 which is in a non emitting state (the last letter is not an $E_{0}$ ). Remark that the non 'emitting period' of the oscillators transmitter last 3 times more than the 'emitting' one. The circle shows the end of the propagation of the train generated by $\mathcal{C}_{4200}$.
(iq1). When a train will cross the $l$ coordinates of another oscillator, it will loose $2 l$ elements and can incorporated at most $\lfloor 2 l / 5\rfloor+8$ others $(\lfloor l / 5\rfloor+4$ at the beginning and $\lfloor l / 5\rfloor+4$ the end of the train). Now consider a sequence of three oscillators $\mathfrak{C}_{l} \mathfrak{C}_{m} \mathfrak{C}_{n}$ where $m$ and $n$ are fixed and $l$ is the minimum size of the first oscillator in order that its train will reach $\mathcal{C}_{n}$. The train of length at most $\lfloor l / 5\rfloor+4$ will gain at most $\lfloor 2 m / 5\rfloor+8$ and will loose at least
$2 m+2(n-(\lfloor m / 5\rfloor+4))$ elements when the head of the train has crossed the two oscillators (we take into consideration that the head of the train may eventually progress of $\lfloor m / 5\rfloor+4$ coordinates when it passes under $\mathcal{C}_{m}$ ). We obtain that $l \geq 6 m+10 n-100$. More generally for all sequences of $n+1$ consecutive oscillators $\mathcal{C}_{l_{n}} \mathfrak{C}_{l_{n-1}} \ldots \mathcal{C}_{l_{0}}$, any change of the state of the first oscillator $\mathcal{C}_{l_{n}}$ will affect the train of 1 situated in the oscillator $\mathcal{C}_{l_{0}}$ if $l_{n} \geq \sum_{i=n-1}^{1} 6 l_{i}+10 l_{0}-80 n+60(\mathrm{iq} 2)$.

Recall that 170 is the minimum size of an oscillator. This minimum size is required to simplify the proof of Proposition 8 which uses quantitative arguments on "flows of trains of 1 ".

### 3.3.4 The topological and measurable properties of $F_{e}$

Proposition 7 The dynamical system $\left(S\left(\mu_{c}\right), F_{e}\right)$ is a $\mu_{c}$-equicontinuous cellular automaton.

## Proof

From the discussion of section 3.3.2, we only need to show that there exists a point $x$ and an integer $m \geq r$ such that $\mu_{I}\left(B_{m}(x)\right)>0$. Remark that since $\mu_{I}(X)=\mu_{I}\left(S^{1} \times^{\infty} 0^{\infty}\right)$ we will take into consideration only configuration in $S^{1} \times^{\infty} 0^{\infty}$ in this proof. Let $x_{0}=\left({ }^{\infty} 0^{\infty},,^{\infty} 0^{\infty}\right)$ and for each integer $k \geq 1$, pick a point $x_{k} \in \overline{\mathfrak{C}_{k}[-k-1-r]} \cap B_{r}\left(x_{0}\right) \cap X^{1} \times^{\infty} 0^{\infty}$. We will prove that there exist integers $k>0$ such that $\mu_{I}\left(B_{r}\left(x_{k}\right)\right)>0$ by showing that $\mu_{I}\left(\overline{\mathcal{C}_{k}[-k-1-r]} \cap B_{r}\left(x_{k}\right)^{\complement}\right)<\mu_{I}\left(\overline{\mathcal{C}_{k}[-k-1-r]}\right)$ (where $B_{r}\left(x_{k}\right)^{\complement}$ is the complement of $\left.B_{r}\left(x_{k}\right)\right)$. The set $\overline{\mathcal{C}_{k}[-k-1-r]} \cap B_{r}\left(x_{k}\right)^{\complement}$ is the set of points that contains oscillators transmitter in the left side of $\overline{\mathcal{C}_{k}[-k-1]}$ that are able to generate trains of 1 which move to the right and cross this void oscillator $\overline{\mathfrak{C}_{k}^{*}[-k-1-r]}$ (the trains of " 1 " may enter in the central coordinates $([-r, r])$ and in this case the point does not belong to $\left.B_{m}\left(x_{0}\right)=B_{m}\left(x_{k}\right)\right)$. Now, consider an oscillator transmitter of size $l$ in position $-p-l-k-r: \mathcal{C}_{l}[-p-$ $l-k-r]$. Denote by $\mathbf{S}(p, k)$ be the minimum size of the oscillator $\mathcal{C}_{l}[-p-$ $l-k-r]$ (whose the right extremity is situated at $p$ coordinates to the left of $\overline{\mathfrak{C}_{k}[-k-1]}$ ) in order that it can produce trains of 1 that cross completely the void oscillator $\overline{\mathfrak{C}_{k}[-k-1]}$. From the discussion in section 3.3.3 about the propagation of train of 1 , it follows that $\overline{\mathfrak{C}_{k}[-k-1-r]} \cap B_{r}\left(x_{k}\right)^{\complement}$ is equal to $\mathbb{S}_{k}=\left\{\cup_{i=\mathbf{S}(0, k)}^{\infty} \mathfrak{C}_{i}[-i-k-1-r] \cup_{p=l_{\mathrm{m}}}^{\infty}\left\{\cup_{j=S(p, k)}^{\infty} \mathcal{C}_{j}[-j-p-k-1-r]\right\}\right\}$ $\cap \overline{\mathfrak{C}_{k}[-k-1-r]}$ where $l_{\mathrm{m}}=170$ be the minimum size of the oscillators in $S\left(\mu_{I}\right)$.

Next we claim that there exists a real $M \geq 0$ such that for all integers $k \geq$ $l_{\mathrm{m}}$ one has $\mu_{I}\left(\overline{\mathcal{C}_{k}[-k-1-r]} \cap B_{r}\left(x_{k}\right)^{\complement}\right)=\mu_{I}\left(\mathbb{S}_{k}\right) \leq \mu_{I}\left(\overline{\left(\overline{\mathcal{C}_{k}}[-k-1-r]\right.}\right) \times$ $\eta(k) \times M$ with $\lim _{k \rightarrow \infty} \eta(k)=0$. Remark that since $\mu_{I}\left(\overline{\mathcal{C}_{k}[-k-1-r]}\right)>0$, the proof of this claim will finish the proof.

From section 3.3.3 a train of 1 generated by an oscillator $\mathcal{C}_{l_{n}}$ will cross the $n-1$ oscillator $\mathfrak{C}_{l_{n-1}} \ldots \mathfrak{C}_{l_{1}}$ and reach $\overline{\mathfrak{C}_{l_{0}}}$ if $l_{n} \geq \sum_{i=n-1}^{1} 6 l_{i}+10 l_{0}-80 n+60$ (iq2). Remark that the last oscillator is $\overline{\mathfrak{C}_{k}[-k-1-r]}$ which implies $l_{0}=k$ and that the number of oscillators $n \leq \frac{p}{l_{\mathrm{m}}}=\frac{p}{170}$. Using (iq2) we obtain that $\mathbf{S}(p, k) \geq 6 p+10 k-80\left(\frac{p}{l_{\mathrm{m}}}\right)+60 \geq 5 p+10 k+60$ (iq 3$)$.

Since $\mu_{I}$ is the product of the Parry measure on the mixing subshift of finite type $S^{1}$ (see [4]) and the Dirac measure on ${ }^{\infty} 0^{\infty}$, then there exist a real $0<q<1$ and a positive integer $\mathcal{L}$ such that $\forall m \geq 0$ one has $\mu_{I}\left(\left[u_{0} \ldots u_{m}\right] \times^{\infty}\right.$ $\left.0^{\infty}\right) \leq q^{\left\lfloor\frac{m+1}{L}\right\rfloor}$.

To prove the claim put $\eta(k)=\mu_{I}\left(\mathcal{C}_{\mathbf{S}_{(0, k)}}[-S(0)-k-1-r]\right)$ and using the lower bound (iq3) for $\mathbf{S}(0, k)$, we obtain that $\eta(k) \leq \mu_{I}\left(\mathrm{C}_{10 k+60}[-10 k-\right.$ $60-k-1-r]) \leq q^{\left\lfloor\frac{\lfloor 1 k+59}{\hbar}\right\rfloor}$ which implies that $\lim _{k \rightarrow \infty} \eta(k)=0$. Using again (iq3) for $\mathbf{S}(p, k)$, we obtain that $\mu_{I}\left(\cup_{i=\mathbf{S}(0, k)}^{\infty} \mathcal{C}_{i}[-i-k-1-r]\right) \leq$ $\eta(k) \sum_{i=1}^{+\infty} q^{\left\lfloor\frac{i}{K}\right\rfloor}$. Remark that the last and the following inequality will give us an upper bound for the measure of $\mathbb{S}_{k}$ :

$$
\mu_{I}\left(\cup_{p=l_{\mathbf{m}}}^{\infty}\left\{\cup_{j=\mathbf{S}(p, k)}^{\infty} \mathcal{C}_{j}[-j-p-k-1-r]\right\}\right) \leq \eta(k) \sum_{i=1}^{+\infty} q^{\left\lfloor\frac{i}{L}\right\rfloor}\left(\sum_{j=1}^{+\infty} q^{\left\lfloor\frac{j}{\kappa}\right\rfloor}\right) .
$$

It follows that we can prove the claim and consequently finish the proof taking

$$
M=\sum_{i=1}^{+\infty} q^{\left\lfloor\frac{i}{\mathcal{L}}\right\rfloor}+\sum_{i=1}^{+\infty} q^{\left\lfloor\frac{i}{\mathcal{L}}\right\rfloor}\left(\sum_{j=1}^{+\infty} q^{\left\lfloor\frac{i}{\mathcal{L}}\right\rfloor}\right)<+\infty .
$$

Proposition 8 The cellular automaton $\left(S\left(\mu_{c}\right), F_{e}\right)$ is sensitive (has no equicontinuous points in the topological support $S\left(\mu_{c}\right)$ ).

Proof
If we suppose that there exists an equicontinuous point $x \in S\left(\mu_{c}\right)$, there must exist an integer $m$ such that $C_{m}(x) \subset B_{0}(x)$. First suppose that if for all positive integers $n$, there exist integers $i>n$, such that $\left(F^{i}(x)\right)_{0}^{2}=0$, then there exists $y \in C_{m}(x)$ such that $\left(F^{i}(y)\right)_{0}^{2}=1$ (it is always possible to
choose a point $y \in C_{m}(x)$ that contains a large enough oscillator transmitter at the left side of $-m$ that sent a train of 1 that can arrived at coordinate 0 after $i$ iterations). This contradict the hypothesis $C_{m}(x) \subset B_{0}(x)$ and it follows that if there exists $x \in X$ and $m>0$ such that $C_{m}(x) \subset B_{0}(x)$, then there exists a positive integer $n$ such that for all $y \in C_{m}(x)$ and $i \in \mathbb{N}$ one has $\left(F^{i+n}(y)\right)_{0}^{2}=1($ condition $(*))$. Then suppose that there exists $m \in \mathbb{N}$ such that $C_{m}(x) \subset B_{0}(x)$. Since the point $z:={ }^{\infty} 0^{\infty}(-\infty,-m-$ 1) $x(-m, \infty)$ belongs to $C_{m}(x)$ and by hypothesis $\left(F^{i+n}(z)\right)_{0}^{2}=1$ for all $i \in \mathbb{N}$, we obtain that the pattern $x(-m,-1)$ contains a finite sequence of consecutive oscillators $\mathcal{C}_{l_{k}}, \mathfrak{C}_{l_{k-1}}, \ldots \mathcal{C}_{l_{0}}$ that generate a "continuous flow of letters 1 ". To finish the proof we need to show that this finite sequence of oscillators does not exists. In order to do that, we will consider the propagation of trains of 1 generated by a finite sequence of $k$ oscillators $(k \in \mathbb{N})$.


Figure 2: The flow of 1 generated by a sequence of $k$ oscillators. The parts not in white color represents the successive flows of letters 1 due to the trains which come from the oscillators transmitters $\mathfrak{C}_{l_{k}}, \mathfrak{C}_{l_{k-1}}, \ldots \mathfrak{C}_{l_{k-4}}$.

Let $P_{i}(0 \leq i \leq k)$ be an upper bound of the number of iterations needed for a train of 1 generated by the oscillator $\mathcal{C}_{l_{i}}$ to cross completely the coordinate 0 . We consider that there is no concatenations with other trains coming from the left (but possible concatenations with trains generated by oscillators situated to the right. The value of $P_{i}$ depends on the difference between the lost and gained letters when the left extremity of the train arrived in coordinate 0 . It follows that $P_{i} \geq\left\lfloor\frac{l_{i}}{5}\right\rfloor+4-s_{i}$ where $s_{i}=\sum_{j=0}^{i-1} 2 l_{j}-$ $\sum_{j=0}^{i-1} 2\left(\left\lfloor\frac{l_{j}}{5}\right\rfloor+4\right)=\sum_{j=0}^{i-1}\left\lfloor\frac{8}{5} l_{j}\right\rfloor-8 i$. Remark that $s_{i} \geq 0$ if $\forall 0 \leq j \leq i-1$,
$l_{j} \geq 5$. As $l_{\mathrm{m}}=170$ we obtain that $P_{i} \leq\left\lfloor\frac{l_{i}}{5}\right\rfloor+4$ for all $0 \leq i \leq k$. Without loosing generalities, we can suppose that the train of 1 generated by the first oscillator of size $l_{k}$ arrive in coordinate 0 at $t=0$ and last at most $P_{k}$ iterations. For time $t=P_{k}+1$ to $4 \times P_{k}$, there is no train of 1 due to this first oscillator that pass through the central coordinate (because $P_{i} \geq\left\lfloor\frac{l_{k}}{5}\right\rfloor+2$ and the oscillator $\mathcal{C}_{l_{k}}$ stop to generate train of 1 for at least $3\left(\left\lfloor\frac{l_{k}}{5}\right\rfloor+2\right)$ iterations). The train of 1 generated by the second oscillator from the left : $\mathcal{C}_{l_{k-1}}$ last at most $P_{k-1}$ iterations and its effect stops for a period of $3 P_{k-1}$ in the interval time $t=P_{k}+1$ to $t=4 P_{k}$. Clearly, if $P_{k-1}$ is small enough, between $t=P_{k}+1$ and $t=4 P_{k}$, there is at least one interval of length at least $3 P_{k-1}$ that will be not affected by the two first oscillators. This interval is minimum when the train of 1 generated by the second oscillator pass exactly in the middle of the interval $\left[P_{k}+1,4 P_{k}\right]$, between 2 trains of the first oscillator. In this case the condition of non existence of "continuous flow" is that $3 P_{k}-(2 \times 3+1)\left(P_{k-1}\right) \geq 0 \Leftrightarrow P_{k-1} \leq \frac{3 P_{k}}{7}$. Hence repeating the same process if $P_{i-1} \leq \frac{3 P_{i}}{7}(0 \leq i \leq k)$, there will always remain a blank interval and the "flow of 1 " would not be continuous. Since for all $0 \leq i \leq k$ one has $P_{i} \leq\left\lfloor\frac{l_{i}}{5}\right\rfloor+4$, it follows that the condition $\left\lfloor\frac{l_{i-1}}{5}\right\rfloor+4 \leq \frac{3}{7}\left(\left\lfloor\frac{l_{i}}{5}\right\rfloor+2\right)$ (iq4) also implies that the "flow of 1 " is not continuous. Remark that we take a lower bound for $P_{i}$ and upper bound for $P_{i-1}$ which leads to a stronger condition on the minimum size $l_{m}$. Then by simplification of (iq4) we get that $3 l_{i}-7 l_{i-1} \geq 110$. Using (iq1) $\equiv\left(l_{i-1} \leq\left\lfloor\frac{3 l_{i}}{10}\right\rfloor+6\right)$ that gives the minimum condition for a train generated by an oscillator $\mathcal{C}_{l_{i}}$ to cross the following one $\mathfrak{C}_{l_{i-1}}$ we obtain that if $3 l_{i}-7\left(\left\lfloor\frac{3 l_{i}}{10}\right\rfloor+6\right) \geq 110$ (iq5) then the condition $3 l_{i}-7 l_{i-1} \geq 110$ (iq4) remains true. The simplification of inequality (iq5) leads to $l_{i} \geq \frac{1520}{9} \approx 168.88$. Since we have chosen $l_{\mathrm{m}}=170$ as the minimum size of the oscillators, there is no equicontinuous points in $\left(S\left(\mu_{c}\right), F_{e}\right)$ which finish the proof.

The dynamical system $\left(X, F_{e}\right)$ has equicontinuous points since patterns of the type $E^{*} 0^{k} E_{0}$ (with $E^{*} \in \hat{E}$ ) will produce a continuous flow of letters 1. Note that we can construct $F_{e}^{\prime}$ a CA similar to $F_{e}$ with a more complex local rule such that the initial measure $\mu_{I}^{\prime}$ is the uniform measure on $X$ and the invariant measure obtained by Cesàro means is similar to the measure $\mu_{c}$ used in our example. In this case $\left(X, F_{e}^{\prime}, \mu_{c}\right)$ and $\left(S\left(\mu_{c}\right), F_{e}^{\prime}, \mu_{c}\right)$ both have $\mu_{c}$-equicontinuous points but no equicontinuous points.

### 3.4 Questions

-Is it possible to find a sensitive, $\mu$-equicontinuous and $\mu$-invariant CA when $\mu$ is the uniform measure? Or more generally when the topological support $S(\mu)$ of the $F$-invariant measure is a mixing subshift of finite type?
-To simplify the proof of Proposition 8, we have taken a lot of upper bounds ( for example the use of (iq1) is very strong). We wonder what is the minimum size for the oscillators such that $\left(S(\mu), F_{e}\right)$ can not produce equicontinuous points?
-From inequality (iq3) in Proposition 7, it is possible to see that for each point $x$ and $m \in \mathbb{N}$ such that $\mu_{c}\left(B_{m}(x)\right)>0$, we have $B_{m}^{\mu_{c}}(x)=B_{m}(x)$. What are the conditions on the minimum size of the counters $l_{m}$ in order to loose this property? In this case is it possible that there is no equicontinuous points? Remark that for some points $x, B_{m}^{\mu}(x) \neq B_{m}(x)$ in the Gilman's example of $\mu$-equicontinuous CA given in the beginning of section 3.3.
-Is there exists a $\mu$-invariant and $\mu$-equicontinuous CA such that $(S(\mu), F)$ has no equicontinuous points and there exists $m \in \mathbb{N}$ and a point $x \in A^{\mathbb{Z}}$ with $B_{m}(x) \neq B_{m}^{\mu}(x)$ and $\mu\left(B_{m}(x)\right)>0$ ?
-More generally, what type of dynamic characterize sensitive and $\mu$-equicontinuous CA $(S(\mu), F)$ and how common is this behavior that seems to appear in different simulations of one dimensional CA?

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