CHARACTERIZATION OF TALAGRAND'S LIKE TRANSPORTATION-COST INEQUALITIES ON THE REAL LINE.

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ABSTRACT. In this paper, we give necessary and sufficient conditions for Talagrand's like transportation cost inequalities on the real line. This brings a new wide class of examples of probability measures enjoying a dimension-free concentration of measure property. Another byproduct is the characterization of modified Log-Sobolev inequalities for Log-concave probability measures on $\mathbb{R}$.

1. INTRODUCTION

1.1. **Transportation-cost inequalities.** This article is devoted to the study of probability measures on the real axis satisfying some kind of transportation-cost inequalities. These inequalities relate two quantities : on the one hand, an optimal transportation cost in the sense of Kantorovich and on the other hand, the relative entropy (also called Kullback-Leibler distance). Let us recall that if $\alpha : \mathbb{R} \to \mathbb{R}^+$ is a continuous even function, the optimal transportation-cost to transport $\nu \in \mathcal{P}(\mathbb{R})$ on $\mu \in \mathcal{P}(\mathbb{R})$ (the set of all probability measures on $\mathbb{R}$) is defined by :

$$(1) \qquad \mathcal{T}_\alpha(\nu, \mu) = \inf_{\pi \in P(\nu,\mu)} \iint_{\mathbb{R} \times \mathbb{R}} \alpha(x - y)\, \pi(dxdy),$$

where $P(\nu,\mu)$ is the set of all the probability measures on $\mathbb{R} \times \mathbb{R}$ such that $\pi(dx \times \mathbb{R}) = \nu$ and $\pi(\mathbb{R} \times dy) = \mu$. The relative entropy of ν with respect to μ is defined by

$$(2) \qquad \mathrm{H}(\nu \mid \mu) = \begin{cases} \int \log \frac{d\nu}{d\mu}\, d\nu & \text{if } \nu \ll \mu \\ +\infty & \text{otherwise} \end{cases}$$

One will say that $\mu \in \mathcal{P}(\mathbb{R})$ satisfies the *transportation-cost inequality with the cost function* $(x,y) \mapsto \alpha(x - y)$ (TCI) if

$$(3) \qquad \forall \nu \in \mathcal{P}(\mathbb{R}), \quad \mathcal{T}_\alpha(\nu, \mu) \le \mathrm{H}(\nu \mid \mu),$$

Transportation-cost inequalities of the form (3) were introduced by K. Marton in [13, 14] and M. Talagrand in [18]. After them, several authors studied inequality (3), possibly in a multidimensional setting, for particular choices of the cost function α (see for example [3], [5], [7], [16] or [9]). The best known example of transportation-cost inequality is the so-called

$\mathbb{T}_2$-inequality (also called Talagrand's inequality). It corresponds to the choice $\alpha(x) = \frac{1}{a}x^2$. One says that μ satisfies $\mathbb{T}_2$ with the constant a if

$$(4) \qquad \forall \nu \in \mathcal{P}(\mathbb{R}), \quad \mathcal{T}_2(\nu, \mu) \leq a \operatorname{H}(\nu \mid \mu),$$

writing $\mathcal{T}_2(\nu, \mu)$ instead of $\mathcal{T}_{x^2}(\nu, \mu)$.

1.2. Links with the concentration of measure phenomenon.

The reason of the increasing interest to TCI is their links with the concentration of measure phenomenon. Roughly speaking, a probability measure which satisfies a TCI, also satisfies a *dimension free* concentration of measure property. This link was first pointed out by K. Marton in [13]. For example, Talagrand's inequality is related to dimension-free gaussian concentration. If μ satisfies (4), then

$$(5)$$
$$\forall n \in \mathbb{N}^*, \forall A \subset \mathbb{R}^n \text{ measurable}, \quad \forall r \geq r_A := \sqrt{-\log \mu^n(A)}, \qquad \mu^n(A^r) \geq 1 - e^{-\frac{1}{a}(r - r_A)^2},$$

where $A^r = \{x \in \mathbb{R}^n \text{ such that } \exists y \in A \text{ with } |x - y|_2 \leq r\}$ and $|\cdot|_2$ is the usual euclidean norm.

Replacing the function x^2 by an other convex function, it is possible to obtain different types of dimension free concentration estimates. For example, if μ is a probability measure which satisfies the transportation cost inequality

$$(6) \qquad \forall \nu \in \mathcal{P}(\mathbb{R}), \quad \mathcal{T}_{\alpha_p}(\nu, \mu) \leq a \operatorname{H}(\nu \mid \mu),$$

where $\alpha_p(x) = \begin{cases} \min(|x|^2, |x|^p) & \text{if } p \in [1, 2[\\ |x|^p & \text{if } p \geq 2 \end{cases}$, $\forall x \in \mathbb{R}$ then, it can be shown that

$$(7)$$
$$\forall n \in \mathbb{N}^*, \forall A \subset \mathbb{R}^n \text{ measurable}, \forall r \geq r_A := \alpha_p^{-1}\left(-\log \mu^n(A)\right), \qquad \mu^n(A^r) \geq 1 - e^{-\frac{1}{a}\alpha_p(r - r_A)},$$

with $A^r = \left\{x \in \mathbb{R}^n \text{ such that } \exists y \in A \text{ with } |x - y|_{\max(p,2)} \leq r\right\}$, denoting $|x|_p = \sqrt[p]{\sum_{i=1}^n |x_i|^p}$. The probability measure $d\mu_p(x) = e^{-|x|^p}\frac{dx}{Z_p}, p \geq 1$ on $\mathbb{R}$ satisfies the TCI (6). The cases $p = 1$ and $p = 2$ were obtained by Talagrand in [18], the case $p \in (1, 2)$ was treated by Gentil, Guillin and Miclo in [9] and the case $p \geq 2$ by Bobkov and Ledoux in [4].

1.3. Strong transportation-cost inequalities.

When dealing with other cost functions than the α_p's, it is convenient to study a stronger form of the transportation-cost inequality (3).

A probability measure μ will be said to satisfy the *strong transportation-cost inequality with cost function* $(x, y) \mapsto \alpha(x - y)$ (strong TCI) if

$$(8) \qquad \forall \nu, \beta \in \mathcal{P}(\mathbb{R}), \quad \mathcal{T}_\alpha(\nu, \beta) \leq \operatorname{H}(\nu \mid \mu) + \operatorname{H}(\beta \mid \mu).$$

Note that this inequality is a sort of symmetrized version of the usual TCI (3). Of course, as $\operatorname{H}(\mu \mid \mu) = 0$,

$$\mu \text{ satisfies (8)} \quad \Rightarrow \quad \mu \text{ satisfies (3)}.$$

When α is convex, these two inequalities are equivalent up to constant factors. Namely, if α is convex one has

$$\mu \text{ satisfies (3)} \quad \Rightarrow \quad \mu \text{ satisfies the strong TCI with the cost function } (x,y) \mapsto 2\alpha\left(\frac{x-y}{2}\right).$$

This elementary fact is proved in Proposition 16.

Strong TCIs are not new. The strong TCI (8) is in fact equivalent to an infimal-convolution inequality. Infimal-convolution inequalities were introduced by B. Maurey in [15]. The translation of (8) in terms of infimal-convolution inequalities will be stated in Theorem 22.

These strong TCIs are powerful tools for deriving general dimension free concentration properties. Indeed, if μ verifies the strong TCI (8), then

$$(9) \qquad \forall n \in \mathbb{N}^*, \forall A \subset \mathbb{R}^n \text{ measurable}, \quad \forall r \geq 0, \qquad \mu^n\left(A_\alpha^r\right) \geq 1 - \frac{1}{\mu^n(A)} e^{-r},$$

where $A_\alpha^r = \left\{ x \in \mathbb{R}^n : \exists y \in A \text{ such that } \sum_{i=1}^n \alpha(|x_i - y_i|) \leq r \right\}.$

Note that in (9), the blow up A_α^r is not generated by a norm. When dealing with the α_p's, one can show that (9) implies (7).

The aim of this paper is to give general criteria guarantying that a probability measure satisfies a (strong) TCI. Before presenting our results let us recall some results of the literature.

1.4. **TCI and Logarithmic-Sobolev type inequalities.** The classical approach to study TCIs is to relate them to other functional inequalities such as Logarithmic-Sobolev inequalities. The main work on the subject is the article by F. Otto and C. Villani on Talagrand's inequality (see [16]). They proved that if $\mu \in \mathcal{P}(\mathbb{R})$ satisfies the Logarithmic-Sobolev inequality

$$\text{Ent}_\mu(f^2) \leq C \int f'^2 \, d\mu, \quad \forall f$$

then it satisfies Talagrand's inequality (4) with the same constant C. In fact, this result is true in a multidimensional setting. Soon after Otto and Villani, S.G. Bobkov, I. Gentil and M. Ledoux provided an other proof of this result (see [5]).

Different authors have tried to generalize this approach to study TCIs associated to other cost functions. Let us summarize these results. Define

$$\theta_p(x) = \begin{cases} x^2 & \text{if } |x| \leq 1 \\ \frac{2}{p}|x|^p + 1 - \frac{2}{p} & \text{if } |x| \geq 1 \end{cases}, \forall p \in [1,2]$$

The function θ_p is just a convex function resembling to the previously defined α_p. Let θ_p^* be the convex conjugate of θ_p, which is defined by

$$\theta_p^*(y) = \sup_{x \in \mathbb{R}} \left\{ xy - \theta_p(x) \right\}.$$

If μ satisfies the following modified Logarithmic-Sobolev inequality

$$(10) \qquad \text{Ent}_\mu(f^2) \leq C \int \theta_p^*\left(\frac{tf'}{f}\right) f^2 \, d\mu, \quad \forall f.$$

for some $C, t > 0$, then μ satisfies the TCI (6) for some constant $a > 0$.

When $p = 2$, one recovers Otto and Villani's result. The case $p = 1$ was treated by Bobkov, Gentil and Ledoux in [5]. Note that in this case, the inequality (10) is equivalent to Poincaré inequality (see [2]). The case $p \in (1, 2)$ is due to I. Gentil, A. Guillin and L. Miclo see [9].

Now the question is to know if the TCI (6) is equivalent to the modified Log-Sobolev (10). Here are some elements of answer :
- This is true for $p = 1$. When $p = 1$, inequalities (6) and (10) are both equivalent to Poincaré inequality (see [2] and [5]).
- This is true as far as Log-concave distributions are concerned (see Corollary 3.1 of [16] or Theorem 2.9 of [9]).
- For $p = 2$, P. Cattiaux and A. Guillin have furnished in [7] an example of a probability measure which does not satisfy the Logarithmic-Sobolev inequality but satisfies Talagrand's inequality. To construct their counterexample, they give an interesting sufficient condition for Talagrand's inequality on the real line. They proved that a probability measure $\mu \in \mathcal{P}(\mathbb{R})$ of the form $d\mu = e^{-V} dx$ satisfies Talagrand's inequality (4) for some constant $a > 0$, as soon as the potential V satisfies the following condition :

$$\text{(11)} \qquad\qquad \limsup_{x \to \pm\infty} \frac{x}{V'(x)} < +\infty.$$

1.5. **Presentation of the results.** In this paper, we will give necessary and sufficient conditions under which a probability measure μ on $\mathbb{R}$ satisfies a strong TCI. We will always assume that μ has no atom ($\mu\{x\} = 0$ for all $x \in \mathbb{R}$) and full support ($\mu(A) > 0$ for all open set $A \subset \mathbb{R}$).

First let us define the set of admissible cost functions. During the paper, $\mathcal{A}$ will be the class of all the functions $\alpha : \mathbb{R} \to \mathbb{R}^+$ such that

- α is even,
- α is a continuous function, nondecreasing on $\mathbb{R}^+$ with $\alpha(0) = 0$,
- α is super-additive on $\mathbb{R}^+$: $\alpha(x + y) \geq \alpha(x) + \alpha(y)$, $\forall x, y \geq 0$,
- α is quadratic near 0 : $\alpha(t) = |t|^2$, $\forall t \in [-1, 1]$.

One will write $\mu \in \mathbb{T}_\alpha(a)$ (resp. $\mu \in \mathbb{ST}_\alpha(a)$) if μ satisfies the TCI (resp. the strong TCI) with the cost function $(x, y) \mapsto \alpha(a(x - y))$.

1.5.1. *The main result.* Our main result (Theorem 80) characterizes the strong TCIs on a large class $\mathcal{L}ip_\sharp\mu_1 \subset \mathcal{P}(\mathbb{R})$. Roughly speaking this set is the class of all probability measures which are Lipschitz deformation of the exponential probability measure $d\mu_1(x) = \frac{1}{2} e^{-|x|} dx$. More precisely μ is in $\mathcal{L}ip_\sharp\mu_1$ if the monotone rearrangement map T transporting μ_1 on μ is Lipschitz. This map T is defined by $T(x) = F^{-1} \circ F_1$, where F (resp. F_1) is the cumulative distribution function of μ (resp. μ_1), and is such that $\mu = T\sharp\mu_1$, where $T\sharp\mu_1$ denotes the image of μ_1 under T.

If μ belongs to $\mathcal{L}ip_\sharp\mu_1$ then one has the following characterization : μ satisfies the strong TCI $\mathbb{ST}_\alpha(a)$ for some constant $a > 0$ if and only if there is some $b > 0$ such that

$$K^+(b) := \sup_{x \geq m} \int e^{\alpha(bu)} d\mu_x^+(u) < +\infty \qquad \text{and} \qquad K^-(b) := \sup_{x \leq m} \int e^{\alpha(bu)} d\mu_x^-(u) < +\infty,$$

where m is the median of μ and where μ_x^+ and μ_x^- are probability measures on $\mathbb{R}^+$ defined as follows :

$$\mu_x^+ = \mathcal{L}(X - x | X \geq x) \qquad \text{and} \qquad \mu_x^- = \mathcal{L}(x - X | X \leq x),$$

with X a random variable of law μ.

The result furnished by Theorem 80 is quite satisfactory. Firstly, though partial, this result covers all the 'regular' cases. Namely, it can be shown that if μ satisfies $\mathbb{ST}_a(a)$ then it satisfies a spectral gap inequality. But the elements of $\mathcal{L}ip_\sharp \mu_1$ satisfy the spectral gap inequality too. Examples of probability measures not belonging to $\mathcal{L}ip_\sharp \mu_1$ but satisfying the spectral gap inequality are known but are rather pathological... Secondly, one can easily derive from the above result an explicit sufficient condition for probability measures $d\mu = e^{-V}\,dx$ with V satisfying a certain regularity condition.

Before stating this explicit sufficient condition, one needs to introduce the class of 'good' potentials V. Let $\mathcal{V}$ be the set of function $f : \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^2$ such that

- there is $x_o > 0$ such that $f' > 0$ on $(-\infty, -x_o] \cup [x_o, +\infty)$,
- $\dfrac{f''(x)}{f'^2(x)} \xrightarrow[x \to \pm\infty]{} 0$.

In Theorem 85, we prove that if $d\mu = e^{-V}\,dx$ with $V \in \mathcal{V}$ and $\alpha \in \mathcal{A} \cap \mathcal{V}$, then μ satisfies $\mathbb{ST}_\alpha(a)$ for some constant $a > 0$ as soon as the following conditions hold :

$$\liminf_{x \to \pm\infty} |V'(x)| > 0 \qquad \text{and} \qquad \exists \lambda > 0 \text{ such that } \limsup_{x \to \pm\infty} \frac{\alpha'(\lambda x)}{V'(x + m)} < +\infty.$$

The first condition guaranties that μ belongs to $\mathcal{L}ip_\sharp \mu_1$ and the second one that $K^+(b) < +\infty$ and $K^-(b) < +\infty$ for some positive b. This sufficient condition completely extends the result by P. Cattiaux and A. Guillin concerning $\mathbb{T}_2$. Our approach is completely different.

1.5.2. *The particular case of Log-concave distributions.* A particularly nice case is when μ is Log-concave. Recall that μ is said to be Log-concave if $\log(1 - F)$ is concave, F being the cumulative distribution function of μ. If μ is Log-concave then it belongs to $\mathcal{L}ip_\sharp \mu_1$. Furthermore, μ satisfies $\mathbb{ST}_\alpha(a)$ for some constant $a > 0$ if and only if

$$(12) \qquad \exists b > 0, \quad \int e^{\alpha(bx)}\,d\mu(x) < +\infty.$$

This result enables us to derive sufficient conditions for modified Logarithmic Sobolev inequalities. Using well known techniques, we prove in Theorem 63 that if μ is a Log-concave distribution which satisfies the inequality $\mathbb{ST}_\alpha(a)$ then μ satisfies the following modified Log-Sobolev inequality

$$(13) \qquad \mathrm{Ent}_\mu(f^2) \leq C \int \alpha^* \left(t \frac{f'}{f} \right) f^2\,d\mu, \quad \forall f,$$

for some $c, t > 0$. Consequently, if the Log-concave distribution μ satisfies the moment condition (12), it satisfies the modified Log-Sobolev inequality (13) (see Theorem 64 and Corollary 66). This extends and completes the results of Gentil, Guillin and Miclo (see [9] and [11]).

1.5.3. *A word on the method.* The originality of this paper is that transportation cost inequalities are studied without the help of Logarithmic-Sobolev inequalities. Our results rely on a simple but powerful perturbation method which is explained in section 3. Roughly speaking, we show that if μ satisfies some (strong) TCI then $T_\sharp\mu$ satisfies a (strong) TCI with a skewed cost function. This principle enables us to derive new (strong) TCIs from old ones. More precisely if μ_{ref} is a known probability measure satisfying some (strong) TCI and if one is able to construct a map T transporting μ_{ref} on an other probability measure μ, then μ will satisfy a (strong) TCI too. This principle is true in any dimension. The reason why this paper deals with dimension one only is that the optimal-transportation of measures is extremely simple in this framework.

Contents

2. Preliminary results

In this section, we are going to recall some well known results on TCI, namely their dual translations, their tensorization properties and their links with the concentration of measure phenomenon.

General Framework : Most of the forthcoming results are available in a very general framework which we shall now describe.

Let $\mathcal{X}$ be a polish space and let $c : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ be a lower semi-continuous function, called the *cost function*. The set of all the probability measures on $\mathcal{X}$ will be denoted by $\mathcal{P}(\mathcal{X})$. The optimal transportation cost between $\nu \in \mathcal{P}(\mathcal{X})$ and $\mu \in \mathcal{P}(\mathcal{X})$ is defined by

$$\mathcal{T}_c(\nu,\mu) = \inf_{\pi \in P(\nu,\mu)} \iint_{\mathcal{X} \times \mathcal{X}} c(x,y)\,\pi(dxdy),$$

where $P(\nu,\mu)$ is the set of all the probability measures on $\mathcal{X} \times \mathcal{X}$ such that $\pi(dx \times \mathcal{X}) = \nu$ and $\pi(\mathcal{X} \times dy) = \mu$.

A probability measure μ is said to satisfy the TCI with the cost function c if

$$(14) \qquad \forall \nu \in \mathcal{P}(\mathcal{X}), \quad \mathcal{T}_c(\nu,\mu) \le \mathrm{H}(\nu \mid \mu).$$

A probability measure μ is said to satisfy the strong TCI with the cost function c if

$$(15) \qquad \forall \nu, \beta \in \mathcal{P}(\mathcal{X}), \quad \mathcal{T}_c(\nu, \beta) \leq \mathrm{H}(\nu \mid \mu) + \mathrm{H}(\beta \mid \mu).$$

2.1. **TCI vs Strong TCI.**

Proposition 16. *Let $\mathcal{X} = \mathbb{R}^p$ and suppose that $c(x, y) = \theta(x - y)$, with $\theta : \mathbb{R}^p \to \mathbb{R}^+$ a convex function such that $\theta(-x) = \theta(x)$. If μ satisfies the TCI with the cost function c then μ satisfies the strong TCI with the cost function $\tilde{c}$ defined by*

$$\tilde{c}(x, y) = 2\theta\left(\frac{x - y}{2}\right), \quad \forall x, y \in \mathbb{R}^p.$$

Proof. Let $\pi_1 \in P(\nu, \mu)$ and $\pi_2 \in P(\mu, \beta)$. One can construct X, Y, Z three random variables such that $\mathcal{L}(X, Y) = \pi_1$ and $\mathcal{L}(Y, Z) = \pi_2$ (see for instance the Gluing Lemma of [19] p. 208). Thus, using the convexity of θ, one has

$$\mathcal{T}_{\tilde{c}}(\nu, \beta) \leq \mathbb{E}\left[2\theta\left(\frac{X - Z}{2}\right)\right] \leq \mathbb{E}\left[\theta\left(X - Y\right)\right] + \mathbb{E}\left[\theta\left(Y - Z\right)\right]$$

$$= \int c(x, y)\, \pi_1(dxdy) + \int c(y, z)\, \pi_2(dydz).$$

Optimizing in π_1 and π_2 yields

$$\mathcal{T}_{\tilde{c}}(\nu, \beta) \leq \mathcal{T}_c(\nu, \mu) + \mathcal{T}_c(\beta, \mu), \quad \forall \nu, \beta \in \mathcal{P}(\mathbb{R}^p).$$

Consequently, if μ satisfies the TCI with the cost function c then μ satisfies the strong TCI with the cost function $\tilde{c}$. $\qquad\square$

Let $\theta : \mathbb{R}^p \to \mathbb{R}^+$ be a symmetric function $(\theta(-x) = \theta(x))$. One will say that a probability measure μ on $\mathbb{R}^p$ satisfies the inequality $\mathbb{ST}_\theta(a)$ if it satisfies the strong TCI with the cost function $c(x, y) = \theta(a(x - y))$.

Lemma 17. *Let θ be as above and suppose that $\theta(kx) \geq k\theta(x), \forall k \in \mathbb{N}, \forall x \in \mathbb{R}^p$. Let $b_1, b_2 > 0$ and define $\tilde{\theta}(x) = b_1\theta(b_2 x), \forall x \in \mathbb{R}^p$. Then, μ satisfies $\mathbb{ST}_\theta(a)$ for some $a > 0$ if and only if μ satisfies $\mathbb{ST}_{\tilde{\theta}}(\tilde{a})$ for some $\tilde{a} > 0$.*

Proof. (See also the proof of Corollary 1.3 of [18]) Suppose that μ satisfies $\mathbb{ST}_\theta(a)$ for some $a > 0$. Let $k \in \mathbb{N}$ such that $k \geq b_1$, then $\theta(ax) \geq k\theta(ax/k) \geq \tilde{\theta}(ax/(b_2 k))$. Hence, μ satisfies $\mathbb{ST}_{\tilde{\theta}}\left(\frac{a}{b_2 k}\right)$. $\qquad\square$

2.2. **Links with the concentration of measure phenomenon.**

The following Theorem explains how to deduce concentration of measure estimates from a strong TCI. The argument used in the proof is due to K. Marton and M. Talagrand see ([13] and the proof of Corollary 1.3 of [18]).

Theorem 18. *Let (X, d) be a polish space and $c : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ be a continuous cost function. Suppose that μ satisfies the strong TCI with the cost function c, then*

$$(19) \qquad \forall A \subset \mathcal{X} \text{ measurable}, \quad \forall r \geq 0, \qquad \mu\left(A_c^r\right) \geq 1 - \frac{1}{\mu(A)} e^{-r},$$

where $A_c^r = \left\{y \in \mathcal{X} : \exists x \in A \text{ such that } c(x, y) \leq r\right\}$.

Proof. Let $A, B \in \mathcal{X}$ and define $\mu_A = \frac{\mu(\cdot \cap A)}{\mu(A)}$ and $\mu_B = \frac{\mu(\cdot \cap B)}{\mu(B)}$. Since μ satisfies the strong TCI, one has :

$$(20) \qquad c(A, B) \leq \mathcal{T}_c(\mu_A, \mu_B) \leq \mathrm{H}\left(\mu_A | \mu\right) + \mathrm{H}\left(\mu_B | \mu\right) = -\log \mu(A) - \log \mu(B),$$

with $c(A, B) = \inf \left\{ c(x, y) : x \in A, y \in B \right\}$. Now taking $B = \mathcal{X} - A_c^r$ in (20) yields the desired result. $\qquad \square$

2.3. Dual translation of transportation-cost inequalities.

2.3.1. *Kantorovich Rubinstein Theorem and its consequences.* According to the celebrated Kantorovich-Rubinstein Theorem, optimal transportation costs admit a dual representation which is the following :

$$(21) \qquad \forall \nu, \mu \in \mathcal{P}(\mathcal{X}), \quad \mathcal{T}_c(\nu, \mu) = \sup_{(\psi, \varphi) \in \Phi_c} \left\{ \int \psi \, d\nu - \int \varphi \, d\mu \right\},$$

where $\Phi_c = \{(\psi, \varphi) \in B(\mathcal{X}) \times B(\mathcal{X}) : \psi(x) - \varphi(y) \leq c(x, y), \forall x, y \in \mathcal{X}\}$ and $B(X)$ is the set of bounded measurable functions on $\mathcal{X}$. The dual representation (21) is in particular true if c is lower semi-continuous function defined on a polish space $\mathcal{X}$ (see for instance Theorem 1.3 [19]). Furthermore, $B(\mathcal{X})$ can be replaced by $C_b(\mathcal{X})$, the set of bounded continuous functions on $\mathcal{X}$.

The infimal-convolution operator Q_c is defined by

$$Q_c \varphi(x) = \inf_y \left\{ \varphi(y) + c(x, y) \right\},$$

for all $\varphi \in B(\mathcal{X})$. If c is continuous, $x \mapsto Q_c \varphi(x)$ is measurable (in fact upper semi continuous) and it can be shown that

$$\forall \nu, \mu \in \mathcal{P}(\mathcal{X}), \quad \mathcal{T}_c(\nu, \mu) = \sup_{\varphi \in C_b(\mathcal{X})} \left\{ \int Q_c \varphi \, d\nu - \int \varphi \, d\mu \right\},$$

$$= \sup_{\varphi \in B(\mathcal{X})} \left\{ \int Q_c \varphi \, d\nu - \int \varphi \, d\mu \right\}.$$

Since optimal transportation costs admit a dual representation, it is natural to ask if TCIs and strong TCIs admit a dual translation too. The answer is given in the following theorem.

Theorem 22. *Suppose that c is a continuous cost-function on the Polish space $\mathcal{X}$.*

(1) *μ satisfies the TCI (14) if and only if*

$$(23) \qquad \forall \varphi \in B(\mathcal{X}), \quad \int e^{Q_c \varphi} \, d\mu \cdot e^{-\int \varphi \, d\mu} \leq 1.$$

(2) *μ satisfies the strong TCI (15) if and only if*

$$(24) \qquad \forall \varphi \in B(\mathcal{X}), \quad \int e^{Q_c \varphi} \, d\mu \cdot \int e^{-\varphi} \, d\mu \leq 1.$$

Furthermore in the preceding statements $B(\mathcal{X})$ can be replaced by $C_b(\mathcal{X})$.

Proof. The first point is due to S.G. Bobkov and F. Götze (see the proof of Theorem 1.3 and (1.7) of [3]). The interested reader can also find an alternative proof of this result in [10] (see Corollary 1). In this latter proof, Large Deviations Theory techniques are used. One can easily adapt the one or the other approach to derive the dual version of strong TCIs (24). This is left to the reader. □

Remark 25. *As mentioned in the introduction, inequalities of the form (24) are called infimal-convolution inequalities. These inequalities were introduced by B. Maurey in [15]. Note that Maurey's work is anterior to the paper [18] and [3]. A good account on infimal-convolution inequalities can be found in M. Ledoux's book [12]. In this article, we have chosen to privilege the strong TCI (15) form, which is the primal form of (24). The reason is that we find (15) more intuitive.*

2.3.2. *Application : strong TCI and integrability.* Let us detail an important application of the infimal-convolution formulation of strong TCI.

Proposition 26. *Let c be a continuous cost function on the Polish space $\mathcal{X}$. Suppose that $\mu \in \mathcal{P}(\mathcal{X})$ satisfies the strong TCI with the cost function c. Let $A \subset \mathcal{X}$ be a measurable set and define $c(x, A) = \inf_{y \in A} c(x, y)$. One has*

$$(27) \qquad \int e^{c(x,A)} \, d\mu(x) \cdot \mu(A) \leq 1.$$

Remark 28. *This integrability property was first noticed by B. Maurey in [15]. Note that the inequality (27) implies the concentration estimate (19).*

Proof. Define, for all $p \in \mathbb{N}$, $\varphi_A^p(x) = \begin{cases} 0 & \text{if } x \in A \\ p & \text{if } x \in A^c \end{cases}$. As φ_A^p is bounded, one can apply (24), this yields

$$\int e^{Q_c \varphi_A^p} \, d\mu \cdot \int e^{-\varphi_A^p} \, d\mu \leq 1.$$

An easy computation shows that $Q_c \varphi_A^p(x) = \min(c(x, A), p) \xrightarrow[p \to +\infty]{} c(x, A)$ and $e^{-\varphi_A^p} \xrightarrow[p \to +\infty]{}$ $\mathbb{1}_A$. Using the monotone convergence theorem, one gets he desired inequality. □

The following Corollary will be very useful in the sequel.

Corollary 29. *Let μ be a probability measure on $\mathbb{R}$ satisfying the strong TCI with the cost function $c(x, y) = \alpha(x - y)$, with α a continuous symmetric non decreasing function. For all $x \in \mathbb{R}$, define*

$$\mu_x^+ = \mathcal{L}(X - x | X \geq x), \qquad \text{and} \qquad \mu_x^- = \mathcal{L}(x - X | X \leq x),$$

where X is a random variable with law μ. Then,

$$\int_0^{+\infty} e^\alpha \, d\mu_x^+ \leq \frac{1}{\mu(-\infty, x]} + 1, \forall x \in \mathbb{R}$$

$$\int_0^{+\infty} e^\alpha \, d\mu_x^- \leq \frac{1}{\mu[x, +\infty)} + 1, \forall x \in \mathbb{R}$$

In particular,

$$\int e^\alpha \, d\mu \le \frac{1}{\mu(\mathbb{R}^+)\mu(\mathbb{R}^-)} - 1.$$

Proof. Let $A = (-\infty, x]$. It is easy to show that $c(y, A) = \alpha(y - x)$ if $y \ge x$ and 0 else. Applying (27) with this A yields

$$\left(\mu(-\infty, x] + \int_x^{+\infty} e^{\alpha(y-x)} \, d\mu(y) \right) \cdot \mu(-\infty, x] \le 1.$$

Rearranging the terms, one gets

$$\int_x^{+\infty} e^{\alpha(y-x)} \, d\mu(y) \le \frac{1 - \mu(-\infty, x]^2}{\mu(-\infty, x]}.$$

Dividing both sides by $\mu[x, +\infty)$ gives the result. Working with $A = [x, +\infty)$ gives the integrability property for μ_x^-. Now,

$$\int e^\alpha \, d\mu = \mu(\mathbb{R}^+) \int_0^{+\infty} e^\alpha \, d\mu_0^+ + \mu(\mathbb{R}^-) \int_0^{+\infty} e^\alpha \, d\mu_0^-$$

$$\le 1 + \frac{\mu(\mathbb{R}^+)}{\mu(\mathbb{R}^-)} + \frac{\mu(\mathbb{R}^-)}{\mu(\mathbb{R}^+)}$$

$$= \frac{1}{\mu(\mathbb{R}^+)\mu(\mathbb{R}^-)} - 1.$$

$\square$

2.4. **Tensorization property of (strong) TCIs.** If μ_1 and μ_2 satisfy a (strong) TCI, does $\mu_1 \otimes \mu_2$ satisfy a (strong) TCI ? The following Theorem gives an answer to this question.

Theorem 30. *Let $(\mathcal{X}_i)_{i=1\ldots n}$ be a family of Polish spaces. Suppose that μ_i is a probability measure on $\mathcal{X}_i$ satisfying a (strong) TCI on $\mathcal{X}_i$ with a continuous cost function c_i such that $c_i(x, x) = 0, \forall x \in \mathcal{X}_i$. Then the probability measure $\mu_1 \otimes \cdots \otimes \mu_n$ satisfies a (strong) TCI on $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ with the cost function $c_1 \oplus \cdots \oplus c_n$ defined as follows :*

$$\forall x, y \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n, \quad c_1 \oplus \cdots \oplus c_n(x, y) = \sum_{i=1}^n c_i(x_i, y_i).$$

Proof. There are two methods to prove this tensorization property. The first one is due to K. Marton and makes use of a coupling argument (the so called *Marton's coupling argument*). It is explained in several places : in Marton's original paper [13], in Talagrand's paper on $\mathbb{T}_2$ [18] or in M. Ledoux book [12] (Chapter 6). The second method uses the dual forms (23) and (24). This approach was originally developed by B. Maurey in [15] for infimal-convolution inequalities (see Lemma 1 of [15]). In the case of TCIs, the proof is given in great details in [10] (see the proof of Theorem 5). $\square$

Remark 31. *Several authors have obtained non-independent tensorization results for transportation-cost inequalities and related inequalities (see [14], [17] or [8]).*

Applying Theorem 30 together with Theorem 18, one obtains the following Corollary :

Corollary 32. *Let c be a continuous cost function on the Polish space $\mathcal{X}$ such that $c(x,x) = 0, \forall x \in \mathcal{X}$. Suppose that $\mu \in \mathcal{P}(\mathcal{X})$ satisfies the strong TCI with the cost function c. Then,*

$$\forall n \in \mathbb{N}^*, \forall A \text{ measurable}, \forall r \geq 0, \quad \mu^n(A_c^r) \geq 1 - \frac{1}{\mu^n(A)} e^{-r},$$

where $A_c^r = \{x \in \mathcal{X}^n : \exists y \in A \text{ such that } \sum_{i=1}^n c(x_i, y_i) \leq r\}$.

3. THE PERTURBATION METHOD FOR (STRONG) TCIs

3.1. **The contraction principle in an abstract setting.**
In the sequel, $\mathcal{X}$ and $\mathcal{Y}$ will be Polish spaces. If μ is a probability measure on $\mathcal{X}$ and $T : \mathcal{X} \to \mathcal{Y}$ is a measurable map, the image of μ under T will be denoted by $T_\sharp \mu$, it is the probability measure on $\mathcal{Y}$ defined by

$$\forall A \subset \mathcal{Y} \text{ measurable}, \quad T_\sharp \mu(A) = \mu\left(T^{-1}(A)\right).$$

In this section, we will explain how a (strong) TCI is modified when the reference probability measure μ is replaced by the image $T_\sharp \mu$ of μ under some map T.

Theorem 33. *Let $T : \mathcal{X} \to \mathcal{Y}$ be a measurable bijection. If μ_{ref} satisfies the (strong) TCI with a cost function c_{ref} on $\mathcal{X}$, then $T_\sharp \mu_{\text{ref}}$ satisfies the (strong) TCI with the cost function c_{ref}^T defined on $\mathcal{Y}$ by*

$$c_{\text{ref}}^T(y_1, y_2) = c_{\text{ref}}(T^{-1}y_1, T^{-1}y_2), \quad \forall y_1, y_2 \in \mathcal{Y}.$$

In other word, $T_\sharp \mu_{\text{ref}}$ satisfies the (strong) TCI with a skewed cost function.

Proof. Let us define $Q(y_1, y_2) = (T^{-1}y_1, T^{-1}y_2), \forall y_1, y_2 \in \mathcal{Y}$. Let $\nu, \beta \in \mathcal{P}(\mathcal{Y})$ and take $\pi \in P(\nu, \beta)$, then $\int c_{\text{ref}}^T(y_1, y_2)\, d\pi = \int c(x,y)\, dQ_\sharp \pi$, so $\mathcal{T}_{c_{\text{ref}}^T}(\nu, \beta) = \inf\limits_{\pi \in Q_\sharp P(\nu, \beta)} \int c(x,y) d\pi$. But it is easily seen that $Q_\sharp P(\nu, \beta) = P(T_\sharp^{-1}\nu, T_\sharp^{-1}\beta)$. Consequently

$$\mathcal{T}_{c_{\text{ref}}^T}(\nu, \beta) = \mathcal{T}_{c_{\text{ref}}}(T_\sharp^{-1}\nu, T_\sharp^{-1}\beta).$$

If μ_{ref} satisfies the strong TCI with the cost function c_{ref}, then

$$\mathcal{T}_{c_{\text{ref}}}(T_\sharp^{-1}\nu, T^{-1}\beta) \leq \text{H}\left(T_\sharp^{-1}\nu \,\middle|\, \mu_{\text{ref}}\right) + \text{H}\left(T_\sharp^{-1}\beta \,\middle|\, \mu_{\text{ref}}\right)$$

But

$$\text{H}\left(T_\sharp^{-1}\nu \,\middle|\, \mu_{\text{ref}}\right) = \text{H}\left(T_\sharp^{-1}\nu \,\middle|\, T_\sharp^{-1}T_\sharp \mu_{\text{ref}}\right) = \text{H}\left(\nu \,\middle|\, T_\sharp \mu_{\text{ref}}\right),$$

where the last equality comes from the following classical invariance property of relative entropy : $\text{H}\left(S_\sharp \nu_1 \,\middle|\, S_\sharp \nu_2\right) = \text{H}\left(\nu_1 \,\middle|\, \nu_2\right)$. Hence

$$\forall \nu, \beta \in \mathcal{P}(\mathcal{Y}), \quad \mathcal{T}_{c_{\text{ref}}^T}(\nu, \beta) \leq \text{H}\left(\nu \,\middle|\, T_\sharp \mu_{\text{ref}}\right) + \text{H}\left(\beta \,\middle|\, T_\sharp \mu_{\text{ref}}\right).$$

$\square$

The Corollary bellow explains the method we will use in the sequel to derive new (strong) TCIs from known ones.

Corollary 34 (Contraction principle). *Let μ_{ref} be a probability measure on $\mathcal{X}$ satisfying a (strong) TCI with a continuous cost function c_{ref}. In order to prove that a probability measure μ on $\mathcal{Y}$ satisfies the (strong) TCI with a continuous cost function c, it is enough to build an application $T : \mathcal{X} \to \mathcal{Y}$ such that $\mu = T_{\sharp}\mu_{\mathrm{ref}}$ and*

$$c(Tx_1, Tx_2) \leq c_{\mathrm{ref}}(x_1, x_2), \quad \forall x_1, x_2 \in \mathcal{X}.$$

This contraction property of strong TCIs (written in their infimal-convolution form) was first observed by B. Maurey (see Lemma 2 of [15]).

Proof. We assume that μ_{ref} satisfies the strong TCI with the cost function c_{ref}. Let $\varphi : \mathcal{Y} \to \mathbb{R}$ be a bounded map. Then, for all $x_1 \in \mathcal{X}$

$$Q_c\varphi(Tx_1) = \inf_{y \in \mathcal{Y}}\{\varphi(y) + c(Tx_1, y)\} \leq \inf_{x_2 \in \mathcal{X}}\{\varphi(Tx_2) + c(Tx_1, Tx_2)\}$$
$$\leq \inf_{x_2 \in \mathcal{X}}\{\varphi \circ T(x_2)) + c_{\mathrm{ref}}(x_1, x_2)\} = Q_{c_{\mathrm{ref}}}(\varphi \circ T).$$

Thus,

$$\int e^{Q_c\varphi}\, d\mu \cdot \int e^{-\varphi}\, d\mu = \int e^{Q_c\varphi} \circ T\, d\mu_{\mathrm{ref}} \cdot \int e^{-\varphi \circ T}\, d\mu_{\mathrm{ref}}$$
$$\leq \int e^{Q_{c_{\mathrm{ref}}}(\varphi \circ T)}\, d\mu_{\mathrm{ref}} \cdot \int e^{-\varphi \circ T}\, d\mu_{\mathrm{ref}}$$
$$\leq 1$$

where the last inequality follows from (24). $\qquad\square$

Remark 35. *If T is invertible, the proof above can be simplified using Theorem 33. Namely, according to Theorem 33, μ satisfies the (strong) TCI with the cost function c_{ref}^T. But, by hypothesis, $c \leq c_{\mathrm{ref}}^T$, so μ satisfies the (strong) TCI with the cost function c.*

3.2. The contraction principle on the real line.

3.2.1. *Monotone rearrangement.* We are going to apply the contraction principle to probability measures on the real line. The reason why dimension one is so easy to handle is the existence of a good map T which pushes forward μ_{ref} on μ : the monotone rearrangement.

Theorem 36 (Monotone rearrangement). *Let μ_{ref} and μ be probability measures on $\mathbb{R}$ and let F_{ref} and F denote their cumulative distribution functions :*

$$F_{\mathrm{ref}}(t) = \mu_{\mathrm{ref}}(-\infty, t], \quad \forall t \in \mathbb{R}, \qquad and \qquad F(t) = \mu(-\infty, t], \quad \forall t \in \mathbb{R}.$$

If F_{ref} and F are continuous and increasing (equivalently μ_{ref} and μ have no atom and full support), then the map $T = F^{-1} \circ F_{\mathrm{ref}}$ transports μ_{ref} on μ, that is $T_{\sharp}\mu_{\mathrm{ref}} = \mu$.

From now on, T will always be the map defined in the preceding Theorem.

3.2.2. *About the exponential distribution.* The reference probability measure μ_{ref} will be the symmetric exponential distribution μ_1 on $\mathbb{R}$:

$$d\mu_{\text{ref}}(x) = d\mu_1(x) := \frac{1}{2}e^{-|x|}\,dx.$$

Theorem 37 (Maurey, Talagrand). *The exponential measure μ_1 satisfies the (strong) TCI with the cost function $\frac{1}{\kappa}c_1$, for some constant $\kappa > 0$, with c_1 defined by*

$$c_1(x,y) := \min(|x-y|, |x-y|^2), \quad \forall x,y \in \mathbb{R}.$$

Remark 38.

(1) *One can take $\kappa = 36$.*
(2) *B. Maurey proved the strong TCI with the sharper cost functions $c(x,y) = \tilde{\alpha}_1(x-y)$, where $\tilde{\alpha}_1(x) = \begin{cases} \frac{1}{36}x^2 & \text{if } |x| \leq 4 \\ \frac{2}{9}(|x|-2) & \text{if } |x| \geq 4 \end{cases}$ (see Proposition 1 of [15]). One can show that $\tilde{\alpha}_1 \geq \frac{1}{36}\alpha_1$.*
(3) *M. Talagrand proved independently that μ_1 satisfies the TCI with the cost functions $c_\lambda(x,y) = \gamma_\lambda(x-y)$ where $\gamma_\lambda(x) = \left(\frac{1}{\lambda}-1\right)\left(e^{-\lambda|x|} - 1 + \lambda|x|\right)$ for all $\lambda \in (0,1)$ (see Theorem 1.2 of [18]).*

Transportation-cost inequalities associated to the cost function c_1 were fully characterized by I. Gentil, M. Ledoux and S . Bobkov in [5] in terms of Poincaré inequalities :

Theorem 39 (Bobkov-Gentil-Ledoux). *A probability measure μ on $\mathbb{R}^p$ satisfies the TCI with the cost function $(x,y) \mapsto \lambda \min(|x-y|_2, |x-y|_2^2)$, for some $\lambda > 0$ if and only if it satisfies a Poincaré inequality, that is if there is some constant $C > 0$ such that*

$$\text{Var}_\mu(f) \leq C \int_{\mathbb{R}^p} |\nabla f|_2^2 \, d\mu, \quad \forall f$$

3.2.3. *Application of the contraction principle on the real line.* A good thong with the exponential distribution is that its cumulative distribution function can be explicitly computed

$$(40) \quad F_1(x) = \begin{cases} 1 - \frac{1}{2}e^{-|x|} & \text{if } x \geq 0 \\ \frac{1}{2}e^{-|x|} & \text{if } x \leq 0 \end{cases} \quad \text{and} \quad F_1^{-1}(t) = \begin{cases} -\log(2(1-t)) & \text{if } t \geq \frac{1}{2} \\ \log(2t) & \text{if } t \leq \frac{1}{2} \end{cases}$$

Suppose that μ is a probability measure on $\mathbb{R}$ having no atom and full support, then its cumulative distribution function F is invertible, and the map T transporting μ_1 on μ can be expressed as follows :

$$(41)$$
$$T(x) = \begin{cases} F^{-1}\left(1 - \frac{1}{2}e^{-|x|}\right) & \text{if } x \geq 0 \\ F^{-1}\left(\frac{1}{2}e^{-|x|}\right) & \text{if } x \leq 0 \end{cases} \quad \text{and} \quad T^{-1}(x) = \begin{cases} -\log(2(1-F(x))) & \text{if } x \geq m \\ \log(2F(x)) & \text{if } x \leq m \end{cases},$$

where m denotes the median of μ.

Let us introduce the following quantity :

$$\omega_\mu(h) = \inf\left\{|T^{-1}x - T^{-1}y| : |x-y| \geq h\right\}, \quad \forall h \geq 0.$$

Proposition 42. *If $\mu \in \mathcal{P}(\mathbb{R})$ is a probability measure with no atom and full support, then μ satisfies the strong TCI with the cost function $c_\mu(x,y) = \frac{1}{\kappa}\alpha_1 \circ \omega_\mu(|x-y|)$, where $\alpha_1(t) = \min(t, t^2), \forall t \geq 0$.*

Proof. By definition of ω_μ,

$$|T^{-1}x - T^{-1}y| \geq \omega_\mu(|x-y|), \quad \forall x, y \in \mathbb{R}.$$

Thus,

$$c_1^T(x,y) \geq \frac{1}{\kappa}\alpha_1\left(\omega_\mu(|x-y|)\right), \quad \forall x, y \in \mathbb{R},$$

and this achieves the proof. $\qquad\square$

To better understand ω_μ it is good to relate it to the continuity modulus of T.

Definition 43 (The class $\mathcal{UC}_\sharp\mu_1$)**.** *The set of all probability measures μ on $\mathbb{R}$, with no atom and full support, such that the monotone rearrangement map transporting the exponential measure $\frac{1}{2}d\mu_1(x) = e^{-|x|}\,dx$ on μ is uniformly continuous is denoted by $\mathcal{UC}_\sharp\mu_1$.*

The proof of the following proposition is left to the reader.

Proposition 44. *Suppose $\mu \in \mathcal{UC}_\sharp\mu_1$, then the continuity modulus Δ_μ of T is defined by $\Delta_\mu(h) = \sup\left\{|Tx - Ty| : |x-y| \leq h\right\}, \forall h \geq 0$. It is a continuous increasing function and*

$$\omega_\mu = \Delta_\mu^{-1}.$$

Remark 45.

(1) *All the elements of $\mathcal{UC}_\sharp\mu_1$ enjoy a dimension free concentration of measure property. Namely, if $\mu \in \mathcal{UC}_\sharp\mu_1$, then μ satisfies the strong TCI with the cost function $c_\mu(x,y) = \alpha_\mu(x-y)$, where $\alpha_\mu(x) = \frac{1}{\kappa}\alpha_1 \circ \omega_\mu(|x|)$. Thus according to Corollary 32, one has*

$$\forall n \in \mathbb{N}^*, \forall A \subset \mathbb{R}^n, \forall r \geq 0, \quad \mu^n(A_{\alpha_\mu}^r) \geq 1 - \frac{1}{\mu(A)}e^{-r},$$

with $A_{\alpha_\mu}^r = \left\{x \in \mathbb{R}^n : \exists y \in A \text{ such that } \sum_{i=1}^n \alpha_\mu(x_i - y_i) \leq r\right\}$.

(2) *The class of all the probability measures on $\mathbb{R}$ satisfying a dimension free concentration of measure property is not yet identified. In [6], S.G. Bobkov and C. Houdré studied probability measures enjoying a weak dimension free concentration property (roughly speaking one can estimate $\mu^n(A_\infty^r)$ independently of the dimension, where A_∞^r denotes the blow-up of A with respect to the norm $|x|_\infty = \max_i |x_i|$). They proved that a probability measure has this weak property if and only if the map T generate a finite modulus, which means that $\Delta_\mu(h) < +\infty$ for some (equivalently for all) $h \in \mathbb{R}$.*

In order to obtain explicit concentration properties, one has to estimate ω_μ.

Proposition 46. *Define*

$$\omega_\mu^+(h) = \inf\left\{|T^{-1}x - T^{-1}y| : |x-y| \geq h, x, y \geq m\right\}$$
$$\omega_\mu^-(h) = \inf\left\{|T^{-1}x - T^{-1}y| : |x-y| \geq h, x, y \leq m\right\}$$

then

$$\omega_\mu(h) \geq \min\left(\omega_\mu^+\left(\frac{h}{2}\right), \omega_\mu^-\left(\frac{h}{2}\right)\right)$$

Proof. Let $x, y \in \mathbb{R}$ with $x \leq m \leq y$ and $y - x \geq h \geq 0$. One has

$$|T^{-1}y - T^{-1}x| = T^{-1}y - T^{-1}x = \left(T^{-1}y - T^{-1}m\right) + \left(T^{-1}m - T^{-1}x\right)$$
$$\geq \omega_\mu^+(y - m) + \omega_\mu^-(m - x).$$

Since $y - x \geq h$ and $m \in [x, y]$, one has $y - m \geq \frac{h}{2}$ or $m - x \geq \frac{h}{2}$, thus

$$|T^{-1}y - T^{-1}x| \geq \min\left(\omega_\mu^+\left(\frac{h}{2}\right), \omega_\mu^-\left(\frac{h}{2}\right)\right).$$

$\square$

Let X be a random variable with law μ and define

$$(47) \qquad \mu_x^+ = \mathcal{L}(X - x \mid X \geq x) \in \mathcal{P}(\mathbb{R}^+), \quad \forall x \geq m$$

$$(48) \qquad \mu_x^- = \mathcal{L}(x - X \mid X \leq x) \in \mathcal{P}(\mathbb{R}^+), \quad \forall x \leq m$$

In the following Proposition, the quantities ω_μ^+ and ω_μ^- are expressed in terms of the cumulative distribution functions of the probability measures μ_x^+ and μ_x^-.

Proposition 49.

$$\begin{array}{c} \omega_\mu^+(h) = \inf\left\{-\log \mu_x^+[h, +\infty) : x \geq m\right\} \\ \omega_\mu^-(h) = \inf\left\{-\log \mu_x^-[h, +\infty) : x \leq m\right\} \end{array}, \quad \forall h \geq 0.$$

Proof. It is easy to see that $\omega_\mu^+(h) = \inf\left\{T^{-1}(x + h) - T^{-1}x : x \geq m\right\}$. Using (41) one sees that

$$T^{-1}(x + h) - T^{-1}x = -\log\left(\frac{1 - F(x + h)}{1 - F(x)}\right) = -\log \mu_x^+[h, +\infty),$$

which gives the result. $\square$

The proof of the following Corollary is immediate.

Corollary 50. *Let* $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ *a continuous non decreasing function with* $\omega(0) = 0$. *In order to show that* $c_\mu(x, y) \geq \frac{1}{\kappa}\alpha_1 \circ \omega\left(\frac{|x-y|}{2}\right)$, *it is enough to show that*

$$(51) \qquad \sup_{x \geq m} \mu_x^+[h, +\infty) \leq e^{-\omega(h)}, \quad \forall h \geq 0$$

$$(52) \qquad \sup_{x \leq m} \mu_x^-[h, +\infty) \leq e^{-\omega(h)}, \quad \forall h \geq 0$$

Remark 53. *The evolution of* μ_x^+ *and* μ_x^- *with x reflects the aging properties of μ. Objects of this type appear naturally in reliability theory. Suppose that X is nonnegative and think of X as the failure time of some engine, then* μ_x^+ *is the law of the failure after time x knowing that the engine works properly at time x.*

Recall that one says that a probability measure ν_1 is stochastically dominated by an other probability measure ν_2 if

$$\nu_1[h, +\infty) \leq \nu_2[h, +\infty), \quad \forall h \geq 0.$$

In the sequel, this will be written $\nu_1 \prec_{st} \nu_2$. If one think of μ_1 and μ_2 as failure time laws, then $\nu_1 \prec_{st} \nu_2$ means that the material modeled by ν_2 is more reliable than the material modeled by ν_1.

It is well known that the following propositions are equivalent :

(1) $\nu_1 \prec_{st} \nu_2$,
(2) $\int f \, d\nu_1 \leq \int f \, d\nu_2$, for all nondecreasing $f : \mathbb{R} \to \mathbb{R}$,
(3) There are X_1, and X_2 two random variables defined on the same probability space, such that $\mathcal{L}(X_1) = \nu_1$, $\mathcal{L}(X_2) = \nu_2$ and $X_1 \leq X_2$ almost surely.

Since every continuous nondecreasing function F with $F(0) = 0$ and $\lim_{x \to +\infty} F(x) = 1$ is the cumulative distribution function of some probability measure on $\mathbb{R}^+$ with no atom, finding a function ω such that (51) and (52) hold is the same as finding some uniform upper bound of the probability measures μ_x^+ and μ_x^- in the sense of stochastic ordering. The preceding Corollary can thus be restated as follows :

Corollary 54. *Let $\mu \in \mathcal{P}(\mathbb{R})$ be a probability measure with no atom and full support. If there is a probability measure $\mu_0 \in \mathcal{P}(\mathbb{R}^+)$ with no atom such that*

$$\mu_x^+ \prec_{st} \mu_0, \quad \forall x \geq m \qquad and \qquad \mu_x^- \prec_{st} \mu_0, \quad \forall x \leq m,$$

then μ satisfies the strong TCI with the cost function c defined by

$$c(x, y) = \frac{1}{\kappa} \alpha_1 \left(-\log \left(1 - F_0 \left(|x - y|/2 \right) \right) \right), \quad \forall x, y \in \mathbb{R},$$

where F_0 denotes the cumulative distribution function of μ_0.

4. (Strong) TCI for Log-concave distributions

Let $\mu \in \mathcal{P}(\mathbb{R})$, let F be its cumulative distribution function and define $\overline{F} = 1 - F$. The probability measure μ is said to be *Log-concave* if $\log \overline{F}$ is concave.

4.1. A natural cost function.
Log-concave are examples of NBU (New Better than Used) distributions. This is explained in the following proposition :

Proposition 55. *If $\mu \in \mathcal{P}(\mathbb{R})$ is a Log-concave distribution, then*

$$\mu_x^+ \prec_{st} \mu_m^+, \quad \forall x \geq m \qquad and \qquad \mu_x^- \prec_{st} \mu_m^-, \quad \forall x \leq m.$$

Proof. Let us show that $\mu_x^+ \prec_{st} \mu_m^+$ for all $x \geq m$. By definition, this means that $\mu_x^+[h, +\infty) \leq \mu_m^+[h, +\infty), \forall h \geq 0, \forall x \geq m$ and this is equivalent to

$$\frac{1 - F(x + h)}{1 - F(x)} \leq \frac{1 - F(m + h)}{1 - F(m)}, \quad \forall h \geq 0, \forall x \geq m.$$

Defining $\overline{F}_m^+(h) = \frac{1 - F(m+h)}{1 - F(m)}, \forall h \geq 0$, the preceding inequality is equivalent to :

$$\overline{F}_m^+(x - m + h) \leq \overline{F}_m^+(x - m) \times \overline{F}_m^+(h), \quad \forall h \geq 0, \forall x \geq m.$$

In other word, $\mu_x^+ \prec_{st} \mu_m^+$ if and only if the function $\log \overline{F}_m^+$ is sub-additive. Since μ is Log-concave, the function $\log \overline{F}_m^+$ is concave. It is easy to check that every concave function defined on $\mathbb{R}^+$ and vanishing at 0 is sub-additive. This achieves the proof. $\square$

Corollary 56. *If $\mu \in \mathcal{P}(\mathbb{R})$ is Log-concave, then it satisfies the strong TCI with the cost function c defined by*

$$c(x,y) = \frac{1}{\kappa}\alpha_1\left(-\log\left(G_0\left(|x-y|/2\right)\right)\right), \quad \forall x,y \in \mathbb{R},$$

with $G_0(h) = 2\max\left(\overline{F}(h+m), F(-h+m)\right), \forall h \geq 0$.
Furthermore, if μ is symmetric, then μ satisfies the strong TCI with the cost function

$$c(x,y) = \frac{1}{\kappa}\alpha_1\left(-\log 2\overline{F}(|x-y|/2)\right), \quad \forall x,y \in \mathbb{R}.$$

Proof. Let μ_0 be the probability measure on $\mathbb{R}^+$ with cumulative distribution function $F_0 = 1 - G_0$. According to Proposition 55, $\mu_x^+ \prec_{\text{st}} \mu_0, \forall x \geq m$ and $\mu_x^- \prec_{\text{st}} \mu_0, \forall x \leq m$. Thus according to Corollary 54, μ satisfies the strong TCI with the cost function

$$c(x,y) = \frac{1}{\kappa}\alpha_1\left(-\log\left(G_0\left(|x-y|/2\right)\right)\right), \quad \forall x,y \in \mathbb{R}.$$

Now, if μ is symmetric, then $m = 0$ and $1 - F(x) = F(-x), \forall x \in \mathbb{R}$. Consequently, $G_0(h) = 2(1 - F(h))$ and the result follows. $\qquad\square$

4.2. Characterization of (strong) TCI for Log-concave measures.
In the sequel, $\mathcal{A}$ will be the class of all the function $\alpha : \mathbb{R} \to \mathbb{R}^+$ such that

- α is even,
- α is continuous, nondecreasing on $\mathbb{R}^+$ and $\alpha(0) = 0$,
- α is super-additive on $\mathbb{R}^+$: $\alpha(x+y) \geq \alpha(x) + \alpha(y), \forall x, y \geq 0$,
- α is quadratic near 0 : $\alpha(t) = t^2, \forall t \in [-1, 1]$.

One will say that μ satisfies the inequality $\mathbb{T}_\alpha(a)$ (resp. $\mathbb{ST}_\alpha(a)$) if μ satisfies the TCI (resp. the strong TCI) with the cost function $c(x,y) = \alpha(a(x-y)), \forall x, y \in \mathbb{R}$.

Theorem 57. *Let $\alpha \in \mathcal{A}$ and $\mu \in \mathcal{P}(\mathbb{R})$ a Log-concave distribution. The following propositions are equivalent*

(1) *There is some constant $a > 0$ such that μ satisfies the inequality $\mathbb{ST}_\alpha(a)$.*
(2) *There is some constant $b > 0$ such that $\int e^{\alpha(bx)}\, d\mu(x) < +\infty$.*

If $\alpha \in \mathcal{A}$ is convex then the same is true for TCI.

Proof.
$[(1) \Rightarrow (2)]$ If μ satisfies $\mathbb{ST}_\alpha(a)$, then according to Corollary 29, one has

$$\int e^{\alpha(az)}\, d\mu(z) < +\infty.$$

Hence, (2) holds with $b = a$.

$[(2) \Rightarrow (1)]$ According to Corollary 56, μ satisfies the strong TCI with the cost function $c(x,y) = \frac{1}{\kappa}\alpha_1\left(-\log\left(G_0\left(|x-y|/2\right)\right)\right)$, where $G_0(h) = 2\max\left(\overline{F}(h+m), F(-h+m)\right), \forall h \geq 0$. If there is some a such that

$$(58) \qquad\qquad \alpha_1\left(-\log G_0(|x|)\right) \geq \alpha(ax), \forall x \in \mathbb{R},$$

then μ satisfies the strong TCI with the cost function $\frac{1}{\kappa}\alpha(a|x-y|/2) \geq \alpha(a|x-y|/(2\kappa))$ (since $\kappa > 1$). Hence it is enough to prove (58). This latter condition is equivalent to

$$(59) \qquad 2\overline{F}(m+h) = \mu_m^+[h, +\infty) \leq e^{-\alpha_1^{-1}\circ\alpha(ah)}, \quad \forall h \geq 0$$

$$(60) \qquad 2F(m-h) = \mu_m^-[h, +\infty) \leq e^{-\alpha_1^{-1}\circ\alpha(ah)}, \quad \forall h \geq 0.$$

We will focus on the condition (59), the same proof will work for (60). Inequality (59) is equivalent to

$$\begin{cases} (i) & \mu_m^+[h, +\infty) \leq e^{-ah}, & \forall h \leq \frac{1}{a} \\ (ii) & \mu_m^+[h, +\infty) \leq e^{-\alpha(ah)}, & \forall h \geq \frac{1}{a} \end{cases}$$

Let us prove that (i) holds for some $a_0 > 0$ and *all* $h \geq 0$. Let $\varphi = \log \overline{F}$. The function φ is concave, so

$$\varphi(m+h) \leq \varphi(m) + \varphi_r'(m)h, \quad \forall h \geq 0,$$

where $\varphi_r'(m)$ is the right derivative of φ at point m. If $\varphi_r'(m) < 0$, then (i) holds with $a_0 = -\varphi_r'(m)$ and for all $h \geq 0$. Since φ is increasing, $\varphi_r'(m) \leq 0$. The function φ being concave, φ_r' is non-increasing. Consequently, if $\varphi_r'(m) = 0$, then $\varphi_r'(x) = 0$, for all $x \leq m$. This would imply that $\overline{F}$ is constant on $(-\infty, m]$, which is absurd.

It is clear that one can find a constant b_0 such that $\int e^{\alpha(b_0 z)} d\mu_m^+(z) < +\infty$. We end the proof applying the following technical result to μ_m^+. $\qquad\square$

Lemma 61. *Let ν be a probability measure on $\mathbb{R}^+$ such that*

$$\nu[h, +\infty) \leq e^{-a_0 h}, \quad \forall h \geq 0,$$

for some $a_0 > 0$. Let $\alpha \in \mathcal{A}$ and suppose that

$$\int e^{\alpha(b_0 z)} d\nu(z) \leq K,$$

for some $b_0 > 0$ and $K > 0$. Then, there is a constant $a > 0$ depending only on a_0, b_0 and K such that

$$\begin{cases} (i) & \nu[h, +\infty) \leq e^{-ah}, & \forall h \leq \frac{1}{a} \\ (ii) & \nu[h, +\infty) \leq e^{-\alpha(ah)}, & \forall h \geq \frac{1}{a} \end{cases}$$

Proof. Using Markov's inequality, one gets

$$K \geq 2e^{\alpha(b_0 h)}\nu[h, +\infty), \quad \forall h \geq 0.$$

Thus, using the super-additivity of α, one has

$$\nu[h, +\infty) \leq Ke^{-\alpha(b_0 h)} \leq \left[Ke^{-\alpha(b_0 h/2)}\right]e^{-\alpha(b_0 h/2)} \leq e^{-\alpha(b_0 h/2)},$$

as soon as $h \geq \frac{2}{b_0}\alpha^{-1}(\log K)$. Since $\nu[h, +\infty) \leq e^{-a_0 h}, \quad \forall h \geq 0$, it is now easy to check that (i) and (ii) hold with $a = \min(a_0, b_0/2, \left[\frac{2}{b_0}\alpha^{-1}(\log K)\right]^{-1})$. $\qquad\square$

4.3. **Links with modified Log-Sobolev inequalities.** Recall the definition of the entropy functional :

$$\text{Ent}_\mu(f) := \int f \log f \, d\mu - \int f \, d\mu \log \int f \, d\mu.$$

Definition 62. *Let $\beta : \mathbb{R} \to \mathbb{R}^+$ be an even convex function with $\beta(0) = 0$. One says that $\mu \in \mathcal{P}(\mathbb{R})$ satisfies the modified Logarithmic-Sobolev inequality $LSI_\beta(C,t)$ if*

$$\text{Ent}_\mu(f^2) \le C \int \beta\left(t\frac{f'}{f}\right) f^2 \, d\mu,$$

for all $f \in \mathcal{C}_c^1$ (the set of continuously differentiable functions having compact support).

Note that if $\beta(x) = x^2$, one recovers the classical Logarithmic-Sobolev inequality. The links between transportation cost inequalities and Logarithmic-Sobolev inequalities have been studied by several authors (see the works by Otto and Villani [16], Bobkov, Gentil and Ledoux [5] and more recently Gentil, Guillin and Miclo [9]). The usual point of view is to prove TCI using Log-Sobolev type inequalities. Here we will do the opposite and derive Log-Sobolev inequalities from TCIs. To this end we will use the following result.

Theorem 63. *Let $\alpha \in \mathcal{A}$ be a convex function. If $\mu = e^{-V} dx \in \mathcal{P}(\mathbb{R})$ with $V : \mathbb{R} \to \mathbb{R}$ a convex function satisfies the inequality $\mathbb{T}_\alpha(a)$, then it satisfies $LSI_{\alpha^*}\left(\frac{\lambda}{1-\lambda}, \frac{1}{a\lambda}\right)$, for all $\lambda \in (0,1)$, where α^* is the convex conjugate of α :*

$$\alpha^*(s) = \sup_{t \in \mathbb{R}} \{st - \alpha(t)\}, \quad \forall s \in \mathbb{R}.$$

Proof. The proof of Theorem 63 can be easily adapted from the one of Theorem 2.9 of [9]. The regularity issue mentioned by the authors during the proof, is irrelevant in our framework. Namely, in dimension one, the Brenier map is simply the monotone rearrangement map, and the regularity of this latter can be easily checked by hand. $\square$

The following result follows immediately from Theorems 57 and 63.

Theorem 64. *Let $\alpha \in \mathcal{A}$ be a convex function and $\mu = e^{-V} dx \in \mathcal{P}(\mathbb{R})$ with V convex. If $\int e^{\alpha(b|x|)} d\mu(x) < +\infty$, for some $b > 0$, then μ satisfies the inequality $LSI_{\alpha^*}(C,t)$ for some $C, t > 0$.*

Remark 65. *Recall that the function θ_p is defined by*

$$\theta_p(x) = \begin{cases} x^2 & \text{if } |x| \le 1 \\ \frac{2}{p}|x|^p + 1 - \frac{2}{p} & \text{if } |x| \ge 1 \end{cases}, \forall p \in [1,2]$$

In [9], I. Gentil, A. Guillin and L. Miclo proved that the measure $d\mu_p(x) = \frac{1}{Z_p} e^{-|x|^p} dx$ with $p \in [1,2]$ satisfies the inequality $LSI_{\theta_p^*}(C,t)$ for some $C, t > 0$.
Using classical tools, one can show that

$$\exists C, t > 0 \text{ s.t. } \mu \text{ satisfies } LSI_{\theta_p^*}(C,t) \quad \Rightarrow \quad \exists b > 0 \text{ s.t. } \int e^{\theta_p(ax)} d\mu(x) < +\infty.$$

Consequently, a Log-concave measure μ satisfies the inequality $LSI_{\theta_p^*}(C,t)$ if and only if there is some $b > 0$ such that $\int e^{\theta_p(ax)} d\mu(x) < +\infty$.

Suppose that $d\mu = e^{-V} dx$ with V a convex and symmetric function. It is tempting to take $\alpha = V$ in the above theorem. To do this one has to modify the potential V near 0. Define

$$\widetilde{V}(x) = \begin{cases} x^2 & \text{if } |x| \leq 1 \\ V(a_0 x) + 1 - V(a_0) & \text{if } |x| \geq 1 \end{cases}$$

Choosing $a_0 > 0$ such that $a_0 V'(a_0) = 2$ (which is always possible), one obtains a convex function. Furthermore, it is clear that one can find some $b > 0$ such that $\int e^{\widetilde{V}(bx)} d\mu(x) < +\infty$. Applying the above Theorem, one obtains the following result

Corollary 66. *With the above notations, μ satisfies the inequality $LSI_{\widetilde{V}^*}(C,t)$ for some $C, t > 0$.*

Remark 67. *In* [11]*, Gentil, Guillin and Miclo have obtained the preceding Corollary under the following additional assumption on V :*

$$\exists \varepsilon \in [0, \frac{1}{2}], \exists M > 0, \forall x \geq M, \quad (1 + \varepsilon)V(x) \leq xV'(x) \leq (2 - \varepsilon)V(x).$$

This hypothesis seems to be useless.

5. Characterization of strong TCI on a larger class of probabilities

In this section, we give a characterization of strong TCI for probability measures belonging to a certain class $\mathcal{Lip}_\sharp \mu_1$ which we shall now define.

5.1. The Lipschitz images of the exponential measure.

Definition 68 (The class $\mathcal{Lip}_\sharp \mu_1$)*. The set of all probability measures μ on $\mathbb{R}$, with no atom and full support, such that the monotone rearrangement map transporting the exponential measure $\frac{1}{2} d\mu_1(x) = e^{-|x|} dx$ on μ is Lipschitz is denoted by $\mathcal{Lip}_\sharp \mu_1$.*

The following Proposition describes the elements of $\mathcal{Lip}_\sharp \mu_1$.

Proposition 69. *Let $\mu \in \mathcal{P}(\mathbb{R})$ with no atom and full support and let T be the monotone rearrangement map between μ_1 and μ. For all $a > 0$, let ν_a be the one sided exponential distribution with parameter a, that is $d\nu_a(z) = ae^{-ay} \mathbb{1}_{[0,+\infty)}(y) \, dy$.*

The following assertions are equivalent

(1) *The map T is $\frac{1}{a}$-Lipschitz.*
(2) *The probability measures $(\mu_x^+)_{x \geq m}$ and $(\mu_x^-)_{x \leq m}$ are stochastically dominated by the exponential measure ν_a : $\mu_x^+ \preceq \nu_a, \forall x \geq m$ and $\mu_x^- \prec_{\mathrm{st}} \nu_a, \forall x \leq m$. In other word, one has*

$$(70) \qquad \sup_{x \geq m} \mu_x^+[h, +\infty) \leq e^{-ah}, \quad \forall h \geq 0$$

$$(71) \qquad \sup_{x \leq m} \mu_x^-[h, +\infty) \leq e^{-ah}, \quad \forall h \geq 0$$

If μ is of the form $d\mu(z) = e^{-V(z)} dz$ where V is a continuous function, then T is $\frac{1}{a}$-Lipschitz if and only if

$$(72) \qquad A^+ := \sup_{x \geq m} \overline{F}(x) e^{V(x)} \leq \frac{1}{a} \qquad and \qquad A^- := \sup_{x \leq m} F(x) e^{V(x)} \leq \frac{1}{a}$$

Furthermore, if V is of class $\mathcal{C}^1$, a sufficient condition for A^+ and A^- to be finite is that

$$(73) \qquad \liminf_{x \to +\infty} V' > 0 \qquad and \qquad \limsup_{x \to -\infty} V' < 0.$$

Proof. It is easy to see that the map T is $\frac{1}{a}$-Lipschitz if and only if

$$(74) \qquad T^{-1}z - T^{-1}y \geq a(z - y), \quad \forall z \geq y.$$

This is equivalent to

$$T^{-1}(x+h) - T^{-1}x \geq ah, \quad \forall x \geq m, \forall h \geq 0 \quad \text{and} \quad T^{-1}x - T^{-1}(x-h) \geq ah, \quad \forall x \leq m, \forall h \geq 0.$$

Using the fact that $T^{-1}(z) = \begin{cases} -\log(2(1 - F(z))) & \text{if } z \geq m \\ \log(2F(z)) & \text{if } z \leq m \end{cases}$, one sees immediately that these conditions are equivalent to (70) and (71).

If $d\mu(z) = e^{-V(z)}\, dz$ with a continuous V, T^{-1} is differentiable. Observe that (74) means that $z \mapsto T^{-1}z - az$ is nondecreasing and this is equivalent to $\sup_{z \in \mathbb{R}} \frac{dT^{-1}}{dz}(z) \leq a$. Computing $\frac{dT^{-1}}{dz}$, one obtains immediately (72).

Finally, let us show that the condition $\liminf_{x \to +\infty} V' > 0$ implies that A^+ is finite. Under this assumption, there is $v_0 > 0$ and $z_0 > m$ such that for all $z \geq z_0$, one has $V'(z) \geq v_0$. If $z \geq z_0$, one thus has

$$e^{-V(z)} = \int_z^{+\infty} V'(y) e^{-V(y)}\, dy \geq v_0 \int_z^{+\infty} e^{-V(y)}\, dy = v_0(1 - F(y)).$$

So, $\sup_{z \geq z_0}(1 - F(z))e^{V(z)} \leq \dfrac{1}{v_0}$. Since $\sup_{m \leq z \leq z_0}(1 - F(z))e^{V(z)} < +\infty$, one concludes that $A^+ < +\infty$. The same reasoning shows that the condition $\limsup_{z \to -\infty} V' < 0$ implies $A^- < +\infty$. $\qquad\square$

Remark 75 ($\mathcal{L}ip_\sharp \mu_1$ and the Poincaré inequality).

(1) *According to Corollary 34, one concludes that a sufficient condition for a probability measure to satisfy the inequality $\mathbb{ST}_{\alpha_1}(a)$ for some constant a is that μ belongs to $\mathcal{L}ip_\sharp \mu_1$.*

(2) *According to Theorem 39, a probability measure satisfies the inequality $\mathbb{ST}_{\alpha_1}(a)$ for some constant $a > 0$ if and only if it satisfies the Poincaré inequality :*

$$(76) \qquad \mathrm{Var}_\mu(f) \leq C \int (f')^2\, d\mu, \quad \forall f,$$

for some constant $C > 0$. Examples of probability measures not belonging to $\mathcal{L}ip_\sharp \mu_1$ but satisfying a Poincaré inequality are known. Thus our perturbation method failed to completely characterize the inequality $\mathbb{ST}_{\alpha_1}$. Nevertheless, the above mentioned counterexamples are rather pathological, and for a large class of probability measures, μ satisfies Poincaré if and only if $\mu \in \mathcal{L}ip_\sharp \mu_1$. This explained in the next proposition.

Definition 77 (Good potentials). *The class $\mathcal{V}$ will be the set of all the functions $f : \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^2$ such that*

(1) *there is $x_o > 0$ such that $f' > 0$ on $(-\infty, -x_o] \cup [x_o, +\infty)$,*

(2) $\dfrac{f''(x)}{f'^2(x)} \xrightarrow[x\to\pm\infty]{} 0.$

Proposition 78. *Let $d\mu = e^{-V}\,dx$ with $V \in \mathcal{V}$, then*

$$\mu \text{ satisfies Poincaré} \Leftrightarrow \liminf_{x\to+\infty} V'(x) > 0 \text{ and } \limsup_{x\to-\infty} V'(x) < 0$$

$$\Leftrightarrow \mu \in \mathcal{L}ip_\sharp \mu_1.$$

Proof. According to the celebrated Muckenhoupt criterion, a probability measure $d\mu = e^{-V}dx$ with a continuous V satisfies (76) for some constant $C > 0$ if and only if

(79) $\quad D^+ := \sup_{x\geq m} \overline{F}(x) \cdot \int_m^x e^V(y)\,dy < +\infty \quad$ and $\quad D^- := \sup_{x\leq m} F(x) \cdot \int_x^m e^V(y)\,dy < +\infty.$

Applying Proposition 86, one shows that

$$\overline{F}(x) \cdot \int_m^x e^V(y)\,dy \sim_{x\to+\infty} \frac{1}{V'^2(x)} \quad \text{and} \quad \overline{F}(x) \cdot e^{V(x)} \sim_{x\to+\infty} \frac{1}{V'(x)}.$$

From this one easily conclude that A^+ and D^+ are finite if and only if $\liminf\limits_{x\to+\infty} V'(x) > 0$. $\quad\square$

5.2. **Characterization of strong TCI on $\mathcal{L}ip_\sharp \mu_1$.**

Theorem 80. *Let $\mu \in \mathcal{L}ip_\sharp \mu_1$ and $\alpha \in \mathcal{A}$. The following assertions are equivalent :*

(1) *There is some $a > 0$ such that μ satisfies $\mathbb{ST}_\alpha(a)$.*
(2) *There is some $b > 0$ such that*

$$K^+ := \sup_{x\geq m} \int e^{\alpha(bz)}\,d\mu_x^+(z) < +\infty \quad \text{and} \quad K^- := \sup_{x\leq m} \int e^{\alpha(bz)}\,d\mu_x^-(z) < +\infty.$$

Proof.
$[(1) \Rightarrow (2)]$ According to Proposition 29, if μ satisfies $\mathbb{ST}_\alpha(a)$, then

$$\int e^{\alpha(bz)}\,d\mu_x^+(z) \leq \frac{1}{\mu(-\infty, x]} + 1 \leq 3, \quad \forall x \geq m$$

$$\int e^{\alpha(bz)}\,d\mu_x^-(z) \leq \frac{1}{\mu[x, +\infty)} + 1 \leq 3, \quad \forall x \leq m.$$

Thus (2) holds true for $b = a$.

$[(2) \Rightarrow (1)]$ According to Corollary 50, if there is some $a > 0$ such that

(81) $$\sup_{x\geq m} \mu_x^+[h, +\infty) \leq e^{-\alpha_1^{-1}\circ\alpha(ah)}, \quad \forall h \geq 0$$

(82) $$\sup_{x\leq m} \mu_x^-[h, +\infty) \leq e^{-\alpha_1^{-1}\circ\alpha(ah)}, \quad \forall h \geq 0,$$

then μ satisfies the strong TCI with the cost function $\frac{1}{\kappa}\alpha(a|x - y|/2) \geq \alpha(a|x - y|/(2\kappa))$. Hence it is enough to prove (81) and (82).

Let us prove (81) (the proof of (82) will be the same). To prove (81), it is enough to find $a > 0$ such that

(83) $$\begin{cases} \mu_x^+[h, +\infty) \leq e^{-ah}, & \forall h \leq \frac{1}{a} \\ \mu_x^+[h, +\infty) \leq e^{-\alpha(ah)}, & \forall h \geq \frac{1}{a} \end{cases},$$

holds for all $x \geq m$. Since $\mu \in \mathcal{L}ip_\sharp \mu_1$, there is $a_0 > 0$ such that

$$\sup_{x \geq m} \mu_x^+[h, +\infty) \leq e^{-a_0 h}, \quad \forall h \geq 0,$$

and by hypothesis there is some $b_0 > 0$ such that $K^+ := \sup_{x \geq m} \int e^{\alpha(b_0 z)} d\mu_x^+(z) < +\infty$. To conclude it suffices to apply Lemma 61 with $\nu = \mu_x^+$ and $K = K^+$. It provides us a constant $a > 0$ depending only on a_0, b_0 and K^+ such that (83) holds true for all $x \geq m$. $\qquad \square$

5.3. Tractable sufficient condition for good probability measures.

Theorem 84. *Let $d\mu = e^{-V} dx$ with $V \in \mathcal{V}$ and $\alpha \in \mathcal{A} \cap \mathcal{V}$. If $\mu \in \mathcal{L}ip_\sharp \mu_1$ and if*

$$(85) \qquad \exists \lambda > 0 \ such \ that \ \limsup_{u \to \pm\infty} \frac{\alpha'(\lambda u)}{V'(u+m)} < +\infty,$$

where m is the median of μ, then μ satisfies the inequality $\mathbb{ST}_\alpha(a)$ for some $a > 0$.

To prove this theorem, we will use the following Lemma.

Lemma 86. *Let $\Phi \in \mathcal{V}$, then*

$$\int_x^{+\infty} e^{-\Phi(t)} \, dt \sim \frac{e^{-\Phi(x)}}{\Phi'(x)} \quad and \quad \int_0^x e^{-\Phi(t)} \, dt \sim \frac{e^{\Phi(x)}}{\Phi'(x)}, \quad as \ x \ goes \ to \ +\infty.$$

Proof. See Corollary 6.4.2 of [1]. $\qquad \square$

Proof of Theorem 84. Let $\tilde{\mu} = \mathcal{L}(X - m)$, where X is a random measure with law μ. The density of $\tilde{\mu}$ with respect to Lebesgues measure is $e^{-\widetilde{V}}$, with $\widetilde{V}(x) = V(x+m), \forall x \in \mathbb{R}$. As $x \mapsto x + m$ is 1-Lipschitz, it follows from Corollary 34 that μ satisfies $\mathbb{ST}_\alpha(a)$ if and only if $\tilde{\mu}$ satisfies $\mathbb{ST}_\alpha(a)$. Observe that $\tilde{\mu} \in \mathcal{L}ip_\sharp \mu_1$. According to Theorem 80, to prove that $\tilde{\mu}$ satisfies $\mathbb{ST}_\alpha(a)$ for some $a > 0$, it suffices to prove that there is $b > 0$ such that

$$K^+(b) = \sup_{x \geq 0} \int e^{\alpha(bu)} d\tilde{\mu}_x^+(u) < +\infty \qquad and \qquad K^-(b) = \sup_{x \leq 0} \int e^{\alpha(bu)} d\tilde{\mu}_x^-(u) < +\infty,$$

where $\tilde{\mu}_x^+ = \mathcal{L}(\widetilde{X} - x | \widetilde{X} \geq x)$ and $\tilde{\mu}_x^- = \mathcal{L}(x - \widetilde{X} | \widetilde{X} \leq x)$ with $\widetilde{X}$ of law $\tilde{\mu}$.

The proof of $K^-(b) < +\infty$ being similar, we will only prove that $K^+(b) < +\infty$ for some $b > 0$. One can suppose without restriction that $\lambda = 1$ in (85). Define

$$K(b, x) = \int e^{\alpha(bt)} d\tilde{\mu}_x^+(t)$$

$$= \frac{\int_x^{+\infty} e^{\alpha(b(u-x))} e^{-\widetilde{V}(u)} \, du}{\int_x^{+\infty} e^{-\widetilde{V}(u)} \, du}, \qquad \forall x \geq 0, \forall b \geq 0.$$

Let us show that there is $k \in \mathbb{N}^*$ such that $K(1/k, x) < +\infty$ for all $x \geq 0$. Since α is super-additive and non decreasing, one gets

$$K(1/k, x) \leq \frac{\int_0^{+\infty} e^{\frac{1}{k}\alpha(u)} e^{-\widetilde{V}(u)} \, du}{\int_x^{+\infty} e^{-\widetilde{V}(u)} \, du}.$$

Since $\limsup\limits_{u\to+\infty}\dfrac{\alpha'(u)}{\widetilde{V}'(u)} < +\infty$, there are $M > 0$ and $u_0 > 0$ such that $\widetilde{V}'(u) \geq M\alpha'(u)$, for all $u \geq u_0$. Integrating yields

$$\widetilde{V}(u) \geq M\alpha(u) + C, \quad \forall u \geq u_0,$$

where C is a constant. Let k_0 be a positive integer such that $k_0 \geq \frac{2}{M}$. Then one has

$$e^{\frac{1}{k_0}\alpha(u)-\widetilde{V}(u)} \leq e^{-\frac{M}{2}\alpha(u)-C} \leq e^{-\frac{M\alpha(1)}{2}(u-1)-C}, \quad \forall u \geq u_0$$

where the last inequality follows from the inequality $\alpha(u) \geq \alpha(1)(u-1), \forall u \geq 0$ which is easy to establish. From this follows easily that $K(1/k_0, x) < +\infty$ for all $x \geq 0$.

Now, let us show that $\sup_{x\geq 0} K(1/k_0, x) < +\infty$. Since the map $x \mapsto K(1/k_0, x)$ is continuous, it suffices to check that $\limsup\limits_{x\to+\infty} K(1/k_0, x) < +\infty$. Using the super-additivity of α, one gets

$$\alpha(u - x) \leq \alpha(u) - \alpha(x), \quad \forall u \geq x \geq 0.$$

So

$$K(1/k_0, x) \leq e^{-1/k_0\alpha(x)} \frac{\int_x^{+\infty} e^{1/k_0\alpha(u)-\widetilde{V}(u)}\, du}{\int_x^{+\infty} e^{-\widetilde{V}(u)}\, du}.$$

Applying Lemma 86, with $\Phi = \widetilde{V} - 1/k_0\alpha$, and then with $\Phi = \widetilde{V}$, one gets

$$e^{-1/k_0\alpha(x)} \frac{\int_x^{+\infty} e^{1/k_0\alpha(u)-\widetilde{V}(u)}\, du}{\int_x^{+\infty} e^{-\widetilde{V}(u)}\, du} \sim e^{-1/k_0\alpha(x)} \frac{e^{1/k_0\alpha(x)-\widetilde{V}(x)}}{\widetilde{V}'(x) - 1/k_0\alpha'(x)} e^{\widetilde{V}(x)}\widetilde{V}'(x) = \frac{1}{1 - 1/k_0\frac{\alpha'(x)}{\widetilde{V}'(x)}}.$$

Since $\limsup\limits_{x\to+\infty} \dfrac{1}{1 - 1/k_0\frac{\alpha'(x)}{\widetilde{V}'(x)}} < +\infty$, one deduces that $\limsup\limits_{x\to+\infty} K(1/k_0, x) < +\infty$, which ends the proof. $\qquad\square$

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