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# Structure of optimal pipe networks subject to a global constraint

Marc Durand

*Matière et Systèmes Complexes, UMR 7057 CNRS & Université Paris 7 - Denis Diderot,  
Tour 33/34 - 2ème étage - case 7056, 2 Place Jussieu - 75251 Paris Cedex 05, France*  
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## Abstract

Structure of pipe networks minimizing the total energy dissipation rate is studied analytically. Among all the pipe networks possible to build with a given amount of total pipe volume (or pipe lateral surface area), the network minimizing the dissipation rate is shown to be loopless. Furthermore, such an optimal network is shown to contain at most  $N - 2$  nodes, in addition to the  $N$  sources plus sinks that it connects. These results are valid whether the possible locations for the additional nodes are chosen freely or from a set of nodes (such as points of a grid). Applications of these results to various physical situations and to efficient computation of optimal pipe networks are also discussed.

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Finding the most efficient transport network is an issue arising in a wide array of different contexts [1, 2]. One can cite, among others, the water, gas and power supply of a city, telecommunication networks, railway or car traffic, and more recently the design of labs-on-chips or microfluidic devices. Besides, this problem also appears in theoretical works intending to describe the architecture of vascular systems of organisms [3, 4]. Generally speaking, consider a set of sources and sinks embedded in a two- or three-dimensional space, their respective number and locations being fixed. Flow rates coming in through each source and going out through each sink are also given. The challenge consists in interconnecting them via possible intermediate junctions, referred to as *additional nodes*, in the most efficient way. That is, to minimize a cost function of general form  $\sum_k w_k f(i_k)$ , where the sum is carried on all the links that compose the network.  $w_k$  is the “weight” associated with the  $k$ th link, and  $f$  is some function of the flow rate  $i_k$  carried by this link (usually,  $f$  is some power law:  $f(i_k) = |i_k|^\gamma$ , with  $\gamma \geq 0$ ). The flow rates are not independent but must satisfy a conservation law (Kirchhoff’s first law) at every source, sink, and additional node of the network.

In the present letter, the structure of pipe networks minimizing the *dissipation rate*  $U = \sum_k r_k i_k^2$  is studied, where the weight  $r_k$  of each pipe is the “flow resistance”, defined as:

$$r_k = \frac{\rho l_k}{s_k^m}, \quad (1)$$

$\rho$  being some positive constant,  $l_k$  and  $s_k$  the respective length and cross-sectional area of each pipe, and  $m$  a parameter characterizing the flow profile (examples of flows are given hereafter). It is shown that – among all the pipe networks possible to build with a given value of total pipe volume (or total lateral surface area) – the network minimizing  $U$  is necessarily loopless. As a consequence, it is also shown that the number of additional nodes in such an optimal network cannot exceed  $N - 2$ , where  $N$  is

the number of initial nodes (sources plus sinks). These results are valid whether the positions of the additional nodes are chosen in a set or free to be adjusted for a further minimization of  $U$ .

Since the minimization is done under a global constraint on the total pipe volume, or pipe surface area, it is convenient to introduce the function  $C_n = \sum_k l_k s_k^n$ , which represents the total volume ( $n = 1$ ) or total lateral surface area ( $n = 1/2$ ) of the network [5]. For most flows encountered in physics,  $m \geq 1$ , ensuring that  $m \geq n$ . In many physical situations also,  $i_k$  derives from a potential function (electrical potential, pressure, concentration, temperature,...) so that potential difference  $v_k$ , flow rate  $i_k$ , and resistance  $r_k$  of the  $k$ th pipe are related by the Ohm’s law  $v_k = r_k i_k$ .

Consider a network of given topology, connecting the sources to the sinks, and whose total pipe volume/surface area is equal to  $C_n$ . The dissipation rate  $U$  is a function of the cross-sections and lengths of all the pipes that compose the network. Pipe cross-sections can be adjusted to minimize the dissipation rate, while preserving the value of  $C_n$ . It has been shown in a previous study [4] that, as a result of this optimization of the network geometry, the flow rate  $i_k$  carried by each pipe in the optimal network scales with its cross-sectional area  $s_k$  as:

$$|i_k| = \kappa I s_k^{(m+n)/2}, \quad (2)$$

where  $I$  is the total flow rate coming in through the sources (and going out through the sinks), and  $\kappa$  is a function depending on  $m$ ,  $n$ , and the geometry and topology of the network. Consequently, the dissipation rate in such a network can be rewritten as:

$$U = (\kappa I)^2 C_n. \quad (3)$$

Similarly, pipe lengths may also be optimized in order to minimize  $U$  (while preserving  $C_n$ ). Actually, coordinates of the additional nodes are the appropriate independent optimization parameters. As a result, the following vector balance is also satisfied at every junction

of the network with optimized cross-sections *and* node locations [4]:

$$\sum_k s_k^n \mathbf{e}_k = \mathbf{0}, \quad (4)$$

where  $\mathbf{e}_k$  is the outward-pointing unit vector along each adjoining pipe. In some situations however, the positions of the additional nodes cannot be freely adjusted, but must be chosen from a set of nodes (such as points of a grid, or some particular cities of a country). In these cases Eq. 4 is usually not satisfied.

Conditions 2 and 4 do not uniquely determine the structure of the optimal network: they do not restrict the number of pipes and junctions, or give indication on the presence or absence of loops in the network. Besides, these are properly necessary conditions for *extrema* of  $U$ . It can be shown (see Appendix) that these extrema are either local minima, or saddle points. Here, it is proven that condition 2 cannot lead to a local minimum of  $U$  in a network containing loops. As a consequence, it is shown that, starting from any network containing loops, it is always possible to build a new loopless network with lower dissipation rate (and with a same value of  $C_n$ ).

Consider a network satisfying Eq. 2 and containing a loop. To go from a given junction  $\mathcal{A}$  to another junction  $\mathcal{B}$  of the loop, there are two different paths, noted  $(\alpha)$  and  $(\beta)$ , as depicted on Fig. 1a. Let us make a shift of material, in such a way that flows in path  $(\alpha)$  tend to be strengthened in one direction (say  $\mathcal{A}$  to  $\mathcal{B}$ ) and flows in path  $(\beta)$  tend to be strengthened in the opposite direction ( $\mathcal{B}$  to  $\mathcal{A}$ ). That is, the new cross-sectional areas  $s'_k$  in the loop are defined as:  $s'_k = s_k^{(m+n)/2} \pm s_0^{(m+n)/2}$  with, for path  $(\alpha)$ , a plus sign if flow rate in pipe  $(i, j)$  is in direction  $\mathcal{A} \rightarrow \mathcal{B}$  and a minus sign if the flow rate is in opposite direction, while signs are inverted for path  $(\beta)$  (see Fig. 1b).  $s_0$  is a positive number smaller than any cross-sectional area  $s_k$  of the original loop. Cross-sections outside the loop remain unaltered ( $s'_k = s_k$ ). Note that flows in a loop cannot turn all clockwise, or counter-clockwise (otherwise, the potential difference  $V_{\mathcal{A}} - V_{\mathcal{B}}$  between nodes  $\mathcal{A}$  and  $\mathcal{B}$ , and thus the flow rates in the loop, would trivially be zero). This guarantees that the cross-sectional areas of the loop did not all simultaneously increase (or decrease).

Such a variation of cross-sectional areas implies a redistribution of flows in the entire network. Let  $\{i'_k\}$  be the new distribution of flow rates in this new geometry,  $r'_k = \rho l_k / s_k^m$  the new resistances, and  $U' = \sum_k r'_k i_k'^2$  the new dissipation rate. Although we do not know the values of the new flow rates, an upper bound on the new dissipation rate  $U'$  can be established, using Thomson's principle [6, 7]. This principle (which happens to be a theorem) states that – among all possible flow rate distributions  $\{j_k\}$  which satisfy conservation of flow at every source, sink, and additional nodes – the actual flow rate

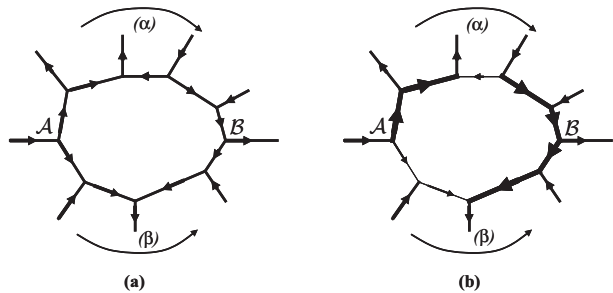


FIG. 1: Shift of material in a loop of the network. (a): the original loop, where flow directions in each pipe are indicated with arrows. (b): the same loop, where cross-sectional area of a pipe is increased when the direction of its carried flow is  $\mathcal{A} \rightarrow \mathcal{B}$  along path  $(\alpha)$  or  $\mathcal{B} \rightarrow \mathcal{A}$  along path  $(\beta)$ , and decreased otherwise. The other cross-sectional areas in the network remain unaltered.

distribution (i.e.: the one satisfying Ohm's law) is the one that makes the function  $\sum_k r_k j_k^2$  an absolute minimum. Consider in particular the flow rate distribution defined as:  $j_k = i_k + i_0$  along path  $(\alpha)$ ,  $j_k = i_k - i_0$  along path  $(\beta)$ , and  $j_k = i_k$  for any pipe outside the loop.  $i_0$  is some positive number, and  $\{i_k\}$  is the actual distribution in the original network, the sign of  $i_k$  being defined in both paths as positive if directed from  $\mathcal{A}$  to  $\mathcal{B}$ . The distribution  $\{j_k\}$  satisfies flow rate conservation at every source, sink, and additional nodes, since the distribution  $\{i_k\}$  does. Besides, by choosing  $i_0 = \kappa I s_0^{(m+n)/2}$  and using Eq. 2, the flow rate distribution  $j_k$  can be rewritten:  $j_k = \text{sgn}(i_k) \kappa I s_k^{(m+n)/2}$ . Thus, according to Thomson's principle:

$$U' \leq (\kappa I)^2 C'_n, \quad (5)$$

with  $C'_n = \sum_k l_k s_k^m$ . Let us now compare the new value of pipe volume/surface area  $C'_n$  with the original value  $C_n$ . This can be done by studying the variation of  $C'_n$  with  $s_0$ . With no difficulty, we obtain the derivative of this function with respect to  $x = s_0^{(m+n)/2}$ :

$$\begin{aligned} \frac{\partial C'_n}{\partial x} &= \frac{2n}{m+n} \sum_{\text{path } \alpha} l_k \left( s_k^{(m+n)/2} + x \right)^{(n-m)/(n+m)} \\ &\quad - \frac{2n}{m+n} \sum_{\text{path } \beta} l_k \left( s_k^{(m+n)/2} - x \right)^{(n-m)/(n+m)}. \end{aligned} \quad (6)$$

Since  $m \geq n$ ,  $\frac{\partial C'_n}{\partial x}$  is a decreasing function of  $x$ , and then:

$$\frac{\partial C'_n}{\partial x} \leq \left( \frac{\partial C'_n}{\partial x} \right)_{x=0}. \quad (7)$$

Therefore, if the bound in inequality 7 is negative,  $C'_n$  is a decreasing function of  $x$ . Using Eqs 1 and 2, this

bound can be rewritten as:

$$\left(\frac{\partial C'_n}{\partial x}\right)_{x=0} = \frac{2n}{m+n} \left( \sum_{path \ \alpha} r_k |i_k| - \sum_{path \ \beta} r_k |i_k| \right). \quad (8)$$

We could have chosen to reinforce flows in direction  $\mathcal{B} \rightarrow \mathcal{A}$  in path ( $\alpha$ ), and  $\mathcal{A} \rightarrow \mathcal{B}$  in path ( $\beta$ ) instead, which comes to swapping ( $\alpha$ ) and ( $\beta$ ) in the calculations above. Inequalities 5 and 7 would still be satisfied for this new geometry, but this time with an opposite sign for  $\left(\frac{\partial C'_n}{\partial x}\right)_{x=0}$  (see Eq. 8). So, necessarily  $\left(\frac{\partial C'_n}{\partial x}\right)_{x=0} \leq 0$  for one of the two shifts, and  $C'_n$  is a decreasing function of  $s_0$  for this particular shift, implying that  $C'_n(s_0) \leq C'_n(0) = C_n$ . From Eqs. 3 and 5, we obtain that the corresponding dissipation rate  $U'$  is lower too:  $U'(s_0) \leq U$  [8].

Afterwards, the total volume/surface area can be increased up to its original value  $C_n$  by increasing any cross-sectional areas in the network. This will imply a further decrease in  $U$  [9]. Thus, we found a small perturbation of the cross-sections such that the dissipation is reduced for a fixed value of  $C_n$ . This proves that condition 2 cannot lead to a local minimum of  $U$  in a network containing loops. Moreover, we can apply the reasoning above with increasingly large values of  $s_0$ , until eventually one of the pipe in the loop has a zero cross-sectional area, so one of the paths is cut of. Possible dead branches can be removed, the equivalent material being shifted to the rest of the network by increasing any other cross-sectional areas again, so that the constraint stays at its initial value while the dissipation rate is subjected to a further decrease. We can repeat the same procedure and eliminate all the duplicate paths until there are no loops in the network. The argument holds even in case of overlapping loops (that is, loops having pipes in common), and more generally for any topology of the original network. Therefore, we conclude that the architecture of the network minimizing  $U$  is necessarily loopless. Note that we do not make use of condition 4 throughout the reasoning, so the demonstration is valid whether or not the positions of additional nodes are optimization parameters.

It must be mentioned that the absence of loops in the least dissipative network has already been speculated, but not formally proved, in the particular case of a constrained total pipe volume [10]. Besides, in order to make connection with recent studies on networks minimizing the cost function  $\sum_k w_k |i_k|^\gamma$  [11, 12], let us rewrite the dissipation rate of a network with optimized cross-sections as  $U = (\kappa I)^\delta \sum_k l_k |i_k|^\gamma$ , where  $\delta = 2m/(m+n)$  and  $\gamma = 2n/(m+n) \leq 1$  (this expression was obtained by using Eqs. 2 and 3). Banavar *et al.* [11] showed that the flow rate distribution satisfying the conservation law at every node, source and sink, and minimizing the cost function  $\sum_k w_k |i_k|^\gamma$  is loopless when  $\gamma < 1$ . In that study however, the whole network struc-

ture (i.e.: its topology and geometry) and the weight  $w_k$  of every link are set. The only optimization parameters are the flow rates  $i_k$  (they do not necessarily obey to Ohm's law). Xue *et al.* [12] obtained a similar result with the cost function  $\sum_k l_k |i_k|^\gamma$ , although the topology of the network is not necessarily settled, since the number and location of additional nodes are free to vary. However, this cost function differs from  $U$ , since  $\kappa$  is not a constant, but depends on the geometry and topology of the network. Therefore, our result on the absence of loops cannot be deduced from those previous studies.

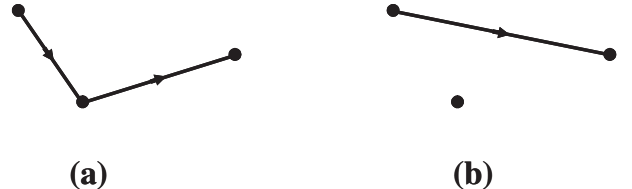


FIG. 2: (a): the two adjoining pipes of a two-fold junction carry flows in opposite directions in order to satisfy flow rate conservation. (b): we can favorably replace the two adjoining pipes with a straight one: since the total channel length is shortened, the dissipation rate will be decreased for a fixed value of  $C_n$ .

Let us now show that the number of additional nodes is at most  $N - 2$  in the optimal network, where  $N$  is the total number of sources plus sinks. This result limits the number of possible topologies for the optimal network. Note first that a loopless network (or tree) has one more node than it has links. So, the number of links in a network with  $A$  additional nodes is  $N + A - 1$ . Since each link has two ends, the number of “incident lines”, summed over all the nodes, is  $2(N + A - 1)$ . This number can be evaluated differently: let  $N_p$  be the number of sources or sinks with  $p$  incident lines. Since each source or sink is linked to the rest of the tree, the smallest value of  $p$  for which  $N_p$  has a nonzero value is 1, so  $\sum_{p \geq 1} p N_p = N$ . Similarly, let  $A_p$  be the number of additional nodes with  $p$  incident lines. By definition, two-fold junctions can exist only if its two links are not parallel. Such junctions cannot exist whenever their locations are optimization parameters (since condition 4 must be satisfied). Two-fold junctions could *a priori* exist if their positions cannot be freely adjusted. However, they can be favorably (i.e.: with no increase of  $U$  and  $C_n$ ) removed and their two adjoining pipes replaced with a straight one, the way it is depicted in Fig. 2. Thus, the smallest value of  $p$  for which  $A_p$  is not zero is  $p = 3$  in both cases, and the total number of incident lines is:  $\sum_{p \geq 1} p N_p + \sum_{p \geq 3} p A_p$ . Comparing these two expressions for the number of incident lines, and considering that  $\sum_{p \geq 1} p N_p \geq N$  and  $\sum_{p \geq 3} p A_p \geq 3A$ , it appears that:

$$A \leq N - 2, \quad (9)$$

which was to be proven.

Because of the broad definition of the flow resistance (Eq. 1), the results presented in this letter can be applied in various situations. For example, the  $m = 1$  case can correspond to electrical current in wires, liquid flow in porous conducts, mass or heat diffusion in bars (provided that for the latter the bar lateral surface is insulated). The  $m = 2$  case corresponds to the laminar Poiseuille flow in hollow pipes. Minimization can be done under a fixed lateral surface area ( $n = 1/2$ ) if one wants to save the material required to build the hollow pipes, or a fixed volume ( $n = 1$ ), if one wants to preserve the amount of liquid flowing through the network.

Unfortunately, the results presented in this letter do not give insights into the way of building the optimal network practically, or even into the uniqueness of such an optimal network. In fact, as for the Steiner tree problem - which consists in finding the tree of minimal length interconnecting a set of given points- this problem is likely to be NP-hard, meaning that the solution cannot be found without an exhaustive search of all the possible topologies. However, the NP-hardness does not exclude the possibility of establishing basic properties on the geometry and topology of Steiner trees [13]. Similarly, we were able to address features on the structure of pipe networks minimizing the total dissipation rate under a global constraint. Specifically, the upper bound on the number of additional nodes restricts the number of possible topologies for the optimal network(s). These results make possible the conception of efficient algorithms for computing the optimal pipe network problem [12]. In many situations however, and especially for vascular systems of organisms, the capacity of the network to resist random injuries may also play a key role in its design. Obviously, a reticulate network containing redundant paths is more adapted than an arborescent one for that purpose. Therefore, it is sometimes essential to look for a compromise between optimization of flow and robustness of the network. Definitely more works need to be conducted in that direction.

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## APPENDIX

In this appendix we prove that conditions 2 and 4 cannot lead to a local maximum of  $U$ . It is more convenient to show first that for a fixed value of  $U$ , a small deviation from condition 4 can lead to an increase of  $C_n$ . Suppose that the location of a given node in the network is moved by a small amount  $\delta\mathbf{r}$ . The new length of each of the adjoining links can be easily evaluated as:  $l'_k = l_k \left(1 - 2\mathbf{e}_k \cdot \delta\mathbf{r}/l_k + (\delta\mathbf{r}/l_k)^2\right)^{1/2}$  (where  $l_k$  is

its original length). Now let us adapt the cross-sectional areas such that each individual resistance is unchanged:  $r'_k = r_k$ . Thus, the dissipation rate is also unchanged, while the value of the total volume (or surface area) is now  $C'_n = \sum s_k^n l_k^{(m+n)/m} / l_k^{n/m}$ , which can be expanded as:

$$C'_n = C_n - \frac{m+n}{m} \sum_j s_j^n \mathbf{e}_j \cdot \delta\mathbf{r} + \frac{m+n}{2m} \sum_j \frac{s_j^n}{l_j} \left( |\delta\mathbf{r}|^2 - \frac{m-n}{m} (\mathbf{e}_j \cdot \delta\mathbf{r})^2 \right) + O(|\delta\mathbf{r}|^3). \quad (10)$$

Hence, when condition 4 is satisfied, we obtain that  $C'_n \geq C_n$  (with the assumption that  $m \geq n$ ). We can then decrease  $C'_n$  down to  $C_n$  by reducing some cross-sectional areas. This can only imply an increase in  $U$ . Thus, we found a small perturbation from the extremum defined by Eqs. 2 and 4 leading to an increase in  $U$  for a fixed value of  $C_n$ . We conclude that this extremum cannot be a local maximum of  $U$ .

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