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STATIONARY AND NONEQUILIBRIUM FLUCTUATIONS IN BOUNDARY DRIVEN EXCLUSION PROCESSES

C. LANDIM, A. MILANÉS, S. OLLA

ABSTRACT. We prove nonequilibrium fluctuations for the boundary driven symmetric simple exclusion process. We deduce from this result the stationary fluctuations.

1. INTRODUCTION

In the last years there has been considerable progress in understanding stationary non equilibrium states (SNS): reversible systems in contact with reservoirs imposing a gradient on the conserved quantities of the system. In particular, large deviation properties has been studied for boundary driven one-dimensional symmetric simple exclusion processes ([1, 4] and references therein).

One of the most striking typical property of SNS is the presence of long-range correlations. For the symmetric simple exclusion this was already shown by H. Spohn in the pioneering paper [10]. But we noticed that a mathematical proof of the convergence of the fluctuation fields to the corresponding Gaussian field was missing from the literature. The purpose of this paper is to fill this gap.

We consider the symmetric exclusion process in an open lattice of length N . Particles jumps to nearest neighbors performing simple symmetric random walks with the exclusion rule: a jump is suppressed if site is already occupied. At the left boundary particles are created with rate α and annihilated with rate $1 - \alpha$. On the right boundary this is done with rates β and $1 - \beta$. To keep notation simple we restrict to the one dimensional nearest neighbor case. Extensions to more dimension and more general jumps rates is straightforward (see remarks 2.2 and 2.3).

If $\alpha = \beta = \rho$, the Bernoulli product measure with probability ρ is stationary and reversible for the dynamics. But when $\alpha \neq \beta$ the stationary measure has correlations and is not explicitly computable. We denote by $\eta(x) = 0$ or 1 the occupation variable of site x . It is easy to prove that $\langle \eta([Nu]) \rangle_{ss} \rightarrow \bar{\rho}(u) = (\beta - \alpha)u + \alpha$.

The fluctuation field is formally defined as the random distribution on $[0, 1]$

$$Y^N(u) = \frac{1}{\sqrt{N}} \sum_{x=1}^{N-1} \delta(u - x/N)(\eta(x) - \bar{\rho}(u)). \quad (1.1)$$

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We prove in this paper that, under the stationary measure for the process, Y^N converges in law to the centered Gaussian field Y on $[0, 1]$ with covariance

$$\langle Y(u)Y(v) \rangle = \chi(\bar{\rho}(u))\delta(u-v) - (\beta - \alpha)^2(-\Delta)^{-1}(u, v), \quad (1.2)$$

where Δ is the Laplace operator with Dirichlet boundary conditions, and $\chi(\rho) = \rho(1-\rho)$. In the 1-dimensional case we have more explicitly $(-\Delta)^{-1}(u, v) = u(1-v)$.

The strategy we use to prove this result is to study first the convergence of the nonstationary fluctuations. If we start with some non-equilibrium density profile $\langle \eta_0([Nu]) \rangle = \rho(0, u)$, then at the diffusive time scale we have $\langle \eta_{N^2t}([Nu]) \rangle \rightarrow \rho(t, u)$, where $\rho(t, u)$ is solution of the heat equation with initial condition $\rho(0, u)$.

We then consider the time-dependent fluctuation field

$$Y^N(t, u) = \frac{1}{\sqrt{N}} \sum_{x=1}^{N-1} \delta(u - x/N)(\eta_{N^2t}(x) - \rho(t, u)). \quad (1.3)$$

The main point of the proof is to show the convergence of $Y^N(u, t)$ to the solution of the stochastic linear partial differential equation

$$\partial_t Y(t, u) = \Delta Y(t, u) - \nabla \left(\sqrt{2\chi(\rho(t, u))} W(t, u) \right), \quad (1.4)$$

where $W(t, u)$ is the standard space-time white noise. If we start in the stationary state, $\rho(t, u) = \bar{\rho}(u)$ for all t . In this case the distribution valued process $Y(t, u)$ is a stationary Gaussian process and its invariant distribution is given by the Gaussian field Y with covariance given by (1.2).

This article presents a rigorous proof of the results described above and presented in [4]. Article [4] also contains the connection between the large deviations and the small fluctuations proved here, showing that the inverse of the covariance (1.2) is given by the second functional derivative of the large deviations rate function.

2. NOTATION AND RESULTS

For $N \geq 1$, let $\Lambda_N = \{1, \dots, N-1\}$. Fix $0 \leq \alpha \leq \beta \leq 1$ and consider the boundary driven symmetric simple exclusion process associated to α, β . This is the Markov process on $\{0, 1\}^{\Lambda_N}$ whose generator L_N is given by

$$\begin{aligned} (L_N f)(\eta) &= \sum_{x=1}^{N-2} \{f(\sigma^{x, x+1}\eta) - f(\eta)\} \\ &+ \left\{ \alpha[1 - \eta(1)] + (1 - \alpha)\eta(1) \right\} \{f(\sigma^1\eta) - f(\eta)\} \\ &+ \left\{ \beta[1 - \eta(N-1)] + (1 - \beta)\eta(N-1) \right\} \{f(\sigma^{N-1}\eta) - f(\eta)\}. \end{aligned}$$

In this formula, $\eta = \{\eta(x), x \in \Lambda_N\}$ is a configuration of the state space $\{0, 1\}^{\Lambda_N}$ so that $\eta(x) = 0$ if and only if site x is vacant for η ; $\sigma^{x, y}\eta$ is the configuration obtained from η by interchanging the occupation variables $\eta(x), \eta(y)$:

$$(\sigma^{x, y}\eta)(z) = \begin{cases} \eta(z) & \text{if } z \neq x, y, \\ \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y; \end{cases}$$

and $\sigma^x\eta$ is the configuration obtained from η by flipping the variable $\eta(x)$:

$$(\sigma^x\eta)(z) = \begin{cases} \eta(z) & \text{if } z \neq x, \\ 1 - \eta(z) & \text{if } z = x. \end{cases}$$

Hence, at rate α (resp. $1 - \alpha$) a particle is created (resp. removed) at the boundary site 1 if this site is vacant (resp. occupied). The same phenomenon occurs at the boundary $x = N - 1$ with β in place of α .

This finite state Markov process is irreducible and has therefore a unique stationary measure, denoted by $\nu_{\alpha,\beta}^N$. For $0 \leq \gamma \leq 1$, denote by ν_γ^N the Bernoulli product measure on $\{0, 1\}^{\Lambda_N}$ with density γ . If $\alpha = \beta$, an elementary computation shows that ν_α^N is the invariant measure and that the process is reversible with respect to this stationary state. On the other hand, if $\alpha < \beta$, it is known since [10] that the invariant state has long range correlations.

Static picture. For $N \geq 1$, denote by π^N the measure on $[0, 1]$ obtained by assigning mass N^{-1} to each particle:

$$\pi^N(\eta) = N^{-1} \sum_{x \in \Lambda_N} \eta(x) \delta_{x/N}, \quad (2.1)$$

where δ_u is the Dirac measure on u . It has been proved in [6] that under the stationary state $\nu_{\alpha,\beta}^N$ the empirical measure π^N converges to the unique solution of the elliptic equation

$$\begin{cases} \Delta \rho = 0, \\ \rho(0) = \alpha, \quad \rho(1) = \beta. \end{cases}$$

We denote the solution of this equation by $\bar{\rho} = \bar{\rho}_{\alpha,\beta}$.

Once a law of large number has been proved for the empirical measure under the stationary state, it is natural to consider the fluctuations around the limit. Let

$$\rho^N(x) = E_{\nu_{\alpha,\beta}^N}[\eta(x)].$$

Since $E_{\nu_{\alpha,\beta}^N}[L_N \eta(x)] = 0$ for all $1 \leq x \leq N - 1$, an elementary computation shows that ρ^N is the solution of

$$\begin{cases} (\Delta_N \rho^N)(x) = 0 & \text{for } 1 \leq x \leq N - 1, \\ \rho^N(0) = \alpha, \quad \rho^N(N) = \beta, \end{cases} \quad (2.2)$$

where Δ_N is the discrete Laplacian: $(\Delta_N H)(x) = N^2 \{H(x+1) + H(x-1) - 2H(x)\}$.

In the case of the symmetric simple exclusion process, $\rho^N(\cdot)$ is just the linear interpolation between $\rho^N(0) = \alpha$, $\rho^N(N) = \beta$.

To define the space in which the fluctuations take place, denote by $C_0^2([0, 1])$ the space of twice continuously differentiable functions on $(0, 1)$ which are continuous on $[0, 1]$ and which vanish at the boundary. Let $-\Delta$ be the positive operator, essentially self-adjoint on $L^2[0, 1]$, defined by

$$\begin{aligned} -\Delta &= -\frac{d^2}{dx^2}, \\ \mathcal{D}(-\Delta) &= C_0^2([0, 1]). \end{aligned}$$

Its eigenvalues and corresponding (normalized) eigenfunctions have the form $\lambda_n = (n\pi)^2$ and $e_n(u) = \sqrt{2} \sin(n\pi u)$ respectively, for any $n \in \mathbb{N}$. By the Sturm-Liouville theory, $\{e_n, n \in \mathbb{N}\}$ forms an orthonormal basis of $L^2[0, 1]$.

We denote with the same symbol the closure of $-\Delta$ in $L^2[0, 1]$. For any nonnegative integer k , we define the Hilbert spaces $\mathcal{H}_k = \mathcal{D}(\{-\Delta\}^{k/2})$, with inner product

$(f, g)_k = (\{-\Delta\}^{k/2} f, \{-\Delta\}^{k/2} g)$, where (\cdot, \cdot) is the inner product in $L^2[0, 1]$. By the spectral theorem for self-adjoint operators,

$$\mathcal{H}_k = \left\{ f \in L^2[0, 1] : \sum_{n=1}^{+\infty} n^{2k} (f, e_n)^2 < \infty \right\},$$

$$(f, g)_k = \sum_{n=1}^{+\infty} (n\pi)^{2k} (f, e_n)(g, e_n).$$

Moreover, if \mathcal{H}_{-k} denotes the topological dual space of \mathcal{H}_k ,

$$\mathcal{H}_{-k} = \left\{ f \in \mathcal{D}'(0, 1) : \sum_{n=1}^{+\infty} n^{-2k} \langle f, e_n \rangle^2 < \infty \right\},$$

$$(f, g)_{-k} = \sum_{n=1}^{+\infty} (n\pi)^{-2k} \langle f, e_n \rangle \langle g, e_n \rangle,$$

where $\langle f, \cdot \rangle$ represents the action of the distribution f over $[0, 1]$ on test functions.

Fix $k > 5/2$ and define the density field Y^N on \mathcal{H}_{-k} by

$$Y^N(H) = N^{-1/2} \sum_{x \in \Lambda_N} H(x/N) \{ \eta(x) - \rho^N(x) \}. \quad (2.3)$$

For $k \geq 1$, denote by q the Gaussian probability measure on \mathcal{H}_{-k} with zero mean and covariance given by

$$E_q[Y(H)Y(G)] = \int_0^1 du \chi(\bar{\rho}(u)) H(u) G(u) - (\beta - \alpha)^2 \int_0^1 du [(-\Delta)^{-1} H(u)] G(u). \quad (2.4)$$

Theorem 2.1. *Fix $k > 5/2$ and denote by q_N the probability measure on \mathcal{H}_{-k} induced by the density field Y^N defined in (2.3) and the stationary measure $\nu_{\alpha, \beta}^N$. As $N \uparrow \infty$, q_N converges to q .*

This result follows from Proposition 3.5 which is proved in Section 3.

Remark 2.2. *The same statement holds in higher dimensions for a symmetric exclusion on the set $\Lambda_N \times \mathbb{T}_N^{d-1}$, where \mathbb{T}_N is the discrete torus of length N . The dynamics is periodic in the $d-1$ directions orthogonal to the gradient of the density. In this case the space correlations are given by (2.4), where Δ is the Laplacian with periodic boundary conditions in the $(d-1)$ dimensions and Dirichlet boundary conditions in the first coordinate. The proof is an elementary extension of the one-dimensional case.*

Remark 2.3. *We may also consider a boundary driven symmetric simple exclusion process in which the occupation variables $\eta(x)$, $\eta(x+y)$ are exchange at rate $p(y)$ for a finite range irreducible probability $p(\cdot)$. In this case, the Laplacian is replaced by the operator $\sum_{i,j=1}^d \sigma_{i,j} \partial_{u_i} \partial_{u_j}$, where $\sigma_{i,j} = \sum_y y_i y_j p(y)$.*

3. NONEQUILIBRIUM FLUCTUATIONS

We prove in this section the dynamical nonequilibrium fluctuations of the boundary driven exclusion process. We start with the law of large numbers.

Fix a density profile $\rho_0 : [0, 1] \rightarrow [0, 1]$. Consider a sequence $\{\mu^N, N \geq 1\}$ of probability measures on $\{0, 1\}^{\Lambda_N}$ such that for every continuous test function $H : [0, 1] \rightarrow \mathbb{R}$ and every $\varepsilon > 0$,

$$\lim_{N \rightarrow \infty} \mu^N \left\{ \left| \pi^N(H) - \int H(u) \rho_0(u) du \right| > \varepsilon \right\} = 0.$$

Denote by \mathbb{P}_{μ^N} the probability on the path space $D(\mathbb{R}_+, \{0, 1\}^{\Lambda_N})$ induced by the Markov process with generator L_N and the initial measure μ^N . Denote by π_t^N the empirical measure associated to the state of the process at time t : $\pi_t^N = N^{-1} \sum_x \eta_{N^2 t}(x) \delta_{x/N}$. It follows from the usual hydrodynamic limits techniques, adapted to the boundary driven context (cf. sections 4 and 5 in [7], and [8]) that for every $t \geq 0$, every continuous test function $H : [0, 1] \rightarrow \mathbb{R}$ and every $\varepsilon > 0$,

$$\lim_{N \rightarrow \infty} \mathbb{P}_{\mu^N} \left\{ \left| \pi_t^N(H) - \int H(u) \rho(t, u) du \right| > \varepsilon \right\} = 0,$$

where $\rho(t, u)$ is the unique solution of the heat equation

$$\begin{cases} \partial_t \rho = \Delta \rho, \\ \rho(0, \cdot) = \rho_0(\cdot), \\ \rho(\cdot, 0) = \alpha, \quad \rho(\cdot, 1) = \beta. \end{cases} \quad (3.1)$$

Furthermore is valid the following *replacement lemma*:

Lemma 3.1. *Let $\Psi(\eta)$ a local function, and $\tilde{\Psi}(\rho) = E_{\nu_\rho}(\Psi)$, where ν_ρ is the Bernoulli measure with probability ρ . Let $G(s, u)$ a continuous function on $\mathbb{R}_+ \times [0, 1]$. Then*

$$\limsup_{N \rightarrow \infty} E \left(\int_0^T ds \left| \frac{1}{N} \sum_{x=1}^{N-1} G(s, \frac{x}{N}) \Psi(\tau_x \eta_{N^2 s}) - \int_0^1 G(s, u) \tilde{\Psi}(\rho(s, u)) du \right| \right) = 0$$

The proof is given in chapter 5 of [7], adapted to the open boundary situation.

We now turn to the fluctuations. Consider a sequence $\{\mu^N : N \geq 1\}$ of probability measures on $\{0, 1\}^{\Lambda_N}$. Let

$$\rho^N(x) = E_{\mu^N}[\eta(x)], \quad \varphi^N(x, y) = E_{\mu^N}[\eta(x); \eta(y)],$$

for x, y in Λ_N , $x < y$. In this formula, $E_\mu[f; g]$ stands for the covariance of f and g : $E_\mu[f; g] = E_\mu[fg] - E_\mu[f]E_\mu[g]$. We extend the definition of ρ^N and φ^N to the boundary of Λ_N by setting

$$\rho^N(0) = \alpha, \quad \rho^N(N) = \beta, \quad \varphi^N(x, y) = 0$$

if x or y does not belong to Λ_N . Assume that there exists a finite constant C_0 such that

$$\sup_{0 \leq x \leq N-1} N |\rho^N(x+1) - \rho^N(x)| \leq C_0, \quad N \max_{\substack{x, y \in \Lambda_N \\ x < y}} |\varphi^N(x, y)| \leq C_0. \quad (3.2)$$

Assume furthermore that ρ^N converges weakly to a profile ρ_0 in the sense that for every continuous function $H : [0, 1] \rightarrow \mathbb{R}$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{x \in \Lambda_N} H(x/N) \rho^N(x) = \int_0^1 du H(u) \rho_0(u). \quad (3.3)$$

It follows from assumptions (3.2), (3.3) and from Chebyshev inequality that under μ^N the empirical measure π^N , defined in (2.1), converges to $\rho_0(u)du$: For every $\delta > 0$ and every continuous function $H : [0, 1] \rightarrow \mathbb{R}$,

$$\lim_{N \rightarrow \infty} \mu^N \left\{ \left| \pi^N(H) - \int_0^1 du H(u) \rho_0(u) \right| > \delta \right\} = 0.$$

In particular, by the law of large numbers stated in the beginning of this section, for every $t > 0$, the empirical measure π_t^N converges to the absolutely continuous measure whose density is the solution of the heat equation (3.1).

We prove in (4.2) that the stationary state $\nu_{\alpha, \beta}^N$ satisfies the assumptions (3.2), (3.3). It is also easy to verify that this property is shared by product measures associated to Lipschitz profiles.

Let ρ_t^N be the solution of the semidiscrete heat equation

$$\begin{cases} \partial_s \rho_s^N(x) = \Delta_N \rho_s(x), & x \in \Lambda_N; \\ \rho_0^N(x) = \rho^N(x), & x \in \Lambda_N; \\ \rho_s^N(0) = \alpha, \quad \rho_s^N(N) = \beta, & s \geq 0. \end{cases} \quad (3.4)$$

Fix $k > 5/2$ and denote by Y_t^N the density fluctuation field which acts on smooth functions H in \mathcal{H}_k as

$$Y_t^N(H) = N^{-1/2} \sum_{x \in \Lambda_N} H(x/N) \{ \eta_{tN^2}(x) - \rho_t^N(x) \}.$$

Notice that time has been speeded up by N^2 . Denote by Q_N the probability measure on $D([0, T], \mathcal{H}_{-k})$ induced by the density fluctuation field Y^N introduced above and the probability measure μ^N .

Assumptions (3.2), (3.3) ensure tightness of the sequence Q_N and permit to describe the asymptotic evolution of the field Y as the sum of two uncorrelated pieces: a deterministic part characterized by the heat kernel and a martingale. This is the content of the first result. Denote by $\{T_s : s \geq 0\}$ the semigroup associated to the operator Δ .

Proposition 3.2. *Fix $T > 0$ and a positive integer $k > 5/2$. The sequence Q_N is tight on $D([0, T], \mathcal{H}_{-k})$ with respect to the uniform topology. All limit points Q^* are concentrated on paths Y_t such that*

$$Y_t(H) = Y_0(T_t H) + W_t(H), \quad (3.5)$$

where $W_t(H)$ is a zero-mean Gaussian variable with variance given by

$$2 \int_0^t ds \int_0^1 du \chi(\rho(s, u)) (T_{t-s} \nabla H)^2(u),$$

and ρ_s is the solution of the heat equation (3.1). Moreover, Y_0 and W_t are uncorrelated in the sense that $E_{Q^*}[Y_0(H)W_t(G)] = 0$ for all functions H, G in $C_0^2([0, 1])$ and all $0 \leq t \leq T$.

Proof. The proof of tightness of the sequence Q_N is left to the end of this section. To check the properties of the limit points, fix a smooth function H in $C_0^2([0, 1])$. An elementary computation shows that

$$(\partial_t + N^2 L_N) Y_t^N(H) = Y_t^N(\Delta_N H).$$

Observe that no boundary term appears in the right hand side of the above equation. In particular, defining

$$\Gamma_s^N(H) = N^2 \{L_N Y_s^N(H)^2 - 2Y_s^N(H)L_N Y_s^N(H)\}, \quad (3.6)$$

it follows that

$$\begin{aligned} M_t^{1,N}(H) &= Y_t^N(H) - Y_0^N(H) - \int_0^t ds Y_s^N(\Delta_N H), \\ M_t^{2,N}(H) &= \{M_t^{1,N}(H)\}^2 - \int_0^t ds \Gamma_s^N(H), \end{aligned} \quad (3.7)$$

are martingales. A simple computation shows that

$$\begin{aligned} \Gamma_s^N(H) &= \frac{1}{N} \sum_{x=1}^{N-2} [\eta_{N^2 s}(x+1) - \eta_{N^2 s}(x)]^2 \{(\nabla_N H)(x/N)\}^2 \\ &+ N^{-1} (\nabla_N H)(0)^2 \{\eta_{N^2 s}(1) - \alpha\}^2 \\ &+ N^{-1} (\nabla_N H)((N-1)/N)^2 \{\eta_{N^2 s}(N-1) - \beta\}^2, \end{aligned} \quad (3.8)$$

where $\nabla_N H(x/N)$ stands for the discrete derivative: $(\nabla_N H)(u) = N\{H(u+N^{-1}) - H(u)\}$. Observe that the last two boundary term on the above equations are of order N^{-1} . By Lemma 3.1, as $N \uparrow \infty$, $\Gamma_s^N(H)$ converges to $2 \int_0^1 \chi(\rho(s,u)) (\nabla H(u))^2 du$, where ρ is the solution of the heat equation (3.1).

Fix a limit point Q^* of the sequence Q_N . It follows from (3.7) that under Q^* for each H in $C_0^2([0, 1])$

$$M_t(H) = Y_t(H) - Y_0(H) - \int_0^t ds Y_s(\Delta H) \quad (3.9)$$

is a martingale with deterministic quadratic variation given by

$$\langle M(H) \rangle_t = 2 \int_0^t ds \int_0^1 du \chi(\rho(s,u)) (\nabla H)^2.$$

In particular, for each H , $M_t(H)$ is Brownian motion changed in time.

Consider the semi-martingale $Y_s(T_{t-s}H)$ for $0 \leq s \leq t$. Apply Ito's formula to derive equation (3.5), with

$$W_t(H) = \int_0^t dM_s(T_{t-s}H).$$

$W_t(H)$ has a Gaussian distribution because the martingales $M_t(H)$ are Gaussian, being a deterministic time-change of a Brownian motion. The expression for the variance of $W_t(H)$ follows from an elementary computation, as well as the fact that $W_t(H)$ and $Y_0(G)$ are uncorrelated. This concludes the proof of the proposition. \square

In view of (3.5), to prove that Y^N converges it remains to guarantee the convergence at the initial time:

Proposition 3.3. *Assume that Y_0^N converges to a zero-mean Gaussian field Y with covariance denoted by $\ll \cdot, \cdot \gg$:*

$$\lim_{N \rightarrow \infty} E_{\mu^N} [Y(H)Y(G)] = E[Y(H)Y(G)] =: \ll H, G \gg.$$

Then, Q^N converges to a generalized Ornstein-Uhlenbeck process with covariances given by

$$E[Y_t(H)Y_s(G)] = \ll T_t H, T_s G \gg + 2 \int_0^s dr \int_0^1 \chi(\rho(r, u)) (\nabla T_{t-r} H)(u) (\nabla T_{s-r} G)(u) \quad (3.10)$$

for all $0 \leq s \leq t \leq T$, H, G in $C_0^2([0, 1])$.

Proof. By Proposition 3.2, the sequence Q^N is tight and all limit points satisfy (3.5). Since W_t and Y_0 are zero-mean Gaussian random variables, so is Y_t . To compute the covariance, it is enough to remind that the variables are uncorrelated. The first piece in formula (3.10) accounts for the covariance between $Y_0(T_t H)$ and $Y_0(T_s G)$, while the last one for the covariance between $W_t(H)$ and $W_s(G)$. \square

A nonequilibrium central limit theorem for the density field follows from the previous two results for processes starting from local Gibbs states. Indeed, fix a Lipschitz profile $\gamma : [0, 1] \rightarrow [0, 1]$ such that $\gamma(0) = \alpha$, $\gamma(1) = \beta$ and denote by $\nu_{\gamma(\cdot)}^N$ the product measure on $\{0, 1\}^{\Lambda_N}$ associated to γ so that

$$\nu_{\gamma(\cdot)}^N \{\eta(x) = 1\} = \gamma(x/N)$$

for x in Λ_N . In this case $\rho^N(x) = \gamma(x/N)$, $\varphi^N(x, y) = 0$ and $\rho_0 = \gamma$. The first hypothesis in (3.2) is satisfied because we assumed γ to be Lipschitz. On the other hand, computing the characteristic functions of Y_0^N , it is easy to show (cf. [7]) that Y_0^N converges to a zero-mean Gaussian field with covariance given by

$$E[Y(H)Y(G)] = \int_0^1 \chi(\gamma(u)) H(u) G(u) du.$$

Therefore, by Propositions 3.2 and 3.3 the density field converges to a generalized Ornstein-Uhlenbeck process:

Corollary 3.4. *Fix $T > 0$ and a positive integer $k > 5/2$. Denote by Q_N the probability measure on $D([0, T], \mathcal{H}_{-k})$ induced by the density fluctuation field Y^N and the probability measure $\nu_{\gamma(\cdot)}^N$. Then, Q^N converges to the centered Gaussian probability measure Q with covariances given by*

$$E[Y_t(H)Y_s(G)] = \int_0^1 du \chi(\gamma(u)) T_t H(u) T_s G(u) + 2 \int_0^s dr \int_0^1 du \chi(\rho(r, u)) (\nabla T_{t-r} H)(u) (\nabla T_{s-r} G)(u)$$

for all $0 \leq s \leq t \leq T$, H, G in $C_0^2([0, 1])$, where ρ_t is the solution of the heat equation with initial condition γ .

A similar result was obtained by De Masi et al. [3], for the one-dimensional symmetric exclusion process in infinite volume. This result was extended to higher dimensions by Ravishankar [9]. Chang and Yau [2] introduced a general method to prove non-stationary fluctuations of one-dimensional interacting particle systems.

In Proposition 3.3, the asymptotic behavior of the covariance (3.10) as $t \uparrow \infty$ can be computed. Indeed, fix a function H in $C_0^2([0, 1])$ and set $G = H$, $s = t$. Since $T_t H$ vanishes as $t \uparrow \infty$, the first part of the covariance converges to 0. Since H vanishes at the boundary, since T_t is the semigroup associated to the Laplacian

and since $2H\nabla H = \nabla H^2$, an integration by parts shows that the second part of the covariance (3.10) is equal to

$$2 \int_0^t ds \int_0^1 du \chi(\rho(s, u)) (T_{t-s}H)(u) (\partial_s T_{t-s}H)(u) \\ - \int_0^t ds \int_0^1 du [\nabla \chi(\rho(s, u))] [\nabla (T_{t-s}H(u))]^2.$$

Since $2G\partial_s G = \partial_s G^2$, integrating by parts in time, since $T_t H$ vanishes in the limit $t \uparrow \infty$ and since the solution of the heat equation converges to the stationary profile $\bar{\rho}$, the previous expression is equal to

$$\int_0^1 du \chi(\bar{\rho}(u))H(u)^2 - \int_0^t ds \int_0^1 du [\partial_s \chi(\rho(s, u))] (T_{t-s}H(u))^2 \\ + \int_0^t ds \int_0^1 du [\Delta \chi(\rho(s, u))] (T_{t-s}H(u))^2$$

plus a term which vanishes in the limit. This sum is equal to

$$\int_0^1 du \chi(\bar{\rho}(u))H(u)^2 - 2 \int_0^t ds \int_0^1 du [\nabla \rho(s, u)]^2 (T_{t-s}H(u))^2$$

because ρ is the solution of the heat equation. As $t \uparrow \infty$, this expression converges to

$$\int_0^1 du \chi(\bar{\rho}(u))H^2(u) - (\beta - \alpha)^2 \int_0^1 du H(u) (-\Delta)^{-1} H(u)$$

because $\nabla \bar{\rho} = \beta - \alpha$. We just recovered the covariance (2.4) of the density field under the stationary state.

We turn now to the proof of Theorem 2.1. Assume that the initial state μ^N is the stationary state $\nu_{\alpha, \beta}^N$. We prove in Section 4 that the second condition in (3.2) is fulfilled.

Fix k in \mathbb{R} and recall the definition of the probability measure q introduced just before (2.4). Let Q be the probability measure on $C([0, T], \mathcal{H}_{-k})$ corresponding to the stationary generalized Ornstein–Uhlenbeck process with mean 0 and covariance given by

$$E_Q \left[Y_t(H) Y_s(G) \right] = E_q \left[Y(T_{t-s}H) Y(G) \right] \quad (3.11)$$

for every $0 \leq s \leq t$ and H, G in \mathcal{H}_k .

Proposition 3.5. *Fix $T > 0$ and a positive integer $k > 5/2$. Denote by Q_N the probability measure on $D([0, T], \mathcal{H}_{-k})$ induced by the density fluctuation field Y^N and the probability measure $\nu_{\alpha, \beta}^N$. The sequence Q_N converges weakly to the probability measure Q .*

Proof. Since $\nu_{\alpha, \beta}^N$ satisfies assumptions (3.2), (3.3), by Proposition 3.2, Q^N is tight and all limit points satisfy (3.5), where $W_t(H)$ is in this stationary context a zero-mean Gaussian variable with variance given by

$$2 \int_0^t ds \int_0^1 du \chi(\bar{\rho}(u)) (T_s \nabla H(u))^2.$$

As $t \uparrow \infty$, $T_t H$ vanishes in $L^2([0, 1])$. On the other hand, the computations performed just before the statement of this lemma show that the variance of $W_t(H)$ converges to (2.4). Therefore, $W_t(H)$ converges in distribution to a zero-mean

Gaussian variable with variance given by (2.4). Since the process is stationary, we just proved that the variables $\{Y_t(H) : t \geq 0, H \in \mathcal{H}_k\}$ have a zero mean Gaussian distribution with covariance given by (2.4).

To compute the covariances $E_{Q^*}[Y_t(H)Y_s(G)] = E_{Q^*}[Y_{t-s}(H)Y_0(G)]$, $0 \leq s \leq t$ it is enough to iterate relation (3.9) to recover formula (3.11). This concludes the proof of the lemma. \square

We conclude this section proving that the sequence of probability measures Q_N is tight and that all limit points are concentrated on continuous paths.

To prove that the sequence Q_N is tight we need to show that for every $0 \leq t \leq T$,

$$\lim_{A \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{P}_{\mu^N} \left[\sup_{0 \leq t \leq T} \|Y_t\|_{-k} > A \right] = 0$$

and that

$$\lim_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{P}_{\mu^N} \left[w_\delta(Y) \geq \varepsilon \right] = 0$$

for every $\varepsilon > 0$. Here $w_\delta(Y)$ stands for the uniform modulus of continuity defined by

$$w_\delta(Y) = \sup_{\substack{|s-t| \leq \delta \\ 0 \leq s, t \leq T}} \|Y_t - Y_s\|_{-k}.$$

We start with a key estimate. Recall the definition of the martingales $M_t^{1,N}(H)$, $M_t^{2,N}(H)$ defined by (3.7).

Lemma 3.6. *Fix a sequence of probability measures $\{\mu^N : N \geq 1\}$ satisfying (3.2), (3.3). There exists a finite constant C_1 , depending only on C_0 , such that for every $j \geq 1$,*

$$\limsup_{N \rightarrow \infty} \mathbb{E}_{\mu^N} \left[\sup_{0 \leq t \leq T} Y_t(e_j)^2 \right] \leq C_1 j^4 (1+T)^2.$$

Proof. Recall (3.7) and write $Y_t^N(e_j)$ as $M_t^{1,N}(e_j) + Y_0^N(e_j) + \int_0^t Y_s^N(\Delta_N e_j) ds$. We estimate these three terms separately.

It follows from (3.2) that $\mathbb{E}_{\mu^N}[Y_0(e_j)^2]$ is bounded by a finite constant C_1 , uniformly in N and j .

Since $M_t^{1,N}(e_j)$ is a martingale, by Doob inequality,

$$\mathbb{E}_{\mu^N} \left[\sup_{0 \leq t \leq T} |M_t^{1,N}(e_j)|^2 \right] \leq 4 \mathbb{E}_{\mu^N} \left[|M_T^{1,N}(e_j)|^2 \right].$$

By definition of the martingale $M_T^{2,N}(e_j)$ and by (3.9), the right hand side is equal to

$$4 \mathbb{E}_{\mu^N} \left[\int_0^T ds \frac{1}{N} \sum_{x=1}^{N-2} [\eta_s(x+1) - \eta_s(x)]^2 \{(\nabla_N e_j)(x/N)\}^2 \right] + O(N^{-1}).$$

By Lemma 3.1, as $N \uparrow \infty$, this expression converges to $8 \int_0^T dt \int_0^1 \chi(\rho(t,u)) (\nabla e_j(u))^2 du$ which is bounded by $C_1 T j^2$.

Finally, by definition of e_j and by Schwarz inequality,

$$\mathbb{E}_{\mu^N} \left[\sup_{0 \leq t \leq T} \left(\int_0^t ds Y_s(\Delta e_j) \right)^2 \right] \leq C_1 j^4 T \mathbb{E}_{\mu^N} \left[\int_0^T ds Y_s(e_j)^2 \right] \quad (3.12)$$

for some finite constant C_1 . The previous expectation can be rewritten as

$$\begin{aligned} & \int_0^T ds \frac{1}{N} \sum_{x \in \Lambda_N} e_j(x/N)^2 \rho_s^N(x) [1 - \rho_s^N(x)] \\ & + \int_0^T ds \frac{1}{N} \sum_{x, y \in \Lambda_N} e_j(x/N) e_j(y/N) \varphi_s^N(x, y), \end{aligned}$$

where

$$\varphi_t^N(x, y) = E_{\mu^N}[\eta_t(x); \eta_t(y)]. \quad (3.13)$$

By Proposition 4.4,

$$\sup_{N \geq 2} \sup_{t \geq 0} N |\varphi_t^N(x, y)| \leq C_1 \quad (3.14)$$

for $C_1 = C_0 + (1/2)C_0^2$. Hence, (3.12) is bounded above by $C_1 T^2 j^4$, which concludes the proof of the lemma. \square

Corollary 3.7. *For $k > 5/2$,*

$$\begin{aligned} (a) \quad & \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu^N} \left[\sup_{0 \leq t \leq T} \|Y_t\|_{-k}^2 \right] < \infty, \\ (b) \quad & \lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu^N} \left[\sup_{0 \leq t \leq T} \sum_{j \geq n} Y_t(e_j)^2 j^{-2k} \right] = 0. \end{aligned}$$

The proof of this result is similar to the one of Corollary XI.3.5 in [7] and therefore omitted.

In view of Lemma 3.6 and part (b) of Corollary 3.7, in order to prove that the sequence Q_N is tight, we only have to show that

$$\lim_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{P}_{\mu^N} \left[\sup_{\substack{0 \leq |s-t| \leq \delta \\ 0 \leq s, t \leq T}} \{Y_t(e_j) - Y_s(e_j)\}^2 > \varepsilon \right] = 0$$

for every $j \geq 1$ and $\varepsilon > 0$. Fix $j \geq 1$ and recall the definition of the martingale $M_t^{1,N}(e_j)$. Since $Y_t^N(e_j) = Y_0^N(e_j) + M_t^{1,N}(e_j) + \int_0^t \Gamma_s^N(e_j) ds$, the previous statement follows from the next two claims: For every function G in $C_0^2([0, 1])$ and every $\varepsilon > 0$,

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{P}_{\mu^N} \left[\sup_{\substack{0 \leq s, t \leq T \\ |t-s| \leq \delta}} |M_t^{1,N}(G) - M_s^{1,N}(G)| > \varepsilon \right] = 0, \\ & \lim_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{P}_{\mu^N} \left[\sup_{\substack{0 \leq s, t \leq T \\ |t-s| \leq \delta}} \left| \int_s^t dr Y_r^N(\Delta_N G) \right| > \varepsilon \right] = 0. \end{aligned}$$

The derivation of these estimates is similar to the proofs of Lemmata XI.3.7 and XI.3.8 in [7] if one keeps in mind the arguments presented in the proof of Lemma 3.3 and the bound (3.14) on the two point correlation function φ_t^N given by (3.13).

4. SEMIDISCRETE HEAT EQUATION

We prove in this section a bound on the two point correlation function $\varphi_t^N(x, y)$ introduced in (3.13). Throughout this section, $N \geq 2$ is fixed.

For the square of points $C = \{0, \dots, N\}^2$, consider the subsets $V = \{(x, y) \in C : 0 < x < y < N\}$ and its boundary $\partial V = \{(x, y) \in C : x = 0 \text{ or } y = N\}$. Let

$\mathcal{M} = \{f : V \cup \partial V \mapsto \mathbb{R} : f|_{\partial V} = 0\}$ and denote by Δ_V^N the discrete Laplacian on \mathcal{M} defined by

$$(\Delta_V^N f)(x, y) = N^2 \{f(x+1, y) + f(x-1, y) + f(x, y-1) + f(x, y+1) - 4f(x, y)\}$$

if $|x - y| > 1$ and

$$(\Delta_V^N f)(x, x+1) = N^2 \{f(x-1, x+1) + f(x, x+2) - 2f(x, x+1)\}.$$

Δ_V^N corresponds to the generator of a symmetric random walk on $V \cup \partial V$ which is absorbed on ∂V .

We start with an explicit formula for the total time spent by the random walk on the diagonal, which is expressed by the Green function or as the solution of the elliptic equation $(-\Delta_V^N \varphi^N)(x, y) = C \delta_{y=x+1}$, C in \mathbb{R} . Let φ^N be the solution of

$$\begin{cases} (-\Delta_N \varphi^N)(x, y) = C \delta_{y=x+1} & \text{for } (x, y) \text{ in } V, \\ \varphi^N(x, y) = 0 & \text{for } (x, y) \text{ in } \partial V. \end{cases} \quad (4.1)$$

An elementary analysis shows that the unique solution of (4.1) is given by

$$\varphi^N(x, y) = \frac{C^2}{N-1} \frac{x}{N} \left(1 - \frac{y}{N}\right). \quad (4.2)$$

We turn now to maximum principles for solutions of homogeneous semidiscrete parabolic equations. Fix a function $\rho^N : \Lambda_N \rightarrow \mathbb{R}$ and let ρ_s^N be the solution of

$$\begin{cases} \partial_s \rho_s^N(x) = \Delta_N \rho_s(x), & x \in \Lambda_N \\ \rho_0^N(x) = \rho^N(x), & x \in \Lambda_N \\ \rho_s^N(0) = \alpha, \quad \rho_s^N(N) = \beta, & s \geq 0. \end{cases} \quad (4.3)$$

Lemma 4.1. *Let ρ_s^N be the solution of (4.3). Then,*

$$\sup_{s \geq 0} \max_{0 \leq x \leq N-1} |(\nabla_N \rho_s^N)(x)| \leq \max_{0 \leq x \leq N-1} |(\nabla_N \rho^N)(x)|.$$

Proof. Fix $T \geq 0$. Let $\gamma_t(x) = \rho_t(x+1) - \rho_t(x)$ for $0 \leq x \leq N-1$. Since $\partial_t \gamma_t(x) = (\Delta_N \gamma_t)(x)$ for $1 \leq x \leq N-2$, by the maximum principle,

$$M = \max_{0 \leq x \leq N-1} \sup_{0 \leq s \leq T} |\gamma_s(x)| = \max \left\{ \max_{0 \leq x \leq N-1} |\rho^N(x+1) - \rho^N(x)|, \sup_{0 \leq t \leq T} |\rho_t(1) - \alpha|, \sup_{0 \leq t \leq T} |\beta - \rho_t(N-1)| \right\}.$$

We claim that the maximum is attained at $t = 0$. To show this assume, without loss of generality, that there exists $t_0 \in (0, T]$ such that $M = |\rho_{t_0}(1) - \alpha|$. By (4.3) with $x = 1$ we have,

$$\partial_s \rho_s(1) = -N^2 \rho_s(1) + N^2 \alpha + N^2 (\rho_s(2) - \rho_s(1))$$

for any $0 < s < T$. Thus, multiplying by e^{sN^2} , grouping the terms conveniently and integrating on $[0, t]$ we get that

$$\rho_t(1) = e^{-tN^2} \rho_0(1) + \alpha(1 - e^{-tN^2}) + \int_0^t e^{-(t-s)N^2} N^2 (\rho_s(2) - \rho_s(1)) ds$$

so that

$$\rho_t(1) - \alpha = e^{-tN^2} (\rho_0(1) - \alpha) + \int_0^t e^{-(t-s)N^2} N^2 (\rho_s(2) - \rho_s(1)) ds.$$

Using the assumption made on $|\rho_{t_0}(1) - \alpha|$, we deduce from this identity that

$$M = |\rho_{t_0}(1) - \alpha| \leq e^{-t_0 N^2} |\rho_0(1) - \alpha| + (1 - e^{-t_0 N^2}) M,$$

which reduces to $M \leq |\rho_0(1) - \alpha|$. This concludes the proof of the lemma. \square

Fix a function $h : V \rightarrow \mathbb{R}$ and denote by f_s the solution of the semidiscrete heat equation

$$\begin{cases} \partial_s f_s(x, y) = \Delta_V^N f_s(x, y), & (x, y) \in V, \\ f_0(x, y) = h(x, y), & (x, y) \in V \\ f_s(x, y) = 0, & 0 \leq s \leq T, \quad (x, y) \in \partial V. \end{cases} \quad (4.4)$$

Lemma 4.2. *Let f satisfy (4.4). Then, the maximum value of f on $[0, T] \times (V \cup \partial V)$ is attained at a point (t_0, x_0, y_0) such that $t_0 = 0$ or $(x_0, y_0) \in \partial V$.*

Proof. The proof is the same as that of the maximum principle for the usual heat equation. It uses that if the maximum is attained at an interior point (t_0, x_0, y_0) then $\partial_{t_0} f_{t_0}(x_0, y_0) \geq 0$ and $\Delta_V^N f_{t_0}(x_0, y_0) \leq 0$. \square

Fix a function $h : V \rightarrow \mathbb{R}$ and a function $g : \mathbb{R}_+ \times V \rightarrow \mathbb{R}$. Consider the following nonhomogeneous parabolic equation,

$$\begin{cases} \partial_s \varphi_s(x, y) = \Delta_V^N \varphi_s(x, y) + g_s(x, y), & (x, y) \in V, \\ \varphi_0(x, y) = h(x, y), & (x, y) \in V, \\ \varphi_s(x, y) = 0, & 0 \leq s \leq T, \quad (x, y) \in \partial V, \end{cases} \quad (4.5)$$

Denote by $\|\cdot\|_{l^\infty(V)}$ the sup norm: $\|h\|_{l^\infty(V)} = \max_{z \in V} |h(z)|$.

Lemma 4.3. *Fix $T > 0$ and assume that the function g_s is supported on the line $y = x + 1$. Then,*

$$\sup_{0 \leq t \leq T} \|\varphi_t\|_{l^\infty(V)} \leq \|h\|_{l^\infty(V)} + \frac{1}{4(N-1)} \sup_{0 \leq t \leq T} \max_{1 \leq x \leq N-2} |g_t(x, x+1)|.$$

Proof. It is not difficult to see that Δ_V^N is a symmetric, negative operator on $\mathcal{M} = \{f : V \cup \partial V \mapsto \mathbb{R} : f|_{\partial V} = 0\}$. It generates in particular a strongly continuous semigroup $\{e^{s\Delta_V^N} : s \geq 0\}$ on \mathcal{M} and the solution φ_s of (4.5) can be written in the form

$$\varphi_s = e^{s\Delta_V^N} h + \int_0^s e^{(s-t)\Delta_V^N} g_t dt.$$

Since $e^{s\Delta_V^N} h$ is the solution of (4.4), by the maximum principle stated in Lemma 4.2,

$$\|e^{s\Delta_V^N} h\|_{l^\infty(V)} \leq \|h\|_{l^\infty(V)}$$

and $e^{s\Delta_V^N}(x, y) \geq 0$. On the other hand, since g_t is supported on the line $y = x + 1$,

$$\begin{aligned} \left| \int_0^s \{e^{(s-t)\Delta_V^N} g_t\}(x, y) dt \right| &\leq \int_0^s \sum_{x'=1}^{N-2} e^{(s-t)\Delta_V^N}(x, y, x', x'+1) |g_t(x', x'+1)| dt \\ &\leq \int_0^s \sum_{x'=1}^{N-2} e^{t\Delta_V^N}(x, y, x', x'+1) dt \sup_{0 \leq t \leq T} \max_{1 \leq x' \leq N-2} |g_t(x', x'+1)| \end{aligned}$$

provided $s \leq T$. To conclude the proof of the lemma it remains to show that the integral is bounded by $(4[N-1])^{-1}$, uniformly in x , y and s . This integral is bounded above by

$$v_N(x, y) = \int_0^\infty \sum_{x'=1}^{N-2} e^{t\Delta_V^N}(x, y, x', x'+1) dt \quad (4.6)$$

which satisfies the equation $\Delta_V^N v_N(x, y) = -\delta_{y=x+1}$. By (4.2),

$$v_N(x, y) = \frac{1}{N-1} \frac{x}{N} \left(1 - \frac{y}{N}\right)$$

and this expression is less than or equal to $(4[N-1])^{-1}$ because $x < y$. This concludes the proof of the lemma. \square

We are now in a position to state the main result of this section.

Proposition 4.4. *Consider a probability measure μ^N on $\{0, 1\}^{\Lambda_N}$ satisfying (3.2). Denote by φ_t^N the two point correlation function:*

$$\varphi_t^N(x, y) = E_{\mu^N} \left[\{\eta_t(x) - \rho_t^N(x)\} \{\eta_t(y) - \rho_t^N(y)\} \right],$$

where ρ_t^N is the solution of (4.3). Then,

$$\sup_{t \geq 0} \|\varphi_t^N\|_{l^\infty(V)} \leq \frac{2C_0 + C_0^2}{2N}$$

for all $N \geq 2$.

Proof. A simple computation shows that $\varphi_t^N(x, y)$ is the solution of (4.5) with $h = E_{\mu^N}[\eta(x); \eta(y)]$ and $g_t(x, y) = -(\nabla_N \rho_t^N)(x)^2 \delta_{y=x+1}$. By Lemma 4.1 and assumption (3.2), g_t , which is supported on the line $y = x+1$, is absolutely bounded by C_0^2 . Therefore, by Lemma 4.3, $N \sup_{t \geq 0} \|\varphi_t^N\|_{l^\infty(V)}$ is less than or equal to $C_0 + C_0^2/2$ since $N \geq 2$. This concludes the proof of the lemma. \square

It remains to check that stationary state $\nu_{\alpha, \beta}^N$ satisfies the assumptions of the previous proposition. Recall the definition of $\rho^N(x)$ given just before (2.2) and denote by $\varphi^N(x, y)$, $1 \leq x < y \leq N-1$ the two point correlation function:

$$\varphi^N(x, y) = E_{\nu_{\alpha, \beta}^N} \left[\{\eta(x) - \rho^N(x)\} \{\eta(y) - \rho^N(y)\} \right]. \quad (4.7)$$

Since $E_{\nu_{\alpha, \beta}^N} [L_N \{\eta(x) - \rho^N(x)\} \{\eta(y) - \rho^N(y)\}] = 0$ for all $1 \leq x < y \leq N-1$, we obtain that $\varphi^N(x, y)$ is the solution of the discrete differential equation (4.1) with $C = \beta - \alpha$. Therefore,

$$\varphi^N(x, y) = \frac{(\beta - \alpha)^2}{N-1} \frac{x}{N} \left(1 - \frac{y}{N}\right).$$

In particular, (3.2) is satisfied and we may apply Proposition 4.4.

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