A characterization of the Tutte polynomial via combinatorial embeddings

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A CHARACTERIZATION OF THE TUTTE POLYNOMIAL VIA COMBINATORIAL EMBEDDINGS

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ABSTRACT. We give a new characterization of the Tutte polynomial of graphs. Our characterization is formally close (but inequivalent) to the original definition given by Tutte as the generating function of spanning trees counted according to *activities*. Tutte's notion of activity requires to choose a *linear order* on the edge set (though the generating function of the activities is, in fact, independent of this order). We define a new notion of activity, the *embedding-activity*, which requires to choose a *combinatorial embedding* of the graph, that is, a cyclic order of the edges around each vertex. We prove that the Tutte polynomial equals the generating function of spanning trees counted according to *embedding-activities* (this generating function being, in fact, independent of the embedding).

1. Introduction

In 1954, Tutte defined a graph invariant that he named *dichromate* because he thought of it as a bivariate generalization of the *chromatic polynomial* [20]. The first definition by Tutte was a generating function of spanning trees counted according to their *activities* (see Theorem 3). Since then, this polynomial, which is now known as the *Tutte polynomial*, has been widely studied (see for instance, [7] and references therein). We refer the reader to [5, Chapter X] for an easy-to-read but comprehensive survey of the properties and applications of the Tutte polynomial.

In this paper, we give a new characterization of the Tutte polynomial of graphs. Our characterization is formally close to the original definition by Tutte (compare Theorem 3 and Theorem 7) in terms of the *activities* of spanning trees. Tutte's notion of activity requires to choose a *linear order* on the edge set. The Tutte polynomial is then the generating function of spanning trees counted according to their (internal and external) activities (this generating function being, in fact, independent of the linear order; see Theorem 3). Our characterization of the Tutte polynomial requires instead to choose an *embedding* of the graph, that is, a cyclic order for the incidences of edges around each vertex. Once the embedding is chosen, one can define the (internal and external) *embedding-activities* of spanning trees. We prove that the Tutte polynomial is equal to the generating function of spanning trees counted according to their (internal and external) embedding-activities (this generating function being, in fact, independent of the embedding; see Theorem 7).

Several other notions of *activities* related to the Tutte polynomial have been introduced. In 1982 Las Vergnas gave another characterization of the Tutte polynomial as the generating function of orientations counted according to their *cyclic-activities* [21]. Las Vergnas' notion of activity requires to linearly order the edge

set. A connection with Tutte's notion of activity was established in [15]. An alternative notion of *external activity* has also been introduced by Gessel and Wang [14] and further investigated in [13, 12]. This notion is related to the depth-first search algorithm and requires to linearly order the vertex set.

This paper is organized as follows. In Section 2, we introduce our notations about graphs, and define embeddings. In Section 3, we recall some classical properties of the Tutte polynomial. In Section 4, we define the *embedding-activities* and prove that the Tutte polynomial equals the generating function of the embedding-activities. Lastly, in Section 5, we mention some possible applications of our characterization of the Tutte polynomial, some of which will be developed in a forthcoming paper [4].

2. **Combinatorial embeddings of graphs**

Graphs. In this paper we consider finite, undirected graphs. Loops and multiple edges are allowed. Formally, a *graph* $G = (V, E)$ is a finite set of *vertices* V, a finite set of *edges* E and a relation of *incidence* in $V \times E$ such that each edge e is incident to either one or two vertices. The *endpoints* of an edge e are the vertices incident to e. A *loop* is an edge with just one endpoint. If e is an edge of G, we denote by $G_{\setminus e}$ the graph obtained by *deleting* e. If e is not a loop, we denote by $G_{/e}$ the graph obtained by *contracting* the edge e, that is, by identifying its two endpoints (see Figure 2). An *isthmus* is an edge whose deletion increases the number of connected components.

Embeddings. We recall the notion of *combinatorial map* introduced by Cori and Machi [8, 9]. A *combinatorial map* (or *map* for short) $\mathcal{G} = (H, \sigma, \alpha)$ is a set of *half-edges* H, a permutation σ and an involution without fixed point α on H such that the group generated by σ and α acts transitively on H. A map is *rooted* if one of the half-edges is distinguished as the *root*. For $h_0 \in H$, we denote by $\mathcal{G} = (H, \sigma, \alpha, h_0)$ the map (H, σ, α) rooted on h_0. Two maps $\mathcal{G} = (H, \sigma, \alpha)$ and $\mathcal{G}' = (H', \sigma', \alpha')$ are *isomorphic* if there is a bijection $\pi : H \mapsto H'$ such that $\pi\sigma = \sigma'\pi$ and $\pi\alpha = \alpha'\pi$. If $\mathcal{G}$ and $\mathcal{G}'$ are rooted on h_0 and h_0' respectively, we also require that $\pi(h_0) = h_0'$.

Given a combinatorial map $\mathcal{G} = (H, \sigma, \alpha)$, we consider the *underlying* graph $G = (V, E)$, where V is the set of cycles of σ, E is the set cycles of α and the incidence relation is to have at least one common half-edge. We represented the underlying graph of the map $\mathcal{G} = (H, \sigma, \alpha)$ where the set of half-edges is $H = \{a, a', b, b', c, c', d, d', e, e', f, f'\}$, the involution α (written as a product of cycles of length 2) is $(a, a')(b, b')(c, c')(d, d')(e, e')(f, f')$ and the permutation σ is (a, f', b, d) $(d')(a', e, f, c)(e', b', c')$ on the left of Figure 1. Graphically, we represent the cycles of σ by the counterclockwise order around each vertex. Hence, our drawing defines the map $\mathcal{G}$ since the order around vertices gives the cycles of σ (the involution α is immediately recovered from the edges). On the right of Figure 1, we represented the map $\mathcal{G}' = (H, \sigma', \alpha)$, where $\sigma' = (a, f', b, d)(d')(a', e, c, f)(e', b', c')$. The maps $\mathcal{G}$ and $\mathcal{G}'$ have isomorphic underlying graphs.

Note that the underlying graph of a map $\mathcal{G} = (H, \sigma, \alpha)$ is always connected since σ and α act transitively on H. A (rooted) *combinatorial embedding* (or *embedding* for short) of a connected graph G is a (rooted) combinatorial map $\mathcal{G}$ whose underlying graph is isomorphic to G. When an embedding $\mathcal{G}$ of G is given we shall write the edges of G as pairs of half-edges (writing for instance $e = \{h, h'\}$).
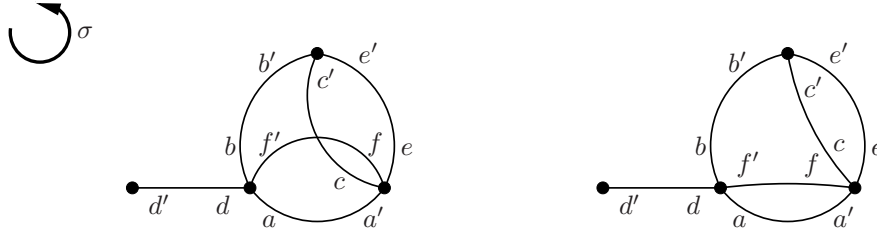
FIGURE 1. Two embeddings of the same graph.

Intuitively, a combinatorial embedding corresponds to the choice a cyclic order of the edges around each vertex. This order can also be seen as a local planar embedding. In fact, combinatorial embeddings are closely related to the *embeddings of graphs on surfaces*. What we call *surfaces* are compact 2-dimensional orientable surfaces without boundary (see [17]). An *embeddings of a graph G in a surface S* is a drawing of G on S without intersecting edges such that each connected component of the complement of the graph (i.e. each *face*) is homeomorphic to a disc. Consider the embeddings of G in a surface S of genus g, defined up to orientation preserving homeomorphisms. An embedding on S gives a cyclic order of the edges around each vertex, hence defines a combinatorial embedding. In fact, this relation is one-to-one. The embeddings of G in a surface S of genus g (defined up to orientation preserving homeomorphisms) are in one-to-one correspondence with the combinatorial embeddings $\mathcal{G} = (H, \sigma, \alpha)$ of G with *Euler characteristic* $\chi_{\mathcal{G}} = 2 - 2g$ (up to isomorphisms) [17, Thm. 3.2.4], where the Euler characteristic $\chi_{\mathcal{G}}$ is defined as the number of cycles of σ, plus the number of cycles of $\sigma\alpha$, minus the number of cycles of α.

In the following, we use the terms *combinatorial maps* and *embedded graphs* interchangeably. *We do not require our embedded graphs to be planar* (i.e. to have an Euler characteristic equal to 2).

We now define the deletion and contraction of edges in embedded graphs. Our definition preserves the *cyclic order* around each vertex. We represented the result of deleting and contracting the edge $e = \{b, b'\}$ in Figure 2.
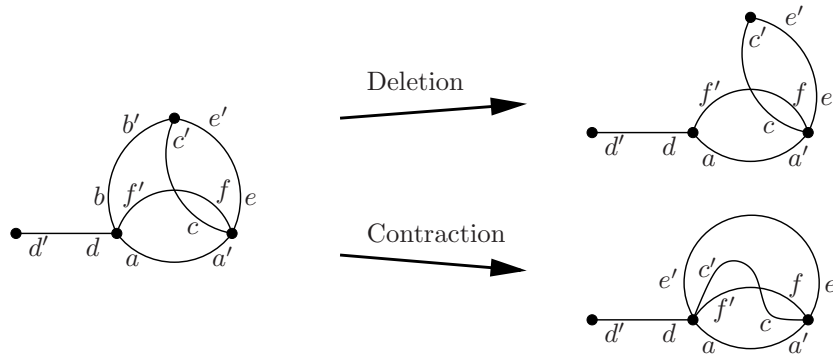

FIGURE 2. Deletion and contraction of the edge $e = \{b, b'\}$.

Let G be a graph and $\mathcal{G} = (H, \sigma, \alpha)$ one of its embeddings. Let $e = \{h_1, h_2\}$ be an edge of G. If e is not an isthmus, we define the embeddings $\mathcal{G}_{\backslash e}$ of $G_{\backslash e}$ by

$\mathcal{G}_{\backslash e} = (H', \sigma', \alpha_{\backslash e})$ where $H' = H \setminus \{h_1, h_2\}$, the involution α' is α restricted to H' and

$$\sigma_{\backslash e}(h) = \begin{vmatrix} \sigma\sigma\sigma(h) & \text{if } (\sigma(h) = h_1 \text{ and } \sigma(h_1) = h_2) \text{ or } (\sigma(h) = h_2 \text{ and } \sigma(h_2) = h_1), \\ \sigma\sigma(h) & \text{if } (\sigma(h) = h_1 \text{ and } \sigma(h_1) \neq h_2) \text{ or } (\sigma(h) = h_2 \text{ and } \sigma(h_2) \neq h_1), \\ \sigma(h) & \text{otherwise.} \end{vmatrix} \quad (1)$$

Similarly, if e is not a loop, we define the embeddings $\mathcal{G}_{/e}$ of $G_{/e}$ by $\mathcal{G}_{/e} = (H', \sigma', \alpha_{/e})$, where $H' = H \setminus \{h_1, h_2\}$, the involution α' is α restricted to H' and

$$\sigma_{/e}(h) = \begin{vmatrix} \sigma\sigma(h) & \text{if } (\sigma(h) = h_1 \text{ and } \sigma(h_2) = h_2) \text{ or } (\sigma(h) = h_2 \text{ and } \sigma(h_1) = h_1), \\ \sigma\alpha\sigma(h) & \text{if } (\sigma(h) = h_1 \text{ and } \sigma(h_2) \neq h_2) \text{ or } (\sigma(h) = h_2 \text{ and } \sigma(h_1) \neq h_1), \\ \sigma(h) & \text{otherwise.} \end{vmatrix} \quad (2)$$

Spanning trees. Let $G = (V, E)$ be a graph. A spanning subgraph of G is a graph $G' = (V, E')$ where $E' \subseteq E$. A spanning subgraph is entirely determined by its edge set and, by convenience, we will often identify the spanning subgraph with its edge set. A *tree* is a connected acyclic graph. A *spanning tree* is a spanning subgraph which is a tree.

Let T be a spanning tree of G. An edge of E is said to be *internal* if it is in T and *external* otherwise. The *fundamental cycle* (resp. *cocycle*) of an external (resp. internal) edge e is the set of edges f such that the spanning subgraph $T \setminus \{f\} \cup \{e\}$ (resp. $T \setminus \{e\} \cup \{f\}$) is a tree. Note that the fundamental cycle (resp. cocycle) of an external (resp. internal) edge e contains only internal (resp. external) edges apart from e. Moreover, if e is internal and f external, then e is in the fundamental cycle of f if and only if f is in the fundamental cocycle of e.

Lastly, for any set S we denote by $|S|$ its cardinality. If $S \subset S'$ and $s \in S'$, we write $S - s$ for $S \setminus \{s\}$ whether s belongs to S or not.

3. **The Tutte polynomial**

Definition 1. The *Tutte polynomial* of a graph $G = (V, E)$ is

$$T_G(x, y) = \sum_{S \subseteq E} (x - 1)^{c(S) - c(G)} (y - 1)^{c(S) + |S| - |V|}, \quad (3)$$

where the sum is over all spanning subgraphs S and $c(S)$ (resp. $c(G)$) denotes the number of connected components of S (resp. of G).

For example, if G is K_3 (the triangle) there are 8 spanning subgraphs. The subgraph with no edge has contribution $(x - 1)^2$, each subgraph with one edge has contribution $(x - 1)$, each subgraph with two edges has contribution 1 and the subgraph with three edges has contributions $(y - 1)$. Summing up these contributions, we get $T_{K_3}(x, y) = (x - 1)^2 + 3(x - 1) + 3 + (y - 1) = x^2 + x + y$.

From Definition 1, it is easy to check that whenever G is the disjoint union of two graphs $G = G_1 \cup G_2$, then

$$T_G(x, y) = T_{G_1}(x, y) \times T_{G_2}(x, y).$$

This relation allows us to restrict our attention to connected graphs. *From now on, all the graphs we consider are connected.*

Let us now recall the relations of induction satisfied by the Tutte polynomial [20]. (These relations reminiscent of the relations of induction of the chromatic polynomial [22] are easy to prove from (3)).

Proposition 2. *(Tutte) Let G be a graph and e be any edge of G. The Tutte polynomial of G satisfies:*

$$T_G(x,y) = \begin{vmatrix} x \cdot T_{G/e}(x,y) & \text{if } e \text{ is an isthmus,} \\ y \cdot T_{G \backslash e}(x,y) & \text{if } e \text{ is a loop,} \\ T_{G/e}(x,y) + T_{G \backslash e}(x,y) & \text{if } e \text{ is neither a loop nor an isthmus.} \end{vmatrix} \tag{4}$$

Before we close this section, we present another characterization of the Tutte polynomial that will have a great similarity with ours. This characterization, given by Tutte in 1954, uses the notion of *activity*. Let $G = (V, E)$ be a graph. Suppose that the edge set E is linearly ordered. Given a spanning tree T, an external (resp. internal) edge is said *active* if it is minimal in its fundamental cycle (resp. cocycle). We recall the following theorem by Tutte [20].

Theorem 3. *(Tutte) Let $G = (V, E)$ be a connected graph. Given any linear order on the edge set E, the Tutte polynomial of G is equal to*

$$T_G(x,y) = \sum_{T \text{ spanning tree}} x^{i(T)} y^{e(T)}, \tag{5}$$

where the sum is over all spanning trees and $i(T)$ (resp. $e(T)$) is the number of active internal (resp. external) edges.

For instance, if G is K_3 there are three spanning trees that we represented in Figure 3. If the linear order on the edge set is $a < b < c$ then the active edges are the one indicated by a $\star$. Hence, the spanning trees (taken from left to right) have respective contributions x^2, x and y and the Tutte polynomial is $T_{K_3}(x,y) = x^2 + x + y$.
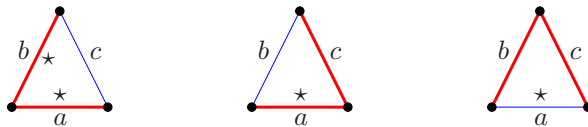

FIGURE 3. The spanning trees of K_3.

Theorem 3 is surprising because it implies that the sum in (5) does not depend on the ordering of the edge set (whereas the summands clearly depends on that order). However, this theorem is easily proved by induction. Indeed, it is simple to prove that the induction relation of Proposition 2 holds *for the edge having the largest label*.

4. The Tutte polynomial of embedded graphs

In this section, we present a new characterization of the Tutte polynomial. This characterization is, just as in Theorem 3, a generating function of the spanning trees counted according to activities (see Theorem 7). We will only change the definition of activities, using the embedding structure instead of a linear order on the edge set.

Let us begin by defining the *tour of a spanning tree*. Informally, the *tour of a tree* is a walk around the tree that follows internal edges and crosses external edges. A graphical representation of this notion is represented in Figure 4.
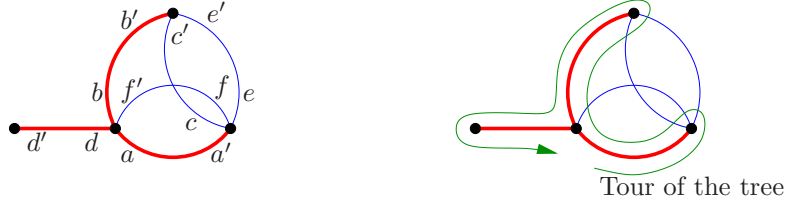

Tour of the tree

FIGURE 4. Intuitive representation of the tour of a spanning tree (indicated by thick lines).

Let $\mathcal{G} = (H, \sigma, \alpha)$ be an embedding of the graph $G = (V, E)$. Given a spanning tree T, we define a function t on the set H of half-edges by :

$$t(h) = \begin{vmatrix} \sigma(h) & \text{if } h \text{ is external,} \\ \sigma\alpha(h) & \text{if } h \text{ is internal.} \end{vmatrix} \tag{6}$$

The mapping t is called the *motion function*. The motion function t is a bijection on H (since the inverse mapping is easily seen to be $t^{-1}(h) = \sigma^{-1}(h)$ if $\sigma^{-1}(h)$ is external and $t^{-1}(h) = \alpha\sigma^{-1}(h)$ if $\sigma^{-1}(h)$ is internal). In fact, as we will prove shortly (Lemma 5), the motion function t is a cyclic permutation. For instance, for the embedded graph of Figure 4, the motion function is the cycle $(a, e, f, c, a', f', b, c', e', b', d, d')$. The cyclic order defined by the motion function t on the set of half-edges is what we call the *tour of the tree T*. Note that turning around a spanning tree and writing down the half-edges in order of appearance gives an encoding of the tree. This encoding is closely related to the encodings of maps with a distinguished spanning tree given by Lehman and Walsh [16].

Before we prove that the motion function t is a cyclic permutation, we describe what happens to this function when an edge is deleted or contracted.

Lemma 4. *Let $\mathcal{G} = (H, \sigma, \alpha)$ be an embedded graph, T a spanning tree and t the corresponding motion function. For all external (resp. internal) edge $e = \{h_1, h_2\}$, the spanning tree T (resp. $T - e$) of $\mathcal{G}_{\backslash e}$ (resp. $\mathcal{G}_{/e}$) defines a motion function t' on $H \setminus \{h_1, h_2\}$ such that*

$$t'(h) = \begin{vmatrix} ttt(h) & \text{if } (t(h) = h_1 \text{ and } t(h_1) = h_2) \text{ or } (t(h) = h_2 \text{ and } t(h_2) = h_1), \\ tt(h) & \text{if } (t(h) = h_1 \text{ and } t(h_1) \neq h_2) \text{ or } (t(h) = h_2 \text{ and } t(h_2) \neq h_1), \\ t(h) & \text{otherwise.} \end{vmatrix}$$

Proof: Lemma 4 follows immediately from the definitions and Equations (1) and (2).   □

Remark: Another way of stating Lemma 4 is to say that the cycles of the permutation t' are obtained from the cycles of t by erasing h_1 and h_2. Consider, for instance, the embedded graph and the spanning tree represented in Figure 4. The motion function is the cycle $t = (a, e, f, c, a', f', b, c', e', b', d, d')$. If we delete the edge the external edge $\{e, e'\}$ (resp. internal edge $\{b, b'\}$), the motion function becomes $t' = (a, f, c, a', f', b, c', b', d, d')$ (resp. $t' = (a, e, f, c, a', f', c', e', d, d')$).

Lemma 5. *For any embedded graph and any spanning tree, the motion function is a cyclic permutation.*

Proof: We prove the lemma by induction on the number of edges of the graph. The property is obviously true for the graph reduced to a loop and the graph reduced to an isthmus. We assume the property holds for all graphs with at most $n \geq 1$ edges and consider an embedded graph $\mathcal{G}$ with $n + 1$ edges. Let T be a spanning tree and t the corresponding motion function. We know that t is a permutation, that is, a product of cycles. We consider an edge $e = \{h_1, h_2\}$.

• *In any cycle of the motion function t, there is a half-edge $h \neq h_1, h_2$.*

First note that $t(h_i) \neq h_i$ for $i = 1, 2$. Indeed, if e is external, this would mean $\sigma(h_i) = h_i$ which is excluded or e would be an isthmus not in the spanning tree. Similarly, if e is internal, we would have $\sigma\alpha(h_i) = h_i$ which is excluded or e would be a loop in the spanning tree. Moreover, we cannot have $(t(h_1) = h_2$ and $t(h_2) = h_1)$. Indeed, this would mean that e is either an isolated loop (if e is external) or an isolated isthmus (if e is internal) contradicting our hypothesis that $\mathcal{G}$ is connected and has more than one edge.

• *The motion function is cyclic.*

If e is external (resp. internal), we consider the motion function t' defined by the spanning tree T (resp. $T - e$) on $\mathcal{G}_{\backslash e}$ (resp. $\mathcal{G}_{/e}$). By Lemma 4, the cycles of t' are the cycles of t where the half-edges h_1, h_2 are erased. Suppose now that t is not cyclic. Then t has at least two cycles each containing a half-edge $h \neq h_1, h_2$. Therefore, t' has at least two non-empty cycles, which contradicts our induction hypothesis. $\square$

Consider a rooted embedded graph $\mathcal{G} = (H, \sigma, \alpha, h)$ and a spanning tree T. We define a linear order on H by $h < t(h) < t^2(h) < \ldots < t^{|H|-1}(h)$, where t is the motion function. (This defines a linear order on the set of half-edges since t is a cyclic permutation.) We call $(\mathcal{G}, T)$-*order* this order. We define a $(\mathcal{G}, T)$-order on the edge set by setting $e = \{h_1, h_2\} < e' = \{h'_1, h'_2\}$ if $\min(h_1, h_2) < \min(h'_1, h'_2)$. Note that this is also a linear order.

We are now ready to define a new notion of activity (that we call *embedding-activity*, so as to differentiate it from Tutte's notion of activity) and state our main theorem.

Definition 6. Let $\mathcal{G}$ be a rooted embedded graph and T be a spanning tree. We say that an external (resp. internal) edge is $(\mathcal{G}, T)$-*active* (or *embedding-active* if $\mathcal{G}$ and T are clear from the context) if it is minimal for the $(\mathcal{G}, T)$-order in its fundamental cycle (resp. cocycle).

Example: Consider the embedded graph $\mathcal{G}$ and spanning tree T represented in Figure 4. Take the half-edge a as the root of $\mathcal{G}$. The $(\mathcal{G}, T)$-order on the half-edges is $a < e < f < c < a' < f' < b < c' < e' < b' < d < d'$. Therefore, the $(\mathcal{G}, T)$-order on the edges is $\{a, a'\} < \{e, e'\} < \{f, f'\} < \{c, c'\} < \{b, b'\} < \{d, d'\}$. The internal active edges are $\{a, a'\}$ and $\{d, d'\}$ and there is no external active edge. For instance, $\{e, e'\}$ is not active since $\{a, a'\}$ is in its fundamental cycle.

Theorem 7. *Let $\mathcal{G}$ be any rooted embedding of the connected graph G (with at least one edge). The Tutte polynomial of G is equal to*

$$T_G(x, y) = \sum_{T \ spanning \ tree} x^{\mathcal{I}(T)} y^{\mathcal{E}(T)}, \tag{7}$$

where the sum is over all spanning trees and $\mathcal{I}(T)$ (resp. $\mathcal{E}(T)$) is the number of embedding-active internal (resp. external) edges.

We represented the spanning trees of K_3 in Figure 5. If the embedding is rooted on the half-edge a, then the embedding-active edges are the one indicated by a $\star$. Hence, the spanning trees (taken from left to right) have respective contributions x, x^2 and y and the Tutte polynomial is $T_{K_3}(x, y) = x^2 + x + y$.
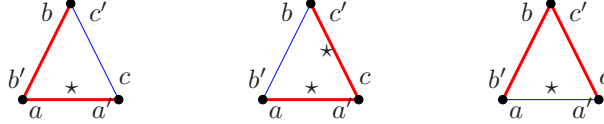

FIGURE 5. The embedding-activities of the spanning trees of K_3.

Let us emphasize that Theorem 7 is *not* a special case of Theorem 3. Indeed, the $(\mathcal{G}, T)$-order defined above is a linear order on the edge set that *depends* on the tree T.

The rest of this section is devoted to the proof of Theorem 7.

Lemma 8. *Let $\mathcal{G}$ be a rooted embedded graph with edge set E and half-edge set H. Let T be a spanning tree and $e = \{h_1, h_2\}$ be an edge not containing the root. If e is external (resp. internal), the $(\mathcal{G}_{\backslash e}, T)$-order (resp. $(\mathcal{G}_{/e}, T - e)$-order) on $H \backslash \{h_1, h_2\}$ and $E - e$ is simply the restriction of the $(\mathcal{G}, T)$-order to these sets.*

Proof of Lemma 8: By Lemma 4, we see that if e is external (resp. internal), the $(\mathcal{G}_{\backslash e}, T)$-order (resp. $(\mathcal{G}_{/e}, T - e)$-order) on the half-edge set $H \backslash \{h_1, h_2\}$ is simply the restriction of the $(\mathcal{G}, T)$-order to this set. The same property follows immediately for the edge set. $\square$

Proof of Theorem 7: We associate to the rooted embedded graph $\mathcal{G}$ the polynomial

$$\mathcal{T}_{\mathcal{G}}(x, y) = \sum_{T \text{ spanning tree}} x^{\mathcal{I}(T)} y^{\mathcal{E}(T)},$$

where $\mathcal{I}(T)$ (resp. $\mathcal{E}(T)$) is the number of embedding-active internal (resp. external) edges. We want to show that the polynomial $\mathcal{T}_{\mathcal{G}}(x, y)$ is equal to the Tutte polynomial $T_G(x, y)$ of G. We proceed by induction on the number of edges, using Proposition 2.

• The graphs with one edge are the graph L reduced to a loop and the graph I reduced to an isthmus. The graph L (resp. I) has a unique rooted embedding $\mathcal{L}$ (resp. $\mathcal{I}$). We check that $T_L(x, y) = y = \mathcal{T}_{\mathcal{L}}(x, y)$ and $T_I(x, y) = x = \mathcal{T}_{\mathcal{I}}(x, y)$.

• We assume the property holds for all (connected) graphs with at most $n \geq 1$ edges and consider a rooted embedding $\mathcal{G} = (H, \sigma, \alpha, h_0)$ of a graph G with $n + 1$ edges. We denote by v_0 the vertex incident to the root h_0 and e_0 the edge containing h_0. We denote by $h_* = \sigma^{-1}(h_0)$ the half-edge preceding h_0 around v_0 and by $e_* = \{h_*, h_*'\}$ the edge containing h_*.

We study separately the 3 different cases of the induction relation (4).

Case 1: *The edge e_* is neither an isthmus nor a loop.*

The set $\mathbb{T}$ of spanning trees of G can be partitioned into $\mathbb{T} = \mathbb{T}_1 \cup \mathbb{T}_2$, where $\mathbb{T}_1$

(resp. $\mathbb{T}_2$) is the set of spanning trees containing (resp. not containing) the edge e_*. The set $\mathbb{T}_1$ (resp. $\mathbb{T}_2$) is in bijection by the mapping $\Phi_1 : T \mapsto T - e_*$ (resp. $\Phi_2 : T \mapsto T$) with the spanning trees of $G_{/e_*}$ (resp. $G_{\backslash e_*}$). We want to show e_* is never embedding-active and that the mappings Φ_i preserve the embedding-activities: for any tree T in $\mathbb{T}_1$ (resp. $\mathbb{T}_2$), an edge is $(\mathcal{G}, T)$-active if and only if it is $(\mathcal{G}_{/e_*}, T - e_*)$-active (resp. $(\mathcal{G}_{\backslash e_*}, T)$-active). We are going to prove successively the following four points:

- *The edges e_* and e_0 are distinct.*
  First note that $h_0 \neq h_*$ or we would have $\sigma(h_*) = h_*$ implying that v_0 has degree one hence that e_* is an isthmus. Also, $h_0 \neq h'_*$ or we would have $\sigma(h_*) = \alpha(h_*)$ implying that e_* is a loop. Thus, $e_* = \{h_*, h'_*\}$ does not contain h_0.

- *Given any spanning tree, the edge e_* is maximal in its fundamental cycle or cocycle.*
  Let T be a spanning tree of G. Suppose first that the edge e_* is internal. In this case, the motion function t satisfies, $t(h'_*) = \sigma\alpha(h'_*) = h_0$. Hence, h'_* is the greatest half-edge for the $(\mathcal{G}, T)$-order. Let C_* be the fundamental cocycle of e_*. Recall that, apart from e_*, the edges in C_* are external. Let e be one of those edges. We want to prove that $e < e_*$ for the $(\mathcal{G}, T)$-order. Consider the embedded graph $\mathcal{G}'$ (with two edges) obtained by deleting all external edges distinct from e and contracting all internal edges distinct from e_*. We apply the same operations to the tree T, thus obtaining $T' = \{e_*\}$. By induction, we see that in $(\mathcal{G}', T')$ the edge e is in the fundamental cocycle of e_*. Hence, the only possibility for $\mathcal{G}'$ is the embedded graph represented on the left of Figure 6. The motion function of $(\mathcal{G}', T')$ is the cycle (h, h_*, h', h'_*) where h, h' are the half-edges of e. By (repeated application of) Lemma 4, the half-edges h, h_*, h', h'_* appear in the same cyclic order around the spanning tree T of $\mathcal{G}$. Moreover, since h'_* is the greatest half-edge for the $(\mathcal{G}, T)$-order, we have $h < h_* < h' < h'_*$ hence $e < e_*$ for the $(\mathcal{G}, T)$-order. Thus, e_* is maximal (for the $(\mathcal{G}, T)$-order) in its fundamental cocycle.
  Similarly, if e_* is external, h_* is the greatest half-edge for the $(\mathcal{G}, T)$-order (since $t(h_*) = \sigma(h_*) = h_0$). Then, we consider the fundamental cycle C_* of e_*. For any edge e in $C_* - e_*$, we consider the embedded graph $\mathcal{G}'$ obtained by deleting all external edges distinct from e_* and contracting all internal edges distinct from e. Applying the same operations to the tree T, we obtain $T' = \{e\}$. We see that the only possibility for $\mathcal{G}'$ is the embedded graph represented on the right of Figure 6. The motion function of $(\mathcal{G}', T')$ is the cycle (h, h'_*, h', h_*) where h, h' are the half-edges of e. We conclude as above.

FIGURE 6. The only possibility for $\mathcal{G}'$ when e_* is internal (left) or external (right).

- *For any tree T in $\mathbb{T}_1$ (resp. $\mathbb{T}_2$), the $(\mathcal{G}, T)$-active and $(\mathcal{G}_{/e_*}, T - e_*)$-active (resp. $(\mathcal{G}_{\backslash e_*}, T)$-active) edges are the same.*
  First note that e_* is never alone in its fundamental cycle or cocycle (or e_* would be a loop or isthmus). Hence, by the preceding point, e_* is never embedding-active. We now look at the embedding-activities of the other edges. Let T be a tree in $\mathbb{T}_1$ (i.e. containing e_*). Let e be an external (resp. internal) edge distinct from e_* and let C be its fundamental cycle (resp. cocycle). The fundamental cycle (resp. cocycle) of e in $(\mathcal{G}_{/e_*}, T - e_*)$ is $C - e_*$. Note that the $(\mathcal{G}, T)$-minimal element of C is in $C - e_*$ (since, if e_* is in C then e is in the fundamental cycle of e_* hence $e < e_*$ for the $(\mathcal{G}, T)$-order). Moreover, by Lemma 8, the $(\mathcal{G}, T)$-order and $(\mathcal{G}_{/e_*}, T - e_*)$-order coincide on $C - e_*$. Hence, the $(\mathcal{G}, T)$-minimal element of C is the $(\mathcal{G}_{/e_*}, T - e_*)$-minimal element in $C - e_*$. Therefore, the edge e is $(\mathcal{G}, T)$-active if and only if it is $(\mathcal{G}_{/e_*}, T - e_*)$-active.
  The case where T is a tree in $\mathbb{T}_2$ (i.e. not containing e_*) is identical.
- *The polynomial $\mathcal{T}_{\mathcal{G}}(x, y)$ is equal to the Tutte polynomial $T_G(x, y)$.*
  From the properties above, we have

  $$
  \begin{aligned}
  \mathcal{T}_{\mathcal{G}}(x, y) \quad &\equiv \sum_{T \text{ spanning tree of } G} x^{\mathcal{I}(T)} y^{\mathcal{E}(T)} \\
  &= \sum_{T \in \mathbb{T}_1} x^{\mathcal{I}(T)} y^{\mathcal{E}(T)} + \sum_{T \in \mathbb{T}_2} x^{\mathcal{I}(T)} y^{\mathcal{E}(T)} \\
  &= \sum_{T \in \mathbb{T}_1} x^{\mathcal{I}'(T - e_*)} y^{\mathcal{E}'(T - e_*)} + \sum_{T \in \mathbb{T}_2} x^{\mathcal{I}''(T)} y^{\mathcal{E}''(T)}
  \end{aligned} \tag{8}
  $$

  where $\mathcal{I}'(T - e_*)$, $\mathcal{E}'(T - e_*)$, $\mathcal{I}''(T)$, $\mathcal{I}''(T)$ are respectively the number of internal $(\mathcal{G}_{/e_*}, T - e_*)$-active, external $(\mathcal{G}_{/e_*}, T - e_*)$-active, $(\mathcal{G}_{\backslash e_*}, T)$-active and external $(\mathcal{G}_{\backslash e_*}, T)$-active edges.
  In the right-hand side of (8) we recognize the polynomials $\mathcal{T}_{\mathcal{G}_{/e_*}}(x, y)$ and $\mathcal{T}_{\mathcal{G}_{\backslash e_*}}(x, y)$. By the induction hypothesis, these polynomials are the Tutte polynomials $T_{G_{/e_*}}(x, y)$ and $T_{G_{\backslash e_*}}(x, y)$. Thus,

  $$
  \mathcal{T}_{\mathcal{G}}(x, y) \;=\; \mathcal{T}_{\mathcal{G}_{/e_*}}(x, y) + \mathcal{T}_{\mathcal{G}_{\backslash e_*}}(x, y) = T_{G_{/e_*}}(x, y) + T_{G_{\backslash e_*}}(x, y). \tag{9}
  $$

  In view of the induction relation of Proposition 2, this is the Tutte polynomial $T_G(x, y)$.

Case 2: *The edge e_* is an isthmus.*
Since e_* is an isthmus, it is in every spanning tree. Moreover, being alone in its fundamental cocycle, it is always active. We want to show that for any spanning tree T, the embedding-activity of any edge other than e_* is the same in $(\mathcal{G}, T)$ and in $(\mathcal{G}_{/e_*}, T - e_*)$. Before we do that, we must cope with a (little) technical difficulty: the edge e_* might be equal to e_0 in which case we should specify how to root the graph $\mathcal{G}_{/e_*}$.
First note that $h_0 \neq h'_*$ or we would have $\sigma(h_*) = \alpha(h_*)$ implying that e_* is a loop. Suppose now that $h_0 = h_*$ (equivalently, $\sigma(h_*) = h_*$). In this case, we define the root of $\mathcal{G}_{/e_*}$ to be $h_1 = \sigma(h'_*)$ (h_1 is not an half-edge of e_* or e_* would be an isolated isthmus).

- *For any spanning tree T of G, the $(\mathcal{G}, T)$-order and the $(\mathcal{G}_{/e_*}, T - e_*)$-order coincide on $E - e_*$.*
  If $e_* \neq e_0$ the property is given by Lemma 8. Now suppose that $e_* = e_0$ (that is $h_* = h_0$). Since e_* is internal, the motion function t satisfies

$t(h'_*) = h_0$ and $t(h_0) = h_1$. Therefore, the $(\mathcal{G}, T)$-order on half-edges is $h_0 = h_* < h_1 < t(h_1) < \ldots < h'_*$. Let us denote by $\mathcal{G}_1$ the embedded graph $\mathcal{G}$ rooted on h_1. The $(\mathcal{G}_1, T)$-order on half-edges is $h_1 < t(h_1) < \ldots < h'_* < h_0 = h_*$. Thus, the $(\mathcal{G}_1, T)$-order and $(\mathcal{G}, T)$-order coincide on $E - e_*$. Moreover, by Lemma 8, the $(\mathcal{G}_1, T)$-order and $(\mathcal{G}_{/e_*}, T)$-order coincide on $E - e_*$.

- *For any spanning tree T, the set of $(\mathcal{G}, T)$-active edges distinct from e_* is the set of $(\mathcal{G}_{/e_*}, T - e_*)$-active edges.*
  For any tree T and any external (resp. internal) edge $e \neq e_*$, the fundamental cycle (resp. cocycle) of e does not contain e_* and is the same in $(\mathcal{G}, T)$ and in $(\mathcal{G}_{/e_*}, T - e_*)$. Since the $(\mathcal{G}, T)$-order and the $(\mathcal{G}_{/e_*}, T - e_*)$-order coincide on $E - e_*$, the edge e is $(\mathcal{G}, T)$-active if and only if it is $(\mathcal{G}_{/e_*}, T - e_*)$-active.

- *The polynomial $\mathcal{T}_\mathcal{G}(x, y)$ is equal to the Tutte polynomial $T_G(x, y)$.*
  From the properties above, we have

$$
\begin{aligned}
\mathcal{T}_\mathcal{G}(x, y) &\equiv \sum_{T \text{ spanning tree of } G} x^{\mathcal{I}(T)} y^{\mathcal{E}(T)} \\
&= \sum_{T \text{ spanning tree of } G} x^{1 + \mathcal{I}'(T - e_*)} y^{\mathcal{E}'(T - e_*)} \\
&= x \cdot \sum_{T \text{ spanning tree of } G} x^{\mathcal{I}'(T - e_*)} y^{\mathcal{E}'(T - e_*)} \quad (10)
\end{aligned}
$$

where $\mathcal{I}'(T - e_*)$ and $\mathcal{E}'(T - e_*)$ are respectively the number of internal $(\mathcal{G}_{/e_*}, T - e_*)$-active and external $(\mathcal{G}_{/e_*}, T - e_*)$-active edges.
In the right-hand side of (10) we recognize the sum as being $\mathcal{T}_{\mathcal{G}_{/e_*}}(x, y)$. By the induction hypothesis, we know this polynomial to be equal to the Tutte polynomial $T_{G_{/e_*}}(x, y)$. Thus,

$$
\mathcal{T}_\mathcal{G}(x, y) = x \cdot \mathcal{T}_{\mathcal{G}_{/e_*}}(x, y) = x \cdot T_{G_{/e_*}}(x, y). \quad (11)
$$

In view of the induction relation of Proposition 2, this is the Tutte polynomial $T_G(x, y)$.

Case 3: *The edge e_* is a loop.*
This case is dual to Case 2.
Since e_* is a loop, it is always external and always active. We want to show that for any spanning tree T, the embedding-activity of any edge other than e_* is the same in $(\mathcal{G}, T)$ and in $(\mathcal{G}_{\backslash e_*}, T)$. Before we do that, we must choose a root for $\mathcal{G}_{\backslash e_*}$ when $e_* = e_0$. We see that $h_0 \neq h_*$ or we would have $\sigma(h_*) = h_*$ implying that e_* is an isthmus. Suppose now that $h_0 = h'_*$ (equivalently, $\alpha(h_*) = \sigma(h_*)$). In this case, we define the root of $\mathcal{G}_{\backslash e_*}$ to be $h_1 = \sigma(h_0)$ (h_1 is not an half-edge of e_* or e_* would be an isolated loop).

- *For any spanning tree T of G, the $(\mathcal{G}, T)$-order and the $(\mathcal{G}_{\backslash e_*}, T)$-order coincide on $E - e_*$.*
  The proof of Case 2 can be copied verbatim except "e_* is internal" is replaced by "e_* is external".

- *For any spanning tree T, the set of $(\mathcal{G}, T)$-active edges distinct from e_* is the set of $(\mathcal{G}_{\backslash e_*}, T)$-active edges.*
  The proof of Case 2 can be copied verbatim.

- *The polynomial $\mathcal{T}_{\mathcal{G}}(x, y)$ is equal to the Tutte polynomial $T_G(x, y)$.*
  From the properties above, we have

$$
\begin{aligned}
\mathcal{T}_{\mathcal{G}}(x, y) &\equiv \sum_{T \text{ spanning tree of } G} x^{\mathcal{I}(T)} y^{\mathcal{E}(T)} \\
&= y \cdot \sum_{T \text{ spanning tree of } G} x^{\mathcal{I}''(T)} y^{\mathcal{E}''(T)} \tag{12}
\end{aligned}
$$

  where $\mathcal{I}''(T)$ and $\mathcal{E}''(T)$ are respectively the number of internal $(\mathcal{G}_{\backslash e_*}, T)$-active and external $(\mathcal{G}_{\backslash e_*}, T)$-active edges.
  In the right-hand side of (12) we recognize the sum as being $\mathcal{T}_{\mathcal{G}_{\backslash e_*}}(x, y)$. By the induction hypothesis, we know this polynomial to be equal to the Tutte polynomial $T_{G_{\backslash e_*}}(x, y)$. Thus,

$$
\mathcal{T}_{\mathcal{G}}(x, y) = y \cdot \mathcal{T}_{\mathcal{G}_{\backslash e_*}}(x, y) = y \cdot T_{G_{\backslash e_*}}(x, y). \tag{13}
$$

  In view of the induction relation of Proposition 2, this is the Tutte polynomial $T_G(x, y)$.

$\square$

5. **Concluding remarks**

We conclude this paper by mentioning some possible applications of our characterization of the Tutte polynomial. Some of them will be developed in a forthcoming paper [4].

The characterization of the Tutte polynomial in terms of the activities of spanning trees is sometimes thought of as slightly unnatural. It is true that the dependence of this characterization on a particular linear ordering of the edge set is a bit puzzling. We want to argue that an embedding may be a less arbitrary structure than a linear order on the edge set. As a matter of fact, there are a number of problems in which the embedding structure is explicitly given.
Firstly, some famous conjectures deal with the Tutte polynomial, or sometimes the chromatic polynomial, of *planar graphs*. A graph is *planar* if and only if can be drawn on the plane without intersecting edges. Equivalently, it has an embedding (H, σ, α) with Euler characteristic equal to 2. For instance, the four color theorem can be stated as: $\mathcal{T}_{\mathcal{G}}(-3, 0) \neq 0$ *for any loopless planar embedding $\mathcal{G}$.*
Another type of problem where the embedding structure is explicitly given appears in mathematical physics, in the study of the *Potts model on random lattices* [2, 6, 10]. The *Potts model* is an important statistical mechanics model for particles interacting in a discrete space (i.e. a graph) [1, 19]. It was shown by Fortuin and Kasteleyn [11] that the partition function of the Potts model on the graph G is equivalent (up to a change of variables) to the Tutte polynomial of G. Studying this model on a *random lattice* means that the underlying space (the graph) is random. Usually, the underlying space is supposed to have a uniform distribution over a class of rooted maps. In this case, the partition function of the model is equivalent to the sum of the Tutte polynomial over the class of maps. For instance, if the *random lattice* is understood as the uniform distribution on the set of rooted planar maps with n edges, we are to study the partition function

$$
Z_n(x, y) = \sum_{\mathcal{G} \text{ rooted map with } n \text{ edges}} \mathcal{T}_{\mathcal{G}}(x, y).
$$

Note that this partition function can also be written as

$$Z_n(x, y) = \sum_{\mathcal{G},\ T} x^{\mathcal{I}(\mathcal{G},T)} y^{\mathcal{E}(\mathcal{G},T)},$$

where the sum is over all rooted maps $\mathcal{G}$ with n edges and all spanning trees T and $\mathcal{I}(\mathcal{G}, T)$ (resp. $\mathcal{E}(\mathcal{G}, T)$) is the number of $(\mathcal{G}, T)$-active internal (resp. external) edges. This last expression of the partition function could be interesting since there are several nice encodings for *rooted maps with a distinguished spanning tree* (a.k.a. *tree-rooted maps*) [3, 16, 18].

Let us also mention that embedding structures can be used to define several bijections between spanning trees and some other structures counted by the Tutte polynomial, including root-connected out-degree sequences and recurrent configurations of the sandpile model [4]. Our characterization of the Tutte polynomial is deeply related to these bijections and gives a combinatorial interpretation of several enumerative results.

References

[1] R.J. Baxter. *Exactly solved model in statistical mechanics*. Academic Press, 1982.

[2] R.J. Baxter. Dichromatic polynomials and Potts models summed over rooted maps. *Ann. Comb.*, 5:17–36, 2001.

[3] O. Bernardi. Bijective counting of tree-rooted maps and shuffles of parenthesis systems. Submitted. See also *ArXiv: math.CO/0601684*.

[4] O. Bernardi. Tutte polynomial, degree sequences and sandpile model: new connections via combinatorial embedding. In preparation.

[5] B. Bollobás. *Modern Graph Theory*. Springer (Graduate Texts in Mathematics), 1998.

[6] G. Bonnet and B. Eynard. The Potts-q random matrix model: loop equations, critical exponents, and rational case. *Phys. Lett. B*, 463:273–279, 1999.

[7] T. Brylawsky and J. Oxley. *The Tutte polynomial and its applications*, chapter 6 in Matroid Applications. Cambridge University Press, 1991.

[8] R. Cori. Un code pour les graphes planaires et ses applications. *Soc. Math. de France*, Astérisque 27, 1975.

[9] R. Cori and A. Machì. Maps, hypermaps and their automorphisms: a survey. I, II, III. *Exposition. Math.*, 10(5):403–467, 1992.

[10] J.M. Daul. Q-states Potts model on a random planar lattice. *ArXiv: hep-th/9502014*, 1995.

[11] C.M. Fortuin and P.W. Kasteleyn. On the random cluster model: I. Introduction and relation to other models. *Physica*, 57:536–564, 1972.

[12] I.M. Gessel. Enumerative applications of a decomposition for graphs and digraphs. *Discrete Math.*, 139:257–271, 1995.

[13] I.M. Gessel and B.E. Sagan. The Tutte polynomial of a graph, depth-first search, and simplicial complex partitions. *Electron. J. Combin.*, 3(2):R9, 1996.

[14] I.M. Gessel and D.L. Wang. Depth-first search as a combinatorial correspondence. *J. Combin. Theory Ser. A*, 26:308–313, 1979.

[15] E. Gioan and M. Las Vergnas. Activity preverving bijections between spanning trees and orientations in graphs. *Discrete Math.*, 298:169–188, 2005.

[16] A.B. Lehman and T.R.S. Walsh. Counting rooted maps by genus II. *J. Combin. Theory Ser. B*, 13:122–141, 1972.

[17] B. Mohar and C. Thomassen. *Graphs on surfaces*. J. Hopkins Univ. Press, 2001.

[18] R.C. Mullin. On the enumeration of tree-rooted maps. *Canad. J. Math.*, 19:174–183, 1967.

[19] A. D. Sokal. The multivariate Tutte polynomial (alias Potts model) for graphs and matroids. *Surveys in Combinatorics (Cambridge University Press)*, pages 173–226, 2005. See also *ArXiv: math.CO/0503607*.

[20] W.T. Tutte. A contribution to the theory of chromatic polynomials. *Canad. J. Math.*, 6:80–91, 1954.

[21] M. Las Vergnas. The Tutte polynomial of a morphism of matroids II. Activities of orienta-
     tions. *Progress in Graph Theory (Waterloo, Ont.)*, pages 367–380, 1982.
[22] H. Whitney. The coloring of graphs. *Ann. Math.*, 33:688–718, 1932.

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