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HYDRODYNAMIC TURBULENCE AND INTERMITTENT RANDOM FIELDS

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Abstract. In this article, we construct two families of multifractal random vector fields with non symmetrical increments. We discuss the use of such families to model the velocity field of turbulent flows.

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1. Introduction

Roughly observed, some random phenomena seem scale invariant. This is the case for the velocity field of turbulent flows or the (logarithm of) evolution in time of the price of a financial asset. However, a more precise empirical study of these phenomena displays in fact a weakened form of scale invariance commonly called multifractal scale invariance or intermittency (the exponent which governs the power law scaling of the process or field is no longer linear). An important question is therefore to construct intermittent random fields which exhibit the observed characteristics.

Following the work of Kolmogorov and Obukhov ([9], [12]) on the energy dissipation in turbulent flows, Mandelbrot introduced in [10] a "limit-lognormal" model to describe turbulent dissipation or the volatility of a financial asset. This model was rigorously defined and studied in a mathematical framework by Kahane in [8]; more precisely, Kahane constructed a random measure called Gaussian multiplicative chaos. A natural extension of this work is to use Gaussian multiplicative chaos to construct a field (or a process in the financial case) which describes the whole phenomenon: the velocity field in turbulent flows (the price of an asset on a financial market). This extension was first performed by Mandelbrot himself who proposed to model the price of a financial asset with a time changed Brownian motion, the time change being random and independant of the Brownian motion. In [2], the

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authors proposed for the time change to take the primitive of multiplicative chaos: this gives the so called multifractal random walk model (MRW) (Bacry and Muzy later generalized the construction of the MRW model in [3]). The obtained process accounts for many observed properties of financial assets.

The drawback of the above construction and of the MRW model is that the laws of the increments are symmetrical. In the case of finance, this is in contradiction with the skewness property observed for certain asset prices. In the case of turbulence, the laws of the increments must be nonsymmetrical: it is a theoretical necessity and stems from the dissipation of the kinetic energy ([7]). In light of these observations, we are naturally led to construct random fields which generalize to any dimension such process and which present multifractal scale invariance as well as nonsymmetrical increments.

We will answer a very natural question: how can one obtain a family of multifractal fields with nonsymmetrical increments by perturbing a given scale invariant Gaussian random field on $\mathbb{R}^d$? Finally, in the last part we will mention the difficulties which arise in trying to construct an incompressible multifractal velocity field that verifies the 4/5-law of Kolmogorov with positive dissipation.

2. Notations and preliminary results

2.1. **The underlying Gaussian field.** Let $dW_0(x)$ denote the Gaussian white noise on $\mathbb{R}^d$ and $\varphi : \mathbb{R}^d \to [0,1]$ denote a C^∞, radially symmetric function worth 1 for $|x| \leqslant 1$ and 0 for $|x| > 2$. We also introduce a fixed correlation scale $R > 0$ and α a number which satisfies

$$d/2 < \alpha < d/2 + 1. \tag{2.1}$$

We define the Gaussian field $\mathcal{X}_g$ by the following formula:

$$\mathcal{X}_g(x) = \int_{\mathbb{R}^d} \varphi_R(x-y) \frac{x-y}{|x-y|^{d-\alpha+1}} dW_0(y), \tag{2.2}$$

where we set the following notation:

$$\varphi_R(x) = R^{d/2-\alpha} \varphi(\frac{x}{R}).$$

Using Kolmogorov's continuity criterion (see the standard book [13]), it is easy to show that (2.2) defines a homogeneous, isotropic gaussian field which is almost surely Hölderian of order $< \alpha - d/2$. Note that condition (2.1) implies that the integrand in (2.2) is square integrable and the $R^{d/2-\alpha}$ factor ensures that the field is dimensionless.

Scaling property. Let e be a unitary vector and $\lambda > 0$. We have the following identity in law:

$$\mathcal{X}_g(x+\lambda e) - \mathcal{X}_g(x) \underset{(law)}{=} \int_{\mathbb{R}^d} (\frac{\varphi_R(y) y}{|y|^{d-\alpha+1}} - \frac{\varphi_R(y-\lambda e)(y-\lambda e)}{|y-\lambda e|^{d-\alpha+1}}) dW_0(y).$$

From the Gaussianity of the above law, we deduce that for all $q > 0$, there exists $c_q > 0$ such that:

$$E(|\mathcal{X}_g(x+\lambda e) - \mathcal{X}_g(x)|^q) = \sigma_{\lambda e}^q c_q,$$

with

$$\begin{aligned}\sigma_{\lambda e}^2 &= \int_{\mathbb{R}^d} (\frac{\varphi_R(y)y}{|y|^{d-\alpha+1}} - \frac{\varphi_R(y-\lambda e)(y-\lambda e)}{|y-\lambda e|^{d-\alpha+1}})^2 dy \\ &= \lambda^{2\alpha-d} \int_{\mathbb{R}^d} (\frac{\varphi_R(\lambda z)z}{|z|^{d-\alpha+1}} - \frac{\varphi_R(\lambda z-\lambda e)(z-e)}{|z-e|^{d-\alpha+1}})^2 dz \\ &\underset{\lambda\to 0}{\sim} (\frac{\lambda}{R})^{2\alpha-d} \int_{\mathbb{R}^d} |\frac{z}{|z|^{d-\alpha+1}} - \frac{z-e}{|z-e|^{d-\alpha+1}}|^2 dz.\end{aligned}$$

We thus derive the following scaling

$$E(|\mathcal{X}_g(x+\lambda e) - \mathcal{X}_g(x)|^q) \underset{\lambda\to 0}{\sim} (\frac{\lambda}{R})^{q(\alpha-d/2)} C_q,$$

where the constant C_q is independent of e. One says that $(\mathcal{X}_g(x))_{x\in\mathbb{R}^d}$ is at small scales monofractal with scaling exponent $\alpha - d/2$.

A homogeneous and isotropic field $(\mathcal{X}(x))_{x\in\mathbb{R}^d}$ is multifractal if there exists a non linear function ζ_q such that:

$$E(|\mathcal{X}(x+\lambda e) - \mathcal{X}(x)|^q) \underset{\lambda\to 0}{\sim} (\frac{\lambda}{R})^{\zeta_q} C_q.$$

We call ζ_q the structural function of the field $(\mathcal{X}(x))_{x\in\mathbb{R}^d}$.

2.2. **Outline of the construction of multifractal vector fields from the field $\mathcal{X}_g$.** Our construction is inspired by the work of Kahane in [8]. Let $\epsilon > 0$ and $X^\epsilon(y)$ be a regular family of scalar Gaussian fields (not necessarily independent of dW_0). We consider a family of fields $\mathcal{X}_\epsilon$ (with scalar components $\mathcal{X}_\epsilon^j$ in the canonical basis) defined by:

$$\mathcal{X}_\epsilon(x) = \int_{\mathbb{R}^d} \varphi_R(x-y) \frac{x-y}{|x-y|_\epsilon^{d-\alpha+1}} e^{X^\epsilon(y)-C_\epsilon} dW_0(y) \tag{2.3}$$

($|x-y|_\epsilon$ is defined in the next subsection and is given by a standard convolution). For an appropriate family X^ϵ, we show that it is possible to find constants C_ϵ such that $\mathcal{X}_\epsilon$ tends to a non trivial field $\mathcal{X}$ (with scalar components $\mathcal{X}^j$ in the canonical basis) as ϵ tends to 0. If one chooses X^ϵ independent of dW_0, we will see that this leads to a field $\mathcal{X}$ that extends the model introduced by Bacry in [2] and that has symmetrical increments. Thus, to obtain nonsymmetrical increments, we must introduce correlation between X^ϵ and dW_0.

2.3. **Notations and construction of the family X^ϵ.** Let k^R be the function

$$k^R(x) = \begin{cases} \frac{1}{|x|^{d/2}} & \text{for } |x| \leqslant R, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\theta(x)$ be a C^∞, non negative and radialy symmetrical function with compact support in $|x| \leqslant 1$ such that

$$\int_{\mathbb{R}^d} \theta(x)dx = 1.$$

We define $\theta^\epsilon = \frac{1}{\epsilon^d}\theta(\frac{\cdot}{\epsilon})$ and the corresponding convolutions:

$$k^R_\epsilon = \theta^\epsilon * k^R, \qquad |.|_\epsilon = \theta^\epsilon * |.|.$$

Let γ be a strictly positive parameter and dW be a gaussian white noise on $\mathbb{R}^d$. We consider the following gaussian field:

$$X^\epsilon(y) = \gamma \int_{\mathbb{R}^d} k^R_\epsilon(y-\sigma)dW(\sigma).$$

Its correlation kernel is given by:

$$E(X^\epsilon(x)X^\epsilon(y)) = \gamma^2 \rho_{\epsilon/R}(\frac{x-y}{R}),$$

where $\rho = k^1 * k^1$ and $\rho_\epsilon = \theta^\epsilon * \theta^\epsilon * \rho$. One can prove the following expansion

$$\rho(x) = \omega_d \ln^+ \frac{1}{|x|} + \phi(x),$$

where ω_d denotes the surface of the unit sphere in $\mathbb{R}^d$ and ϕ is a continuous function that vanishes for $|x| \geqslant 2$. We will note $|.|_* = \inf(1, |.|)$ and, with this definition, the previous expansion is equivalent to:

$$e^{\rho(x)} = \frac{e^{\phi(x)}}{|x|_*^{\omega_d}}.$$

One can also prove the following expansions with respect to ϵ for $\epsilon < R$:

$$k^R_\epsilon(0) = \frac{C_0}{\epsilon^{d/2}} \tag{2.4}$$

with $C_0 = \int_{|u| \leqslant 1} \frac{\theta(u)}{|u|^{d/2}}du$ and there exists a constant C_1 such that

$$\rho_{\epsilon/R}(0) = \omega_d \ln \frac{R}{\epsilon} + C_1 + o(\epsilon). \tag{2.5}$$

In the sequel, we will consider the case

$$\gamma dW = \gamma_0(\epsilon)dW_0 + \gamma_1 dW_1,$$

where dW_1 is a white noise independant of dW_0 and $\gamma_0(\epsilon)$ is a function of ϵ that will be defined later. Note that the integral in formula (2.3) has a meaning since dW_0 can be viewed as a random distribution.

2.4. **Preliminary technical results.** We remind the following integration by parts formula for gaussian vectors (cf. lemma 1.2.1 in [11]):

Lemma 2.1. *Let $(g, g_1, \dots, g_n)$ be a centered gaussian vector and $G : \mathbb{R}^n \to \mathbb{R}$ a C^1 function such that its partial derivatives have at most exponential growth. Then we have:*

$$E(gG(g_1, \dots, g_n)) = \sum_{i=1}^{n} E(gg_i)E(\frac{\partial G}{\partial x_i}(g_1, \dots, g_n)). \tag{2.6}$$

From the above formula, one can easily deduce by induction the following lemma which will be frequently used in the sequel:

Lemma 2.2. *Let $l \in N^*$ be some positive integer and $(g, g_1, \dots, g_{2l})$ a centered gaussian vector. Then:*

$$E(g_1 \dots g_{2l} e^g) = (\sum_{k=0}^{l} S_{k,l}) e^{\frac{1}{2}E(g^2)},$$

where

$$S_{k,l} = \sum_{\{i_1,\dots,i_{2k}\}\subset\{1,\dots,2l\}} \sum E(gg_{i_1}) \dots E(gg_{i_{2k}}) E(g_{i_{2k+1}} g_{i_{2k+2}}) \dots E(g_{i_{2l-1}} g_{i_{2l}}),$$

where the second sum is taken over all partitions of $\{1, \dots, 2l\} \backslash \{i_1, \dots, i_{2k}\}$ in subsets of two elements $\{i_{2p+1}, i_{2p+2}\}$.

Similarly, we get the following formula:

$$E(g_1 \dots g_{2l+1} e^g) = (\sum_{k=0}^{l} \widetilde{S}_{k,l}) e^{\frac{1}{2}E(g^2)},$$

where

$$\widetilde{S}_{k,l} = \sum_{\{i_1,\dots,i_{2k+1}\}\subset\{1,\dots,2l+1\}} \sum E(gg_{i_1}) \dots E(gg_{i_{2k+1}}) E(g_{i_{2k+2}} g_{i_{2k+3}}) \dots E(g_{i_{2l}} g_{i_{2l+1}}),$$

Remark 2.3. *In $S_{k,l}$ ($\widetilde{S}_{k,l}$), the summation is made of $\frac{2l!}{2k!2^{l-k}(l-k)!}$ ($\frac{(2l+1)!}{(2k+1)!2^{l-k}(l-k)!}$) terms, number we will denote by $\alpha_{k,l}$ ($\widetilde{\alpha}_{k,l}$).*

We will also use the following lemma essentially due to Kahane ([8]).

Lemma 2.4. *Let (T, d) be a metric space and σ a finite positive measure on T equiped with the borelian σ-field induced by d.*

Let $q : T \times T :\to \mathbb{R}_+$ a symmetric application and m a positive integer. Then we have the following inequalities:

$$\int_{T^{2m}} e^{\sum_{1 \leqslant j<k \leqslant 2m} q(t_j,t_k)} d\sigma(t_1) \dots d\sigma(t_{2m}) \leqslant \sigma(T)(\sup_{s\in T} \int_T e^{mq(t,s)} d\sigma(t))^{2m-1}, \tag{2.7}$$

$$\int_{T^{2m+1}} e^{\sum_{1\leqslant j<k\leqslant 2m+1} q(t_j,t_k)} d\sigma(t_1)\dots d\sigma(t_{2m+1})$$
$$\leqslant \sigma(T)\sup_{s,\widetilde{s}}(\int_T e^{q(\widetilde{s},t)}d\sigma(t))(\int_T e^{mq(s,t)}e^{q(\widetilde{s},t)}d\sigma(t))^{2m-1}. \tag{2.8}$$

Proof. The proof of (2.7) can be found in [8]. Thus we just prove how to derive inequality (2.8) from (2.7). By integrating with respect to the first $2m$ variables and applying (2.7) with the measure $e^{q(t,t_{2m+1})}d\sigma(t)$, we get:

$$\int_{T^{2m+1}} e^{\sum_{1\leqslant j<k\leqslant 2m+1} q(t_j,t_k)} d\sigma(t_1)\dots d\sigma(t_{2m+1})$$
$$= \int_T d\sigma(t_{2m+1})\int_{T^{2m}} e^{\sum_{1\leqslant j<k\leqslant 2m} q(t_j,t_k)}\prod_{j=1}^{2m} e^{q(t_j,t_{2m+1})}d\sigma(t_1)\dots d\sigma(t_{2m})$$
$$\underset{(2.7)}{\leqslant} \int_T d\sigma(t_{2m+1})(\int_T e^{q(t,t_{2m+1})}d\sigma(t))(\sup_s\int_T e^{mq(s,t)}e^{q(t,t_{2m+1})}d\sigma(t))^{2m-1}$$
$$\leqslant \sigma(T)\sup_{s,\widetilde{s}}(\int_T e^{q(\widetilde{s},t)}d\sigma(t))(\int_T e^{mq(s,t)}e^{q(\widetilde{s},t)}d\sigma(t))^{2m-1}.$$

3. Construction of a four parameter family of multifractal, homogeneous, isotropic vector fields with non symmetrical increments

In this section, we will suppose that $d/2<\alpha<(d/2+1)\wedge d$ **and** $\omega_d\gamma_1^2<d$. We consider the field $\mathcal{X}^\epsilon$ defined by formula (2.3) with

$$X^\epsilon(y)=\gamma_0(\epsilon)X_0^\epsilon(y)+\gamma_1 X_1^\epsilon(y),$$

where

$$X_i^\epsilon(y)=\int_{\mathbb{R}^d} k_\epsilon^R(y-\sigma)dW_i(\sigma), \qquad i=0,1.$$

We set also

$$C_\epsilon=((\gamma_0(\epsilon))^2+\gamma_1^2)\rho_{\epsilon/R}(0).$$

and

$$\gamma_0(\epsilon)=\gamma_0^*(\frac{\epsilon}{R})^{\frac{d-\omega_d\gamma_1^2}{2}}.$$

Therefore, we introduce a slight correlation between X^ϵ and dW_0 ($\gamma_0(\epsilon)$ tends to 0 as ϵ goes to 0).

3.1. **Multiplicative chaos in dimension** d**.** Multiplicative chaos or the "limit-lognormal" model introduced by Mandelbrot is a generalization of the exponential of a gaussian process. As mentioned in the introduction, it was defined rigorously by Kahane in [8]. The construction of Kahane was based on the theory of martingales and thus the generalized correlation kernel (here $\rho(t-s)$) had to verify a condition hard to verify practically (the σ-positivity condition). Our construction is based on L^2-theory and can be carried out without this condition.

We will construct the multiplicative chaos associated to the generalized correlation kernel $\rho(\frac{x-y}{R})$ defined in 2.3 and to some (positive) intermittency parameter γ_1 such that $\gamma_1^2\omega_d < d$.

Let ϵ be a positive number. Let $\mathcal{B}(\mathbb{R}^d)$ denote the standard borelian σ-field; we want to consider the limit as ϵ goes to 0 of the random measures Q^{ϵ,γ_1} defined by:

$$\begin{aligned} Q^{\epsilon,\gamma_1}(dy) &= e^{\gamma_1 X_1^\epsilon(y) - \frac{\gamma_1^2}{2}E((X_1^\epsilon(y))^2)}dy \\ &= e^{\gamma_1 X_1^\epsilon(y) - \frac{1}{2}\gamma_1^2\rho_{\epsilon/R}(0)}dy. \end{aligned} \tag{3.1}$$

This leads us to state the following proposition:

Proposition 3.1 (Multiplicative chaos of order γ_1)**.** *There exists a positive random measure $Q^{\gamma_1}(dy)$ independent of the regularizing function θ such that:*

1. *for all A bounded in $\mathcal{B}(\mathbb{R}^d)$, $E(Q^{\gamma_1}(A)) = |A|$.*
2. *Q^{γ_1} has almost surely no atoms.*
3. *Almost surely, Q^{γ_1} is singular with respect to the Lebesgue measure on all set A (with positive measure).*

If q is some positive integer and $f : \mathbb{R}^d \to \mathbb{R}$ a deterministic function that satisfies the following condition:

$$\int_{(\mathbb{R}^d)^{2q}} |f(y_1)|\dots|f(y_{2q})| \prod_{1\leqslant i<j\leqslant 2q} \frac{1}{|\frac{y_i-y_j}{R}|_*^{\gamma_1^2\omega_d}} dy_1\dots dy_{2q} < \infty, \tag{3.2}$$

then we have the following convergence:

$$\int_{\mathbb{R}^d} f(y)Q^{\epsilon,\gamma_1}(dy) \underset{\epsilon\to 0}{\overset{L^{2q}}{\longrightarrow}} \int_{\mathbb{R}^d} f(y)Q^{\gamma_1}(dy).$$

We also have the following expression for the moments of $\int_{\mathbb{R}^d} f(y)Q^{\gamma_1}(dy)$:

$$\forall k \leqslant 2q, \quad E\left((\int_{\mathbb{R}^d} f(y)Q^{\gamma_1}(dy))^k\right) = \int_{(\mathbb{R}^d)^k} f(y_1)\dots f(y_k) \prod_{1\leqslant i<j\leqslant k} \frac{e^{\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{\gamma_1^2\omega_d}} dy_1\dots dy_k. \tag{3.3}$$

Moreover, the above formula (3.3) extends straightforwardly to the case of two functions f, g and two intermittency parameters γ_1, γ_2 giving:

$$\begin{aligned} &E\left((\int_{\mathbb{R}^d} f(y)Q^{\gamma_1}(dy))^k(\int_{\mathbb{R}^d} g(y)Q^{\gamma_2}(dy))^l\right) \\ &= \int_{(\mathbb{R}^d)^{k+l}} f(y_1)\dots f(y_k)g(y_{k+1})\dots g(y_{k+l}) \prod_{1\leqslant i<j\leqslant k} \frac{e^{\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{\gamma_1^2\omega_d}} \times \\ &\prod_{1\leqslant i\leqslant k,\, j>k} \frac{e^{\gamma_1\gamma_2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{\gamma_1\gamma_2\omega_d}} \prod_{k+1\leqslant i<j\leqslant k+l} \frac{e^{\gamma_2^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{\gamma_2^2\omega_d}} dy_1\dots dy_{k+l}. \end{aligned}$$

We will call $Q^{\gamma_1}(dy)$ multiplicative chaos of order γ_1.

Proof. We first start by considering a positive integer q and a function f that satisfies the corresponding integrability condition (3.2). Let ϵ, ϵ' be two positive numbers. By using Fubini, we get for all $j \leqslant 2q$:

$$\begin{aligned}
&E((\int_{\mathbb{R}^d} f(y)Q^{\epsilon,\gamma_1}(dy))^j(\int_{\mathbb{R}^d} f(y)Q^{\epsilon',\gamma_1}(dy))^{2q-j}) \\
&= e^{-\frac{j}{2}\gamma_1^2\rho_{\epsilon/R}(0)-\frac{2q-j}{2}\gamma_1^2\rho_{\epsilon'/R}(0)} \int_{(\mathbb{R}^d)^{2q}} f(y_1)\dots f(y_{2q})\times \\
&\quad e^{\frac{1}{2}\gamma_1^2 E((\sum_{i=1}^j X_1^\epsilon(y_i)+\sum_{i=j+1}^{2q} X_1^{\epsilon'}(y_i))^2)} dy_1\dots dy_{2q} \\
&\underset{\epsilon,\epsilon'\to 0}{\to} \int_{(\mathbb{R}^d)^{2q}} f(y_1)\dots f(y_{2q}) \prod_{1\leqslant i<j\leqslant 2q} \frac{e^{\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{\gamma_1^2\omega_d}} dy_1\dots dy_{2q},
\end{aligned}$$

since $\rho_{\epsilon/R}(x) \to \rho(x)$ as ϵ goes to 0.

From this, we deduce that:

$$E((\int_{\mathbb{R}^d} f(y)Q^{\epsilon,\gamma_1}(dy) - \int_{\mathbb{R}^d} f(y)Q^{\epsilon',\gamma_1}(dy))^{2q}) \underset{\epsilon,\epsilon'\to 0}{\to} 0$$

and therefore that $\int_{\mathbb{R}^d} f(y)Q^{\epsilon,\gamma_1}(dy)$ is a Cauchy sequence in L^{2q} that converges to some random variable $\widetilde{Q}^{\gamma_1}(f)$. For $k \leqslant 2q$, the moment $E((\widetilde{Q}^{\gamma_1}(f))^k)$ is the limit as ϵ goes to 0 of $E((Q^{\epsilon,\gamma_1}(f))^k)$; from this one can deduce that the moments of $\widetilde{Q}^{\gamma_1}(f)$ are given by formula (3.3).

For any bounded set A in $\mathcal{B}(\mathbb{R}^d)$, consider $f = 1_A$ and $q = 1$. Since $\gamma_1^2\omega_d < d$, we deduce from lemma 2.4 that the integrability condition (3.2) is satisfied. Thus it follows from the proof above that $Q^{\epsilon,\gamma_1}(A)$ converges in L^2 to some random variable $\widetilde{Q}^{\gamma_1}(A)$. This defines a family of random variables (indexed by the bounded Borelian sets) that satisfies the following properties:

(1) For all disjoint and bounded sets A_1, A_2 in $\mathcal{B}(\mathbb{R}^d)$,

$$\widetilde{Q}^{\gamma_1}(A_1 \cup A_2) = \widetilde{Q}^{\gamma_1}(A_1) + \widetilde{Q}^{\gamma_1}(A_2) \qquad a.s.$$

(2) For any bounded sequence $(A_n)_{n\geqslant 1}$ decreasing to $\varnothing$:

$$\widetilde{Q}^{\gamma_1}(A_n) \underset{n\to\infty}{\to} 0 \quad a.s.$$

By theorem 6.1.VI. in [5], there exists a random measure Q^{γ_1} such that for all bounded A in $\mathcal{B}(\mathbb{R}^d)$ we have:

$$Q^{\gamma_1}(A) = \widetilde{Q}^{\gamma_1}(A) \quad a.s.$$

Finally, one can easily show that the limit random variable $\widetilde{Q}^{\gamma_1}(f)$ is almost surely equal to $\int_{\mathbb{R}^d} f(y)Q^{\gamma_1}(dy)$. □

3.2. **Convergence of $\mathcal{X}_\epsilon$ towards a field $\mathcal{X}$.**

In the sequel, $(e_j)_j$ will denote the canonical basis (whereas $(e^j)_j$ denotes the components of a vector e). In this subsection, we will prove the following proposition:

Proposition 3.2. *Let α be such that $d/2 < \alpha < (d/2+1) \wedge d$ and γ_1 such that $2\gamma_1^2\omega_d < \alpha - d/2$. There exists a field $(\mathcal{X}(x))_{x\in\mathbb{R}^d}$ such that for all k and $x_1,\dots,x_k \in \mathbb{R}^d$ the following convergence in law holds:*

$$(\mathcal{X}_\epsilon(x_1),\dots,\mathcal{X}_\epsilon(x_k)) \underset{\epsilon\to 0}{\Rightarrow} (\mathcal{X}(x_1),\dots,\mathcal{X}(x_k)). \tag{3.4}$$

Let l be an integer such that one of the following conditions hold:

(1) *l is even and $l\gamma_1^2\omega_d < \alpha - d/2$.*
(2) *l is odd and $(l+1)\gamma_1^2\omega_d < \alpha - d/2$.*

Let F_R^j be defined by $F_R^j(y) = \varphi_R(y)\frac{y^j}{|y|^{d-\alpha+1}}$. Then there exists C such that, for all x in $\mathbb{R}^d$, the random variables $\mathcal{X}^j(x)$ have a moment of order $2l$ given by the following expression:

$$E((\mathcal{X}^j(x))^{2l}) = \sum_{k=0}^{l} \alpha_{k,l} C^{2k} \int_{(\mathbb{R}^d)^{k+l}} F_R^j(y_1)\dots F_R^j(y_{2k})(F_R^j(y_{2k+1}))^2\dots(F_R^j(y_{k+l}))^2$$
$$\prod_{1\leqslant i<j\leqslant 2k} \frac{e^{\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{\gamma_1^2\omega_d}} \prod_{\substack{1\leqslant i\leqslant 2k\\ j>2k}} \frac{e^{2\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{2\gamma_1^2\omega_d}} \prod_{2k+1\leqslant i<j\leqslant k+l} \frac{e^{4\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{4\gamma_1^2\omega_d}} dy_1\dots dy_{k+l}. \tag{3.5}$$

We also have:

$$E((\mathcal{X}^j(x+h)-\mathcal{X}^j(x))^{2l}) = \sum_{k=0}^{l} \alpha_{k,l} C^{2k} \int_{(\mathbb{R}^d)^{k+l}} (F_R^j(y_1)-F_R^j(y_1-h))\dots$$
$$(F_R^j(y_{2k})-F_R^j(y_{2k}-h))(F_R^j(y_{2k+1})-F_R^j(y_{2k+1}-h))^2\dots(F_R^j(y_{k+l})-F_R^j(y_{k+l}-h))^2$$
$$\prod_{1\leqslant i<j\leqslant 2k} \frac{e^{\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{\gamma_1^2\omega_d}} \prod_{\substack{1\leqslant i\leqslant 2k\\ j>2k}} \frac{e^{2\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{2\gamma_1^2\omega_d}} \prod_{2k+1\leqslant i<j\leqslant k+l} \frac{e^{4\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{4\gamma_1^2\omega_d}} dy_1\dots dy_{k+l}. \tag{3.6}$$

Proof. Let γ_1 be such that $2\gamma_1^2\omega_d < \alpha - d/2$. We set

$$C = \frac{\gamma_0^* C_0 e^{-1/2\gamma_1^2 C_1}}{R^{d/2}}.$$

and define two auxiliary fields $\mathcal{Y}_\epsilon, \mathcal{Z}_\epsilon$ by the following expressions:

$$\mathcal{Y}_\epsilon(x) = \int_{\mathbb{R}^d} \varphi_R(x-y)\frac{x-y}{|x-y|^{d-\alpha+1}} Q^{\epsilon,\gamma_1}(dy) \tag{3.7}$$

and

$$\mathcal{Z}_\epsilon(x) = \int_{\mathbb{R}^d} \varphi_R(x-y)\frac{x-y}{|x-y|^{d-\alpha+1}} e^{\gamma_1 X_1^\epsilon(y)-C_\epsilon} dW_0(y). \tag{3.8}$$

Note that $\mathcal{Z}_\epsilon(x)$ exists since X_1^ϵ and dW_0 are independent with:

$$E(\int_{\mathbb{R}^d} \frac{\varphi_R(x-y)^2}{|x-y|^{2(d-\alpha)}} e^{2\gamma_1 X_1^\epsilon(y)-2C_\epsilon} dy) < \infty. \tag{3.9}$$

We can compute, for all x in $\mathbb{R}^d$, $E(|\mathcal{X}_\epsilon(x) - C\mathcal{Y}_\epsilon(x) - \mathcal{Z}_\epsilon(x)|^2)$ (cf. the more complicated computations in the proof of proposition 3.7) and derive the following limit:

$$\mathcal{X}_\epsilon(x) - (C\mathcal{Y}_\epsilon(x) + \mathcal{Z}_\epsilon(x)) \underset{\epsilon\to 0}{\overset{L^2}{\to}} 0.$$

Thus, we must show that the finite dimensional distributions of the field $C\mathcal{Y}_\epsilon + \mathcal{Z}_\epsilon$ converge in law. Let k be some positive integer and $x_1, \ldots, x_k$ points in $\mathbb{R}^d$. For all $\xi = (\xi_1, \ldots, \xi_k)$ in $(\mathbb{R}^d)^k$, we compute the characteristic function of $(C\mathcal{Y}_\epsilon(x_1) + \mathcal{Z}_\epsilon(x_1), \ldots, C\mathcal{Y}_\epsilon(x_k) + \mathcal{Z}_\epsilon(x_k))$:

$$\mathcal{C}_\epsilon(\xi) = E(e^{i\sum_{j=1}^k \xi_j.(C\mathcal{Y}_\epsilon(x_j)+\mathcal{Z}_\epsilon(x_j))}).$$

By conditioning on the field generated by the white noise dW_1, we get:

$$\begin{aligned}\mathcal{C}_\epsilon(\xi) &= E(e^{iC\sum_{j=1}^k \xi_j.\mathcal{Y}_\epsilon(x_j)} e^{-\frac{1}{2}\int(\sum_{j=1}^k \xi_j.F_R(x_j-y))^2 e^{2\gamma_1 X_1^\epsilon(y)-2C_\epsilon} dy}) \\ &= E(e^{iC\sum_{j=1}^k \int \xi_j.F_R(x_j-y)Q^{\epsilon,\gamma_1}(dy)} e^{-\frac{1}{2}e^{-2\gamma_0^2\rho_{\epsilon/R}(0)}\int(\sum_{j=1}^k \xi_j.F_R(x_j-y))^2 Q^{\epsilon,2\gamma_1}(dy)})\end{aligned}$$

Now, using proposition (3.1), we have:

$$\sum_{j=1}^k \int \xi_j.F_R(x_j-y)Q^{\epsilon,\gamma_1}(dy) \underset{L^2}{\to} \sum_{j=1}^k \int \xi_j.F_R(x_j-y)Q^{\gamma_1}(dy),$$

$$\int (\sum_{j=1}^k \xi_j.F_R(x_j-y))^2 Q^{\epsilon,2\gamma_1}(dy) \underset{L^2}{\to} \int (\sum_{j=1}^k \xi_j.F_R(x_j-y))^2 Q^{2\gamma_1}(dy),$$

from where:

$$\mathcal{C}_\epsilon(\xi) \underset{\epsilon\to 0}{\to} \mathcal{C}(\xi) = E(e^{iC\sum_{j=1}^k \int \xi_j.F_R(x_j-y)Q^{\gamma_1}(dy)} e^{-\frac{1}{2}\int(\sum_{j=1}^k \xi_j.F_R(x_j-y))^2 Q^{2\gamma_1}(dy)}).$$

Thus, by applying Levy's theorem, we conclude that the finite dimensional distributions of the field $C\mathcal{Y}_\epsilon + \mathcal{Z}_\epsilon$ converge in law to those of a field $\mathcal{X}$ whose finite dimensional distributions are given by:

$$E(e^{i\sum_{j=1}^k \xi_j.\mathcal{X}(x_j)}) = \mathcal{C}(\xi).$$

Suppose that l is a positive integer that satisfies the condition of the proposition. For all ξ in $\mathbb{R}^d$, we have:

$$E(e^{i\xi.\mathcal{X}(x)}) = E(e^{-iC\int \xi.F_R(y)Q^{\gamma_1}(dy)} e^{-\frac{1}{2}\int(\xi.F_R(y))^2 Q^{2\gamma_1}(dy)}).$$

We derive expression (3.5) by computing $\frac{\partial^{2l}}{(\partial\xi^j)^{2l}} E(e^{i\xi.\mathcal{X}(x)})|_{\xi=0}$ using proposition 3.1. We derive (3.6) similarly.

3.3. **Scaling of $\mathcal{X}$.** The purpose of this subsection is to show that the field $(\mathcal{X}(x))_{x\in\mathbb{R}^d}$ satisfies the multifractal scaling relation (this is what propositions 3.5 and 3.6 below assert).

We first state two preliminary lemmas we will use in the rest of the paper.

Lemma 3.3. *Let δ be some real number such that $0 \leqslant \delta < \alpha$ and $\delta \neq \alpha - 1$. There exists $C = C(\delta)$ such that we have the following inequality for $|h| \leqslant R$:*

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |F_R(y) - F_R(y-h)| \frac{1}{|\frac{x-y}{R}|_*^\delta} dy \leqslant R^{d/2} C |\frac{h}{R}|^{(\alpha-\delta)\wedge 1}. \tag{3.10}$$

Proof. By homogenity, we suppose that $R = 1$ and for simplicity, we suppose $d \geqslant 2$. Since $\frac{1}{|x|_*^\delta} \leqslant \frac{1}{|x|^\delta} + 1$ and the right hand side of (3.10) increases with δ, we have to show that for $\delta \in [0, \alpha[$ and $|h| \leqslant 1$:

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |F_1(y) - F_1(y-h)| \frac{1}{|x-y|^\delta} dy \leqslant C|h|^{(\alpha-\delta)\wedge 1}.$$

Indeed, this would imply that for $|h| \leqslant 1$:

$$\begin{aligned}\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |F_1(y) - F_1(y-h)| \frac{1}{|x-y|_*^\delta} dy &\leqslant C|h|^{(\alpha-\delta)\wedge 1} + C|h|^{\alpha\wedge 1} \\ &\leqslant 2C|h|^{(\alpha-\delta)\wedge 1}.\end{aligned}$$

There exists C such that for all y and h, we have:

$$|\varphi(y-h) - \varphi(y)| \leqslant C|h| \text{ and } \varphi(y) \leqslant C 1_{|y| \leqslant 2}. \tag{3.11}$$

We set

$$I(x) = \int_{\mathbb{R}^d} |F_1(y) - F_1(y-h)| \frac{1}{|x-y|^\delta} dy.$$

Therefore we get

$$\begin{aligned} I(x) &\leqslant C|h| \int_{|y| \leqslant 3} \frac{1}{|y|^{d-\alpha}} \frac{1}{|x-y|^\delta} dy \qquad (3.12)\\ &\quad + C \int_{|y| \leqslant 3} |\frac{y}{|y|^{d-\alpha+1}} - \frac{y-h}{|y-h|^{d-\alpha+1}}| \frac{1}{|x-y|^\delta} dy \\ &\leqslant C|h| + C \int_{|y| \leqslant 3} |\frac{y}{|y|^{d-\alpha+1}} - \frac{y-h}{|y-h|^{d-\alpha+1}}| \frac{1}{|x-y|^\delta} dy, \qquad (3.13)\end{aligned}$$

where we denote by C different constants.

First case: $\delta < \alpha - 1$.

Plugging inequality

$$|\frac{y}{|y|^{d-\alpha+1}} - \frac{y-h}{|y-h|^{d-\alpha+1}}| \leqslant \frac{(d-\alpha+1)|h|}{|y-h|^{d-\alpha+1} \wedge |y|^{d-\alpha+1}}$$

in (3.13), we get

$$I(x) \leqslant C|h| \int_{|y| \leqslant 3} \frac{1}{|y-h|^{d-\alpha+1} \wedge |y|^{d-\alpha+1}} \frac{1}{|x-y|^\delta} dy.$$

We have:

$$\begin{aligned}
&\int_{|y|\leqslant 3} \frac{1}{|y-h|^{d-\alpha+1}\wedge |y|^{d-\alpha+1}}\frac{1}{|x-y|^{\delta}}dy\\
&\leqslant \int_{|y|\leqslant 3} \frac{1}{|y-h|^{d-\alpha+1}}\frac{1}{|x-y|^{\delta}}dy + \int_{|y|\leqslant 3} \frac{1}{|y|^{d-\alpha+1}}\frac{1}{|x-y|^{\delta}}dy\\
&\leqslant 2\sup_x \int_{|y|\leqslant 4} \frac{1}{|y|^{d-\alpha+1}}\frac{1}{|x-y|^{\delta}}dy,
\end{aligned}$$

which concludes the proof.

Second case: $\delta > \alpha - 1$.

By the change of variable $y = |h|u$ and setting $h = |h|e$ with $|e| = 1$, we get:

$$\begin{aligned}
&\int_{|y|\leqslant 3} |\frac{y-h}{|y-h|^{d-\alpha+1}} - \frac{y}{|y|^{d-\alpha+1}}|\frac{1}{|x-y|^{\delta}}dy\\
&= |h|^{\alpha-\delta}\int_{|u|\leqslant \frac{3}{|h|}} |\frac{u-e}{|u-e|^{d-\alpha+1}} - \frac{u}{|u|^{d-\alpha+1}}|\frac{1}{|x/|h|-u|^{\delta}}du\\
&\leqslant |h|^{\alpha-\delta}\sup_{a\in\mathbb{R}^d}\int_{\mathbb{R}^d} |\frac{u-e}{|u-e|^{d-\alpha+1}} - \frac{u}{|u|^{d-\alpha+1}}|\frac{1}{|a-u|^{\delta}}du.
\end{aligned}$$

□

Lemma 3.4. *Let δ be some real number such that $0 \leqslant \delta < 2\alpha - d$. There exists $C = C(\delta)$ such that we have the following inequality for $|h| \leqslant R$:*

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d} |F_R(y) - F_R(y-h)|^2 \frac{1}{|\frac{x-y}{R}|_*^{\delta}}dy \leqslant C|\frac{h}{R}|^{2\alpha-d-\delta}. \tag{3.14}$$

Proof. As in the proof above, we can replace $|.|_*$ by $|.|$ and suppose that $R = 1$; thus we have to show inequality (3.14) with $J(x)$ where we set:

$$J(x) = \int_{\mathbb{R}^d} |F_R(y) - F_R(y-h)|^2 \frac{1}{|x-y|^{\delta}}dy.$$

Using inequality (3.11), we get

$$\begin{aligned}
J(x) &\leqslant C|h|^2\int_{|y|\leqslant 3} \frac{1}{|y|^{2(d-\alpha)}}\frac{1}{|x-y|^{\delta}}dy\\
&\quad + C\int_{|y|\leqslant 3} |\frac{y-h}{|y-h|^{d-\alpha+1}} - \frac{y}{|y|^{d-\alpha+1}}|^2\frac{1}{|x-y|^{\delta}}dy\\
&\leqslant C|h|^2 + C\int_{|y|\leqslant 3} |\frac{y-h}{|y-h|^{d-\alpha+1}} - \frac{y}{|y|^{d-\alpha+1}}|^2\frac{1}{|x-y|^{\delta}}dy.
\end{aligned} \tag{3.15}$$

Since $2 > 2\alpha - d - \delta$, we only have to consider the second term in inequality (3.15). By the change of variable $y = |h|u$ and setting $h = |h|e$ with $|e| = 1$, we get:

$$\begin{aligned}
&\int_{|y| \leqslant 3} |\frac{y-h}{|y-h|^{d-\alpha+1}} - \frac{y}{|y|^{d-\alpha+1}}|^2 \frac{1}{|x-y|^\delta} dy \\
&= |h|^{2\alpha-d-\delta} \int_{|u| \leqslant \frac{3}{|h|}} |\frac{u-e}{|u-e|^{d-\alpha+1}} - \frac{u}{|u|^{d-\alpha+1}}|^2 \frac{1}{|x/|h| - u|^\delta} du \\
&\leqslant |h|^{2\alpha-d-\delta} \sup_{a \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\frac{u-e}{|u-e|^{d-\alpha+1}} - \frac{u}{|u|^{d-\alpha+1}}|^2 \frac{1}{|a-u|^\delta} du.
\end{aligned}$$

□

Proposition 3.5. *(Scaling along the even integers)*

Let l be an integer such that one of the following conditions hold:

(1) *l is even and $l\gamma_1^2\omega_d < \alpha - d/2$*
(2) *l is odd and $(l+1)\gamma_1^2\omega_d < \alpha - d/2$.*

Let e be a unit vector ($|e| = 1$). Then there exists $C_l^j(e) > 0$ such that the following scaling relation holds:

$$E((\mathcal{X}^j(x+\lambda e) - \mathcal{X}^j(x))^{2l}) \underset{\lambda \to 0}{\sim} C_l^j(e)(\frac{\lambda}{R})^{\zeta_{2l}}, \tag{3.16}$$

where we have

$$\zeta_{2l} = l(2\alpha - d) - 2\gamma_1^2\omega_d l(l-1). \tag{3.17}$$

Proof. For simplicity, we will suppose that l is even and that $l\gamma_1^2\omega_d < \alpha - d/2$. We introduce the following notation:

$$f_h(y) = F_R^j(y) - F_R^j(y-h).$$

We shall see that the scaling at small scale of the sum (3.6) is given by the term $k = 0$. Indeed for all $k \geqslant 1$ let us consider the integral

$$\begin{aligned}
&\int_{(\mathbb{R}^d)^{k+l}} f_h(y_1)\dots f_h(y_{2k})(f_h(y_{2k+1}))^2 \dots (f_h(y_{k+l}))^2 \prod_{1 \leqslant i<j \leqslant 2k} \frac{e^{\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{\gamma_1^2\omega_d}} \times \\
&\prod_{\substack{1 \leqslant i \leqslant 2k \\ j>2k}} \frac{e^{2\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{2\gamma_1^2\omega_d}} \prod_{2k+1 \leqslant i<j \leqslant k+l} \frac{e^{4\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{4\gamma_1^2\omega_d}} dy_1 \dots dy_{k+l}. \\
&\leqslant I_k J_{k,l},
\end{aligned} \tag{3.18}$$

where we set

$$\begin{aligned}
I_k = &\sup_{y_{2k+1},\dots,y_{k+l}} \int_{(\mathbb{R}^d)^{2k}} |f_h(y_1)|\dots|f_h(y_{2k})| \prod_{\substack{1 \leqslant i \leqslant 2k \\ j>2k}} \frac{e^{2\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{2\gamma_1^2\omega_d}} \times \\
&\prod_{1 \leqslant i<j \leqslant 2k} \frac{e^{\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{\gamma_1^2\omega_d}} dy_1 \dots dy_{2k}.
\end{aligned}$$

and

$$J_{k,l} = \int_{(\mathbb{R}^d)^{l-k}} (f_h(y_{2k+1}))^2 \dots (f_h(y_{k+l}))^2 \prod_{2k+1 \leqslant i<j \leqslant k+l} \frac{e^{4\gamma_1^2 \phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{4\gamma_1^2\omega_d}} dy_{2k+1} \dots dy_{k+l}.$$

By using the estimates (3.10),(3.14) and the inequalities (2.7), (2.8), one can show that for all $k \geqslant 1$, we have

$$I_k J_{k,l} \leqslant C R^{dk} |\frac{h}{R}|^{c_{k,l}},$$

with

$$c_{k,l} = (\alpha - 2(l-k)\gamma_1^2\omega_d) \wedge 1 + ((\alpha - (2l-k)\gamma_1^2\omega_d) \wedge 1)(2k-1) + (2\alpha - d)(l-k) - 2\gamma_1^2\omega_d(l-k)(l-k-1).$$

If $\alpha - 2(l-k)\gamma_1^2\omega_d < 1$, then $c_{k,l} = \zeta_{2l} + k(d - \gamma_1^2\omega_d)$; If $\alpha - 2(l-k)\gamma_1^2\omega_d \geqslant 1$ and $\alpha - (2l-k)\gamma_1^2\omega_d < 1$, then $c_{k,l} = \zeta_{2l} + 1 - \alpha + dk + (2l-3k)\gamma_1^2\omega_d$; otherwise $c_{k,l} = 2k + (2\alpha - d)(l-k) - 2\gamma_1^2\omega_d(l-k)(l-k-1)$. In all cases, it is easy to show that $c_{k,l} > \zeta_{2l}$ under the conditions of the proposition.

Finally, we study the term where $k = 0$. We get for $h = \lambda e$ with $|e| = 1$:

$$\begin{aligned}
&\int_{(\mathbb{R}^d)^l} (f_h(y_1))^2 \dots (f_h(y_l))^2 \prod_{1 \leqslant i<n \leqslant l} \frac{e^{4\gamma_1^2\phi(\frac{y_i-y_n}{R})}}{|\frac{y_i-y_n}{R}|_*^{4\gamma_1^2\omega_d}} dy_1 \dots dy_l. \\
&\underset{y_i=\lambda u_i}{=} (\frac{\lambda}{R})^{l(2\alpha-d)} \int_{(\mathbb{R}^d)^l} (\varphi(\frac{\lambda}{R}(u_1 - e)) \frac{u_1^j - e^j}{|u_1 - e|^{d-\alpha+1}} - \varphi(\frac{\lambda}{R}u_1) \frac{u_1^j}{|u_1|^{d-\alpha+1}})^2 \dots \times \\
&\quad (\varphi(\frac{\lambda}{R}(u_l - e)) \frac{u_l^j - e^j}{|u_l - e|^{d-\alpha+1}} - \varphi(\frac{\lambda}{R}u_l) \frac{u_l^j}{|u_l|^{d-\alpha+1}})^2 \prod_{1 \leqslant i<n \leqslant l} \frac{e^{4\gamma_1^2\phi(\frac{\lambda(u_i-u_n)}{R})}}{|\frac{\lambda(u_i-u_n)}{R}|_*^{4\gamma_1^2\omega_d}} du_1 \dots du_l. \\
&\underset{\lambda \to 0}{\sim} e^{2l(l-1)\gamma_1^2\phi(0)} (\frac{\lambda}{R})^{\zeta_{2l}} \int_{(\mathbb{R}^d)^l} (\frac{u_1^j - e^j}{|u_1 - e|^{d-\alpha+1}} - \frac{u_1^j}{|u_1|^{d-\alpha+1}})^2 \dots (\frac{u_l^j - e^j}{|u_l - e|^{d-\alpha+1}} - \frac{u_l^j}{|u_l|^{d-\alpha+1}})^2 \times \\
&\quad \prod_{1 \leqslant i<n \leqslant l} \frac{1}{|u_i - u_n|^{4\gamma_1^2\omega_d}} du_1 \dots du_l,
\end{aligned}$$

and inequality (2.7) shows that this integral is finite when $l\gamma_1^2\omega_d < \alpha - d/2$.

□

In the next proposition, we state the scaling relations of $\mathcal{X}$ along the odd integers. We define $I_l^j(e)$ by:

$$\begin{aligned}
I_l^j(e) = &\int_{(\mathbb{R}^d)^l} (\frac{u_1^j - e^j}{|u_1 - e|^{d-\alpha+1}} - \frac{u_1^j}{|u_1|^{d-\alpha+1}})^2 \dots (\frac{u_l^j - e^j}{|u_l - e|^{d-\alpha+1}} - \frac{u_l^j}{|u_l|^{d-\alpha+1}})^2 \times \\
&\prod_{1 \leqslant i<j \leqslant l} \frac{1}{|u_i - u_j|^{4\gamma_1^2\omega_d}} du_1 \dots du_l.
\end{aligned}$$

Proposition 3.6. *(Scaling along the odd integers) Let l be an integer satisfying the conditions in proposition 3.5 and $1 + 2l\gamma_1^2\omega_d < \alpha$.*

Let e be a unit vector ($|e|=1$). Then we have the following scaling relation:

$$E((\mathcal{X}^j(x+\lambda e)-\mathcal{X}^j(x))^{2l+1}) \underset{\lambda\to 0}{\sim} \Sigma_l^j(e)(\frac{\lambda}{R})^{\widetilde{\zeta}_{2l+1}}, \tag{3.19}$$

where we have

$$\widetilde{\zeta}_{2l+1} = l(2\alpha-d)-2\gamma_1^2\omega_d l(l-1)+1, \tag{3.20}$$

and

$$\Sigma_l^j(e) = \gamma_0^* C(l,\gamma_1) I_l^j(e) e^j, \quad C(l,\gamma_1)>0.$$

Proof. As in proposition 3.2, setting $C=\frac{\gamma_0^* C_0 e^{-1/2\gamma_1^2 C_1}}{R^{d/2}}$, it is possible to show that:

$$E((\mathcal{X}^j(x+h)-\mathcal{X}^j(x))^{2l+1})$$
$$=\sum_{k=0}^{l}\widetilde{\alpha}_{k,l}C^{2k+1}\int_{(\mathbb{R}^d)^{k+l+1}} f_h(y_1)\dots f_h(y_{2k+1})(f_h(y_{2k+2}))^2\dots(f_h(y_{k+l+1}))^2\times$$
$$\prod_{1\leqslant i<j\leqslant 2k+1}\frac{e^{\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{\gamma_1^2\omega_d}}\prod_{\substack{1\leqslant i\leqslant 2k+1\\ j>2k+1}}\frac{e^{2\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{2\gamma_1^2\omega_d}}\prod_{2k+2\leqslant i<j\leqslant k+l+1}\frac{e^{4\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{4\gamma_1^2\omega_d}}dy_1\dots dy_{k+l+1},$$

where, as usual, we set:

$$f_h(y)=F_R^j(y)-F_R^j(y-h).$$

Similarly to proposition 3.5, to get the main contribution as $|h|$ goes to 0, we examine the term $k=0$. We introduce $\mathcal{I}$:

$$\mathcal{I}=\int_{(\mathbb{R}^d)^{l+1}} f_h(y_1)(f_h(y_2))^2\dots(f_h(y_{l+1}))^2\prod_{j\geqslant 2}e^{2\gamma_1^2\rho(\frac{y_1-y_j}{R})}\times$$
$$\prod_{2\leqslant i<j\leqslant l+1}\frac{e^{4\gamma_1^2\phi(\frac{y_i-y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{4\gamma_1^2\omega_d}}dy_1\dots dy_{l+1}.$$

Putting $h=\lambda e$ ($|e|=1$), $y_1=Ru_1$, $y_i=\lambda u_i$ ($i\geqslant 2$), we get:

$$\mathcal{I}\underset{\lambda\to 0}{\sim} R^{d/2}e^{2l(l-1)\gamma_1^2\phi(0)}(\frac{\lambda}{R})^{\widetilde{\zeta}_{2l+1}}I_l^j(e)e.\int\nabla\psi^j(y)e^{2l\gamma_1^2\rho(y)}dy,$$

where $\psi^j(y)=\varphi(y)\frac{y^j}{|y|^{d-\alpha+1}}$. Now a calculation gives:

$$\int\nabla\psi^j(y)e^{2l\gamma_1^2\rho(y)}dy=-\frac{2l\gamma_1^2\omega_d}{d}(\int_0^\infty r^{\alpha-1}\varphi(r)e^{2l\gamma_1^2\rho(r)}\frac{d\rho}{dr}dr)e_j,$$

the last integral being negative since $\rho(r)$ is a strictly decreasing function on the interval $]0,2[$: the result follows. Notice that the condition $1+2l\gamma_1^2<\alpha$ implies that this integral is finite while the other conditions imply that $I_l^j(e)$ is finite.

□

From proposition 3.6, we deduce readily that for λ small the law of $\mathcal{X}(x+\lambda e)-\mathcal{X}(x)$ is nonsymmetrical (for $\gamma_0^*\neq 0$). Indeed, by isotropy, we have :

$$(\mathcal{X}(x+\lambda e)-\mathcal{X}(x)).e \underset{law}{=} \mathcal{X}^j(x+\lambda e_j)-\mathcal{X}^j(x)$$

and $\Sigma_l^j(e_j)=\gamma_0^* C(l,\gamma_1)I_l^j(e_j)>0$.

3.4. **Tightness of $\mathcal{X}_\epsilon$ and regularity of $\mathcal{X}$.** In this section, we prove that the convergence in law of $\mathcal{X}_\epsilon$ towards $\mathcal{X}$ given by proposition 3.2 holds in a functional sense and that the field $\mathcal{X}$ is locally Hölderian. The straight way to do so is to prove the tightness of the sequence $\mathcal{X}_\epsilon$ by means of a Kolmogorov estimate (cf. chapter 13 of [13]).

Proposition 3.7. *(Tightness) Let l be some positive integer that satisfies the condition of proposition 3.5 and γ a positive parameter such that $\gamma_1^2<\gamma^2$. Then there exists $\epsilon_0>0$ and C independent of ϵ such that for $\epsilon<\epsilon_0$ and $|h|\leqslant R$:*

$$\forall x, \qquad E((\mathcal{X}_\epsilon(x+h)-\mathcal{X}_\epsilon(x))^{2l})\leqslant C|h|^{l(2\alpha-d)-2\gamma^2\omega_d l(l-1)}, \tag{3.21}$$

and

$$E((\mathcal{X}_\epsilon(0))^{2l})\leqslant C. \tag{3.22}$$

Proof. We only prove (3.21) (the proof of 3.22 is similar). We are going to compute the moment

$$E((\mathcal{X}_\epsilon^j(x+h)-\mathcal{X}_\epsilon^j(x))^{2l})=E((\int_{\mathbb{R}^d} f_{\epsilon,h}(y)e^{X^\epsilon(y)-C_\epsilon}dW_0(y))^{2l})$$

where we set:

$$f_{\epsilon,h}(y)=\frac{\varphi_R(y)y^j}{|y|_\epsilon^{d-\alpha+1}}-\frac{\varphi_R(y-h)(y^j-h^j)}{|y-h|_\epsilon^{d-\alpha+1}}.$$

We get:

$$E((\int_{\mathbb{R}^d} f_{\epsilon,h}(y)e^{X^\epsilon(y)-C_\epsilon}dW_0(y))^{2l})=e^{-2lC_\epsilon}\int f_{\epsilon,h}(y_1)\dots f_{\epsilon,h}(y_{2l})E(e^{\hat{X}^\epsilon}dW_0(y_1)\dots dW_0(y_{2l})), \tag{3.23}$$

where

$$\hat{X}^\epsilon=X^\epsilon(y_1)+\dots+X^\epsilon(y_{2l}).$$

The rest of the computation can be performed rigorously by regularizing the white noise dW_0, using lemma 2.2 and going to the limit. It is easy to see that we obtain the same result by introducing the following formal rules:

$$E(dW_0(y)dW_0(y'))=\delta_{y-y'}dy \tag{3.24}$$

and

$$E(dW_0(y)X^\epsilon(y'))=\gamma_0(\epsilon)k_\epsilon^R(y'-y)dy \tag{3.25}$$

As a consequence of lemma 2.2, $E(e^{\hat{X}^\epsilon}dW_0(y_1)\dots dW_0(y_{2l}))$ is the sum of terms of the form

$$E(dW_0(y_1)\hat{X}^\epsilon)\dots E(dW_0(y_k)\hat{X}^\epsilon)E(dW_0(y_{k+1})dW_0(y_{k+2}))\dots E(dW_0(y_{q-1})dW_0(y_{2l}))e^{\frac{1}{2}E((\hat{X}^\epsilon)^2)}. \tag{3.26}$$

We will compute the limit of each one of these terms. By using (3.25), we get

$$\begin{aligned} E(dW_0(y_k)\hat{X}^\epsilon) &= \gamma_0(\epsilon)(\sum_{i=1}^{2l} k_\epsilon^R(y_i - y_k))dy_k \\ &= \gamma_0(\epsilon)k_\epsilon^R(0)(1 + Q_k^\epsilon)dy_k \end{aligned}$$

where

$$Q_k^\epsilon = \frac{1}{k_\epsilon^R(0)}(\sum_{i\neq k} k_\epsilon^R(y_i - y_k)).$$

We also have from the definition of X^ϵ:

$$e^{\frac{1}{2}E((\hat{X}^\epsilon)^2)} = e^{(l\rho_{\epsilon/R}(0)+\sum_{i<j}\rho_{\epsilon/R}(\frac{y_i-y_j}{R}))(\gamma_0(\epsilon)^2+\gamma_1^2)}.$$

By using lemma 2.2, expression (3.23) and the rules above, we get:

$$\begin{aligned} &E((\int_{\mathbb{R}^d} f_{\epsilon,h}(y)e^{X^\epsilon(y)-C_\epsilon}dW_0(y))^{2l}) = \sum_{k=0}^{l}\alpha_{k,l}(\gamma_0(\epsilon))^{2k}(k_\epsilon^R(0))^{2k}e^{(2l-k)((\gamma_0(\epsilon))^2+\gamma_1^2)\rho_{\epsilon/R}(0)-2lC_\epsilon} \\ &\int_{(\mathbb{R}^d)^{k+l}} f_{\epsilon,h}(y_1)\dots f_{\epsilon,h}(y_{2k})(f_{\epsilon,h}(y_{2k+1}))^2\dots(f_{\epsilon,h}(y_{k+l}))^2\prod_{i=1}^{2k}(1+Q_{i,k,l}^\epsilon)e^{S_{k,l}^\epsilon}dy_1\dots dy_{k+l} \end{aligned}$$

where

$$Q_{i,k,l}^\epsilon = \frac{1}{k_\epsilon^R(0)}(\sum_{\substack{1\leqslant j\leqslant 2k\\ j\neq i}} k_\epsilon^R(y_i - y_j) + 2\sum_{j>2k} k_\epsilon^R(y_i - y_j))$$

and

$$\begin{aligned} S_{k,l}^\epsilon &= ((\gamma_0(\epsilon))^2 + \gamma_1^2)(\sum_{1\leqslant i<j\leqslant 2k}\rho_{\epsilon/R}(\frac{y_i - y_j}{R}) + 2\sum_{1\leqslant i\leqslant 2k}\sum_{j>2k}\rho_{\epsilon/R}(\frac{y_i - y_j}{R}) \\ &\quad +4\sum_{2k+1\leqslant i<j\leqslant k+l}\rho_{\epsilon/R}(\frac{y_i - y_j}{R})). \end{aligned}$$

We first take care of the normalizing constant outside each integral:

$$(\gamma_0(\epsilon)k_\epsilon^R(0)e^{-1/2((\gamma_0(\epsilon))^2+\gamma_1^2)\rho_{\epsilon/R}(0)})^{2k}e^{2l((\gamma_0(\epsilon))^2+\gamma_1^2)\rho_{\epsilon/R}(0)-2lC_\epsilon}.$$

By the choice of C_ϵ, we have $e^{2l((\gamma_0(\epsilon))^2+\gamma_1^2)\rho_{\epsilon/R}(0)-2lC_\epsilon} = 1$. Using expansions (2.4) and (2.5), we derive the following limit:

$$\gamma_0(\epsilon)k_\epsilon^R(0)e^{-1/2((\gamma_0(\epsilon))^2+\gamma_1^2)\rho_{\epsilon/R}(0)} \underset{\epsilon\to 0}{\longrightarrow} \frac{\gamma_0^* C_0 e^{-1/2\gamma_1^2 C_1}}{R^{d/2}}.$$

In conclusion, the constant outside the integral of term k in the above sum converges to $\alpha_{k,l}(\frac{\gamma_0^* C_0 e^{-1/2\gamma_1^2 C_1}}{R^{d/2}})^{2k}$.

Let γ be such that $\gamma_1^2 < \gamma^2$. One can choose $\epsilon_0 > 0$ such that $\gamma_0(\epsilon_0))^2 + \gamma_1^2 < \gamma^2$. Using the fact that, for all y, $\rho_{\epsilon/R}(y/R) \leqslant \omega_d \ln^+ \frac{R}{|y|} + C$ with C independent of ϵ,

we get:

$$e^{S^{\epsilon}_{k,l}} \leqslant C \prod_{1 \leqslant i<j \leqslant 2k} \frac{1}{|\frac{y_i-y_j}{R}|_*^{\gamma^2\omega_d}} \prod_{\substack{1 \leqslant i \leqslant 2k \\ j>2k}} \frac{1}{|\frac{y_i-y_j}{R}|_*^{2\gamma^2\omega_d}} \times$$
$$\prod_{2k+1 \leqslant i<j \leqslant k+l} \frac{1}{|\frac{y_i-y_j}{R}|_*^{4\gamma^2\omega_d}}. \tag{3.27}$$

Finally, we conclude by using the fact that $|Q^{\epsilon}_{i,k,l}|$ is bounded by a constant independent of ϵ, inequality (3.10) and (3.14) similarly as in the proof of proposition 3.5.

□

Corollary 3.8. *One can easily deduce from this that for γ_1^2 sufficiently small, by Kolmogorov's compacity theorem ([13]), $\mathcal{X}_\epsilon$ tends to $\mathcal{X}$ in the functional sense and that $\mathcal{X}$ is locally Hölderian.*

Comment 3.9. *Starting with a two parameter (R, α) monofractal Gaussian field, we constructed a four parameter $(R, \alpha, \gamma_1, \gamma_0^*)$ multifractal field with nonsymmetrical increments. This family has it's own interest. As we shall see in the next section, this family is too restricted to take into account all the constraints needed for a satisfactory model of turbulent flows.*

In the case where $\gamma_0^* = 0$, we obtain symmetrical random fields which extend to higher dimensions the model introduced in [2].

In the next section, we will study a multifractal field which is not in this family but that can be seen as a limit case where $\gamma_1 = 0$ and γ_0 is constant (independent of ϵ). As we will see, this family will be compatible with the 4/5-law.

4. A step towards a model of the velocity field of turbulent flows

An acceptable solution to the problem of hydrodynamical turbulence in dimension 3 would be to construct a random velocity field U solution to the dynamics (Euler or Navier Stokes typically) that is stationnary, incompressible, space-homogeneous, isotropic and that satisfies the main statistical properties of the velocity field of turbulent flows. Two main properties are:

(1) The 4/5-law of Kolmogorov that links the energy dissipation of the turbulent flow to the statistics of the increments of the velocity. This law is widely accepted since it is the only one that can be proven with the dynamics ([6], [7], [14]). More precisely, this law states:

$$E((U(x+\xi) - U(x).\frac{\xi}{|\xi|})^3) = -\frac{4}{5}D|\xi|. \tag{4.1}$$

In the above formula, D denotes the average dissipation of the kinetic energy per unit mass in the fluid.

Remark 4.1. *To obtain this law, it is sufficient to suppose that the field U is space homogeneous and isotropic.*

(2) The intermittency of the field U:

$$E((U(x+\xi)-U(x).\frac{\xi}{|\xi|})^q) \underset{|\xi|\to 0}{\sim} C_q|\xi|^{\zeta_q}, \tag{4.2}$$

where ζ_q is a well known concave structure function (cf. [7]).

It is a very challenging task to construct a field with all the aforementioned properties, especially because this field must be invariant by the Euler or Navier-Stokes equation.

Nevertheless, one can in the first place forget the invariance by the dynamics and simply try to construct a field that satisfies the other properties. The 4/5-law shows that the nonsymmetry of the increments is an essential feature. Let us consider the family $\mathcal{X}$ constructed in the previous section ($d=3$). By proposition 3.6, we have:

$$E(((\mathcal{X}(x+\lambda e)-\mathcal{X}(x)).e)^3) \underset{\lambda\to 0}{\sim} C_3(\frac{\lambda}{R})^{\widetilde{\zeta}_3}, \quad C_3>0,$$

with $\widetilde{\zeta}_3 = 2\alpha-2$. To satisfy the 4/5 law one should have $\widetilde{\zeta}_3 = 1$, which gives $\alpha = 3/2$. This is incompatible with the constraint $3/2<\alpha<5/2$. Thus we have now to modify the family $\mathcal{X}$ to reach the limit case $\widetilde{\zeta}_3 = 1$. In this aim, we will construct a new (three parameter) family $\mathcal{X}_0$ corresponding to the limit case $\gamma_1 = 0$, γ_0 constant >0.

4.1. **Construction of the field $\mathcal{X}_0$.** In this section, we only outline the main steps of the construction of $\mathcal{X}_0$. The field $\mathcal{X}_{0,\epsilon}$ is given by formula (2.3) where X^ϵ is now defined by:

$$X^\epsilon(y) = \gamma_0 \int_{\mathbb{R}^d} k_\epsilon^R(y-\sigma)dW_0(\sigma).$$

We suppose that α is in the interval $]0,1[$. We choose the normalizing constant C_ϵ such that:

$$\gamma_0 k_\epsilon^R(0)e^{-C_\epsilon+\frac{1}{2}\gamma_0^2\rho_{\epsilon/R}(0)} = 1.$$

We start by stating a lemma we will use in the proof of the proposition below:

Lemma 4.2. *let δ be some real number different from d. Then there exists $C = C(\delta)>0$ with:*

$$\int_{|u|\leqslant R}\frac{du}{|u|_\epsilon^\delta} \leqslant C\epsilon^{(d-\delta)\wedge 0}. \tag{4.3}$$

Proof. We suppose $\delta>d$, the other case being obvious. We have:

$$\begin{aligned}\int_{|u|\leqslant R}\frac{du}{|u|_\epsilon^\delta} &\underset{u=\epsilon\widetilde{u}}{=} \epsilon^{d-\delta}\int_{|u|\leqslant R/\epsilon}\frac{d\widetilde{u}}{(\int_{|v|\leqslant 1}\theta(v)|v+\widetilde{u}|dv)^\delta}\\ &\leqslant \epsilon^{d-\delta}\int_{\mathbb{R}^d}\frac{d\widetilde{u}}{(\int_{|v|\leqslant 1}\theta(v)|v+\widetilde{u}|dv)^\delta}.\end{aligned}$$

□

We can now state the following proposition:

Proposition 4.3. *Let l be an integer $\geqslant 1$ and γ_0 such that:*

(1) $\gamma_0^2\omega_d < \alpha$ *if* $l = 1$.
(2) $(2l - 3/2)\gamma_0^2\omega_d < \alpha \wedge d/2$ *if* $l > 1$.

Then for all x, $\mathcal{X}_{0,\epsilon}(x)$ *converges in* L^{2l} *to a random vector* $\mathcal{X}_0(x)$ *(i.e.* $E((\mathcal{X}_{0,\epsilon}(x) - \mathcal{X}_0(x))^{2l}) \to 0$*). The random vector field* $\mathcal{X}_0(x)$ *satisfies the following scaling: For* e *(*$|e| = 1$*), and for all* $q \leqslant 2l$*:*

$$E((\mathcal{X}_0^j(x + \lambda e) - \mathcal{X}_0^j(x))^q) \underset{\lambda \to 0}{\sim} C_q^j(e)(\frac{\lambda}{R})^{\zeta_q}, \tag{4.4}$$

where $\zeta_q = q\alpha - \frac{1}{2}q(q-1)\gamma_0^2\omega_d$ *and*

$$C_q^j(e) = R^{qd/2} e^{\frac{q(q-1)}{2}\gamma_0^2\phi(0)} \int_{(\mathbb{R}^d)^q} \prod_{1 \leqslant i<j \leqslant q} \frac{1}{|u_i - u_j|^{\gamma_0^2\omega_d}} \prod_{1 \leqslant i \leqslant q} (\frac{u_i^j}{|u_i|^{d-\alpha+1}} - \frac{u_i^j - e^j}{|u_i - e|^{d-\alpha+1}}) du_1 \dots du_q. \tag{4.5}$$

Proof. We will first prove that:

$$E((\mathcal{X}_{0,\epsilon}^j(x))^{2l}) \underset{\epsilon \to 0}{\to} \int_{(\mathbb{R}^d)^{2l}} \prod_{1 \leqslant i<j \leqslant 2l} \frac{e^{\gamma_0^2\phi(\frac{y_i - y_j}{R})}}{|\frac{y_i - y_j}{R}|_*^{\gamma_0^2\omega_d}} \prod_{1 \leqslant i \leqslant 2l} \frac{\varphi_R(y_i) y_i^j}{|y_i|^{d-\alpha+1}} dy_1 \dots dy_{2l}. \tag{4.6}$$

We remind that the right hand side of the above limit exists by lemma 2.4. In order to prove the above relation, we develop $E((\mathcal{X}_{0,\epsilon}^j(x))^{2l})$ in $l+1$ terms similarly as in the proof of proposition 3.7; then, using formula (2.4) and the fact that, for all y, $\rho_{\epsilon/R}(y) \leqslant \omega_d \ln \frac{R}{\epsilon} + C$, we are led to show that, for all $k \leqslant l-1$, we have the following convergence:

$$\epsilon^{(l-k)(d-\gamma_0^2\omega_d)} \epsilon^{-2(l-k)(l-k-1)\gamma_0^2\omega_d} \epsilon^{-4k(l-k)\gamma_0^2\omega_d} \int_{(\mathbb{R}^d)^{k+l}} \frac{\varphi_R(y_1) y_1^j}{|y_1|_\epsilon^{d-\alpha+1}} \dots \frac{\varphi_R(y_{2k}) y_{2k}^j}{|y_{2k}|_\epsilon^{d-\alpha+1}} \times$$
$$\frac{(\varphi_R(y_{2k+1}) y_{2k+1}^j)^2}{|y_{2k+1}|_\epsilon^{2(d-\alpha+1)}} \dots \frac{(\varphi_R(y_{k+l}) y_{k+l}^j)^2}{|y_{k+l}|_\epsilon^{2(d-\alpha+1)}} \prod_{1 \leqslant i<j \leqslant 2k} \frac{e^{\gamma_0^2\phi(\frac{y_i - y_j}{R})}}{|\frac{y_i - y_j}{R}|_*^{\gamma_0^2\omega_d}} dy_1 \dots dy_{k+l} \underset{\epsilon \to 0}{\to} 0.$$

We apply inequality (4.3) and obtain (if $\alpha = d/2$, one can work with $\alpha - \eta$ for $\eta > 0$ sufficiently small) :

$$\int_{\mathbb{R}^d} \frac{\varphi_R(y)^2}{|y|_\epsilon^{2(d-\alpha)}} dy \leqslant C\epsilon^{(2\alpha - d)\wedge 0}$$

Therefore the above convergence to 0 amounts to showing that, for all $k \leqslant l-1$, we have the following inequality:

$$d + (2\alpha - d) \wedge 0 - \gamma_0^2\omega_d > 2(l-k-1)\gamma_0^2\omega_d + 4k\gamma_0^2\omega_d.$$

This is equivalent to $(2l - \frac{3}{2})\gamma_0^2\omega_d < \alpha \wedge \frac{d}{2}$. One can show, for all x, that $(\mathcal{X}_{0,\epsilon}(x))_{\epsilon>0}$ is a Cauchy sequence in L^{2l} by computing $E((\mathcal{X}_{0,\epsilon}^j(x) - \mathcal{X}_{0,\epsilon'}^j(x))^{2l})$ and letting ϵ, ϵ' go to 0. Thus, $E((\mathcal{X}_0^j(x))^{2l})$ is given by the left hand side of (4.6).

To show the scaling (4.4), observe that we can prove the following analogue to (4.6) for any $q \leqslant 2l$:

$$E((\mathcal{X}_0^j(x+\lambda e) - \mathcal{X}_0^j(x))^q) = \int_{(\mathbb{R}^d)^q} \prod_{1 \leqslant i<j \leqslant q} \frac{e^{\gamma_0^2 \phi(\frac{y_i - y_j}{R})}}{|\frac{y_i-y_j}{R}|_*^{\gamma_0^2 \omega_d}} \prod_{1 \leqslant i \leqslant q} f_{\lambda e}(y_i) dy_1 \dots dy_q, \tag{4.7}$$

where

$$f_{\lambda e}(y) = \frac{\varphi_R(y) y^j}{|y|^{d-\alpha+1}} - \frac{\varphi_R(y-\lambda e)(y^j - \lambda e^j)}{|y-\lambda e|^{d-\alpha+1}}. \tag{4.8}$$

By setting $y_i = \lambda u_i$ in the integral of (4.7), we deduce easily (4.4). □

Remark 4.4. *It is not obvious why in the above proposition the coefficients $C_q^j(e)$ are different from 0 (cf. appendix).*

Remark 4.5. *Similarly as in the previous section, for γ_0 sufficiently small, $\mathcal{X}_{0,\epsilon}$ converges in law to $\mathcal{X}_0$ in the space of continuous fields.*

4.2. **Nonsymmetry of the increments of $\mathcal{X}_0$.** By isotropy, we have:

$$(\mathcal{X}_0(x+\lambda e) - \mathcal{X}_0(x)).e \underset{law}{=} \mathcal{X}_0^j(x+\lambda e_j) - \mathcal{X}_0^j(x)$$

One can show that for λ small the law is nonsymmetrical by showing that the third moment is $\neq 0$, that is $C_3^j(e_j) \neq 0$ (see appendix).

4.3. **Towards a model of the turbulent velocity field.** In dimension 3, for the field $\mathcal{X}_0$, we have:

$$\zeta_q = q\alpha - 2\pi q(q-1)\gamma_0^2.$$

Thus, for $\alpha = 1/3 + 4\pi\gamma_0^2$, we have $\zeta_3 = 1$, which means that for this choice the associated fields $\mathcal{X}_0$ satisfy at small scale the 4/5 law with a non zero finite dissipation. Unfortunately, the fields in this family are not incompressible. The incompressible case (at small scale) would correspond to the choice $\alpha = 1$, a limit case which is excluded by the constraint $0 < \alpha < 1$ needed for the validity of the scaling of proposition 4.3. There is another severe obstacle for the choice $\alpha = 1$. Indeed, for the field $\mathcal{X}_0$ above, we have:

$$\zeta_q = (1/3 + 6\pi\gamma_0^2)q - 2\pi\gamma_0^2 q^2.$$

One can easily identify the intermittency parameter $4\pi\gamma_0^2$ using the experimental curve given in [1] (cf. fig 8.8 p.132 in [7]). With this data, we find $4\pi\gamma_0^2 = 0.023$. With this small intermittency parameter, we would get $\alpha \sim 0.35$ which is not close to the incompressible value $\alpha = 1$. So, in spite of its qualitative interest, this model cannot reach quantitative adequacy.

Another natural way to get incompressible fields is to use a Biot-Savart like formula and take the limit as ϵ goes to 0 of fields of the form:

$$U^\epsilon(x) = \int_{\mathbb{R}^3} \varphi_R(x-y) \frac{x-y}{|x-y|_\epsilon^{d-\alpha+1}} \wedge d\Omega^\epsilon,$$

where $d\Omega^\epsilon$ is an isotropic random field. For example, we can take:

$$d\Omega^\epsilon = e^{X^\epsilon(y) - C_\epsilon} dW(y),$$

where $dW(y) = (dW_1(y), dW_2(y), dW_3(y))$ denotes a three dimensional white noise and X^ϵ is defined by the following formula:

$$X^\epsilon(y) = \gamma \int_{\mathbb{R}^3} K_\epsilon^R(y-\sigma).dW(\sigma),$$

with $K^R(x) = \frac{x}{|x|^{1+d/2}} 1_{|x| \leqslant R}$.

As for the case of $\mathcal{X}_0$, we choose the constant C_ϵ such that U^ϵ converges to a non trivial field U as ϵ goes to zero. The vector field U we obtain is incompressible, homogeneous, isotropic and intermittent with structural exponents ζ_q given by:

$$\zeta_q = q\alpha - 2\pi\gamma^2 q(q-1).$$

Unfortunately, since the field $d\Omega^\epsilon(y)$ is isotropic with respect to all unitary transformations (and not just the rotations) we get for U the symmetry:

$$U(-x) - U(0) \underset{law}{=} U(x) - U(0)$$

so that the dissipation is equal to 0. Thus the construction of an homogeneous, isotropic, intermittent and incompressible vector field with positive finite dissipation remains an open question.

Comment 4.6. *In our approach, we perturb a Gaussian field to get multifractality and we further introduce some dependency to obtain also dissymmetry. This can make one think that in turbulence dissipation is linked to intermittency. This is a rather intricate issue. On one hand, only dissymmetry seems to be needed to get energy dissipation (see the 4/5 law). On the other hand, it is well known experimentally that dissipation is not homogeneously distributed in the fluid but rather follows the lognormal distribution described by Kolmogorov and Obukhov by which it appears linked to intermittency.*

In dimension $d = 1$, our model displays some kind of (non causal) leverage effect. To get a realistic model for finance, with causal leverage effect, we have to make some specific change in the construction. This issue will be addressed in a forthcoming paper.

5. APPENDIX

In this appendix, we prove that for q even $C_q^1(e)$, given by equation (4.5), is different from 0 outside a countable set and in the neighbourhood of 0. We also show the same result for $C_3^j(e_j)$.

Consider first the case q even; we set $q = 2l$ with l greater or equal to 1 and we introduce the following function F:

$$F(\gamma) = \int_{(\mathbb{R}^d)^{2l}} \prod_{1 \leqslant i<j \leqslant 2l} \frac{1}{|u_i - u_j|^\gamma} \prod_{1 \leqslant i \leqslant 2l} f(u_i) du_1 \dots du_{2l},$$

where f is the real function defined by:

$$f(u) = \frac{u^j + e^j/2}{|u + e/2|^{d-\alpha+1}} - \frac{u^j - e^j/2}{|u - e/2|^{d-\alpha+1}}.$$

The function F is analytical in a neighbourhood of 0; therefore, in order to obtain the desired result, we have to prove that F is not identically equal to 0. One can show that, for all $i < l$, $F^{(i)}(0) = 0$ and that:

$$F^{(l)}(0) = \frac{2l!}{2^l}(\int_{(\mathbb{R}^d)^2} \ln(\frac{1}{|u_1 - u_2|}) f(u_1) f(u_2) du_1 du_2)^l.$$

The Fourier transform of $\ln(\frac{1}{|u|})$ is $a_d Pf(\frac{1}{|\xi|^d}) + b_d \delta_0$ where $a_d > 0$ and b_d are two constants that depend only on the dimension and Pf is Hadamard's finite part (see p.258 in [15]). Since $\int_{\mathbb{R}^d} f(u)du = 0$, we get:

$$\int_{(\mathbb{R}^d)^2} \ln(\frac{1}{|u_1 - u_2|}) f(u_1) f(u_2) du_1 du_2 = a_d \int_{\mathbb{R}^d} \frac{\hat{f}(\xi)^2}{|\xi|^d} d\xi,$$

thus $F^{(l)}(0) > 0$.

Let us now consider $C_3^j(e_j)$ and the corresponding function:

$$F(\gamma) = \int_{(\mathbb{R}^d)^3} \frac{1}{|u_1 - u_2|^\gamma |u_1 - u_3|^\gamma |u_2 - u_3|^\gamma} f(u_1) f(u_2) f(u_3) du_1 du_2 du_3.$$

We obviously have $F(0) = 0$, $F'(0) = 0$ and:

$$\frac{1}{6} F''(0) = \mathcal{I} = \int_{(\mathbb{R}^d)^3} \ln(|u_1 - u_2|) \ln(|u_1 - u_3|) f(u_1) f(u_2) f(u_3) du_1 du_2 du_3$$

so that $\mathcal{I} = \int_{\mathbb{R}^d} f(x) \Theta(x)^2 dx$ where:

$$\Theta(x) = \int_{\mathbb{R}^d} \ln(|x - y|) f(y) dy.$$

Now we prove that there exists some real constant c different from 0 such that:

$$\Theta(x) = c(\frac{x^j + 1/2}{|x + e_j/2|^{1-\alpha}} - \frac{x^j - 1/2}{|x - e_j/2|^{1-\alpha}}). \tag{5.1}$$

Indeed, we have (in what follows, c denotes different real constants that are not equal to 0):

$$\hat{\Theta}(\xi) = c \frac{\hat{f}(\xi)}{|\xi|^d}$$

and $\hat{f}(\xi) = c \sin(\pi \xi^j) \frac{\xi^j}{|\xi|^{\alpha+1}}$ thus

$$\hat{\Theta}(\xi) = c \sin(\pi \xi^j) \frac{\xi^j}{|\xi|^{d+\alpha+1}}.$$

The above expression (5.1) now follows from:

$$\widehat{(\frac{x^j}{|x|^{1-\alpha}})}(\xi) = ci \frac{\xi^j}{|\xi|^{d+\alpha+1}}$$

and $2i \sin(\pi \xi^j) = \widehat{\delta_{-e_j/2}}(\xi) - \widehat{\delta_{e_j/2}}(\xi)$.

Now let us denote $x = x^j e_j + \widetilde{x}$ and

$$\phi(p, y, a) = \frac{y + 1/2}{((y + 1/2)^2 + a)^p} - \frac{y - 1/2}{((y - 1/2)^2 + a)^p}.$$

If we set:

$$\varphi(x^j) = \phi(\frac{d - \alpha + 1}{2}, x^j, |\widetilde{x}|^2)$$

and

$$\psi(x^j) = \phi(\frac{1 - \alpha}{2}, x^j, |\widetilde{x}|^2),$$

we get:

$$\mathcal{I} = c^2 \int_{\mathbb{R}^{d-1}} d\widetilde{x} \int_{\mathbb{R}} \varphi(x^j)\psi(x^j)^2 dx^j$$

Since $0 < \alpha < 1$, it is easy to check that $\psi(z)$ is a positive function of z, decreasing on $[0, \infty[$. One can also check that there exists some $z^* > 1/2$ such that $\varphi(z)$ is positive on $[0, z^*[$ and negative on $]z^*, \infty[$. Since φ and ψ are even and $\int_0^\infty \varphi(z)dz = 0$, one can derive the following:

$$\begin{aligned}
\int_{\mathbb{R}} \varphi(z)\psi(z)^2 dz &= 2\int_0^\infty \varphi(z)\psi(z)^2 dz \\
&= 2\int_0^{z^*} \varphi(z)\psi(z)^2 dz + 2\int_{z^*}^\infty \varphi(z)\psi(z)^2 dz \\
&\geqslant 2\int_0^{z^*} \varphi(z)\psi(z)^2 dz + 2\psi(z^*)^2 \int_{z^*}^\infty \varphi(z) dz \\
&= 2\int_0^{z^*} \varphi(z)(\psi(z)^2 - \psi(z^*)^2) dz \\
&> 0.
\end{aligned}$$

It follows that $F''(0) > 0$.

□

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