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Pathwise asymptotic behavior of random determinants in the Jacobi ensemble

A. Rouault *

July 29, 2006

Summary. This is a companion paper of [Rou05]. It concentrates on asymptotic properties of determinants of some random matrices in the Jacobi ensemble. Let $M \in \mathcal{M}_{n_1+n_2,r}(\mathbb{R})$ (with $r \leq n_1 + n_2$) be a random matrix whose entries are standard i.i.d. Gaussian. We can decompose $M^T = (M_1^T, M_2^T)$ with $M_1 \in \mathcal{M}_{n_1,r}$ and $M_2 \in \mathcal{M}_{n_2,r}$. Then, $W_1 := M_1^T M_1$ and $W_2 := M_2^T M_2$ are independent $r \times r$ Wishart matrices with parameters n_1 and n_2 and $M^T M = W_1 + W_2$ is Wishart with parameter $n_1 + n_2$. The matrix

$$\mathcal{Z} := (W_1 + W_2)^{-1/2} W_1 (W_1 + W_2)^{-1/2}$$

has a Beta matrix variate distribution with parameters $n_1/2, n_2/2$ (sometimes called the Jacobi distribution). We set $n_1 = \lfloor n\tau_1 \rfloor$, $n_2 = \lfloor n\tau_2 \rfloor$, $r = \lfloor nt \rfloor$ $t \in [0, \tau_1]$; we define $\mathcal{Z}_n(t)$ as the corresponding matrix and $\Theta_n(t) := |\mathcal{Z}_n(t)|$ as its determinant. In the Jacobi ensemble, the Kshirsagar's theorem decomposes $\Theta_n(t)$ into a product of independent beta distributed random variables. It is very reminiscent to the Bartlett's decomposition of determinants in the Wishart ensemble. This allows us to study the process $(n^{-1} \log \Theta_n(t), t \in [0, \tau_1])$ and the asymptotic behavior of the sequence $\{n^{-1} \log \Theta_n\}_n$ as $n \rightarrow \infty$ with τ_1 and τ_2 fixed : a.s. convergence, fluctuations, large deviations. We connect the results for marginals (fixed t) with those obtained by the study of the empirical spectral distribution. In the whole paper, we consider the problem of general β , where the particular cases $\beta = 1, 2, 4$ correspond to real, complex, and quaternionic matrices.

Key words. Random matrices, Wishart ensemble, Jacobi ensemble, determinant, invariance principle, large deviations, projections.

A.M.S. Classification. 15 A 52, 15 A 15, 60F 10, 60F 17, 62 H 10

1 Introduction

Let us first define some matrix models with elements in X which means \mathbb{R}, \mathbb{C} or the division algebra \mathbb{H} . Recall that an element of \mathbb{H} (called quaternion) may be viewed as a

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2×2 matrix of the form

$$q = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$$

where z and w are complex numbers. Its dual (or conjugate) is

$$\bar{q} = \begin{pmatrix} \bar{z} & -w \\ \bar{w} & z \end{pmatrix}.$$

For $n, r \in \mathbb{N}$ such that $r \leq n$, let $\mathcal{M}_{n,r}(\mathbb{X})$ be the set of $n \times r$ matrices with entries in \mathbb{X} . For a matrix M and p, q integers, let us denote by $M^{[p,q]}$ the upper-left corner of size $p \times q$ of M , and $M^{[p]}$ for $M^{[p,p]}$. Finally, if M is a square matrix, $|M|$ denotes the determinant of M .

We say that a random vector Y in \mathbb{X}^N has the standard normal distribution, if its coordinates are independent and $\mathcal{N}(0, 1)$. If $\mathbb{X} = \mathbb{C}$, the $\mathcal{N}(0, 1)$ distribution has density

$$z \in \mathbb{C} \mapsto \frac{1}{\pi} e^{-|z|^2}$$

and if $\mathbb{X} = \mathbb{H}$ the $\mathcal{N}(0, 1)$ distribution has density

$$(z, w) \in \mathbb{C} \times \mathbb{C} \mapsto \frac{2}{\pi} e^{-2(|z|^2 + |w|^2)}.$$

Let Y_1, \dots, Y_p be independent vectors on \mathbb{X}^N with identical standard normal distribution and set $S = Y_1 \cdot Y_1^* + \dots + Y_p \cdot Y_p^*$. Then S is said to follow the Wishart distribution $W(N, p, \mathbb{X})$.

1.1 Real matrices

See [OR64], [Mui82] p.108. Let $W_1, W_2 \in \mathcal{M}_{r,r}(\mathbb{R})$ independent random Wishart matrices of distribution $W(r, n_1, \mathbb{R})$ and $W(r, n_2, \mathbb{R})$ such that $r \leq n_1 + n_2$. Then it is well known that $W_1 + W_2$ is $W(r, n_1 + n_2, \mathbb{R})$ distributed and a.s. invertible. Let

$$\mathcal{Z} = (W_1 + W_2)^{-1/2} W_1 (W_1 + W_2)^{-1/2}.$$

If $r \leq \min(n_1, n_2)$, the distribution of \mathcal{Z} has a density with respect to the Lebesgue measure on \mathcal{S}_r (the set of real symmetric matrices) which is

$$\frac{1}{\beta_r \left(\frac{n_1}{2}, \frac{n_2}{2} \right)} |\mathcal{Z}|^{\frac{n_1-r-1}{2}} |I_r - \mathcal{Z}|^{\frac{n_2-r-1}{2}} \mathbb{1}_{0 < \mathcal{Z} < I_r}, \quad (1)$$

where $|\cdot|$ means determinant and

$$\beta_r \left(\frac{n_1}{2}, \frac{n_2}{2} \right) = \frac{\Gamma_r \left(\frac{n_1}{2} \right) \Gamma_r \left(\frac{n_2}{2} \right)}{\Gamma_r \left(\frac{n_1+n_2}{2} \right)},$$

(see for example Muirhead [Mui82] th.3.3.1), where Γ_r is the multivariate Gamma function. It is the matrix variate extension of the beta distribution. Recall that, for $\alpha > 0$, $\beta > 0$ the beta(α, β) (real) distribution has density

$$\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \quad (x > 0). \quad (2)$$

If \mathcal{Z} is distributed as in (1), the density of eigenvalues is given by

$$\frac{1}{Z_r(n_1, n_2)} \prod_{i=1}^r \lambda_i^{\frac{n_1-r-1}{2}} (1-\lambda_i)^{\frac{n_2-r-1}{2}} \prod_{1 \leq i < j \leq r} (\lambda_j - \lambda_i), \quad (3)$$

on the chamber $(1 > \lambda_r > \dots > \lambda_1 > 0)$, where Z_r , the normalizing constant is

$$Z_r(n_1, n_2) = \pi^{-\frac{r^2}{2}} \beta_r\left(\frac{n_1}{2}, \frac{n_2}{2}\right) \Gamma_r\left(\frac{r}{2}\right) \frac{\Gamma_r\left(\frac{n_1}{2}\right) \Gamma_r\left(\frac{n_2}{2}\right) \Gamma_r\left(\frac{r}{2}\right)}{\Gamma_r\left(\frac{n_1+n_2}{2}\right)}.$$

(see for example [Mui82] th. 3.3.4).

It should be noticed that for $n_2 < r < n_1$ the matrix W_2 is singular, and then the Jacobi matrix $(W_1 + W_2)^{-1/2} W_1 (W_1 + W_2)^{-1/2}$ has 1 as an eigenvalue with multiplicity $r - n_2$. The distribution of \mathcal{Z} has no density. Nevertheless we may study its determinant. Indeed, the matrix $(W_1 + W_2)^{-1/2} W_2 (W_1 + W_2)^{-1/2}$ has 0 as a multiple eigenvalue with multiplicity $r - n_2$. Actually the density of the law of the non-zero eigenvalues of this matrix is known (see Srivastava [Sri03] and Diaz-Garcia and Gutierrez Jaimez [DGGJ97]), so that the non-one eigenvalues of \mathcal{Z} have the density

$$\frac{1}{\tilde{Z}_r(n_1, n_2)} \prod_{i=1}^{n_2} \lambda_i^{\frac{n_1-r-1}{2}} (1-\lambda_i)^{\frac{r-n_2-1}{2}} \prod_{1 \leq i < j \leq n_2} (\lambda_j - \lambda_i), \quad (4)$$

on the chamber $(1 > \lambda_{n_2} > \dots > \lambda_1 > 0)$, where the normalizing constant is

$$\tilde{Z}_r(n_1, n_2) = Z_{n_2}(n_1 + n_2 - r, r).$$

1.2 Complex and symplectic matrices

We now consider matrices with elements in $\mathbb{X} = \mathbb{C}$ or \mathbb{H} . Actually it is usual to embed the three models (real, complex and quaternionic) in the same framework, with a parameter $\beta = 1, 2, 4$. When $r \leq \min(n_1, n_2)$, the matrix variate distribution of \mathcal{Z} has the density

$$\frac{1}{B_r^{(\beta)}\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} |\mathcal{Z}|^{\frac{\beta(n_1-r+1)}{2}-1} |I_r - \mathcal{Z}|^{\frac{\beta(n_2-r+1)}{2}-1} \mathbb{1}_{0 < \mathcal{Z} < I_r}.$$

The distribution of the ordered eigenvalues of \mathcal{Z} has the density

$$f_{r, n_1, n_2}(\lambda_1, \dots, \lambda_r) = \frac{1}{Z_r^{(\beta)}(n_1, n_2)} \prod_{i=1}^r \lambda_i^{\frac{\beta(n_1-r+1)}{2}-1} (1-\lambda_i)^{\frac{\beta(n_2-r+1)}{2}-1} \prod_{1 \leq i < j \leq r} (\lambda_j - \lambda_i)^\beta \quad (5)$$

on the chamber $(1 > \lambda_r > \dots > \lambda_1 > 0)$, where (Selberg's integral)

$$Z_r^{(\beta)}(n_1, n_2) = \prod_{j=1}^r \frac{\Gamma\left(1 + \frac{\beta j}{2}\right) \Gamma\left(\frac{\beta(n_1+j-r)}{2}\right) \Gamma\left(\frac{\beta(n_2+j-r)}{2}\right)}{\Gamma\left(1 + \frac{\beta}{2}\right) \Gamma\left(\frac{\beta(n_1+n_2+j-r)}{2}\right)}. \quad (6)$$

(see also Edelman and Rao [ER05] 4.7 and 4.8 and references therein).

In the "singular" case $(n_2 \leq r \leq n_1)$, the density of the non-one eigenvalues is $f_{n_2, n_1+n_2-r, r}$ on the chamber $(1 > \lambda_{n_2} > \dots > \lambda_1 > 0)$.

We say that this model corresponds to the Jacobi ensemble (orthogonal for $\beta = 1$, unitary for $\beta = 2$ or symplectic for $\beta = 4$), see [For] chap. 2.

1.3 General β framework

We consider an extension of the above models. The idea of extending the range of β to $(0, \infty)$ is quite natural. It has been shown recently that it corresponds to other matrix models (see [DE03], [KN04]). For every $\beta > 0$, we define a family of distribution densities \mathbf{f}_{r, n_1, n_2} :

$$\mathbf{f}_{r, n_1, n_2} = \begin{cases} f_{r, n_1, n_2} & \text{if } r \leq \min(n_1, n_2) \\ f_{n_2, n_1+n_2-r, r} & \text{if } n_2 \leq r \leq n_1 \end{cases} \quad (7)$$

It is a density on the Weyl chamber with $\min(n_2, r)$ variables.

We set by convention

$$|\mathcal{Z}_{n_1, n_2, r}| = \prod_{i=1}^{\min(n_2, r)} \lambda_i$$

in all cases, and we call it the determinant, even if we do not define any matrix.

2 Determinant

2.1 Determinant and decomposition

The following result is the companion of the Bartlett's decomposition of Wishart determinants.

Proposition 2.1 *If $1 \leq r \leq n_1$, in the real case, let us write $\mathcal{Z} = T'T$ where T is uppertriangular. Then the random variables $T_{j,j}^2$, $j = 1, \dots, r$ are independent and*

$$T_{j,j}^2 \stackrel{(d)}{=} \text{beta}\left(\frac{n_1 - j + 1}{2}, \frac{n_2}{2}\right).$$

As an easy consequence, we have

$$|\mathcal{Z}| = \prod_{j=1}^r T_{j,j}^2, \quad (8)$$

where the independent variables $T_{j,j}^2$, $j = 1, \dots, r$ are as above.

This proposition, due to Kshirsagar, is quoted and proved in [Mui82] p.110 under the assumption $r \leq n_1, n_2$ and in [Rao73] p. 541 under the only assumption $r \leq n_1$. Actually,

$$T_{jj}^2 = \frac{|W_1^{[j]}|}{|W_1^{[j]} + W_2^{[j]}|} \times \frac{|W_1^{[j-1]} + W_2^{[j-1]}|}{|W_1^{[j-1]}|}. \quad (9)$$

Some proofs use probabilistic arguments (as [Rao73]) or Jacobian arguments (as in [GN00] th. 5.3.24 p. 181).

The fact (8) may be proved directly using Mellin's transform as in [Mat99] th.2. From (2), it is clear that if $X \stackrel{(d)}{=} \text{beta}(a, b)$ then

$$\mathbb{E}X^\mu = \frac{\Gamma(a + \mu)\Gamma(a + b)}{\Gamma(a)\Gamma(a + b + \mu)}, \quad \theta > -a. \quad (10)$$

It is the starting point for an extension to the general β case.

Proposition 2.2 *In the general β case, for $r \leq n_1$ we have*

$$|\mathcal{Z}_{n_1, n_2, r}| \stackrel{(d)}{=} \prod_{j=1}^r \tilde{T}_{jj}^2,$$

where $\tilde{T}_{j,j}^2$, $j = 1, \dots, r$ are independent and

$$\tilde{T}_{j,j}^2 \stackrel{(d)}{=} \text{beta}\left(\frac{\beta(n_1 - j + 1)}{2}, \frac{\beta n_2}{2}\right).$$

Proof: For $r \leq n_1, n_2$, using (5) and (6) we get

$$\mathbb{E}(|\mathcal{Z}_{n_1, n_2, r}|^\theta) = \frac{Z_r^{(\beta)}\left(n_1 + \frac{2\theta}{\beta}, n_2, r\right)}{Z_r^{(\beta)}(n_1, n_2, r)} = \prod_{j=1}^r \frac{\Gamma\left(\frac{\beta(n_1 + n_2 + j - r)}{2}\right) \Gamma\left(\frac{\beta(n_1 + j - r)}{2} + \theta\right)}{\Gamma\left(\frac{\beta(n_1 + j - r)}{2}\right) \Gamma\left(\frac{\beta(n_1 + n_2 + j - r)}{2} + \theta\right)}, \quad (11)$$

which proves the proposition, owing to the expression of Mellin transform (10).

If $n_2 < r \leq n_1$ we start directly from (7) and (6) we have

$$\mathbb{E}(|\mathcal{Z}_{n_1, n_2, r}|^\theta) = \frac{Z_{n_2}^{(\beta)}\left(n_1 + n_2 - r + \frac{2\theta}{\beta}, r, n_2\right)}{\tilde{Z}_r^{(\beta)}(n_1 + n_2 - r, r, n_2)} = \prod_{j=1}^{n_2} \frac{\Gamma\left(\frac{\beta(n_1 + j)}{2}\right) \Gamma\left(\frac{\beta(n_1 + j - r)}{2} + \theta\right)}{\Gamma\left(\frac{\beta(n_1 + j - r)}{2}\right) \Gamma\left(\frac{\beta(n_1 + j)}{2} + \theta\right)} \quad (12)$$

Multiplying up and down by $\prod_{k=n_2+1}^r \Gamma\left(\frac{\beta(n_1 + k - r)}{2}\right) \Gamma\left(\frac{\beta(n_1 + k - r)}{2} + \theta\right)$ we get again (11). ■

2.2 Processes

In the real case ($\beta = 1$) we see from (9) that for $1 \leq r \leq n_1$

$$\log |\mathcal{Z}_{n_1, n_2, r}| = \sum_{k=1}^r \log T_{k,k}^2 = \log \frac{|W_1^{[r]}|}{|W_1^{[r]} + W_2^{[r]}|} \quad (13)$$

On the probability space where the Wishart matrices are defined, when n_1 and n_2 are fixed, we have then an array of independent random variables $(\log T_{k,k}^2, k = 1, \dots, n_1)$, so that it is meaningful to consider the process (indexed by r) of partial sums. A now classical asymptotic regime is $n_1, n_2, r \rightarrow \infty$ with fixed ratios. To be more precise, we fix τ_1 and $\tau_2 > 0$, set $n_1 = \lfloor n\tau_1 \rfloor, n_2 = \lfloor n\tau_2 \rfloor$, define the process

$$\Theta_n(t) := \sum_{k=1}^{\lfloor nt \rfloor} \log T_{k,k}^2, \quad t \in [0, \tau_1],$$

and let $n \rightarrow \infty$. It means that we consider the asymptotic behavior of Jacobi determinants in a dynamic (or pathwise) way.

In the general β case, we take $\tau_1, \tau_2, n, n_1, n_2$ as above and consider $T_1^{(n)}, \dots, T_{n_1}^{(n)}$ independent random variables such that

$$T_j^{(n)} \stackrel{(d)}{=} \text{beta} \left(\frac{\beta(n_1 - j + 1)}{2}, \frac{\beta n_2}{2} \right), \quad j = 1, \dots, n_1. \quad (14)$$

We set $\rho_j^{(n)} = \log T_j^{(n)}$, $j = 1, \dots, n_1$ and

$$\Theta_n(t) := \sum_{j=1}^{\lfloor nt \rfloor} \rho_j^{(n)}, \quad t \in [0, \tau_1]. \quad (15)$$

The main tool of our study is the family of (rescaled) cumulant generating functions :

$$\Lambda_{j,n}^J(\theta) := \log \mathbb{E} \exp \frac{\beta\theta}{2} \rho_j^{(n)} = \log \mathbb{E} (T_j^{(n)})^{\beta\theta/2}. \quad (16)$$

Using the notation $\ell = \log \Gamma$ and (10), this function can be written

$$\begin{aligned} \Lambda_{j,n}^J(\theta) &= \ell \left(\frac{\beta(n_1 - j + 1 + \theta)}{2} \right) - \ell \left(\frac{\beta(n_1 - j + 1)}{2} \right) \\ &+ \ell \left(\frac{\beta(n_1 + n_2 - j + 1)}{2} \right) - \ell \left(\frac{\beta(n_1 + n_2 - j + 1 + \theta)}{2} \right). \end{aligned} \quad (17)$$

Remark 2.3 *In the real case, let us discuss another occurrence of the Jacobi ensemble, (see Doumerc [Dou05], Collins [Col05]). Let $M \in \mathcal{M}_{r, n_1+n_2}(\mathbb{R})$ ($r \leq n := n_1 + n_2$) be a random matrix whose entries are standard i.i.d. Gaussian. We can decompose $M = (M_1, M_2)$ with $M_1 \in \mathcal{M}_{r, n_1}$ and $M_2 \in \mathcal{M}_{r, n_2}$. Then, $W_1 := M_1 M_1^*$ and $W_2 := M_2 M_2^*$ are independent*

Wishart matrices with parameters n_1 and n_2 and then $MM^* = W_1 + W_2$ is Wishart with parameter $n_1 + n_2$. The matrix

$$\mathcal{Z} := (W_1 + W_2)^{-1/2}W_1(W_1 + W_2)^{-1/2}$$

is in the Jacobi ensemble. The singular value decomposition of M is $M = UDV$ with $U \in \mathcal{O}_r$, $V \in \mathcal{O}_n$, $D = (\Delta, 0) \in \mathcal{M}_{r,n}$ and Δ diagonal in $\mathcal{M}_{r,r}$ with nonnegative entries. Although U and V are not uniquely determined, one can choose U and V Haar-distributed and U, V, Δ independent. Then $MM^* = U\Delta^2U^*$ and

$$(W_1 + W_2)^{1/2} = (MM^*)^{1/2} = U\Delta U^* .$$

Let $X_r := V^{[r,n_1]}$, the $r \times n_1$ upper-left corner of V . Actually $M_1 = U\Delta X$ and

$$M_1M_1^* = U\Delta X_r X_r^* U^* = (MM^*)^{1/2}(UX_r X_r^* U^*)(MM^*)^{1/2}$$

and then $\mathcal{Z} = (UX_r)(UX_r)^* \stackrel{(d)}{=} X_r X_r^*$.

In particular $|\mathcal{Z}| = |X_r X_r^*|$, or in other words

$$\frac{|W_1|}{|W_1 + W_2|} = |V^{[r,n_1]} (V^{[r,n_1]})^*|$$

We stress that for every $t \in [0, \tau_1]$, $\Theta_n(t)$ has the same distribution as $|X_{[nt]} X_{[nt]}^*|$. Nevertheless we conjecture that the two processes are not equal in distribution, since in general for $s < r$

$$\frac{|W_1^{[s]}|}{|W_1^{[s]} + W_2^{[s]}|} \neq |V^{[s,n_1]} (V^{[s,n_1]})^*| .$$

2.3 The spectral method

Beside the "decomposition method" of Section 2.1, we may use the direct "spectral" approach which we describe now. The empirical distribution is defined by

$$\mu_{n_1, n_2, r} = \frac{1}{\min(r, n_2)} \sum_{k=1}^{\min(r, n_2)} \delta_{\lambda_k}.$$

In our models, the λ_k are a.s. positive and we can write

$$\log |\mathcal{Z}_{n_1, n_2, r}| = \sum_{k=1}^{\min(r, n_2)} \log \lambda_k = \min(r, n_2) \int (\log x) \mu_{n_1, n_2, r}(dx). \quad (18)$$

It is then possible to carry asymptotical results about this empirical distribution to $\log |\mathcal{Z}_{n_1, n_2, r}|$.

In the matrix models ($\beta = 1, 2$ or 4) let $\lambda_k, k = 1, \dots, r$ be the eigenvalues of $\mathcal{Z}_{n_1, n_2, r}$ (they are real nonnegative). The empirical spectral distribution (ESD) is

$$\nu_{n_1, n_2, r} = \frac{1}{r} \sum_{k=1}^r \delta_{\lambda_k}.$$

When $r \leq n_1 \leq n_2$ we have a.s. $\nu_{n_1, n_2, r} = \mu_{n_1, n_2, r}$ and when $n_2 \leq r \leq n_1$ we have

$$\nu_{n_1, n_2, r} = \frac{n_2 \mu_{n_1, n_2, r} + (r - n_2) \delta_1}{r}.$$

In Capitaine-Casalis [CC04], the asymptotical regime is $n_1/r \rightarrow \alpha', n_2/r \rightarrow \beta'$ with $\alpha' + \beta' \geq 1$. They prove¹ in the complex case, that $\mathbb{E}\nu_{n_1, n_2, r}$ converges (in moments hence) in distribution:

$$\mathbb{E}\nu_{n_1, n_2, r} \Rightarrow CC_{\alpha', \beta'}. \quad (19)$$

where $CC_{\alpha', \beta'}$ is a probability distribution defined below in (27), and for which details and comments are given in the following section.

Remark 2.4 *The case ($\beta' < 1$) corresponds to $r > n_2$, the second matrix W_2 is singular and the case ($\beta' \geq 1$) corresponds to $r \leq n_2$, the second matrix is non-singular.*

3 The generalized Kesten-McKay distributions

3.1 Definition

For $0 < a_- < a_+ < 1$ let π_{a_-, a_+} be the probability distribution on \mathbb{R} defined by

$$\pi_{a_-, a_+}(dx) = C_{a_-, a_+} \frac{\sqrt{(x - a_-)(a_+ - x)}}{2\pi x(1 - x)} \mathbb{1}_{[a_-, a_+]}(x) dx. \quad (20)$$

¹we use the notation α' and β' not to confuse with β already defined.

where C_{a_-,a_+} is the normalization constant. For particular values of the parameters and up to an affine change to make the distribution symmetric, the latter was introduced by Kesten as limit distribution for random walk on some classical groups [Kes59]. It was (independently) introduced by McKay [McK81] as a limit distribution in a graph problem.

Since we found some mistakes in the literature, let us compute explicitly the constant $C_{a,b}$. From the obvious decomposition

$$\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$$

we get

$$(C_{a_-,a_+})^{-1} = \int_{a_-}^{a_+} \frac{\sqrt{(x-a_-)(a_+-x)}}{2\pi x} dx + \int_{1-a_+}^{1-a_-} \frac{\sqrt{(y-(1-a_+))(1-a_- - y)}}{2\pi y} dy \quad (21)$$

and from the classical formula

$$\int_u^v \frac{\sqrt{(x-u)(v-x)}}{2\pi x} dx = \frac{1}{4} [u+v-2\sqrt{uv}] , \quad (22)$$

(for $0 < u < v$) we get

$$(C_{a_-,a_+})^{-1} = \frac{1}{2} \left[1 - \sqrt{a_-a_+} - \sqrt{(1-a_-)(1-a_+)} \right] . \quad (23)$$

In order to cite known results with coherent notation, we will use in the following, four functions :

for any pair $0 < b, c < 1$ we set

$$\sigma_{\pm}(b, c) = \frac{1}{2} \left[1 + \sqrt{bc} \pm \sqrt{(1-b)(1-c)} \right] . \quad (24)$$

and for any pair $0 < x, y < 1$ we set

$$\begin{aligned} a_{\pm}(x, y) &= (1-x-y+2xy) \pm 2\sqrt{x(1-x)y(1-y)} \\ &= \left(\sqrt{(1-x)(1-y)} \pm \sqrt{xy} \right)^2 . \end{aligned} \quad (25)$$

The mappings σ_{\pm} and a_{\pm} are inverse in the following sense. We have the equivalence

$$\begin{aligned} 0 < x < y < 1 , \quad x + y > 1 & , \quad \sigma_-(a, b) = x , \quad \sigma_+(a, b) = y \\ & \Updownarrow \\ 0 < b < c < 1 & , \quad a_-(x, y) = b , \quad a_+(x, y) = c . \end{aligned} \quad (26)$$

3.2 Some families of distributions

For α' and β' positive numbers² such that $\alpha' + \beta' > 1$, Capitaine and Casalis [CC04] defined the probability measure

$$CC_{\alpha',\beta'} := (1-\alpha')^+ \delta_0 + (1-\beta')^+ \delta_1 + [1 - (1-\alpha')^+ - (1-\beta')^+] \pi_{a_-,a_+} , \quad (27)$$

²we use the symbol β' (hence α') not to confuse with β already defined.

where

$$(a_-, a_+) = a_{\pm} \left(\frac{\alpha'}{\alpha' + \beta'}, 1 - \frac{1}{\alpha' + \beta'} \right). \quad (28)$$

For $k \neq 0$ let D_k the dilatation operator by factor k . For $p \leq 1$ let \mathbf{b}_p denotes the Bernoulli distribution of parameter p . At last \boxplus denotes the additive free convolution, and \boxtimes the multiplicative free convolution. Then rewriting [Dem], we find four cases

- Situation *I* : $\min(\alpha', \beta') \geq 1$, no Dirac mass,

$$\sigma_- = \frac{\alpha'}{\alpha' + \beta'}, \quad \sigma_+ = 1 - \frac{1}{\alpha' + \beta'}, \quad \alpha' = \frac{\sigma_-}{1 - \sigma_+}, \quad \beta' = \frac{1 - \sigma_-}{1 - \sigma_+}$$

$$CC_{\alpha', \beta'} = \pi_{a_-, a_+} = D_{1 - \sigma_+}(\mathbf{b}_{\sigma_-}) \boxplus_{1 - \sigma_+} \quad (29)$$

- Situation *II* : $\alpha' < 1 \leq \beta'$, one Dirac mass at 0

$$\sigma_- = \frac{1}{\alpha' + \beta'}, \quad \sigma_+ = 1 - \frac{\alpha'}{\alpha' + \beta'}, \quad \alpha' = \frac{1 - \sigma_+}{\sigma_-}, \quad \beta' = \frac{\sigma_+}{\sigma_-}$$

$$\begin{aligned} CC_{\alpha', \beta'} &= (1 - \alpha')\delta_0 + \alpha'\pi_{a_-, a_+} \\ &= D_{\sigma_-}(\mathbf{b}_{1 - \sigma_+}) \boxplus_{\sigma_-} \end{aligned} \quad (30)$$

- Situation *III* : $\beta' < 1 \leq \alpha'$, one Dirac mass at 1

$$\sigma_- = 1 - \frac{1}{\alpha' + \beta'}, \quad \sigma_+ = \frac{\alpha'}{\alpha' + \beta'}, \quad \alpha' = \frac{\sigma_+}{1 - \sigma_-}, \quad \beta' = \frac{1 - \sigma_+}{1 - \sigma_-}$$

$$\begin{aligned} CC_{\alpha', \beta'} &= (1 - \beta')\delta_1 + \beta'\pi_{a_-, a_+} \\ &= D_{1 - \sigma_-}(\mathbf{b}_{\sigma_+}) \boxplus_{1 - \sigma_-} \end{aligned} \quad (31)$$

- Situation *IV* : $\max(\alpha', \beta') < 1$, two Dirac masses (at 0 and at 1)

$$\sigma_- = 1 - \frac{\alpha'}{\alpha' + \beta'}, \quad \sigma_+ = \frac{1}{\alpha' + \beta'}, \quad \alpha' = \frac{1 - \sigma_-}{\sigma_+}, \quad \beta' = \frac{\sigma_-}{\sigma_+}$$

$$CC_{\alpha', \beta'} = (1 - \alpha')\delta_0 + (1 - \beta')\delta_1 + (\alpha' + \beta' - 1)\pi_{a_-, a_+} \quad (32)$$

$$= D_{\sigma_+}(\mathbf{b}_{1 - \sigma_-}) \boxplus_{\sigma_+} \quad (33)$$

There is a connection with the family of free Meixner law ([BW05], [Bry06],[BI05]). Indeed, computing the mean m and variance V of the distribution $CC_{\alpha', \beta'}$, we get

- Situation *I*

$$m = \sigma_-, \quad V = \sigma_-(1 - \sigma_-)(1 - \sigma_+)$$

- Situation *II*

$$m = (1 - \sigma_+) , V = \sigma_- \sigma_+ (1 - \sigma_+)$$

- Situation *III*

$$m = \sigma_+ , V = (1 - \sigma_-) \sigma_+ (1 - \sigma_+)$$

- Situation *IV*

$$m = 1 - \sigma_- , V = \sigma_- \sigma_+ (1 - \sigma_-)$$

so that, in all cases

$$m = \frac{\alpha'}{\alpha' + \beta'} \quad V = \frac{\alpha' \beta'}{(\alpha' + \beta')^3} .$$

We see that fixing $\alpha' + \beta' = s^{-1}$, we get $V = s^2 m (1 - m)$, and then up to an affine transformation we find the "free binomial type law" of [BI05] example vi p.18 or [BW05] example 6 p.8. It could also be seen starting from the above formulas using dilatations and free convolutions and comparing with formula (7) page 6 in [BW05].

Finally, we quote the correspondance with the results of Collins [Col05] who claimed that for $0 < p_- < p_+ < 1$

$$\mathbf{b}_{p_-} \boxtimes \mathbf{b}_{p_+} = (1 - p_-) \delta_0 + (p_- + p_+ - 1)^+ \delta_1 + C_{a_-, a_+}^{-1} \pi_{a_-, a_+} \quad (34)$$

where $a_{\pm} = a_{\pm}(1 - p_-, p_+)$. In [HP05], formula (2.8) the authors consider the same distribution.

- Situation *II* : $p_- + p_+ - 1 < 0$, $\sigma_- = p_+$, $\sigma_+ = 1 - p_-$

$$\begin{aligned} \mathbf{b}_{p_-} \boxtimes \mathbf{b}_{p_+} &= \sigma_+ \delta_0 + (1 - \sigma_+) \pi_{a_-, a_+} \\ &= \sigma_- C C_{\alpha', \beta'} + (1 - \sigma_-) \delta_0 \end{aligned} \quad (35)$$

- Situation *IV* : $p_- + p_+ - 1 > 0$, $\sigma_- = 1 - p_-$, $\sigma_+ = p_+$

$$\begin{aligned} \mathbf{b}_{p_-} \boxtimes \mathbf{b}_{p_+} &= \sigma_- \delta_0 + (\sigma_+ - \sigma_-) \delta_1 + (1 - \sigma_+) \pi_{a_-, a_+} \\ &= \sigma_+ C C_{\alpha', \beta'} + (1 - \sigma_+) \delta_0 \end{aligned} \quad (36)$$

3.3 Logarithm

In view of comparing (18) with its limit we need the integral of the logarithm.

Lemma 3.1 *For $0 < a_- < a_+ < 1$, we have*

$$\int (\log x) \pi_{a_-, a_+}(dx) = \frac{\sigma_+ \log \sigma_+ + \sigma_- \log \sigma_- - (\sigma_+ + \sigma_- - 1) \log(\sigma_+ + \sigma_- - 1)}{1 - \sigma_+} \quad (37)$$

where σ_{\pm} are given by (24).

After some attempts to find a short direct proof, we choose to consider this result as a by-product of random matrix theory and then we give a short indirect proof in Section 9. Actually there is also another proof in [Dem].

4 Law of large numbers and fluctuations

In this section, we use the Gamma function, its logarithm $\ell = \log \Gamma$, and the Binet formula (see Appendix 1) with the function f defined in (119). We use also the following entropy function

$$\mathcal{J}(u) = \begin{cases} u \log u - u + 1 & \text{if } u > 0 \\ 1 & \text{if } u = 0 \\ +\infty & \text{if } u < 0 \end{cases} \quad (38)$$

and the function defined, for $x \geq 0$, $x + y \geq 0$, and $z \leq \min\{x, x + y\}$, by

$$\mathcal{E}(x, y, z) = \mathcal{I}(x) - \mathcal{I}(x - z) - \mathcal{I}(x + y) + \mathcal{I}(x + y - z). \quad (39)$$

It can be noticed that \mathcal{E} is homogeneous: for every $\lambda > 0$

$$\mathcal{E}(\lambda x, \lambda y, \lambda z) = \lambda \mathcal{E}(x, y, z), \quad (40)$$

and satisfies

$$\mathcal{E}(x, y, z) = \mathcal{E}(x + y - z, z, y). \quad (41)$$

Finally, we use the partial derivative of \mathcal{E} with respect to x :

$$\mathcal{E}_1(x, y, z) := \frac{\partial}{\partial x} \mathcal{E}(x, y, z) = \log \frac{x(x + y - z)}{(x - z)(x + y)}. \quad (42)$$

4.1 Law of large numbers

Starting from (15), (16) and (17) and differentiating once, we get

$$\mathbb{E}\Theta_n(t) = \sum_{j=1}^r \Psi\left(\frac{\beta(n_1 - j + 1)}{2}\right) - \Psi\left(\frac{\beta(n_1 + n_2 - j + 1)}{2}\right) \quad (43)$$

where $r = \lfloor nt \rfloor$. Using the Binet's formula (see Appendix 1)

$$\mathbb{E}\Theta_n(t) \cong \sum_{j=1}^r \log \frac{n_1 - j + 1}{n_1 + n_2 - j + 1} \cong \sum_{j=1}^r \log \frac{\tau_1 - \frac{j}{n}}{\tau_1 + \tau_2 - \frac{j}{n}} \quad (44)$$

so that, using Riemann sums, we may conjecture,

$$\lim_n \frac{1}{n} \mathbb{E}\Theta_n(t) = \int_0^t \log \frac{\tau_1 - s}{\tau_1 + \tau_2 - s} ds = \mathcal{E}(\tau_1, \tau_2, t). \quad (45)$$

In the following theorems, we state a strong form of (45) and study the asymptotic behavior of variance and fluctuations. Let us stress that for fixed t , the convergence (45) may be deduced from (19) and (37). Although the function $x \mapsto \log x$ is not bounded, the control of extremal eigenvalues by [Col05] Proposition 4.8 allows to conclude easily.

Theorem 4.1 1. As $n \rightarrow \infty$,

$$\sup_{t \in [0, \tau_1]} \left| \frac{1}{n} \mathbb{E} \Theta_n(t) - \mathcal{E}(\tau_1, \tau_2, t) \right| \rightarrow 0. \quad (46)$$

2. For every $t \in [0, \tau_1)$, as $n \rightarrow \infty$,

$$\mathbb{E} \Theta_n(t) - \mathcal{E}(\lfloor \tau_1 n \rfloor, \lfloor \tau_2 n \rfloor, \lfloor tn \rfloor) \longrightarrow \mathbf{d}_J(t) = \left(\frac{1}{2} - \frac{1}{\beta} \right) \mathcal{E}_1(\tau_1, \tau_2, t), \quad (47)$$

and

$$\mathbb{E} \Theta_n(\tau_1) - \mathcal{E}(\lfloor \tau_1 n \rfloor, \lfloor \tau_2 n \rfloor, \lfloor \tau_1 n \rfloor) + \left(\frac{1}{\beta} - \frac{1}{2} \right) \log n \rightarrow \left(\frac{1}{2} - \frac{1}{\beta} \right) \log \frac{\tau_1 \tau_2}{\tau_1 + \tau_2} + K_\beta^1 \quad (48)$$

where

$$K_\beta^1 := \frac{1}{2} \log(2\pi) - \frac{\gamma}{\beta} + \int_0^\infty \frac{sf(s)}{e^{\beta s/2} - 1} ds, \quad (49)$$

and $\gamma = -\Gamma'(1)$ is the Euler constant.

3. For every $t \in [0, \tau_1)$, as $n \rightarrow \infty$,

$$\text{Var } \Theta_n(t) \rightarrow \frac{2}{\beta} \mathcal{E}_1(\tau_1, \tau_2, t) \quad (50)$$

and

$$\text{Var } \Theta_n(\tau_1) - \frac{2}{\beta} \log n \longrightarrow \frac{2}{\beta} \left(\log \frac{\tau_1 \tau_2}{\tau_1 + \tau_2} \right) + K_\beta^2, \quad (51)$$

where

$$K_\beta^2 := \frac{2\gamma}{\beta} + \int_0^\infty \frac{s(sf(s) + \frac{1}{2})}{e^{\beta s/2} - 1} ds. \quad (52)$$

4. As $n \rightarrow \infty$,

$$\sup_{t \in [0, \tau_1]} \left| \frac{1}{n} \Theta_n(t) - \mathcal{E}(\tau_1, \tau_2, t) \right| \rightarrow 0 \quad (53)$$

in probability.

Remark 4.2 For $\beta = 1, 2, 4$, when all the variables are on the same probability space, the convergence in 4) above may be strengthened to almost sure convergence.

4.2 Fluctuations

For $T < \tau_1$ set $D_T = \{v \in \mathbf{ID}([0, T]) : v(0) = 0\}$ and $D = \{v \in \mathbf{ID}([0, \tau_1]) : v(0) = 0\}$ the set of càdlàg functions on $[0, T]$ and $[0, \tau_1)$, respectively, starting from 0.

Let for $0 \leq t < \tau_1$

$$\sigma^2(t) := \frac{\partial}{\partial t} \mathcal{E}_1(\tau_1, \tau_2, t) = \frac{\tau_2}{(\tau_1 - t)(\tau_1 + \tau_2 - t)}. \quad (54)$$

Theorem 4.3 1. Let for $n \geq 1$

$$\eta_n(t) := \Theta_n(t) - \mathcal{E}(\lfloor \tau_1 n \rfloor, \lfloor \tau_2 n \rfloor, \lfloor tn \rfloor) \quad , \quad t \in [0, \tau_1).$$

Then, as $n \rightarrow \infty$

$$(\eta_n(t); t \in [0, \tau_1)) \Rightarrow (J_t; t \in [0, \tau_1)) \quad , \quad (55)$$

where J is the (Gaussian) diffusion solution of the stochastic differential equation :

$$dJ_t = \left(\frac{1}{2} - \frac{1}{\beta} \right) \sigma^2(t) dt + \sqrt{\frac{2}{\beta}} \sigma(t) d\mathbf{B}_t, \quad (56)$$

with $J_0 = 0$, \mathbf{B} is a standard Brownian motion and \Rightarrow stands for the weak convergence of distributions in D provided with the Skorokhod topology.

2. Let

$$\widehat{\eta}_n^J(\tau_1) := \left(\frac{2}{\beta} \log n \right)^{-1/2} \left[\Theta_n(\tau_1) - n\mathcal{E}(\tau_1, \tau_2, \tau_1) + \left(\frac{1}{2} - \frac{1}{\beta} \right) \log n \right].$$

Then as $n \rightarrow \infty$, $\widehat{\eta}_n^J(1) \Rightarrow N$ where N is a standard normal variable independent of \mathbf{B} , (and \Rightarrow stands for the weak convergence of distribution in \mathbb{R}).

5 Large Deviations

All along this section, we use the notation of Dembo-Zeitouni [DZ98]. In particular we write LDP for Large Deviation Principle. The reader may have some interest in consulting [DG05] where a similar method is used for a different model, but here we use a slightly different topology to be able to catch the marginals in T .

5.1 Large deviations

5.1.1 Large deviations for paths

For $T < \tau_1$, let \mathbf{M}_T be the set of signed measures on $[0, T]$ and let $\mathbf{M}_<$ be the set of measures whose support is a compact subset of $[0, \tau_1)$.

We provide D with the weakened topology $\sigma(D, \mathbf{M}_<)$. So, D is the projective limit of the family, indexed by $T < 1$ of topological spaces $(D_T, \sigma(D_T, \mathbf{M}_T))$.

Let V_ℓ (resp. V_r) be the set of functions from $[0, \tau_1]$ to \mathbb{R} which are left (resp. right) continuous functions with bounded variations. We put a superscript T to specify the functions on $[0, T]$. There is a one-to-one correspondence between V_r^T and \mathbf{M}_T :

- for any $v \in V_r^T$, there exists a unique $\mu \in \mathbf{M}_T$ such that $v = \mu([0, \cdot])$; we denote it by \dot{v}
- for any $\mu \in \mathbf{M}_T$, $v = \mu([0, \cdot])$ stands in V_r^T .

For the following statement, we need some notation. Let \mathbf{H} be the entropy function :

$$\mathbf{H}(x|p) = x \log \frac{x}{p} + (1-x) \log \frac{1-x}{1-p}, \quad (57)$$

and set³, for $t < \tau_1$,

$$L_a^J(t, y) = (\tau_1 + \tau_2 - t) \mathbf{H} \left(\frac{\tau_1 - t}{\tau_1 + \tau_2 - t} \mid e^y \right) \delta(y | (-\infty, 0)) \quad (58)$$

$$L_s^J(t, y) = -(\tau_1 - t)y \delta(y | (-\infty, 0)). \quad (59)$$

For $v \in D$, let $\dot{v} = \dot{v}_a + \dot{v}_s$ be the Lebesgue decomposition of the measure $\dot{v} \in \mathbf{M}([0, \tau_1])$ in absolutely continuous and singular parts with respect to the Lebesgue measure and μ is any bounded positive measure dominating \dot{v}_s .

For $A \subset [0, \tau_1)$ and $v \in V_r$ let

$$I_A^J(v) = \int_A L_a^J \left(t, \frac{d\dot{v}_a}{dt}(t) \right) dt + \int_A L_s^J \left(t, \frac{d\dot{v}_s}{d\mu}(t) \right) d\mu(t) \quad \text{if } v \in V_r, \quad (60)$$

For $v \notin V_r$ let $I_A^J(v) = \infty$.

Theorem 5.1 *The sequence $\{n^{-1}\Theta_n\}_n$ satisfies a LDP in $(D, \sigma(D, \mathbf{M}_<))$ at scale $2\beta^{-1}n^{-2}$ with good rate function $I_{[0, \tau_1)}$.*

The proof, in Section 6.3 needs several steps. First we show that $\{n^{-1}\dot{\Theta}_n\}$ satisfies a LDP in \mathcal{M}_T provided with the topology $\sigma(\mathcal{M}_T, V_\ell)$. Then we carry the LDP to $(D_T, \sigma(D_T, \mathcal{M}_T))$ with good rate function:

$$I_{[0, T]}^J(v) = \int_{[0, T]} L_a^J \left(t, \frac{d\dot{v}_a}{dt}(t) \right) dt + \int_{[0, T]} L_s^J \left(t, \frac{d\dot{v}_s}{d\mu}(t) \right) d\mu(t). \quad (61)$$

To end the proof we apply the Dawson-Gärtner theorem on projective limits ([DZ98] Theorem 4.6.1, see also [Léo00] Proposition A2).

Let us notice that $I_{[0, T]}^J$ vanishes only for

$$\frac{d\dot{v}_a}{dt}(t) = \log \frac{\tau_1 - t}{\tau_1 + \tau_2 - t}, \quad \frac{d\dot{v}_s}{d\mu}(t) = 0,$$

(essentially) i.e. for $v(t) = \mathcal{E}(\tau_1, \tau_2, t)$, which is consistent with 53.

³we set $\delta(y|A) = 0$ if $y \in A$ and $= \infty$ if $y \notin A$

5.1.2 ESD

The logarithmic entropy of a probability measure μ on \mathbb{R} is

$$\Sigma(\mu) := \iint \log|x - y| d\mu(x)d\mu(y). \quad (62)$$

In the complex case ($\beta = 2$), Hiai et Petz [HP05] proved that if $n \rightarrow \infty, n_1/n \rightarrow \tau_1, n_2/n \rightarrow \tau_2 > \tau_1, r/n \rightarrow T < \tau_1$, then $\{\mu_{n_1, n_2, r}\}_n$ satisfies a LDP in $\mathcal{M}_1([0, 1])$, the set of probability measures on $[0, 1]$ endowed with the weak convergence topology, at scale n^{-2} , with the good rate function

$$\begin{aligned} I_T^{spJ}(\mu) := & - T^2 \Sigma(\mu) - T \int_0^1 ((\tau_1 - T) \log x + (\tau_2 - T) \log(1 - x)) d\mu(x) \\ & + T^2 B\left(\frac{\tau_1 - T}{T}, \frac{\tau_2 - T}{T}\right), \end{aligned} \quad (63)$$

where B is the function defined in [HP05], for $s, t \geq 0$:

$$\begin{aligned} B(s, t) := & \frac{(1+s)^2}{2} \log(1+s) - \frac{s^2}{2} \log s + \frac{(1+t)^2}{2} \log(1+t) - \frac{t^2}{2} \log t \\ & - \frac{(2+s+t)^2}{2} \log(2+s+t) + \frac{(1+s+t)^2}{2} \log(1+s+t). \end{aligned} \quad (64)$$

A computation similar to p.10 of [HP05] gives the same result for general β .

Proposition 5.2 *If $T < \tau_1 \leq \tau_2$, the family $\{\mu_{\lfloor n\tau_1 \rfloor, \lfloor n\tau_2 \rfloor, \lfloor nT \rfloor}\}$ satisfies a LDP in $\mathcal{M}_1([0, 1])$ at scale $2\beta^{-1}n^{-2}$ and good rate function I_T^{spJ} .*

The above function B is the limiting free energy, since we have (by a small calculation from (6))

$$\lim_n \frac{1}{\beta n^2} \log Z_r(\beta, n_1, n_2) = \frac{T^2}{2} B\left(\frac{\tau_1 - T}{T}, \frac{\tau_2 - T}{T}\right). \quad (65)$$

5.1.3 LDP for one-dimensional marginals

The LDP for $\{\Theta_n(T)\}_n$ is obtained by contraction $m \mapsto m([0, T])$ from Theorem (5.1). Let F be the primitive of \mathcal{I} :

$$F(t) = \int_0^t \mathcal{J}(u) du = \frac{t^2}{2} \log t - \frac{3t^2}{4} + t, \quad (t \geq 0).$$

Actually, we have

$$B(s, t) = F(1+s) - F(s) + F(1+t) - F(t) - F(2+s+t) + F(1+s+t) - \frac{7}{4}. \quad (66)$$

For $T > 0$, set

$$\xi_J^T = \mathcal{J}(\tau_2) + \mathcal{J}(T) - \mathcal{J}(T + \tau_2) - 1, \quad \mathcal{D}_T = [\xi_J^T, 0]. \quad (67)$$

Theorem 5.3 *Let $T < \tau_1$.*

1. *The sequence $\{\frac{1}{n}\Theta_n(T)\}_n$ satisfies a LDP in \mathbb{R} at scale $2\beta^{-1}n^{-2}$ with good rate function I_T^J defined by*

$$I_T^J(\xi) := \inf\{I_{[0,T]}^J(v) ; v(T) = \xi\}. \quad (68)$$

2. *If $\xi \in \mathcal{D}_T$ the equation*

$$\mathcal{E}(\theta + \tau_1, \tau_2, T) = \xi \quad (69)$$

has a unique solution $\theta \in [T - \tau_1, \infty)$, and we have

$$\begin{aligned} I_T^J(\xi) = \theta\xi & - [F(\theta + \tau_1) - F(\theta + \tau_1 - T)] - [F(\tau_1 + \tau_2) - F(\tau_1 + \tau_2 - t)] \\ & + [F(\tau_1) - F(\tau_1 - T)] + [F(\theta + \tau_1 + \tau_2) - F(\theta + \tau_1 + \tau_2 - T)]. \end{aligned} \quad (70)$$

3. *If $\xi \notin \mathcal{D}_T$, we have*

$$I_T^J(\xi) = I_T^J(\xi_T^J) + (\xi_T^J - \xi)(\tau_1 - T). \quad (71)$$

4. *If $\xi \geq 0$, then $I_T^J(\xi) = \infty$.*

5.1.4 Contraction from ESD

If the contraction $\mu \mapsto \int \log x d\mu(x)$ from $\mathcal{M}_1([0, 1])$ to \mathbb{R} was continuous, we would claim that $\{\frac{1}{n}\Theta_n(T)\}_n$ satisfies a LDP with good rate function $\frac{\beta}{2}\tilde{I}_T^J$ where

$$\tilde{I}_T^J(\xi) = \inf\left\{I_T^{(spJ)}(\mu) ; \mu \in \mathcal{A}(\xi T^{-1})\right\} \quad (72)$$

with

$$\mathcal{A}(u) = \left\{\mu : \int \log x d\mu(x) = u\right\}.$$

We do not know if the contraction $\mu \mapsto \int \log x d\mu(x)$ works, although not continuous. However we will prove the following result.

Proposition 5.4 *Let $T < \min(\tau_1, \tau_2)$, $\xi \in \mathcal{D}_T$ and θ solution of (69). Then the infimum of $I_T^{(spJ)}(\mu)$ on $\mathcal{A}(\xi T^{-1})$ is uniquely achieved at $\mu = \pi_{\tilde{a}_-, \tilde{a}_+}$ where*

$$(\tilde{a}_-, \tilde{a}_+) = \lambda_{\pm}(\tilde{s}_-, \tilde{s}_+)$$

with

$$\tilde{s}_- = \frac{\tau_1 + \theta}{\tau_1 + \tau_2 + \theta}, \quad \tilde{s}_+ = \frac{\tau_1 + \theta + \tau_2 - t}{\tau_1 + \tau_2 + \theta}, \quad (73)$$

and

$$I_T^J(\xi) = I_T^{(spJ)}(\pi_{\tilde{a}_-, \tilde{a}_+}) = \inf\{I_T^{(spJ)}(\mu); \mu \in \mathcal{A}(\xi T^{-1})\}. \quad (74)$$

Remark 5.5 The endpoint is ξ_T^J , which corresponds to $\theta = T - \tau_1$, i.e.

$$\tilde{a}_- = 0 \quad , \quad \tilde{a}_+ = \frac{4\tau_2 T}{(\tau_2 + T)^2}.$$

For $\xi < \xi_T^J$ we do not know what happens. We can imagine that the infimum in (72) has a solution in some extended space.

Remark 5.6 In the range $\tau_2 \leq T < \tau_1$ we have a similar result, exchanging \tilde{s}_- and \tilde{s}_+ in (73). We omit the details.

6 Proofs

6.1 Proof of Theorem 4.1

We start from

$$\mathbb{E} \log |\mathcal{Z}_{n_1, n_2, r}| = \sum_{j=1}^r \Psi \left(\frac{\beta(n_1 - j + 1)}{2} \right) - \Psi \left(\frac{\beta(n_1 + n_2 - j + 1)}{2} \right)$$

and owing to formula (121) we have the decomposition

$$\begin{aligned} \mathbb{E} \log |\mathcal{Z}_{n_1, n_2, r}| &= \log \frac{(n_1)_r}{(n_1 + n_2)_r} \\ &+ \frac{1}{\beta} (H_{n_1 + n_2} - H_{n_1 + n_2 - r} - H_{n_1} + H_{n_1 - r}) \\ &- \delta_{n_1}^1 + \delta_{n_1 - r}^1 + \delta_{n_1 + n_2}^1 - \delta_{n_1 + n_2 - r}^1, \end{aligned} \quad (75)$$

in which

- 1) $(p)_r = p(p-1)\cdots(p-r+1)$ is the falling factorial
- 2) $H_0 = 0$ and $H_p = 1 + \frac{1}{2} + \cdots + \frac{1}{p}$ are the harmonic numbers
- 3) the remaining terms are

$$\delta_0^1 = 0 \quad , \quad \delta_p^1 = \int_0^\infty sf(s) \sum_{k=1}^p e^{-\frac{\beta ks}{2}} ds = \int_0^\infty sf(s) \frac{1 - e^{-\beta ps/2}}{e^{\beta s/2} - 1} ds. \quad (76)$$

- 1) The logarithmic contribution.

Using the Stirling's formula

$$\log(p)_r = p \log p - p + \frac{1}{2} \log(2\pi p) - \ell(p-r+1) + o(1)$$

where $o(1)$ is uniform in r , as $p \rightarrow \infty$. This yields

$$\begin{aligned} \log \frac{(n_1)_r}{(n_1 + n_2)_r} &= n_1 \log n_1 - (n_1 + n_2) \log(n_1 + n_2) + n_2 + \frac{1}{2} \log \frac{n_1}{n_1 + n_2} \\ &- \ell(n_1 - r + 1) + \ell(n_1 + n_2 - r + 1) + o(1) \end{aligned}$$

Now we use

$$\begin{aligned}
\ell(n_1 + n_2 - r + 1) - \ell(n_1 - r + 1) &= \ell(n_1 + n_2 - r) - \ell(n_1 - r) + \log \frac{n_1 + n_2 - r}{n_1 - r} \\
&= (n_1 + n_2 - r) \log(n_1 + n_2 - r) - (n_1 - r) \log(n_1 - r) - n_2 \\
&\quad + \frac{1}{2} \log \frac{n_1 + n_2 - r}{n_1 - r} + \int_0^\infty f(s) [e^{-(n_1+n_2-r)s} - e^{-(n_1-r)s}] ds
\end{aligned}$$

Besides, using (39) we have

$$\log \frac{(n_1)_r}{(n_1 + n_2)_r} - n\mathcal{E} \left(\frac{n_1}{n}, \frac{n_2}{n}, \frac{r}{n} \right) = \frac{1}{2} \log \frac{n_1(n_1 + n_2 - r)}{(n_1 - r)(n_1 + n_2)} \quad (77)$$

$$+ \int_0^\infty f(s) [e^{-(n_1+n_2-r)s} - e^{-(n_1-r)s}] ds + o(1) \quad (78)$$

for $r < n_1$ and for $r = n_1$,

$$\log \frac{(n_1)_r}{(n_1 + n_2)_r} - n\mathcal{E} \left(\frac{n_1}{n}, \frac{n_2}{n}, \frac{r}{n} \right) = \frac{1}{2} \log(2\pi) + \frac{1}{2} \log \frac{n_1 n_2}{n_1 + n_2} + o(1)$$

The integral term in (78) is negative and bounded below by -1 . In the above asymptotics it tends to 0. Summarizing we have the bound

$$0 \leq \sup_{r \leq n} \left\{ \log \frac{(n_1)_r}{(n_1 + n_2)_r} - n\mathcal{E} \left(\frac{n_1}{n}, \frac{n_2}{n}, \frac{r}{n} \right) \right\} = O(\log n) \quad (79)$$

and the limits

$$\lim_n \left\{ \log \frac{(n_1)_r}{(n_1 + n_2)_r} - n\mathcal{E} \left(\frac{n_1}{n}, \frac{n_2}{n}, \frac{r}{n} \right) \right\} = \frac{1}{2} \log \frac{\tau_1(\tau_1 + \tau_2 - t)}{(\tau_1 - t)(\tau_1 + \tau_2)} = \frac{1}{2} \mathcal{E}_1(\tau_1, \tau_2, t) \quad (80)$$

when $t < \tau_1$ and

$$\lim_n \left\{ \log \frac{(n_1)_r}{(n_1 + n_2)_r} - n\mathcal{E} \left(\frac{n_1}{n}, \frac{n_2}{n}, \frac{n_1}{n} \right) - \frac{1}{2} \log n \right\} = \frac{1}{2} \log(2\pi) + \frac{1}{2} \log \frac{\tau_1 \tau_2}{(\tau_1 + \tau_2)} \quad (81)$$

when $r = n_1$ i.e. $t = \tau_1$.

2) The (opposite of the) harmonic contribution

$$(\Delta H)_{n_1, n_2, r} := H_{n_1} - H_{n_1-r} + H_{n_1+n_2-r} - H_{n_1+n_2} = n_2 \sum_{j=n_1-r+1}^{n_1} \frac{1}{j(n_1+j)}$$

satisfies

$$0 \leq \sup_{r \leq n_1} (\Delta H)_{n_1, n_2, r} \leq \frac{n_2}{n_1} (1 + \log n_1) = O(n) \quad (82)$$

Besides, in the above asymptotics, for $0 \leq t < \tau_1$

$$(\Delta H)_{n_1, n_2, r} \rightarrow \mathcal{E}_1(\tau_1, \tau_2, t) \quad (83)$$

and for $t = \tau_1$

$$(\Delta H)_{n_1, n_2, n_1} - \log n \rightarrow \gamma + \log \frac{\tau_1 \tau_2}{\tau_1 + \tau_2} \quad (84)$$

3) The δ contribution may be written

$$\begin{aligned} (\Delta \delta)_{n_1, n_2, r} &:= -\delta_{n_1} + \delta_{n_1-r} - \delta_{n_1+n_2-r} + \delta_{n_1+n_2} \\ &= \int_0^\infty \frac{sf(s)}{e^{\beta s/2} - 1} \left[e^{-\beta(n_1+n_2)s/2} - e^{-\beta(n_1+n_2-r)s/2} - e^{-\beta n_1 s/2} + e^{-\beta(n_1-r)s/2} \right] ds \end{aligned} \quad (85)$$

The integrand is positive, increases with r and its supremum (reached for $r = n_1$) is less than 1. We conclude

$$0 \leq \sup_{r \leq n_1} (\Delta \delta)_{n_1, n_2, r} \leq K_\beta := \int_0^\infty \frac{sf(s)}{e^{\beta s/2} - 1} ds \quad (86)$$

In the above asymptotics, for fixed $0 \leq t < \tau_1$, the integrand converges to 0 for every s and then the dominated convergence theorem yields

$$(\Delta \delta)_{n_1, n_2, r} \rightarrow 0. \quad (87)$$

If $t = \tau_1$, the limit of the integrand is 1, so we get

$$(\Delta \delta)_{n_1, n_2, n_1} \rightarrow K_\beta \quad (88)$$

We now claim that the "mean" part of the theorem⁴ is proved, since

- a) (79), (82) and (86) yields (46)
- b) (80), (83) and (87) yields (47)
- c) (81), (84) and (88) yields (48).

Let us try the computation of the variance. From (124) and ... we have

$$\left| \text{Var } |\mathcal{Z}| - \frac{2}{\beta} (\Delta H)_{n_1, n_2, r} \right| \leq \sum_{j=n_1-r+1}^{n_1+n_2-r+1} \frac{1}{k^2}. \quad (89)$$

This yields easily (50), with the help of (83). For $r = n_1$, it remains to work a little. A first part of the limit comes from the harmonic contribution. The last part comes from the δ contribution. A little computation (with the help of the dominated convergence theorem) gives (52).

Part 4) comes from Doob's inequality and the variance estimate (51).

⁴When checking the result for $t = \tau_1$ with the Wishart machinery [Rou05] Prop. 5.1, we see that actually

$$K_1^1 = \frac{-\log(2\pi) - \gamma - \log 2 + 1}{2}$$

6.2 Proof of Theorem 4.3

We follow the same scheme as in the Wishart case ([Rou05] Section ?). We just give some slight modifications. Let

$$\tilde{\rho}_{k,n} = \rho_{k,n} - \mathbb{E}\rho_{k,n}, \quad k = 1, \dots, \quad (90)$$

With the help of the Lindeberg-Lévy-Lyapunov theorem, it is enough to prove that

$$\lim_n \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}\tilde{\rho}_{n,k}^4 = 0 \quad (91)$$

From (16)

$$\mathbb{E}\tilde{\rho}_{n,k}^4 = (\Lambda_{k,n}^J)^{(4)}(0) + 3[\Lambda_{k,n}^J]^{(2)}(0)^2 \quad (92)$$

Actually

$$\frac{2^2}{\beta^2} (\Lambda_{n,k}^J)^{(2)}(0) = \Psi' \left(\frac{\beta(n_1 - k + 1)}{2} \right) - \Psi' \left(\frac{\beta(n_1 + n_2 - k + 1)}{2} \right). \quad (93)$$

Since Ψ' is nonincreasing and positive

$$\sup_{k \leq r} (\Lambda_{n,k}^J)^{(2)}(0) \leq \frac{\beta^2}{4} \Psi' \left(\frac{\beta(n_1 - r + 1)}{2} \right) \rightarrow 0. \quad (94)$$

Moreover

$$\frac{2^4}{\beta^4} (\Lambda_{n,k}^J)^{(4)}(0) = \Psi^{(3)} \left(\frac{\beta(n_1 - k + 1 + \theta)}{2} \right) - \Psi^{(3)} \left(\frac{\beta(n_1 + n_2 - k + 1 + \theta)}{2} \right) \quad (95)$$

hence

$$\left| \sum_{k=1}^r (\Lambda_{n,k}^J)^{(4)}(0) - 3\beta \sum_{k=1}^r \left[\frac{1}{(n_1 - k + 1)^3} - \frac{1}{(n_1 + n_2 - k + 1)^3} \right] \right| \leq \quad (96)$$

$$\leq 12 \sum_{k=1}^r \frac{1}{(n_1 - k + 1)^4} \rightarrow 0. \quad (97)$$

6.3 Proof of Theorem 5.1

We do not give details since it is similar to the Wishart case ([Rou05], Section 7.4) and based on the ideas of Baldi's theorem ([DZ98]). The main tool is the normalized cumulant generating function

$$\mathcal{L}_{n, \lfloor nT \rfloor}^J(\varphi) = \frac{2}{\beta n^2} \log \mathbb{E} \exp n \langle \dot{\Theta}_n, \varphi \rangle = \frac{2}{\beta n^2} \sum_{k=1}^{\lfloor nt \rfloor} \Lambda_{n,k}^J(n\varphi(k/n)), \quad (98)$$

(from (17)) and a variational formula.

6.3.1 Convergence of the n.c.g.f.

Lemma 6.1 *If $\varphi \in V_\ell$ satisfies $\varphi(s) + \tau_1 - s > 0$ for every $s \in (0, T]$, then*

$$\lim_n \mathcal{L}_{n, [nT]}^J(\varphi) = \Lambda_{[0, T]}^J(\varphi) := \int_0^T g^J(s, \varphi(s)) ds, \quad (99)$$

where, for $\theta + \tau_1 - s > 0$

$$g^J(s, \theta) = \mathcal{E}(\tau_1 - s + \theta, \tau_2, \theta). \quad (100)$$

Proof: We use the Binet formula and the convergence of Riemann sums. We use the following scheme

$$\begin{aligned} \ell\left(\frac{n\beta}{2}\psi_n(j/n)\right) &= \frac{n\beta}{2} \mathcal{I} \circ \psi_n(j/n) \\ &+ \frac{n\beta}{2} \psi_n(j/n) \log\left(\frac{n\beta}{2}\right) \end{aligned} \quad (101)$$

$$- \frac{1}{2} \log\left(\frac{n\beta}{2}\psi_n(j/n)\right) + 1 - \frac{n\beta}{2} + R_{j,n} \quad (102)$$

where $R_{j,n}$ is some remaining term. We take successively

$$\begin{aligned} \psi_n^1(s) &= \frac{\lfloor n\tau_1 \rfloor - \lfloor ns \rfloor + 1}{n} - \varphi(s) \rightarrow \psi^1(s) = \tau_1 - s + \varphi(s) \\ \psi_n^2(s) &= \frac{\lfloor n\tau_1 \rfloor - \lfloor ns \rfloor + 1}{n} \rightarrow \psi^2(s) = \tau_1 - s \\ \psi_n^3(s) &= \frac{\lfloor n\tau_1 \rfloor + \lfloor n\tau_2 \rfloor - \lfloor ns \rfloor + 1}{n} + \varphi(s) \rightarrow \psi^3(s) = \tau_1 + \tau_2 - s + \varphi(s) \\ \psi_n^4(s) &= \frac{\lfloor n\tau_1 \rfloor + \lfloor n\tau_2 \rfloor - \lfloor ns \rfloor + 1}{n} \rightarrow \psi^4(s) = \tau_1 + \tau_2 - s \end{aligned}$$

The term (101) vanishes by compensation, and the term (102) vanishes when n goes to infinity. It remains the limit of Riemann sums

$$\int_0^T \{\mathcal{I} \circ \psi^1(s) - \mathcal{I} \circ \psi^2(s) - \mathcal{I} \circ \psi^3(s) + \mathcal{I} \circ \psi^4(s)\} ds = \int_0^T g^J(s, \varphi(s)) ds$$

where this last equality comes from (39) and (100). ■

6.3.2 Duality

Let us define $\Lambda_{[0, T]}^J(\varphi) = +\infty$ if φ does not satisfy the assumption of Lemma 6.1. The dual of $\Lambda_{[0, T]}^J$ is then

$$\left(\Lambda_{[0, T]}^J\right)^*(\nu) = \sup_{\varphi \in V_\ell} \{\langle \nu, \varphi \rangle - \Lambda_{[0, T]}^J(\varphi)\} \quad (103)$$

for $\nu \in V_\ell^*$. Mimicking the method of Léonard ([Léo00] p. 112-113), we get

$$(\Lambda_{[0,T]}^J)^*(\nu) = \sup_{\varphi \in \mathcal{C}} \{ \langle \nu, \varphi \rangle - \Lambda_{[0,T]}^J(\varphi) \} \quad (104)$$

where \mathcal{C} is the set of continuous functions from $[0, T]$ into \mathbb{R} vanishing at 0. Then we apply Theorem 5 of Rockafellar [Roc71]. We get

$$(\Lambda_{[0,T]}^J)^*(\nu) = \int_0^T (g^J)^* \left(t, \frac{d\nu_a}{dt} \right) dt + \int_0^T r^J \left(t, \frac{d\nu_s}{d\mu}(t) \right) d\mu(t) \quad (105)$$

where

$$(g^J)^*(s, y) = \sup_{\lambda} \{ \lambda y - g^J(s, \lambda) \delta(\lambda | (-\tau_1, \infty)) \} . \quad (106)$$

This supremum is achieved in

$$\lambda^J(s, y) = -(\tau_1 - s) + \frac{\tau_2}{e^{-y} - 1} \quad (107)$$

and we have

$$(g^J)^*(s, y) = \lambda^J(s, y)y - g^J(s, \lambda^J(s, y)) \quad (108)$$

$$= (\tau_1 + \tau_2 - s) \mathbf{H} \left(\frac{\tau_1 - s}{\tau_1 + \tau_2 - s} \mid e^y \right) . \quad (109)$$

The recession is $r^J(s; y) = -(\tau_1 - s)y$ if $y < 0$.

6.4 Proof of Theorem 5.3

We use the contraction from the LDP for paths. Since the mapping $m \mapsto m([0, T])$ is continuous from D to \mathbb{R} , the family $\{\Theta_n(T)\}$ satisfies the LDP with good rate function given by (68). Since the process Θ_n takes its values in $(-\infty, 0]$, it is clear that $I_T^J(\xi) = \infty$ for $\xi > 0$. Fixing $\xi < 0$, we can look for optimal v , i.e. a path $(v(t), t \in [0, T])$ such that $v(T) = \xi$ and v achieves the infimum in (68). Fix $\theta \geq T - \tau_1$ (playing the role of a Lagrange multiplier). In view of (105), (106) and (107), it is clear that (in the generic case) the Euler-Lagrange equation is

$$\begin{aligned} \lambda^J(s, \phi'(s)) &= \theta \\ \phi(0) &= 0. \end{aligned} \quad (110)$$

This ordinary differential equation admits for unique solution in $\mathcal{C}^1([0, T])$

$$s \mapsto \phi^J(\theta; s) := \mathcal{E}(\theta + \tau_1, \tau_2, s) .$$

To know if the path ϕ^J may have ξ as its terminal value, look at

$$\mathcal{E}'(\theta + \tau_1, \tau_2, T) = \frac{\partial}{\partial \tau} \mathcal{E}(\tau, \tau_2, T) |_{\tau=\theta+\tau_1} = \log \left(1 - \frac{T}{\theta + \tau_1 + \tau_2} \right) - \log \left(1 - \frac{T}{\theta + \tau_1} \right) ;$$

since it is positive, we see that the mapping

$$\theta \longmapsto \mathcal{E}(\theta + \tau_1, \tau_2, T)$$

is continuous and increasing from $[T - \tau_1, \infty)$ onto $\mathcal{D}_T = [\xi_T^J, 0)$. If $\xi \in [\xi_T^J, 0)$, we call θ_ξ the unique solution of $\phi^J(\theta, T) = \xi$ or in other words,

$$\mathcal{E}(\theta_\xi + \tau_1, \tau_2, T) = \xi, \quad (111)$$

and we set $v^\xi := \phi^J(\theta_\xi, \cdot)$.

To end the proof, let us now consider some inequalities. The duality property (106) gives, for every v and t

$$(g^J)^\star \left(t, \frac{d\dot{v}_a}{dt}(t) \right) \geq \theta \frac{d\dot{v}_a}{dt}(t) - g^J(t, \theta). \quad (112)$$

Setting

$$L_T^J(\theta) := \int_0^T g^J(t, \theta) dt, \quad (113)$$

integrating (112) and using (60), (58) and (113) we get

$$I_{[0, T]}^J(v) \geq \theta \dot{v}_a([0, T]) - L_T^J(\theta) - \int_0^T (\tau_1 - t) d\dot{v}_s(t). \quad (114)$$

For every v such that $v(T) = \xi$ it turns out that

$$I_{[0, T]}^J(v) \geq \theta \xi - L_T^J(\theta) - \int_0^T (\tau_1 - T + \theta) d\dot{v}_s(t) \geq \theta \xi - L_T^J(\theta). \quad (115)$$

There are three cases.

- If $\xi \in [\xi_T^J, 0)$, we get

$$I_{[0, T]}^J(v^\xi) = \theta_\xi \xi - L_T^J(\theta_\xi)$$

so that v^ξ realizes the infimum in (68). A simple computation leads to (70) which ends the proof of the first statement of Theorem 5.3.

Let us notice that at the end point $\xi = \xi_T^J$, we have

$$\theta_\xi = (T - \tau_1), \quad v^\xi(t) = \mathcal{E}(T, \tau_2, t), \quad (v^\xi)'(t) = \log \frac{T - t}{\tau_1 + \tau_2 - t}.$$

- Let us assume $\xi = \xi_T^J - \varepsilon$ with $\varepsilon > 0$. Plugging $\theta = T - \tau_1$ in (115) yields, for every v such that $v(T) = \xi$

$$I_{[0, T]}^J(v) \geq (T - \tau_1)\xi_T^J - L_T^J(T - \tau_1) - \varepsilon(T - \tau_1) = I_T^J(\xi_T^J) - \varepsilon(T - \tau_1).$$

Moreover this lower bound is achieved by the measure $\tilde{v} = \left(v^{\xi_T^J}\right)'(t)dt - \varepsilon d\delta_T(t)$, since

$$\int_0^T L_a^J \left(t, \left(v^{\xi_T^J}\right)'(t) \right) dt = I_T^J(\xi_T^J) \quad , \quad \int_0^T (\tau_1 - t)\varepsilon d\delta_T(t) = (\tau_1 - T)\varepsilon .$$

• It remains to look at $\xi = 0$. Taking $\xi = 0$ in (115), we get $I_{[0,T]}^J(0) \geq -L_T^J(\theta)$ for every $\theta \geq T - \tau_1$. Now, from (100) and (113), we may write (after some calculation)

$$\begin{aligned} -L_T^J(\theta) &= \int_0^T -\mathcal{E}(\tau_1 - t + \theta, \tau_2, \theta) dt = \int_0^T \int_0^\theta \log \left(1 + \frac{\tau_2}{\tau_1 - t + s} \right) ds \\ &\geq T \int_0^\theta \log \left(1 + \frac{\tau_2}{\tau_1 + s} \right) ds , \end{aligned}$$

which tends to infinity as $\theta \rightarrow \infty$. We conclude $I_{[0,T]}^J(0) = \infty$. ■

6.5 Proof of Proposition 5.4

Lemma 6.2 *For $\zeta_1, \zeta_2 > 0$, the infimum of*

$$-\Sigma(\mu) - 2\zeta_1 \int \log x d\mu(x) - 2\zeta_2 \int \log(1 - x) d\mu(x)$$

among the probability measures μ on $[0, 1]$ is achieved at π_{a_-, a_+} where

$$(a_-, a_+) = \lambda_\pm(s_-, s_+) \tag{116}$$

with

$$s_- = \frac{1 + 2\zeta_1}{2(1 + \zeta_1 + \zeta_2)} \quad , \quad s_+ = \frac{1 + 2\zeta_1 + 2\zeta_2}{2(1 + \zeta_1 + \zeta_2)} .$$

Proof: In [ST97] p.241 it is proved that the infimum of

$$\int \int -\log |x - y| d\mu(x) d\mu(y) - 2\zeta_1 \int \log(1 - x) d\mu(x) - 2\zeta_2 \int \log(1 + x) d\mu(x)$$

among the probability measures μ on $[-1, +1]$ is achieved at

$$d\mu(y) = K(b_-, b_+) \frac{\sqrt{(y - b_-)(b_+ - y)}}{2\pi(1 - y^2)} \mathbb{1}_{[b_-, b_+]}(y) dy ,$$

where $b_\pm = \theta_2^2 - \theta_1^2 \pm \sqrt{\Delta}$ with

$$\theta_i = \frac{\zeta_i}{1 + \zeta_1 + \zeta_2} \quad , \quad i = 1, 2 \quad , \quad \Delta = [1 - (\theta_1 + \theta_2)^2] [1 - (\theta_1 - \theta_2)^2] .$$

and $K(b_-, b_+)$ is a constant. With the push forward $x \rightarrow (x + 1)/2$, we get the result. ■

Let $\theta < t - \tau_1$ (Lagrangian multiplier). We begin by minimizing

$$I_t^{spJ}(\mu) - \theta t \int \log x \, d\mu(x) = t^2 \left[-\Sigma(\mu) - 2\zeta_1 \int \log x \, d\mu(x) - 2\zeta_2 \int \log(1-x) \, d\mu(x) \right] + C$$

where

$$2\zeta_1 = \frac{\tau_1 + \theta - t}{t}, \quad 2\zeta_2 = \frac{\tau_2 - t}{t} \quad \text{and} \quad C = t^2 B \left(\frac{\tau_1 - t}{t}, \frac{\tau_2 - t}{t} \right).$$

From Lemma 6.2, this infimum is achieved at $\pi_{\tilde{\xi}, \tilde{\eta}}$, where

$$(\tilde{\xi}, \tilde{\eta}) = \lambda_{\pm}(\tilde{s}_-, \tilde{s}_+), \quad \tilde{s}_- = \frac{\tau_1 + \theta}{\tau_1 + \tau_2 + \theta}, \quad \tilde{s}_+ = \frac{\tau_1 + \theta + \tau_2 - t}{\tau_1 + \tau_2 + \theta}.$$

It should be clear that

$$\Sigma(\pi_{\tilde{\xi}, \tilde{\eta}}) + \frac{\tau_1 + \theta - t}{t} \int \log x \, d\pi_{\tilde{\xi}, \tilde{\eta}}(x) + \frac{\tau_2 - t}{t} \int \log(1-x) \, d\pi_{\tilde{\xi}, \tilde{\eta}}(x) = B \left(\frac{\tau_1 + \theta - t}{t}, \frac{\tau_2 - t}{t} \right)$$

and then, on $\mathcal{A}(\xi T^{-1})$ the infimum is uniquely realized in $\pi_{\tilde{\xi}, \tilde{\eta}}$ and its value is

$$\theta \xi + t^2 \left[B \left(\frac{\tau_1 - t}{t}, \frac{\tau_2 - t}{t} \right) - B \left(\frac{\tau_1 + \theta - t}{t}, \frac{\tau_2 - t}{t} \right) \right].$$

Finally a small computation leads to (70) and (74). ■

7 Appendix 1: Some properties of $\ell = \log \Gamma$ and Ψ

From the Binet's formula ([AS72] [EMOT81] p.21), we have

$$\ell(x) = \left(x - \frac{1}{2}\right) \log x - x + 1 + \int_0^{\infty} f(s) [e^{-sx} - e^{-s}] \, ds \quad (117)$$

$$= \left(x - \frac{1}{2}\right) \log x - x + \frac{1}{2} \log(2\pi) + \int_0^{\infty} f(s) e^{-sx} \, ds. \quad (118)$$

where the function f is defined by

$$f(s) = \left[\frac{1}{2} - \frac{1}{s} + \frac{1}{e^s - 1} \right] \frac{1}{s} = 2 \sum_{k=1}^{\infty} \frac{1}{s^2 + 4\pi^2 k^2}, \quad (119)$$

and satisfies for every $s \geq 0$

$$0 < f(s) \leq f(0) = 1/12, \quad 0 < \left(s f(s) + \frac{1}{2} \right) < 1. \quad (120)$$

By differentiation

$$\Psi(x) = \log x - \frac{1}{2x} - \int_0^{\infty} s f(s) e^{-sx} \, ds = \log x - \int_0^{\infty} e^{-sx} \left(s f(s) + \frac{1}{2} \right) \, ds, \quad (121)$$

As easy consequences, we have, for every $x > 0$

$$0 < x (\log x - \Psi(x)) \leq 1 \quad , \quad 0 < x^2 \left(\log x - \Psi(x) - \frac{1}{2x} \right) \leq \frac{1}{12}. \quad (122)$$

Differentiating again we see that for $q \geq 1$

$$\Psi^{(q)}(z) = (-1)^{q-1} q! z^{-q} + (-1)^{q-1} \int_0^\infty e^{-sz} s^q \left(sf(s) + \frac{1}{2} \right) ds \quad (123)$$

and then

$$|\Psi^{(q)}(z) - (-1)^{q-1} q! z^{-q}| \leq z^{-q-1} q!. \quad (124)$$

8 Appendix 2 : Proof of Lemma 3.1

From the convergence (46), the weak convergence of the ESD ([CC04]) and the control on the extremal eigenvalues ([Led04], [Col05] and references therein), we get, if $\alpha \geq 1$

$$\lim_n \frac{1}{r} \log |\mathcal{Z}_{n_1, n_2, r}| = \int (\log x) \text{LSD}_{\alpha', \beta'}(dx) = \min(\beta', 1) \int (\log x) \pi_{\lambda_-, \lambda_+}(dx) \quad (125)$$

The correspondence with our notation is the following

$$\alpha' = \tau_1/t, \beta' = \tau_2/t. \quad (126)$$

From (45),

$$\lim_n \frac{1}{r} \log |\mathcal{Z}_{n_1, n_2, r}| = \frac{1}{t} \lim_n \frac{1}{n} \log |\mathcal{Z}_{n_1, n_2, r}| = \frac{1}{t} \mathcal{E}(\tau_1, \tau_2, t) \quad (127)$$

By homogeneity (40), this yields

$$\min(\beta, 1) \int (\log x) \pi_{\xi, \eta}(dx) = \mathcal{E}(\alpha', \beta', 1). \quad (128)$$

From (26) we see that if $\alpha \geq 1$ then

$$\{\sigma_-, \sigma_+\} = \left\{ \frac{\alpha'}{\alpha' + \beta'}, \frac{\alpha' + \beta' - 1}{\alpha' + \beta'} \right\}$$

We have two cases. When $\beta' > 1$, we get

$$\sigma_- = \frac{\alpha'}{\alpha' + \beta'} \quad , \quad \sigma_+ = \frac{\alpha' + \beta' - 1}{\alpha' + \beta'},$$

so that (128) yields

$$\int (\log x) \pi_{\xi, \eta}(dx) = \mathcal{E}(\alpha', \beta', 1) = \mathcal{E} \left(\frac{\sigma_-}{1 - \sigma_+}, \frac{1 - \sigma_-}{1 - \sigma_+}, 1 \right) \quad (129)$$

When $\beta < 1$, we get

$$\sigma_+ = \frac{\alpha'}{\alpha' + \beta'} \quad , \quad \sigma_- = \frac{\alpha' + \beta' - 1}{\alpha' + \beta'} \quad ,$$

so that (128) yields

$$\int (\log x) \pi_{\xi, \eta}(dx) = \frac{1}{\beta} \mathcal{E}(\alpha', \beta', 1) = \frac{1 - \sigma_-}{1 - \sigma_+} \mathcal{E}\left(\frac{\sigma_+}{1 - \sigma_-}, \frac{1 - \sigma_+}{1 - \sigma_-}, 1\right) \quad (130)$$

and both displays (129) and (130) provide (37).

This ends the proof. Let us remark that the first case above ($\beta' < 1$) corresponds to $t > \tau_2$ (i.e. $r > n_2$, the second matrix W_2 is singular) and the second one ($\beta \geq 1$) corresponds to $t \leq \tau_2$ (i.e. $r \leq n_2$, the second matrix is non-singular). ■

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