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The functor of units of Burnside rings for p -groups

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Abstract: In this note I describe the structure of the biset functor B^\times sending a p -group P to the group of units of its Burnside ring $B(P)$. In particular, I show that B^\times is a rational biset functor. It follows that if P is a p -group, the structure of $B^\times(P)$ can be read from a genetic basis of P : the group $B^\times(P)$ is an elementary abelian 2-group of rank equal to the number isomorphism classes of rational irreducible representations of P whose type is trivial, cyclic of order 2, or dihedral.

1. Introduction

If G is a finite group, denote by $B(G)$ the Burnside ring of G , i.e. the Grothendieck ring of the category of finite G -sets (see e.g. [2]). The question of structure of the multiplicative group $B^\times(G)$ has been studied by T. tom Dieck ([13]), T. Matsuda ([11]), T. Matsuda and T. Miyata ([12]), T. Yoshida ([16]), by geometric and algebraic methods.

Recently, E. Yalçın wrote a very nice paper ([14]), in which he proves an induction theorem for B^\times for 2-groups, which says that if P is a 2-group, then any element of $B^\times(P)$ is a sum of elements obtained by inflation and tensor induction from sections (T, S) of P , such that T/S is trivial or dihedral.

The main theorem of the present paper implies a more precise form of Yalçın's Theorem, but the proof is independent, and uses entirely different methods. In particular, the biset functor techniques developed in [1], [4] and [6], lead to a precise description of $B^\times(P)$, when P is a 2-group (actually also for arbitrary p -groups, but the case p odd is known to be rather trivial). The main ingredient consists to show that B^\times is a *rational* biset functor, and this is done by showing that the functor B^\times (restricted to p -groups) is a subfunctor of the functor $\mathbb{F}_2 R_{\mathbb{Q}}^*$. This leads to a description of $B^\times(P)$ in

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terms of a *genetic basis* of P , or equivalently, in terms of rational irreducible representations of P .

The paper is organized as follows : in Section 2, I recall the main definitions and notation on biset functors. Section 3 deals with genetic subgroups and rational biset functors. Section 4 gives a natural exposition of the biset functor structure of B^\times . In Section 5, I state results about faithful elements in $B^\times(P)$ for some specific p -groups P . In Section 6, I introduce a natural transformation of biset functors from B^\times to $\mathbb{F}_2 B^*$. This transformation is injective, and in Section 7, I show that the image of its restriction to p -groups is contained in the subfunctor $\mathbb{F}_2 R_{\mathbb{Q}}^*$ of $\mathbb{F}_2 B^*$. This is the key result, leading in Section 8 to a description of the lattice of subfunctors of the restriction of B^\times to p -groups : it is always a uniserial p -biset functor (even simple if p is odd). This also provides an answer to the question, raised by Yalçın ([14]), of the surjectivity of the exponential map $B(P) \rightarrow B^\times(P)$ for a 2-group P .

2. Biset functors

2.1. Notation and Definition : *Denote by \mathcal{C} the following category :*

- *The objects of \mathcal{C} are the finite groups.*
- *If G and H are finite p -groups, then $\text{Hom}_{\mathcal{C}}(G, H) = B(H \times G^{op})$ is the Burnside group of finite (H, G) -bisets. An element of this group is called a virtual (H, G) -biset.*
- *The composition of morphisms is \mathbb{Z} -bilinear, and if G, H, K are finite groups, if U is a finite (H, G) -biset, and V is a finite (K, H) -biset, then the composition of (the isomorphism classes of) V and U is the (isomorphism class) of $V \times_H U$. The identity morphism Id_G of the group G is the class of the set G , with left and right action by multiplication.*

If p is a prime number, denote by \mathcal{C}_p the full subcategory of \mathcal{C} whose objects are finite p -groups.

Let \mathcal{F} denote the category of additive functors from \mathcal{C} to the category $\mathbb{Z}\text{-Mod}$ of abelian groups. An object of \mathcal{F} is called a biset functor. Similarly, denote by \mathcal{F}_p the category of additive functors from \mathcal{C}_p to $\mathbb{Z}\text{-Mod}$. An object of \mathcal{F}_p will be called a p -biset functor.

If F is an object of \mathcal{F} , if G and H are finite groups, and if $\varphi \in \text{Hom}_{\mathcal{C}}(G, H)$, then the image of $w \in F(G)$ by the map $F(\varphi)$ will generally be denoted by $\varphi(w)$. The composition $\psi \circ \varphi$ of morphisms $\varphi \in \text{Hom}_{\mathcal{C}}(G, H)$ and $\psi \in \text{Hom}_{\mathcal{C}}(H, K)$ will also be denoted by $\psi \times_H \varphi$.

2.2. Notation : The Burnside biset functor (defined e.g. as the Yoneda functor $\text{Hom}_{\mathcal{C}}(\mathbf{1}, -)$), will be denoted by B . The functor of rational representations (see Section 1 of [4]) will be denoted by $R_{\mathbb{Q}}$. The restriction of B and $R_{\mathbb{Q}}$ to \mathcal{C}_p will also be denoted by B and $R_{\mathbb{Q}}$.

2.3. Examples : Recall that this formalism of bisets gives a single framework for the usual operations of induction, restriction, inflation, deflation, and transport by isomorphism via the following correspondences :

- If H is a subgroup of G , then let $\text{Ind}_H^G \in \text{Hom}_{\mathcal{C}}(H, G)$ denote the set G , with left action of G and right action of H by multiplication.
- If H is a subgroup of G , then let $\text{Res}_H^G \in \text{Hom}_{\mathcal{C}}(G, H)$ denote the set G , with left action of H and right action of G by multiplication.
- If $N \trianglelefteq G$, and $H = G/N$, then let $\text{Inf}_H^G \in \text{Hom}_{\mathcal{C}}(H, G)$ denote the set H , with left action of G by projection and multiplication, and right action of H by multiplication.
- If $N \trianglelefteq G$, and $H = G/N$, then let $\text{Def}_H^G \in \text{Hom}_{\mathcal{C}}(G, H)$ denote the set H , with left action of H by multiplication, and right action of G by projection and multiplication.
- If $\varphi : G \rightarrow H$ is a group isomorphism, then let $\text{Iso}_G^H = \text{Iso}_G^H(\varphi) \in \text{Hom}_{\mathcal{C}}(G, H)$ denote the set H , with left action of H by multiplication, and right action of G by taking image by φ , and then multiplying in H .

2.4. Definition : A section of the group G is a pair (T, S) of subgroups of G such that $S \trianglelefteq T$.

2.5. Notation : If (T, S) is a section of G , set

$$\text{Indinf}_{T/S}^G = \text{Ind}_T^G \text{Inf}_{T/S}^T \quad \text{and} \quad \text{Defres}_{T/S}^G = \text{Def}_{T/S}^T \text{Res}_T^G \quad .$$

Then $\text{Indinf}_{T/S}^G \cong G/S$ as $(G, T/S)$ -biset, and $\text{Defres}_{T/S}^G \cong S \backslash G$ as $(T/S, G)$ -biset.

2.6. Notation : Let G and H be groups, let U be an (H, G) -biset, and let $u \in U$. If T is a subgroup of H , set

$$T^u = \{g \in G \mid \exists t \in T, tu = ug\} \quad .$$

This is a subgroup of G . Similarly, if S is a subgroup of G , set

$${}^u S = \{h \in H \mid \exists s \in S, us = hu\} \quad .$$

This is a subgroup of H .

2.7. Lemma : Let G and H be groups, let U be an (H, G) -biset, and let S be a subgroup of G . Then there is an isomorphism of H -sets

$$U/G = \bigsqcup_{u \in [H \backslash U/S]} H/{}^u S \quad ,$$

where $[H \backslash U/S]$ is a set of representatives of (H, S) -orbits on U .

Proof: Indeed $H \backslash U/S$ is the set of orbits of H on U/S , and ${}^u S$ is the stabilizer of uS in H .

2.8. Opposite bisets : If G and H are finite groups, and if U is a finite (H, G) -biset, then let U^{op} denote the opposite biset : as a set, it is equal to U , and it is a (G, H) -biset for the following action

$$\forall h \in H, \forall u \in U, \forall g \in G, g.u.h \text{ (in } U^{op}) = h^{-1}ug^{-1} \text{ (in } U) \quad .$$

This definition can be extended by linearity, to give an isomorphism

$$\varphi \mapsto \varphi^{op} : \text{Hom}_{\mathcal{C}}(G, H) \rightarrow \text{Hom}_{\mathcal{C}}(H, G) \quad .$$

It is easy to check that $(\varphi \circ \psi)^{op} = \psi^{op} \circ \varphi^{op}$, with obvious notation, and the functor

$$\left\{ \begin{array}{l} G \mapsto G \\ \varphi \mapsto \varphi^{op} \end{array} \right.$$

is an equivalence of categories from \mathcal{C} to the dual category, which restricts to an equivalence of \mathcal{C}_p to its dual category.

2.9. Example : if G is a finite group, and (T, S) is a section of G , then

$$(\text{Indinf}_{T/S}^G)^{op} \cong \text{Defres}_{T/S}^G$$

as $(T/S, G)$ -bisets.

2.10. Definition and Notation : If F is a biset functor, the dual biset functor F^* is defined by

$$F^*(G) = \text{Hom}_{\mathbb{Z}}(F(G), \mathbb{Z}) \quad ,$$

for a finite group G , and by

$$F^*(\varphi)(\alpha) = \alpha \circ F(\varphi^{op}) \quad ,$$

for any $\alpha \in F^*(G)$, any finite group H , and any $\varphi \in \text{Hom}_{\mathcal{C}}(G, H)$.

2.11. Some idempotents in $\text{End}_C(G)$: Let G be a finite group, and let $N \trianglelefteq G$. Then it is clear from the definitions that

$$\text{Def}_{G/N}^G \circ \text{Inf}_{G/N}^G = (G/N) \times_G (G/N) = \text{Id}_{G/N} \quad .$$

It follows that the composition $e_N^G = \text{Inf}_{G/N}^G \circ \text{Def}_{G/N}^G$ is an idempotent in $\text{End}_C(G)$. Moreover, if M and N are normal subgroups of G , then $e_N^G \circ e_M^G = e_{NM}^G$. Moreover $e_1^G = \text{Id}_G$.

2.12. Lemma : ([6] Lemma 2.5) *If $N \trianglelefteq G$, define $f_N^G \in \text{End}_C(G)$ by*

$$f_N^G = \sum_{\substack{M \trianglelefteq G \\ N \subseteq M}} \mu_{\trianglelefteq G}(N, M) e_M^G \quad ,$$

where $\mu_{\trianglelefteq G}$ denotes the Möbius function of the poset of normal subgroups of G . Then the elements f_N^G , for $N \trianglelefteq G$, are orthogonal idempotents of $\text{End}_C(G)$, and their sum is equal to Id_G .

Moreover, it is easy to check from the definition that for $N \trianglelefteq G$,

$$(2.13) \quad f_N^G = \text{Inf}_{G/N}^G \circ f_1^{G/N} \circ \text{Def}_{G/N}^G \quad ,$$

and

$$e_N^G = \text{Inf}_{G/N}^G \circ \text{Def}_{G/N}^G = \sum_{\substack{M \trianglelefteq G \\ M \supseteq N}} f_M^G \quad .$$

2.14. Lemma : *If N is a non trivial normal subgroup of G , then*

$$f_1^G \circ \text{Inf}_{G/N}^G = 0 \quad \text{and} \quad \text{Def}_{G/N}^G \circ f_1^G = 0 \quad .$$

Proof: Indeed by 2.13

$$\begin{aligned} f_1^G \circ \text{Inf}_{G/N}^G &= f_1^G \circ \text{Inf}_{G/N}^G \circ \text{Def}_{G/N}^G \circ \text{Inf}_{G/N}^G \\ &= \sum_{\substack{M \trianglelefteq N \\ M \supseteq N}} f_1^G f_M^G \text{Inf}_{G/N}^G = 0 \quad , \end{aligned}$$

since $M \neq \mathbf{1}$ when $M \supseteq N$. The other equality of the lemma follows by taking opposite bisets. \square

2.15. Remark : It was also shown in Section 2.7 of [6] that if P is a p -group, then

$$f_1^P = \sum_{N \subseteq \Omega_1 Z(P)} \mu(\mathbf{1}, N) P/N \quad ,$$

where μ is the Möbius function of the poset of subgroups of N , and $\Omega_1 Z(P)$ is the subgroup of the centre of P consisting of elements of order at most p .

2.16. Notation and Definition : *If F is a biset functor, and if G is a finite group, then the idempotent f_1^G of $\text{End}_{\mathcal{C}}(G)$ acts on $F(G)$. Its image*

$$\partial F(G) = f_1^G F(G)$$

is a direct summand of $F(G)$ as \mathbb{Z} -module : it will be called the set of faithful elements of $F(G)$.

The reason for this name is that any element $u \in F(G)$ which is inflated from a proper quotient of G is such that $F(f_1^G)u = 0$. From Lemma 2.14, it is also clear that

$$\partial F(G) = \bigcap_{\mathbf{1} \neq N \trianglelefteq G} \text{Ker Def}_{G/N}^G .$$

3. Genetic subgroups and rational p -biset functors

The following definitions are essentially taken from Section 2 of [7] :

3.1. Definition and Notation : *Let P be a finite p -group. If Q is a subgroup of P , denote by $Z_P(Q)$ the subgroup of P defined by*

$$Z_P(Q)/Q = Z(N_P(Q)/Q) .$$

A subgroup Q of P is called genetic if it satisfies the following two conditions :

1. *The group $N_P(Q)/Q$ has normal p -rank 1.*
2. *If $x \in P$, then $Q^x \cap Z_P(Q) \subseteq Q$ if and only if $Q^x = Q$.*

Two genetic subgroups Q and R are said to be linked modulo P (notation $Q \text{ ---}_P R$), if there exist elements x and y in P such that $Q^x \cap Z_P(R) \subseteq R$ and $R^y \cap Z_P(Q) \subseteq Q$.

This relation is an equivalence relation on the set of genetic subgroups of P . The set of equivalence classes is in one to one correspondence with the set of isomorphism classes of rational irreducible representations of P . A genetic basis of P is a set of representatives of these equivalence classes.

If V is an irreducible representation of P , then the *type* of V is the isomorphism class of the group $N_P(Q)/Q$, where Q is a genetic subgroup of P in the equivalence class corresponding to V by the above bijection.

3.2. Remark : The definition of the relation ---_P given here is different from Definition 2.9 of [7], but it is equivalent to it, by Lemma 4.5 of [6].

The following is Theorem 3.2 of [6], in a slightly different form :

3.3. Theorem : *Let P be a finite p -group, and \mathcal{G} be a genetic basis of P . Let F be a p -biset functor. Then the map*

$$\mathcal{I}_{\mathcal{G}} = \bigoplus_{Q \in \mathcal{G}} \text{Indinf}_{N_P(Q)/Q}^P : \bigoplus_{Q \in \mathcal{G}} \partial F(N_P(Q)/Q) \rightarrow F(P)$$

is split injective.

3.4. Remark : There are two differences with the initial statement of Theorem 3.2 of [6] : here I use genetic *subgroups* instead of genetic *sections*, because these two notions are equivalent by Proposition 4.4 of [6]. Also the definition of the map $\mathcal{I}_{\mathcal{G}}$ is apparently different : with the notation of [6], the map $\mathcal{I}_{\mathcal{G}}$ is the sum of the maps $F(a_Q)$, where a_Q is the trivial $(P, P/P)$ -biset if $Q = P$, and a_Q is the virtual $(P, N_P(Q)/Q)$ -biset $P/Q - P/\hat{Q}$ if $Q \neq P$, where \hat{Q} is the unique subgroup of $Z_P(Q)$ containing Q , and such that $|\hat{Q} : Q| = p$. But it is easy to see that the restriction of the map $F(P/\hat{Q})$ to $\partial F(N_P(Q)/Q)$ is actually 0. Moreover, the map $F(a_Q)$ is equal to $\text{Indinf}_{N_P(Q)/Q}^P$. So in fact, the above map $\mathcal{I}_{\mathcal{G}}$ is the same as the one defined in Theorem 3.2 of [6].

3.5. Definition : *A p -biset functor F is called rational if for any finite p -group P and any genetic basis \mathcal{G} of P , the map $\mathcal{I}_{\mathcal{G}}$ is an isomorphism.*

It was shown in Proposition 7.4 of [6] that subfunctors, quotient functors, and dual functors of rational p -biset functors are rational.

4. The functor of units of the Burnside ring

4.1. Notation : *If G is a finite group, let $B^\times(G)$ denote the group of units of the Burnside ring $B(G)$.*

If G and H are finite groups, if U is a finite (H, G) -biset, recall that U^{op} denotes the (G, H) -biset obtained from U by reversing the actions. If X is a finite G -set, then $T_U(X) = \text{Hom}_G(U^{op}, X)$ is a finite H -set. The correspondence $X \mapsto T_U(X)$ can be extended to a correspondence $T_U : B(G) \rightarrow B(H)$, which is multiplicative (i.e. $T_U(ab) = T_U(a)T_U(b)$ for any $a, b \in B(G)$), and preserves identity elements (i.e. $T_U(G/G) = H/H$). This extension to $B(G)$ can be built by different means, and the following is described in Section 4.1 of [3] : if a is an element of $B(G)$, then there exists a finite G -poset X such that a is equal to the Lefschetz invariant Λ_X . Now $\text{Hom}_G(U^{op}, X)$ has a natural structure of H -poset, and one can set $T_U(a) = \Lambda_{\text{Hom}_G(U^{op}, X)}$. It is an element of $B(H)$, which does not depend of the choice of the poset X

such that $a = \Lambda_X$, because with Notation 2.6 and Lemma 2.7, for any subgroup T of H the Euler-Poincaré characteristics $\chi(\text{Hom}_G(U^{op}, X)^T)$ can be computed by

$$\chi(\text{Hom}_G(U^{op}, X)^T) = \prod_{u \in T \backslash U/G} \chi(X^{T^u}) \quad ,$$

and the latter only depends on the element Λ_X of $B(G)$. As a consequence, one has that

$$|T_U(a)^T| = \prod_{u \in T \backslash U/G} |a^{T^u}| \quad .$$

It follows in particular that $T_U(B^\times(G)) \subseteq B^\times(H)$. Moreover, it is easy to check that $T_U = T_{U'}$ if U and U' are isomorphic (H, G) -bisets, that $T_{U_1 \sqcup U_2}(a) = T_{U_1}(a)T_{U_2}(a)$ for any (H, G) -bisets U_1 and U_2 , and any $a \in B(G)$.

It follows that there is a well defined bilinear pairing

$$B(H \times G^{op}) \times B^\times(G) \rightarrow B^\times(H) \quad ,$$

extending the correspondence $(U, a) \mapsto T_U(a)$. If $f \in B(H \times G^{op})$ (i.e. if f is a virtual (H, G) -biset), the corresponding group homomorphism $B^\times(G) \rightarrow B^\times(H)$ will be denoted by $B^\times(f)$.

Now let K be a third group, and V be a finite (K, H) -set. If X is a finite G -set, there is a canonical isomorphism of K -sets

$$\text{Hom}_H(V^{op}, \text{Hom}_G(U^{op}, X)) \cong \text{Hom}_G((V \times_H U)^{op}, X) \quad ,$$

showing that $T_V \circ T_U = T_{V \times_H U}$.

It follows more generally that $B^\times(g) \circ B^\times(f) = B^\times(g \times_H f)$ for any $g \in B(K \times H^{op})$ and any $f \in B(H \times G^{op})$. Finally this shows :

4.2. Proposition : *The correspondence sending a finite group G to $B^\times(G)$, and an homomorphism f in \mathcal{C} to $B^\times(f)$, is a biset functor.*

4.3. Remark and Notation : The restriction and inflation maps for the functor B^\times are the usual ones for the functor B . The deflation map $\text{Def}_{G/N}^G$ corresponds to taking fixed points under N (so it *does not coincide* with the usual deflation map for B , which consist in taking *orbits under* N).

Similarly, if H is a subgroup of G , the induction map from H to G for the functor B^\times is sometimes called *multiplicative induction*. I will call it *tensor induction*, and denote it by Ten_H^G . If (T, S) is a section of G , I will also set $\text{Teninf}_{T/S}^P = \text{Ten}_T^P \text{Inf}_{T/S}^T$.

5. Faithful elements in $B^\times(G)$

5.1. Notation and definition : Let G be a finite group. Denote by $[s_G]$ a set of representatives of conjugacy classes of subgroups of G . Then the elements G/L , for $L \in [s_G]$, form a basis of $B(G)$ over \mathbb{Z} , called the canonical basis of $B(G)$.

The primitive idempotents of $\mathbb{Q}B(G)$ are also indexed by $[s_G]$: if $H \in [s_G]$, the correspondent idempotent e_H^G is equal to

$$e_H^G = \frac{1}{|N_G(H)|} \sum_{K \subseteq H} |K| \mu(K, H) G/K \quad ,$$

where $\mu(K, H)$ denotes the Möbius function of the poset of subgroups of G , ordered by inclusion (see [10], [15], or [2]).

Recall that if $a \in B(G)$, then $a \cdot e_H^G = |a^H| e_H^G$ so that a can be written as

$$a = \sum_{H \in [s_G]} |a^H| e_H^G \quad .$$

Now $a \in B^\times(G)$ if and only if $a \in B(G)$ and $|a^H| \in \{\pm 1\}$ for any $H \in [s_G]$, or equivalently if $a^2 = G/G$. If now P is a p -group, and if $p \neq 2$, since $|a^H| \equiv |a| \pmod{p}$ for any subgroup $|H|$ of P , it follows that $|a^H| = |a|$ for any H , thus $a = \pm P/P$. This shows the following well know

5.2. Lemma : *If P is an odd order p -group, then $B^\times(P) = \{\pm P/P\}$.*

5.3. Remark : So in the sequel, when considering p -groups, the only really non-trivial case will occur for $p = 2$. However, some statements will be given for arbitrary p -groups.

5.4. Notation : *If G is a finite group, denote by F_G the set of subgroups H of G such that $H \cap Z(G) = \mathbf{1}$, and set $[F_G] = F_G \cap [s_G]$.*

5.5. Lemma : *Let G be a finite group. If $|Z(G)| > 2$, then $\partial B^\times(G)$ is trivial.*

Proof: Indeed let $a \in \partial B^\times(G)$. Then $\text{Def}_{G/N}^G a$ is the identity element of $B^\times(G/N)$, for any non-trivial normal subgroup N of G . Now suppose that H is a subgroup of G containing N . Then

$$|a^H| = |\text{Def}_{N_G(H)/H}^G a| = |\text{Iso}_{N_G/N(H/N)/(H/N)}^{N_G(H)/H} \text{Def}_{N_G/N(H/N)}^{G/N} \text{Def}_{G/N}^G a| = 1 \quad .$$

In particular $|a^H| = 1$ if $H \cap Z(G) \neq \mathbf{1}$. It follows that there exists a subset A of $[F_G]$ such that

$$a = G/G - 2 \sum_{H \in A} e_H^G \quad .$$

If $A \neq \emptyset$, i.e. if $a \neq G/G$, let L be a maximal element of A . Then $L \neq G$, because $Z(G) \neq \mathbf{1}$. The coefficient of G/L in the expression of a in the canonical basis of $B(G)$ is equal to

$$-2 \frac{|L|\mu(L, L)}{|N_G(L)|} = -2 \frac{|L|}{|N_G(L)|} .$$

This is moreover an integer, since $a \in B^\times(G)$. It follows that $|N_G(L) : L|$ is equal to 1 or 2. But since $L \cap Z(G) = \mathbf{1}$, the group $Z(G)$ embeds into the group $N_G(L)/L$. Hence $|N_G(L) : L| \geq 3$, and this contradiction shows that $A = \emptyset$, thus $a = G/G$. \square

5.6. Lemma : *Let P be a finite 2-group, of order at least 4, and suppose that the maximal elements of F_P have order 2. If $|P| \geq 2|F_P|$, then $\partial B^\times(P)$ is trivial.*

Proof: Let $a \in \partial B^\times(P)$. By the argument of the previous proof, there exists a subset A of $[F_P]$ such that

$$a = P/P - 2 \sum_{H \in A} e_H^P .$$

The hypothesis implies that $\mu(\mathbf{1}, H) = -1$ for any non-trivial element H of $[F_P]$. Now if $\mathbf{1} \in A$, the coefficient of $P/1$ in the expression of a in the canonical basis of $B(P)$ is equal to

$$-2 \frac{1}{|P|} + 2 \sum_{H \in A - \{\mathbf{1}\}} \frac{1}{|N_P(H)|} = -2 \frac{1}{|P|} + 2 \sum_{H \in \bar{A} - \{\mathbf{1}\}} \frac{1}{|P|} = \frac{-4 + 2|\bar{A}|}{|P|} ,$$

where \bar{A} is the set of subgroups of P which are conjugate to some element of A . This coefficient is an integer if $a \in B(P)$, so $|P|$ divides $2|\bar{A}| - 4$. But $|\bar{A}|$ is always odd, since the trivial subgroup is the only normal subgroup of P which is in \bar{A} in this case. Thus $2|\bar{A}| - 4$ is congruent to 2 modulo 4, and cannot be divisible by $|P|$, since $|P| \geq 4$.

So $\mathbf{1} \notin A$, and the coefficient of $P/1$ in the expression of a is equal to

$$2 \sum_{H \in A} \frac{1}{|N_P(H)|} = \frac{2|\bar{A}|}{|P|} .$$

Now this is an integer, so $2|\bar{A}|$ is congruent to 0 or 1 modulo the order of P , which is even since $|P| \geq 2|F_P| \geq 2$. Thus $\mathbf{1} \notin A$, and $2|\bar{A}|$ is a multiple of $|P|$. But $2|\bar{A}| < 2|F_P|$ since $\mathbf{1} \notin A$. So if $2|F_P| \leq |P|$, it follows that \bar{A} is empty, and A is empty. Hence $a = P/P$, as was to be shown. \square

5.7. Corollary : *Let P be a finite 2-group. Then the group $\partial B^\times(P)$ is trivial in each of the following cases :*

1. P is abelian of order at least 3.
2. P is generalized quaternion or semi-dihedral.

5.8. Remark : Case 1 follows easily from Matsuda's Theorem ([11]). Case 2 follows from Lemma 4.6 of Yalçın ([14]).

Proof: Case 1 follows from Lemma 5.5. In Case 2, if P is generalized quaternion, then $F_P = \{\mathbf{1}\}$, thus $|P| \geq 2|F_P|$. And if P is semidihedral, then there is a unique conjugacy class of non-trivial subgroups H of P such that $H \cap Z(P) = \mathbf{1}$. Such a group has order 2, and $N_P(H) = HZ(P)$ has order 4. Thus $|F_P| = 1 + \frac{|P|}{4}$, and $|P| \geq 2|F_P|$ also in this case. \square

5.9. Corollary : [Yalçın [14] Lemma 4.6 and Lemma 5.2] *Let P be a p -group of normal p -rank 1. Then $\partial B^\times(P)$ is trivial, except if P is*

- *the trivial group, and $\partial B^\times(P)$ is the group of order 2 generated by $v_P = -P/P$.*
- *cyclic of order 2, and $\partial B^\times(P)$ is the group of order 2 generated by*

$$v_P = P/P - P/\mathbf{1} \quad .$$

- *dihedral of order at least 16, and then $\partial B^\times(P)$ is the group of order 2 generated by the element*

$$v_P = P/P + P/1 - P/I - P/J \quad ,$$

where I and J are non-central subgroups of order 2 of P , not conjugate in P .

Proof: Lemma 5.2 and Lemma 5.5 show that $\partial B^\times(P)$ is trivial, when P has normal p -rank 1, and P is not trivial, cyclic of order 2, or dihedral : indeed then, the group P is cyclic of order at least 3, or generalized quaternion, or semi-dihedral.

Now if P is trivial, then obviously $B(P) = \mathbb{Z}$, so $B^\times(P) = \partial B^\times(P) = \{\pm P/P\}$. If P has order 2, then clearly $B^\times(P)$ consists of $\pm P/P$ and $\pm(P/P - P/\mathbf{1})$, and $\partial B^\times(P) = \{P/P, P/P - P/\mathbf{1}\}$. Finally, if P is dihedral, the set F_P consists of the trivial group, and of two conjugacy classes of subgroups H of order 2 of P , and $N_P(H) = HZ$ for each of these, where Z is the centre of P . Thus

$$|F_P| = 1 + 2\frac{|P|}{4} = 1 + \frac{|P|}{2} \quad .$$

Now with the notation of the proof of Lemma 5.6, one has that $2|\bar{A}| \equiv 0 \pmod{|P|}$, and $2|\bar{A}| < |F_P| = 2 + |P|$. So either $A = \emptyset$, and in this case $a = P/P$, or $2|\bar{A}| = |P|$, which means that \bar{A} is the whole set of non-trivial elements of F_P . In this case

$$a = P/P - 2(e_I^P + e_J^P) \quad ,$$

where I and J are non-central subgroups of order 2 of P , not conjugate in P . It is then easy to check that

$$a = P/P + P/1 - (P/I + P/J) \quad ,$$

so a is indeed in $B(P)$, hence in $B^\times(P)$. Moreover $\text{Def}_{P/Z}^P a$ is the identity element of $B^\times(P/Z)$, so $a = f_1^P a$, and $a \in \partial B^\times(P)$. This completes the proof. \square

6. A morphism of biset functors

If k is any commutative ring, there is an obvious isomorphism of biset functor from $kB^* = k \otimes_{\mathbb{Z}} B^*$ to $\text{Hom}(B, k)$, which is defined for a group G by sending the element $\alpha = \sum_i \alpha_i \otimes \psi_i$, where $\alpha_i \in k$ and $\psi_i \in B^*(G)$, to the linear form $\tilde{\alpha} : B(G) \rightarrow k$ defined by $\tilde{\alpha}(G/H) = \sum_i \psi_i(G/H) \alpha_i$.

6.1. Notation : Let $\{\pm 1\} = \mathbb{Z}^\times$ be the group of units of the ring \mathbb{Z} . The unique group isomorphism from $\{\pm 1\}$ to $\mathbb{Z}/2\mathbb{Z}$ will be denoted by $u \mapsto u_+$.

If G is a finite group, and if $a \in B^\times(G)$, then recall that for each subgroup S of G , the integer $|a^S|$ is equal to ± 1 . Define a map $\epsilon_G : B^\times(G) \rightarrow \mathbb{F}_2 B^*(G)$ by setting $\epsilon_G(a)(G/S) = |a^S|_+$, for any $a \in B^\times(G)$ and any subgroup S of G .

6.2. Proposition : The maps ϵ_G define a injective morphism of biset functors

$$\epsilon : B^\times \rightarrow \mathbb{F}_2 B^* \quad .$$

Proof: The injectivity of the map ϵ_G is obvious. Now let G and H be finite groups, and let U be a finite (H, G) -biset. Also denote by U the corresponding element of $B(H \times G^{op})$. If $a \in B^\times(G)$, and if T is a subgroup of H , then

$$|B^\times(U)(a)^T| = \prod_{u \in T \backslash U/G} |a^{Tu}| \quad .$$

Thus

$$\begin{aligned}
\epsilon_H (B^\times(U)(a)) (H/T) &= \left(\prod_{u \in T \backslash U/G} |a^{T^u}| \right)_+ \\
&= \sum_{u \in T \backslash U/G} |a^{T^u}|_+ \\
&= \sum_{u \in T \backslash U/G} \epsilon_G(a)(G/T^u) \\
&= \epsilon_G(a)(U^{op}/T) \\
&= \epsilon_G(a)(U^{op} \times_H H/T) \\
&= \mathbb{F}_2 B^*(U)(\epsilon_G(a))(H/T)
\end{aligned}$$

thus $\epsilon_H \circ B^\times(U) = \mathbb{F}_2 B^*(U) \circ \epsilon_G$. Since both sides are additive with respect to U , the same equality holds when U is an arbitrary element of $B(H \times G^{op})$, completing the proof. \square

7. Restriction to p -groups

The additional result that holds for finite p -groups (and not for arbitrary finite groups) is the Ritter-Segal theorem, which says that the natural transformation $B \rightarrow R_{\mathbb{Q}}$ of biset functors for p -groups, is surjective. By duality, it follows that the natural transformation $i : kR_{\mathbb{Q}}^* \rightarrow kB^*$ is injective, for any commutative ring k . The following gives a characterization of the image $i(kR_{\mathbb{Q}}^*)$ inside kB^* :

7.1. Proposition : *Let p be a prime number, let P be a p -group, let k be a commutative ring. Then the element $\varphi \in kB^*(P)$ lies in $i(kR_{\mathbb{Q}}^*(P))$ if and only if the element $\text{Defres}_{T/S}^P \varphi$ lies in $i(kR_{\mathbb{Q}}^*(T/S))$, for any section T/S of P which is*

- elementary abelian of rank 2, or non-abelian of order p^3 and exponent p , if $p \neq 2$.
- elementary abelian of rank 2, or dihedral of order at least 8, if $p = 2$.

Proof: Since the image of $kR_{\mathbb{Q}}^*$ is a subfunctor of kB^* , if $\varphi \in i(kR_{\mathbb{Q}}^*(P))$, then $\text{Defres}_{T/S}^P \varphi \in i(kR_{\mathbb{Q}}^*(T/S))$, for any section (T, S) of P .

Conversely, consider the exact sequence of biset functors over p -groups

$$0 \rightarrow K \rightarrow B \rightarrow R_{\mathbb{Q}} \rightarrow 0 \quad .$$

Every evaluation of this sequence at a particular p -group is a split exact sequence of (free) abelian groups. Hence by duality, for any ring k , there is an exact sequence

$$0 \rightarrow kR_{\mathbb{Q}}^* \rightarrow kB^* \rightarrow kK^* \rightarrow 0 \quad .$$

With the identification $kB^* \cong \text{Hom}_{\mathbb{Z}}(B, k)$, this means that if P is a p -group, the element $\varphi \in RB^*(P)$ lies in $i(kR_{\mathbb{Q}}^*(P))$ if and only if $\varphi(K(P)) = 0$. Now by Corollary 6.16 of [7], the group $K(P)$ is the set of linear combinations of elements of the form $\text{Indinf}_{T/S}^P \theta(\kappa)$, where T/S is a section of P , and θ is a group isomorphism from one of the group listed in the proposition to T/S , and κ is a specific element of $K(T/S)$ in each case. The proposition follows, because

$$\varphi(\text{Indinf}_{T/S}^P \theta(\kappa)) = (\text{Defres}_{T/S}^P \varphi)(\theta(\kappa)) \quad ,$$

and this is zero if $\text{Defres}_{T/S}^P \varphi$ lies in $i(kR_{\mathbb{Q}}^*(T/S))$. \square

7.2. Theorem : *Let p be a prime number, and P be a finite p -group. The image of the map ϵ_P is contained in $i(\mathbb{F}_2 R_{\mathbb{Q}}^*(P))$.*

Proof: Let $a \in B^\times(P)$, and let T/S be any section of P . Since

$$\text{Defres}_{T/S}^P i_P(a) = i_{T/S} \text{Defres}_{T/S}^P a \quad ,$$

by Proposition 7.1, it is enough to check that the image of ϵ_P is contained in $i(\mathbb{F}_2 R_{\mathbb{Q}}^*(P))$, when P is elementary abelian of rank 2 or non-abelian of order p^3 and exponent p if p is odd, or when P is elementary abelian of rank 2 or dihedral if $p = 2$.

Now if N is a normal subgroup of P , one has that

$$f_N^P i_P(a) = \text{Inf}_{P/N}^P \left(i_{P/N} (f_1^{P/N} \text{Def}_{P/N}^P a) \right) \quad .$$

Thus by induction on the order of P , one can suppose $a \in \partial B^\times(P)$. But if P is elementary abelian of rank 2, or if P has odd order, then $\partial B^\times(P)$ is trivial, by Lemma 5.2 and Corollary 5.7. Hence there is nothing more to prove if p is odd. And for $p = 2$, the only case left is when P is dihedral. In that case by Corollary 5.9, the group $\partial B^\times(P)$ has order 2, generated by the element

$$v_P = \sum_{H \in [s_P] - \{I, J\}} e_H^P - (e_I^P + e_J^P) \quad ,$$

where $[s_P]$ is a set of representatives of conjugacy classes of subgroups of P , and where I and J are the elements of $[s_P]$ which have order 2, and are non central in P . Moreover the element $\theta(\kappa)$ mentioned above is equal to

$$(P/I' - P/I'Z) - (P/J' - P/J'Z) \quad ,$$

where Z is the centre of P , and I' and J' are non-central subgroups of order 2 of P , not conjugate in P . Hence up to sign $\theta(\kappa)$ is equal to

$$\delta_P = (P/I - P/IZ) - (P/J - P/JZ) \quad .$$

Since $\epsilon_P(v_P)(P/H)$ is equal to zero, except if H is conjugate to I or J , and then $\epsilon_P(v_P)(P/H) = 1$, it follows that $\epsilon_P(v_P)(\delta_P) = 1 - 1 = 0$, as was to be shown. This completes the proof. \square

7.3. Corollary : *The p -biset functor B^\times is rational.*

Proof: Indeed, it is isomorphic to a subfunctor of $\mathbb{F}_2 R_{\mathbb{Q}}^* \cong \text{Hom}_{\mathbb{Z}}(R_{\mathbb{Q}}, \mathbb{F}_2)$, which is rational by Proposition 7.4 of [6]. \square

7.4. Theorem : *Let P be a p -group. Then $B^\times(P)$ is an elementary abelian 2-group of rank equal to the number of isomorphism classes of rational irreducible representations of P whose type is trivial, cyclic of order 2, or dihedral. More precisely :*

1. *If $p \neq 2$, then $B^\times(P) = \{\pm 1\}$.*
2. *If $p = 2$, then let \mathcal{G} be a genetic basis of P , and let \mathcal{H} be the subset of \mathcal{G} consisting of elements Q such that $N_P(Q)/Q$ is trivial, cyclic of order 2, or dihedral. If $Q \in \mathcal{H}$, then $\partial B^\times(N_P(Q)/Q)$ has order 2, generated by $v_{N_P(Q)/Q}$. Then the set*

$$\{\text{Teninf}_{N_P(Q)/Q}^P v_{N_P(Q)/Q} \mid Q \in \mathcal{H}\}$$

is an \mathbb{F}_2 -basis of $B^\times(P)$.

Proof: This follows from the definition of a rational biset functor, and from Corollary 5.9. \square

7.5. Remark : If P is abelian, then there is a unique genetic basis of P , consisting of subgroups Q such that P/Q is cyclic. So in that case, the rank of $B^\times(P)$ is equal 1 plus the number of subgroups of index 2 in P : this gives a new proof of Matsuda's Theorem ([11]).

8. The functorial structure of B^\times for p -groups

In this section, I will describe the lattice of subfunctors of the p -biset functor B^\times .

8.1. The case $p \neq 2$. If $p \neq 2$, there is not much to say, since $B^\times(P) \cong \mathbb{F}_2$ for any p -group P . In this case, the functor B^\times is the constant functor $\Gamma_{\mathbb{F}_2}$ introduced in Corollary 8.4 of [8]. It is also isomorphic to the simple functor S_{1, \mathbb{F}_2} . In this case, the results of [6] and [7] lead to the following remarkable version of Theorem 11.2 of [8]:

8.2. Proposition : *If $p \neq 2$, the inclusion $B^\times \rightarrow \mathbb{F}_2 R_{\mathbb{Q}}^*$ leads to a short exact sequence of p -biset functors*

$$0 \rightarrow B^\times \rightarrow \mathbb{F}_2 R_{\mathbb{Q}}^* \rightarrow D_{tors} \rightarrow 0 \quad ,$$

where D_{tors} is the torsion part of the Dade p -biset functor.

8.3. The case $p = 2$. There is a bilinear pairing

$$\langle \ , \ \rangle : \mathbb{F}_2 R_{\mathbb{Q}}^* \times \mathbb{F}_2 R_{\mathbb{Q}} \rightarrow \mathbb{F}_2 \quad .$$

This means that for each 2-group P , there is a bilinear form

$$\langle \ , \ \rangle_P : \mathbb{F}_2 R_{\mathbb{Q}}^*(P) \times \mathbb{F}_2 R_{\mathbb{Q}}(P) \rightarrow \mathbb{F}_2 \quad ,$$

with the property that for any 2-group Q , for any $f \in \text{Hom}_{C_p}(P, Q)$, for any $a \in \mathbb{F}_2 R_{\mathbb{Q}}^*(P)$ and any $b \in \mathbb{F}_2 R_{\mathbb{Q}}(Q)$, one has that

$$\langle \mathbb{F}_2 R_{\mathbb{Q}}^*(f)(a), b \rangle_Q = \langle a, \mathbb{F}_2 R_{\mathbb{Q}}(f^{op})(b) \rangle_P \quad .$$

Moreover this pairing is non-degenerate : this means that for any 2-group P , the pairing $\langle \ , \ \rangle_P$ is non-degenerate. In particular, each subfunctor F of $\mathbb{F}_2 R_{\mathbb{Q}}^*$ is isomorphic to $\mathbb{F}_2 R_{\mathbb{Q}}/F^\perp$, where F^\perp is the orthogonal of F for the pairing $\langle \ , \ \rangle$.

In particular, the lattice of subfunctors of $\mathbb{F}_2 R_{\mathbb{Q}}^*$ is isomorphic to the opposite lattice of subfunctors of $\mathbb{F}_2 R_{\mathbb{Q}}$. Now since B^\times is isomorphic to a subfunctor of $\mathbb{F}_2 R_{\mathbb{Q}}$, its lattice of subfunctors is isomorphic to the opposite lattice of subfunctors of $\mathbb{F}_2 R_{\mathbb{Q}}$ containing $B^\sharp = (B^\times)^\perp$. By Theorem 4.4 of [4], any subfunctor L of $\mathbb{F}_2 R_{\mathbb{Q}}$ is equal to the sum of subfunctors H_Q it contains, where Q is a 2-group of normal 2-rank 1, and H_Q is the subfunctor of $\mathbb{F}_2 R_{\mathbb{Q}}$ generated by the image $\bar{\Phi}_Q$ of the unique (up to isomorphism) irreducible rational faithful $\mathbb{Q}\mathbb{Q}$ -module Φ_Q in $\mathbb{F}_2 R_{\mathbb{Q}}$.

In particular B^\sharp is the sum of the subfunctors H_Q , where Q is a 2-group of normal 2-rank 1 such that $\bar{\Phi}_Q \in B^\sharp(Q)$. This means that $\langle a, \bar{\Phi}_Q \rangle_Q = 0$, for any $a \in B^\times(Q)$. Now $\Phi_Q = f_1 \bar{\Phi}_Q$ since Φ_Q is faithful, so

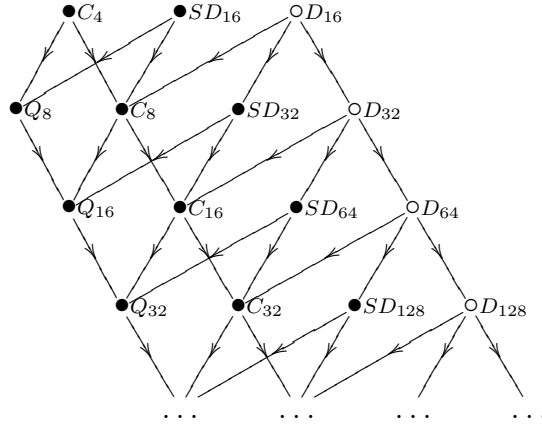
$$\langle a, \bar{\Phi}_Q \rangle_Q = \langle a, f_1^Q \bar{\Phi}_Q \rangle_Q = \langle f_1^Q a, \bar{\Phi}_Q \rangle_Q \quad ,$$

because $f_1^Q = (f_1^Q)^{op}$. Thus $\bar{\Phi}_Q \in B^\sharp(Q)$ if and only if $\bar{\Phi}_Q$ is orthogonal to $\partial B^\times(Q)$. Since Q has normal 2-rank 1, this is always the case by Corollary 5.9, except maybe if Q is trivial, cyclic of order 2, or dihedral (of order at least 16). Now $H_1 = H_{C_2} = \mathbb{F}_2 R_{\mathbb{Q}}$ by Theorem 5.6 of [4]. Since B^\times is not the zero subfunctor of $\mathbb{F}_2 R_{\mathbb{Q}}$, it follows that $H_Q \not\subseteq B^\sharp$, if Q is trivial or cyclic of order 2. Now if Q is dihedral, then Φ_Q is equal to $\mathbb{Q}Q/I - \mathbb{Q}Q/IZ$, where I is a non-central subgroup of order 2 of Q , and Z is the centre of Q . Now

$$\epsilon_Q(v_Q)(i(\bar{\Phi}_Q)) = \epsilon_Q(v_Q)(Q/I - Q/IZ) = 1 - 0 = 1 \quad ,$$

It follows that $H_Q \not\subseteq B^\sharp$ if Q is dihedral. Finally B^\sharp is the sum of all subfunctors H_Q , when Q is cyclic of order at least 4, or generalized quaternion, or semi-dihedral.

Recall from Theorem 6.2 of [4] that the poset of proper subfunctors of $\mathbb{F}_2 R_{\mathbb{Q}}$ is isomorphic to the poset of closed subsets of the following graph :



The vertices of this graph are the isomorphism classes of groups of normal 2-rank 1 and order at least 4, and there is an arrow from vertex Q to vertex R if and only if $H_R \subseteq H_Q$. The vertices with a filled \bullet are exactly labelled by the groups Q for which $H_Q \subseteq B^\sharp$, and the vertices with a \circ are labelled by dihedral groups.

By the above remarks, the lattice of subobjects of B^\times is isomorphic to the opposite lattice of subfunctors of $\mathbb{F}_2 R_{\mathbb{Q}}$ containing B^\sharp . Thus :

8.4. Theorem : *The p -biset functor B^\times is uniserial. It has an infinite strictly increasing series of proper subfunctors*

$$0 \subset L_0 \subset L_1 \cdots \subset L_n \subset \cdots$$

where L_0 is generated by the element v_1 , and L_i , for $i > 0$, is generated by the element $v_{D_{2^i+3}}$ of $B^\times(D_{2^i+3})$. The functor L_0 is isomorphic to the simple

functor $S_{\mathbf{1}, \mathbb{F}_2}$, and the quotient L_i/L_{i-1} , for $i \geq 1$, is isomorphic to the simple functor $S_{D_{2^{i+3}}, \mathbb{F}_2}$.

Proof: Indeed $L_0^\perp = B^\sharp + H_{D_{16}}$ is the unique maximal proper subfunctor of $\mathbb{F}_2 R_{\mathbb{Q}}$. Thus L_0 is isomorphic to the unique simple quotient of $\mathbb{F}_2 R_{\mathbb{Q}}$, which is $S_{\mathbf{1}, \mathbb{F}_2}$ by Proposition 5.1 of [4]. Similarly for $i \geq 1$, the simple quotient L_i/L_{i-1} is isomorphic to the quotient

$$(B^\sharp + H_{D_{2^{i+3}}}) / (B^\sharp + H_{D_{2^{i+4}}}) \quad ,$$

which is a quotient of

$$(B^\sharp + H_{D_{2^{i+3}}}) / B^\sharp \cong H_{D_{2^{i+3}}} / (B^\sharp \cap H_{D_{2^{i+3}}}) \quad .$$

But the only simple quotient of $H_{D_{2^{i+3}}}$ is $S_{D_{2^{i+3}}, \mathbb{F}_2}$, by Proposition 5.1 of [4] again. \square

8.5. Remark : Let P be a 2-group. By Theorem 5.12 of [4], the \mathbb{F}_2 -dimension of $S_{\mathbf{1}, \mathbb{F}_2}(P)$ is equal to the number of isomorphism classes of rational irreducible representations of P whose type is $\mathbf{1}$ or C_2 , whereas the \mathbb{F}_2 -dimension of $S_{D_{2^{i+3}}, \mathbb{F}_2}(P)$ is the number of isomorphism classes of rational irreducible representations of P whose type is isomorphic to $D_{2^{i+3}}$. This gives a way to recover Theorem 7.4 : the \mathbb{F}_2 -dimension of $B^\times(P)$ is equal to the number of isomorphism classes of rational irreducible representations of P whose type is trivial, cyclic of order 2, or dihedral.

8.6. The surjectivity of the exponential map. Let G be a finite group. The exponential map $\exp_G : B(G) \rightarrow B^\times(G)$ is defined in Section 7 of Yalçın's paper ([14]) by

$$\exp_G(x) = (-1) \uparrow x \quad ,$$

where $-1 = -\mathbf{1}/\mathbf{1} \in B^\times(\mathbf{1})$, and where the exponentiation

$$(y, x) \in B^\times(G) \times B(G) \rightarrow B^\times(G)$$

is defined by extending the usual exponential map $(Y, X) \mapsto Y^X$, where X and Y are G -sets, and Y^X is the set of maps from X to Y , with G -action given by $(g \cdot f)(x) = gf(g^{-1}x)$.

It is possible to give another interpretation of this map : indeed $B(G)$ is naturally isomorphic to $\text{Hom}_{\mathcal{C}}(\mathbf{1}, G)$, by considering any G -set as a $(G, \mathbf{1})$ -biset. It is clear that if X is a finite G -set, and Y is a finite set, then

$$T_X(Y) = Y^X \quad .$$

This can be extended by linearity, to show that for any $x \in B(G)$

$$(-1)^x = B^\times(x)(-1) \quad .$$

In particular the image $\text{Im}(\exp_G)$ of the exponential map \exp_G is equal to $\text{Hom}_{\mathcal{C}}(\mathbf{1}, G)(-1)$. Denoting by I the sub-biset functor of B^\times generated by $-1 \in B^\times(\mathbf{1})$, it is now clear that $\text{Im}(\exp_G) = I(G)$ for any finite group G .

Now the restriction of the functor I to the category \mathcal{C}_2 is equal to L_0 , which is isomorphic to the simple functor S_{1, \mathbb{F}_2} . Using Remark 5.13 of [4], this shows finally the following :

8.7. Proposition : *Let P be a finite 2-group. Then :*

1. *The \mathbb{F}_2 -dimension of the image of the exponential map*

$$\exp_P : B(P) \rightarrow B^\times(P)$$

is equal to the number of isomorphism classes of absolutely irreducible rational representations of P .

2. *The map \exp_P is surjective if and only if the group P has no irreducible rational representation of dihedral type, or equivalently, no genetic subgroup Q such that $N_P(Q)/Q$ is dihedral.*

8.8. Proposition : *Let p be a prime number. There is an exact sequence of p -biset functors :*

$$0 \rightarrow B^\times \rightarrow \mathbb{F}_2 R_{\mathbb{Q}}^* \rightarrow \mathbb{F}_2 D_{tors}^\Omega \rightarrow 0 \quad ,$$

where D_{tors}^Ω is the torsion part of the functor D^Ω of relative syzygies in the Dade group.

Proof: In the case $p \neq 2$, this proposition is equivalent to Proposition 8.2, because $\mathbb{F}_2 D_{tors}^\Omega = \mathbb{F}_2 D_{tors} \cong D_{tors}$ in this case. And for $p = 2$, the 2-functor D_{tors}^Ω is a quotient of the functor $R_{\mathbb{Q}}^*$, by Corollary 7.5 of [6] : there is a surjective map $\pi : R_{\mathbb{Q}}^* \rightarrow D_{tors}^\Omega$, which is the restriction to $R_{\mathbb{Q}}^*$ of the surjection $\Theta : B^* \rightarrow D^\Omega$ introduced in Theorem 1.7 of [5]. The \mathbb{F}_2 -reduction of π is a surjective map

$$\mathbb{F}_2 \pi : \mathbb{F}_2 R_{\mathbb{Q}}^* \rightarrow \mathbb{F}_2 D_{tors}^\Omega \quad .$$

To prove the proposition in this case, it is enough to show that the image of B^\times in $\mathbb{F}_2 R_{\mathbb{Q}}^*$ is contained in the kernel of $\mathbb{F}_2 \pi$, and that for any 2-group P , the \mathbb{F}_2 -dimension of $\mathbb{F}_2 R_{\mathbb{Q}}^*(P)$ is equal to the sum of the \mathbb{F}_2 -dimensions of

$B^\times(P)$ and $\mathbb{F}_2 D_{tors}^\Omega(P)$: but by Corollary 7.6 of [6], there is a group isomorphism

$$D_{tors}^\Omega(P) \cong (\mathbb{Z}/4\mathbb{Z})^{a_P} \oplus (\mathbb{Z}/2\mathbb{Z})^{b_P} \quad ,$$

where a_P is equal to the number of isomorphism classes of rational irreducible representations of P whose type is generalized quaternion, and b_P equal to the number of isomorphism classes of rational irreducible representations of P whose type is cyclic of order at least 3, or semi-dihedral. Thus

$$\dim_{\mathbb{F}_2} \mathbb{F}_2 D_{tors}^\Omega(P) = a_P + b_P \quad .$$

Now since $\dim_{\mathbb{F}_2} B^\times(P)$ is equal to the number of isomorphism classes of rational irreducible representations of P whose type is cyclic of order at most 2, or dihedral, it follows that $\dim_{\mathbb{F}_2} \mathbb{F}_2 D_{tors}^\Omega(P) + \dim_{\mathbb{F}_2} B^\times(P)$ is equal to the number of isomorphism classes of rational irreducible representations of P , i.e. to $\dim_{\mathbb{F}_2} \mathbb{F}_2 R_{\mathbb{Q}}^*(P)$.

So the only thing to check to complete the proof, is that the image of B^\times in $\mathbb{F}_2 R_{\mathbb{Q}}^*$ is contained in the kernel of $\mathbb{F}_2 \pi$. Since B^\times , $\mathbb{F}_2 R_{\mathbb{Q}}^*$ and $\mathbb{F}_2 D_{tors}^\Omega$ are rational 2-biset functors, it suffices to check that if P is a 2-group of normal 2-rank 1, and $a \in \partial B^\times(P)$, then the image of a in $\partial \mathbb{F}_2 R_{\mathbb{Q}}^*(P)$ lies in the kernel of $\mathbb{F}_2 \pi$. There is nothing to do if P is generalized quaternion, or semi-dihedral, or cyclic of order at least 3, for in this case $\partial B^\times(P) = 0$ by Corollary 5.7. Now if P is cyclic of order at most 2, then $D^\Omega(P) = \{0\}$, and the result follows. And if P is dihedral, then $D^\Omega(P)$ is torsion free by Theorem 10.3 of [9], so $D_{tors}^\Omega(P) = \{0\}$ again. \square

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