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Infinite products of 2×2 matrices and the Gibbs properties of Bernoulli convolutions

by ERIC OLIVIER & ALAIN THOMAS

Abstract.— We consider the infinite sequences $(A_n)_{n \in \mathbb{N}}$ of 2×2 matrices with nonnegative entries, where the A_n are taken in a finite set of matrices. Given a vector $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ with $v_1, v_2 > 0$, we give a necessary and sufficient condition for $\frac{A_1 \dots A_n V}{\|A_1 \dots A_n V\|}$ to converge uniformly. In application we prove that the Bernoulli convolutions related to the numeration in Pisot quadratic bases are weak Gibbs.

Key-words: Infinite products of matrices, weak Gibbs measures, Bernoulli convolutions, Pisot numbers, β -numeration.

2000 Mathematics Subject Classification: 28A12, 11A67, 15A48.

Introduction

Let $\mathcal{M} = \{M_0, \dots, M_{s-1}\}$ be a finite subset of the set – stable by matrix multiplication – of nonnegative and column-allowable $d \times d$ matrices (i.e., the matrices with nonnegative entries and without null column). We associate to any sequence $(\omega_n)_{n \in \mathbb{N}}$ with terms in $\mathcal{S} := \{0, 1, \dots, s-1\}$, the sequence of product matrices

$$P_n(\omega) = M_{\omega_1} M_{\omega_2} \dots M_{\omega_n}.$$

Experimentally, in most cases each normalized column of $P_n(\omega)$ converges when $n \rightarrow \infty$ to a limit-vector, which depends on $\omega \in \mathcal{S}^{\mathbb{N}}$ and may depend on the index of the column. Nevertheless the normalized rows of $P_n(\omega)$ in general do not converge: suppose for instance that all the matrices in \mathcal{M} are positive but do not have the same positive normalized left-eigenvector, let L_k such that $L_k M_k = \rho_k L_k$. For any positive matrix M , the normalized rows of MM_0^n converge to L_0 and the ones of MM_1^n to L_1 . Consequently we can choose the sequence $(n_k)_{k \in \mathbb{N}}$ sufficiently increasing such that the normalized rows of $M_0^{n_1} M_1^{n_2} \dots M_0^{n_{2k-1}}$ converge to L_0 while the ones of $M_0^{n_1} M_1^{n_2} \dots M_0^{n_{2k-1}} M_1^{n_{2k}}$ converge to L_1 . This proves – if $L_0 \neq L_1$ – that the normalized rows in $P_n(\omega)$ do not converge when $\omega = 0^{n_1} 1^{n_2} 0^{n_3} 1^{n_4} \dots$

Now in case \mathcal{M} is a set of positive matrices it is clear that, if both normalized columns and normalized rows in $P_n(\omega)$ converge then – after replacing each matrix M_k by $\frac{1}{\rho_k}M_k$ – the matrix $P_n(\omega)$ itself converges: the previous counterexample proves that the matrices $P_n(\omega)$ have a common left-eigenvector for any n , and a straightforward computation (using the limits of the normalized columns in $P_n(\omega)$) proves the existence of $\lim_{n \rightarrow \infty} P_n(\omega)$.

The existence of a common left-eigenvector is settled in a more general context by L. Elsner and S. Friedland ([5, Theorem 1]), in case \mathcal{M} is a finite set of matrices with entries in \mathbb{C} . This theorem means (after transposition of the matrices) that if $P_n(\omega)$ converges to a non-null limit, then there exists $N \in \mathbb{N}$ such that the matrices M_{ω_n} for $n \geq N$ have a common left-eigenvector for the eigenvalue 1. Now, L. Elsner & S. Friedland (in [5, Main Theorem]) and I. Daubechies & J. C. Lagarias (in [2, Theorem 5.1] (resp. [1, Theorem 4.2])) give necessary and sufficient conditions for $P_n(\omega)$ to converge for any $\omega \in \mathcal{S}^{\mathbb{N}}$ (resp., to converge to a continuous map).

By these theorems we see that the problem of the convergence of the normalized columns in $P_n(\omega)$ is very different from the problem of the convergence of $P_n(\omega)$ itself. Let for instance $M_0 = \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$ and $M_1 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$; then the normalized columns in

$$P_n(\omega) = \begin{cases} \frac{1}{10} \cdot \begin{pmatrix} 4 + 6 \cdot 6^{-n} & 6 - 6 \cdot 6^{-n} \\ 4 - 4 \cdot 6^{-n} & 6 + 4 \cdot 6^{-n} \end{pmatrix} & \text{if } \omega_1 \dots \omega_n = 0 \dots 0 \\ \frac{1}{10} \cdot \begin{pmatrix} 4 + 6^{-h} & 6 - 6^{-h} \\ 4 + 6^{-h} & 6 - 6^{-h} \end{pmatrix} & \text{if } \omega_1 \dots \omega_n = \omega_1 \dots \omega_{n-h-1} 10 \dots 0 \end{cases}$$

converge to $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ for any $\omega \in \{0, 1\}^{\mathbb{N}}$, but $P_n(\omega)$ diverges (although it is bounded) if ω is not eventually constant.

In Section 1 we study the uniform convergence – in direction – of $P_n(\omega)V$ in case the M_k are 2×2 nonnegative column-allowable matrices and $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ a positive vector (Theorem 1.1). Notice that the convergence in direction of the columns of $P_n(\omega)$, to a same vector, implies the ones of $P_n(\omega)V$, but the converse is not true: see for instance the case $\mathcal{M} = \left\{ \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \right\}$.

The second section is devoted to the *Bernoulli convolutions* [4], which have been studied since the early 1930's (see [8] for the other references). We give a matricial relation for such measures.

In the third section we apply more precisely Theorem 1.1 to prove that certain Bernoulli convolutions are weak Gibbs in the following sense (see [10]): given a system of affine contractions $\mathbb{S}_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ such that the intervals $\mathbb{S}_\varepsilon([0, 1])$ make a partition of $[0, 1]$ for $\varepsilon \in \mathcal{S} = \{0, 1, \dots, s-1\}$, a measure η supported by $[0, 1]$ is weak Gibbs w.r.t. $\{\mathbb{S}_\varepsilon\}_{\varepsilon=0}^{s-1}$ if there exists a map $\Phi : \mathcal{S}^{\mathbb{N}} \rightarrow \mathbb{R}$, continuous for the product topology, such that

$$\lim_{n \rightarrow \infty} \left(\frac{\eta[\xi_1 \dots \xi_n]}{\exp \left(\sum_{k=0}^{n-1} \Phi(\sigma^k \xi) \right)} \right)^{1/n} = 1 \quad \text{uniformly on } \xi \in \mathcal{S}^{\mathbb{N}}, \quad (1)$$

where $[\xi_1 \dots \xi_n] := \mathbb{S}_{\xi_1} \circ \dots \circ \mathbb{S}_{\xi_n}([0, 1])$ and σ is the shift on $\mathcal{S}^{\mathbb{N}}$. Let us give a sufficient condition for η to be weak Gibbs. For each $\xi \in \mathcal{S}^{\mathbb{N}}$ we put $\phi_1(\xi) = \log \eta[\xi_1]$ and for $n \geq 2$,

$$\phi_n(\xi) = \log \left(\frac{\eta[\xi_1 \dots \xi_n]}{\eta[\xi_2 \dots \xi_n]} \right). \quad (2)$$

The continuous map $\phi_n : \mathcal{S}^{\mathbb{N}} \rightarrow \mathbb{R}$ ($n \geq 1$) is the n -step potential of η . Assume the existence of the uniform limit $\Phi = \lim_{n \rightarrow \infty} \phi_n$; it is then straightforward that for $n \geq 1$,

$$\frac{1}{K_n} \leq \frac{\eta[\xi_1 \dots \xi_n]}{\exp \left(\sum_{k=0}^{n-1} \Phi(\sigma^k \xi) \right)} \leq K_n \quad \text{with} \quad K_n = \exp \left(\sum_{k=1}^n \|\Phi - \phi_n\|_\infty \right). \quad (3)$$

By a well known lemma on the Cesàro sums, K_1, K_2, \dots form a subexponential sequence of positive real numbers, that is $\lim_{n \rightarrow \infty} (K_n)^{1/n} = 1$ and thus, (3) means η is weak Gibbs w.r.t. $\{\mathbb{S}_\varepsilon\}_{\varepsilon=0}^{s-1}$.

Now the weak Gibbs property can be proved for certain Bernoulli convolutions by computing the n -step potential by means of products of matrices (see [6] for the Bernoulli convolution associated with the golden ratio $\beta = \frac{1 + \sqrt{5}}{2}$ – called the *Erdős measure* – and the application to the multifractal analysis). In Theorem 3.1 we generalize this result in case $\beta > 1$ is a quadratic number with conjugate $\beta' \in]-1, 0[$.

1 Infinite product of 2×2 matrices

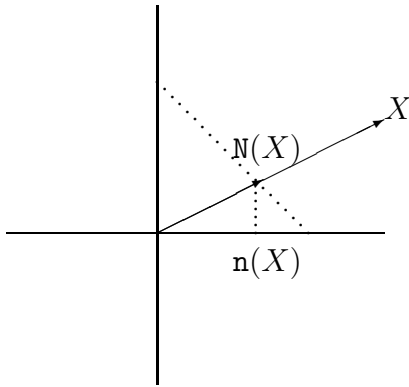
From now the vectors $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and the matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we consider are supposed to be nonnegative and column-allowable that is, x_1, x_2, a, b, c, d are nonnegative

and $x_1 + x_2, a + c, b + d$ are positive. In particular we suppose that the matrices in $\mathcal{M} = \{M_0, \dots, M_{s-1}\}$ satisfy these conditions. We associate to X the normalized vector:

$$\mathbf{N}(X) := \begin{pmatrix} \frac{x_1}{x_1+x_2} \\ \frac{x_2}{x_1+x_2} \end{pmatrix} = \begin{pmatrix} \mathbf{n}(X) \\ 1 - \mathbf{n}(X) \end{pmatrix} \quad \text{where} \quad \mathbf{n}(X) := \frac{x_1}{x_1 + x_2}$$

and define the distance between the column of A (or the rows of tA):

$$d_{\text{columns}}(A) := \left| \mathbf{n} \left(\begin{pmatrix} a \\ c \end{pmatrix} \right) - \mathbf{n} \left(\begin{pmatrix} b \\ d \end{pmatrix} \right) \right| = \frac{|\det A|}{(a+c)(b+d)} =: d_{\text{rows}}({}^tA).$$



THEOREM 1.1 Given $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ with $v_1, v_2 > 0$, the sequence of vectors $\mathbf{N}(P_n(\omega)V)$ converges uniformly for $\omega \in \mathcal{S}^{\mathbb{N}}$ only in the five following cases:

Case 1: $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{M} \Rightarrow a \geq d$; $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \mathcal{M} \Rightarrow a \leq d$;

$\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}, \begin{pmatrix} 0 & b' \\ c' & d' \end{pmatrix} \in \mathcal{M} \Rightarrow bc' \geq b'c$; no matrix in \mathcal{M} has the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ or $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$.

Case 2: $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{M} \Rightarrow a < d$; $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \mathcal{M} \Rightarrow a > d$;

$\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}, \begin{pmatrix} 0 & b' \\ c' & d' \end{pmatrix} \in \mathcal{M} \Rightarrow bc' < b'c$.

Case 3: $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{M} \Rightarrow a \geq d$; $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \mathcal{M} \Rightarrow a > d$;

no matrix in \mathcal{M} has the form $\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$.

Case 4: $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{M} \Rightarrow a < d$; $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \mathcal{M} \Rightarrow a \leq d$;

no matrix in \mathcal{M} has the form $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$;

Case 5: V is an eigenvector of all the matrices in \mathcal{M} .

COROLLARY 1.2 *If $\mathbb{N}(P_n(\cdot)V)$ converges uniformly on $\mathcal{S}^{\mathbb{N}}$, the limit do not depend on the positive vector V , except in the fifth case of Theorem 1.1.*

Proof. Suppose that \mathcal{M} satisfies the conditions of the case 1,2,3 or 4 in Theorem 1.1 and let V, W be two positive vectors. Then the following set \mathcal{M}' also do:

$\mathcal{M}' := \mathcal{M} \cup \{M_s\}$, where M_s is the matrix whose both columns are W .

Denoting by $\omega' = \omega_1 \dots \omega_n \bar{s}$ the sequence defined by $\omega'_i = \begin{cases} \omega_i & \text{if } i \leq n \\ s & \text{if } i > n \end{cases}$ one has for any $\omega \in \mathcal{S}^{\mathbb{N}}$

$$\mathbb{N}(P_n(\omega)V) - \mathbb{N}(P_n(\omega)W) = \mathbb{N}(P_n(\omega')V) - \mathbb{N}(P_{n+1}(\omega')V)$$

and this tends to 0, according to the uniform Cauchy property of the sequence $\mathbb{N}(P_n(\cdot)V)$. ■

Nevertheless, this limit may depend of V if one assume only that V is nonnegative. For instance, if $\mathcal{M} = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ then $\lim_{n \rightarrow \infty} \mathbb{N} \left(P_n(\omega) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ differs from $\lim_{n \rightarrow \infty} \mathbb{N} \left(P_n(\omega) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$ iff $\omega = \bar{0}$ (implying the second limit is not uniform on $\mathcal{S}^{\mathbb{N}}$).

1.1 Geometric considerations

We follow the ideas of E. Seneta about products of nonnegative matrices in Section 3 of [9], or stochastic matrices in Section 4. In what follows we denote the matrices by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ or $A_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ for $n \in \mathbb{N}$, and we suppose they are nonnegative and column-allowable. We define the coefficient

$$\tau(A) := \sup_{d_{\text{columns}}(A') \neq 0} \frac{d_{\text{columns}}(A'A)}{d_{\text{columns}}(A')}.$$

The straightforward formula

$$d_{\text{columns}} \left(\prod_{k=1}^n (A_k) \right) \leq d_{\text{columns}}(A_1) \prod_{k=2}^n \tau(A_k) \quad (4)$$

is of use to prove Theorem 1.1 because, according to the following proposition one has $\tau(A) < 1$ if A is positive.

PROPOSITION 1.3

$$\tau(A) = \begin{cases} \frac{|\sqrt{ad} - \sqrt{bc}|}{\sqrt{ad} + \sqrt{bc}} & \text{if } A \text{ do not have any null row} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. $\frac{d_{\text{columns}}(A'A)}{d_{\text{columns}}(A')} = \frac{|\det A|}{(a + c/x)(bx + d)}$ (where $x = \frac{a' + c'}{b' + d'}$) is maximal for $x = \sqrt{\frac{cd}{ab}}$. ■

REMARK 1.4 One can consider – instead of d_{columns} – the angle between the columns of A :

$$\alpha(A) := \left| \arctan \frac{a}{c} - \arctan \frac{b}{d} \right|,$$

or the Hilbert distance between the columns of a positive matrix A :

$$d_{\text{Hilbert}}(A) := \left| \log \frac{a}{c} - \log \frac{b}{d} \right|.$$

This last can be interpreted either as the distance between the columns or the rows of A , because $d_{\text{Hilbert}}(A) = d_{\text{Hilbert}}({}^t A)$. The Birkhoff coefficient $\tau_{\text{Birkhoff}}(A) := \sup_{d_{\text{Hilbert}}(A') \neq 0} \frac{d_{\text{Hilbert}}(A'A)}{d_{\text{Hilbert}}(A')}$ has – from [9, Theorem (3.12)] – the same value as $\tau(A)$ in Proposition 1.3, and probably as a large class of coefficients defined in this way.

In the following proposition we list the properties of d_{columns} that are required for proving Theorem 1.1.

PROPOSITION 1.5 (i) $\sup_{d_{\text{columns}}(A') \neq 0} \frac{d_{\text{columns}}(AA')}{d_{\text{columns}}(A')} = \frac{|\det A|}{\min((a+c)^2, (b+d)^2)} =: \tau_1(A)$.

(ii) If A is positive then $\sup_{d_{\text{columns}}(A') \neq 0} \frac{d_{\text{rows}}(AA')}{d_{\text{columns}}(A')} \leq \frac{|\det A|}{\min(a, b) \cdot \min(c, d)} =: \tau_2(A)$.

(iii) If $\lim_{n \rightarrow \infty} d_{\text{columns}}(A_n) = 0$ then $\lim_{n \rightarrow \infty} d_{\text{columns}}(AA_n A') = 0$ and, assuming that A is positive, $\lim_{n \rightarrow \infty} d_{\text{rows}}(AA_n A') = 0$.

(iv) Suppose the matrices A_n are upper-triangular. If $\inf_{k \in \mathbb{N}} \frac{a_k}{d_k} \geq 1$ and $\sum_{k \in \mathbb{N}} \frac{b_k}{d_k} = \infty$ then

$$\lim_{n \rightarrow \infty} d_{\text{columns}}(A_1 \dots A_n) = \lim_{n \rightarrow \infty} d_{\text{columns}}(A_n \dots A_1) = 0.$$

(v) Suppose the matrices A_n are lower-triangular. If $\inf_{k \in \mathbb{N}} \frac{d_k}{a_k} \geq 1$ and $\sum_{k \in \mathbb{N}} \frac{c_k}{a_k} = \infty$ then

$$\lim_{n \rightarrow \infty} d_{\text{columns}}(A_1 \dots A_n) = \lim_{n \rightarrow \infty} d_{\text{columns}}(A_n \dots A_1) = 0.$$

Proof. (i) and (ii) are obtained from the formula

$$d_{\text{columns}}(AA') = \frac{\det A \cdot \det A'}{((a+c)a' + (b+d)c') \cdot ((a+c)b' + (b+d)d')},$$

and the relation $d_{\text{rows}}(AA') = d_{\text{columns}}({}^t A' {}^t A)$.

(iii) is due to the fact that the inequalities of items (i), (ii) and (4) imply $d_{\text{columns}}(AA_n A') \leq \tau_1(A)d_{\text{columns}}(A_n)\tau(A')$ and – if A is positive – $d_{\text{rows}}(AA_n A') \leq \tau_2(A)d_{\text{columns}}(A_n)\tau(A')$.

(iv) follows from the formula

$$A_1 \dots A_n = \begin{pmatrix} a_1 \dots a_n & s_n \\ 0 & d_1 \dots d_n \end{pmatrix},$$

where $s_n = \sum_{k=1}^n a_1 \dots a_{k-1} b_k d_{k+1} \dots d_n \geq d_1 \dots d_n \sum_{k=1}^n \frac{b_k}{d_k}$.

(v) can be deduced from (iv) by using the relation $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ 0 & a \end{pmatrix}$. ■

We need also the following:

PROPOSITION 1.6 *Let V_A be a nonnegative eigenvector associated to the maximal eigenvalue of A , and C a cone of nonnegative vectors containing V_A . If $\det A \geq 0$ then C is stable by left-multiplication by A .*

Proof. The discriminant of the characteristic polynomial of A is $(a-d)^2 + 4bc$. In case this discriminant is null the proof is obtained by direct computation, because $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ or $\begin{pmatrix} a & 0 \\ c & a \end{pmatrix}$. Otherwise A has two eigenvalues $\lambda > \lambda'$ and, given a nonnegative vector X , there exists a real α and an eigenvector W_A (associated to λ') such that

$$X = \alpha V_A + W_A \quad \text{and} \quad AX = \lambda \alpha V_A + \lambda' W_A = \lambda' X + (\lambda - \lambda') \alpha V_A.$$

Notice that $\alpha \geq 0$ (because the nonnegative vector $A^n X = \lambda^n \alpha V_A + \lambda'^n W_A$ converges in direction to αV_A) and $\lambda' \geq 0$ (from the hypothesis $\det A \geq 0$). Hence AX is a nonnegative linear combination of X and V_A ; if X belongs to C then AX also do. ■

1.2 How pointwise convergence implies uniform convergence

Let m and M be the bounds of $\mathbf{n}(P_n(\omega)V)$ for $n \in \mathbb{N}$ and $\omega \in \mathcal{S}^{\mathbb{N}}$, and let $M_V := \begin{pmatrix} m & M \\ 1-m & 1-M \end{pmatrix}$. Each real $x \in [m, M]$ can be written $x = mx_1 + Mx_2$ with $x_1, x_2 \geq 0$

and $x_1 + x_2 = 1$; in particular the real $x = \mathbf{n}(P_n(\omega)V)$ can be written in this form, hence

$$\forall \omega \in \mathcal{S}^{\mathbb{N}}, \exists t_1, t_2 \geq 0, P_n(\omega)V = M_V \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}. \quad (5)$$

PROPOSITION 1.7 *If $d_{\text{columns}}(P_n(\cdot)M_V)$ converges pointwise to 0 when $n \rightarrow \infty$, then $\mathbb{N}(P_n(\cdot)V)$ converges uniformly on $\mathcal{S}^{\mathbb{N}}$.*

Proof. Suppose the pointwise convergence holds. Given $\omega \in \mathcal{S}^{\mathbb{N}}$ and $\varepsilon > 0$, there exists the integer $n = n(\omega, \varepsilon)$ such that $d_{\text{columns}}(P_n(\omega)M_V) \leq \varepsilon$. The family of cylinders $C(\omega, \varepsilon) := \llbracket \omega_1 \dots \omega_{n(\omega, \varepsilon)} \rrbracket$, for ω running over $\mathcal{S}^{\mathbb{N}}$, is a covering of the compact $\mathcal{S}^{\mathbb{N}}$; hence there exists a finite subset $X \subset \mathcal{S}^{\mathbb{N}}$ such that $\mathcal{S}^{\mathbb{N}} = \bigcup_{\omega \in X} C(\omega, \varepsilon)$. Let $p > q \geq n_\varepsilon := \max\{n(\omega, \varepsilon) ; \omega \in X\}$. For any $\xi \in \mathcal{S}^{\mathbb{N}}$, there exists $\zeta \in X$ such that $\xi \in C(\zeta, \varepsilon)$ that is, $\xi_k = \zeta_k$ for any $k \leq n = n(\zeta, \varepsilon)$. From (5) there exists two nonnegative vectors V_p and V_q such that $P_p(\xi)V = P_n(\zeta)M_V V_p$ and $P_q(\xi)V = P_n(\zeta)M_V V_q$. Denoting by $M(p, q)$ the column-allowable matrix whose columns are V_p and V_q we have – in view of (4)

$$\begin{aligned} |\mathbf{n}(P_p(\xi)V) - \mathbf{n}(P_q(\xi)V)| &= d_{\text{columns}}(P_n(\zeta)M_V M(p, q)) \\ &\leq d_{\text{columns}}(P_n(\zeta)M_V) \\ &\leq \varepsilon, \end{aligned}$$

implying the uniform Cauchy property for $\mathbb{N}(P_n(\cdot)V)$. ■

1.3 Proof of the uniform convergence of $\mathbb{N}(P_n(\cdot)V)$

According to Proposition 1.7 it is sufficient to prove that $\lim_{n \rightarrow \infty} d_{\text{columns}}(P_n(\omega)M_V) = 0$ for each $\omega \in \mathcal{S}^{\mathbb{N}}$. This convergence is obvious in the following cases:

- If there exists N such that M_{ω_N} has rank 1, then $P_n(\omega)M_V$ has rank 1 for $n \geq N$ and

$$\forall n \geq N, d_{\text{columns}}(P_n(\omega)M_V) = 0.$$

- If there exists infinitely many integers n such that M_{ω_n} is a positive matrix, one has $\tau(M_{\omega_n}) \leq \rho := \max_{M \in \mathcal{M}, M > 0} \tau(M) < 1$, and the formula (4) implies

$$\lim_{n \rightarrow \infty} d_{\text{columns}}(P_n(\omega)M_V) = 0.$$

- Similarly, this limit is null also in case there exists infinitely many integers n such that $M_{\omega_n} M_{\omega_{n+1}}$ is a positive matrix.

So we can make from now the following hypotheses on the sequence ω under consideration:

(H): $\det M_{\omega_n} \neq 0$ for any $n \in \mathbb{N}$, and there exists N such that the matrix $M_{\omega_n} M_{\omega_{n+1}}$ has at least one null entrie for any $n > N$.

Proof in the case 1: Since the couples of matrices $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}, \begin{pmatrix} 0 & b' \\ c' & d' \end{pmatrix} \in \mathcal{M}$ satisfy $\frac{b}{c} \geq \frac{b'}{c'}$, there exists a real α such that

$$\forall \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}, \begin{pmatrix} 0 & b' \\ c' & d' \end{pmatrix} \in \mathcal{M}, \quad \frac{b}{c} \geq \alpha \geq \frac{b'}{c'}.$$

Let $\Delta = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$. We denote by \mathcal{P} the set of 2×2 matrices with nonnegative determinant and by $\tilde{\mathcal{M}}$ the subset of \mathcal{P} defined as follows:

$$\tilde{\mathcal{M}} := \{\Delta^{-1}M, M\Delta ; M \in \mathcal{M} \setminus \mathcal{P}\} \cup \{M, \Delta^{-1}M\Delta ; M \in \mathcal{M} \cap \mathcal{P}\}.$$

This set of matrices also satisfies the conditions mentionned in the case 1: for instance if $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \in \mathcal{M}$, the matrix $\Delta^{-1} \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} = \begin{pmatrix} c & 0 \\ a/\alpha & b/\alpha \end{pmatrix}$ satisfies $c \leq b/\alpha$, and so one. For any sequence $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots$ of elements of $\{0, 1\}$ such that $\varepsilon_0 = 0$ we can write

$$\begin{aligned} P_n(\omega) &= M_{\omega_1} M_{\omega_2} \dots M_{\omega_n} \\ &= (\Delta^{-\varepsilon_0} M_{\omega_1} \Delta^{\varepsilon_1}) \cdot (\Delta^{-\varepsilon_1} M_{\omega_2} \Delta^{\varepsilon_2}) \dots \cdot (\Delta^{-\varepsilon_{n-1}} M_{\omega_n} \Delta^{\varepsilon_n}) \cdot \Delta^{-\varepsilon_n} \\ &= A_1 A_2 \dots A_n \Delta^{-\varepsilon_n} \end{aligned} \quad (6)$$

where $A_n := \Delta^{-\varepsilon_{n-1}} M_{\omega_n} \Delta^{\varepsilon_n}$ for any $n \in \mathbb{N}$. By the following choice of the sequence $(\varepsilon_n)_{n \in \mathbb{N}}$, the matrices A_n belong to $\tilde{\mathcal{M}}$:

$$\varepsilon_n = \begin{cases} \varepsilon_{n-1} & \text{if } \det M_{\omega_n} > 0 \\ 1 - \varepsilon_{n-1} & \text{otherwise.} \end{cases}$$

The hypotheses (H) imply that either all the matrices A_n for $n > N$ are upper-triangular, or all of them are lower-triangular (otherwise $M_{\omega_n} M_{\omega_{n+1}} = \Delta^{\varepsilon_{n-1}} A_n A_{n+1} \Delta^{-\varepsilon_{n+1}}$ is positive for some $n > N$). By Proposition 1.5 (iv) and (v),

$$\lim_{n \rightarrow \infty} d_{\text{columns}}(A_{N+1} \dots A_n) = 0.$$

From (6) and Proposition 1.5 (iii), $\lim_{n \rightarrow \infty} d_{\text{columns}}(P_n(\omega) M_V) = 0$.

Proof in the case 2: We use the matrix Δ and the set of matrices $\tilde{\mathcal{M}}$ defined in the previous case; here the real α is supposed such that $\frac{b}{c} \leq \alpha \leq \frac{b'}{c'}$ for any $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}, \begin{pmatrix} 0 & b' \\ c' & d' \end{pmatrix} \in \mathcal{M}$,

and consequently $\tilde{\mathcal{M}}$ satisfies the hypotheses of the case 2. This imply that each matrix in $\tilde{\mathcal{M}}$ has a positive eigenvector. Let C be the (minimal) cone containing V , $\Delta^{-1}V$ and the positive eigenvectors of the matrices in $\tilde{\mathcal{M}}$. From (6) and Proposition 1.6, $P_n(\omega)V$ belongs to this cone for any $\omega \in \mathcal{S}^{\mathbb{N}}$ hence M_V is positive.

Using again the relation (6) we have

$$d_{\text{columns}}(P_n(\omega)M_V) = d_{\text{rows}}({}^tM_V {}^t\Delta^{-\varepsilon_n} {}^tA_n \dots {}^tA_1). \quad (7)$$

Each matrix tA_n for $n > N$ satisfy $a > d$ if tA_n is upper-triangular, and $a < d$ if it is lower-triangular. By Proposition 1.5 (iv) and (v), $\lim_{n \rightarrow \infty} d_{\text{columns}}({}^tA_n \dots {}^tA_{N+1}) = 0$. This implies $\lim_{n \rightarrow \infty} d_{\text{columns}}(P_n(\omega)M_V) = 0$ by applying Proposition 1.5 (iii) to the r.h.s. in (7).

Proof in the case 3: Let C' be the (minimal) cone containing V , the nonnegative eigenvectors (associated to the maximal eigenvalues) of the matrices in $\mathcal{M} \cap \mathcal{P}$, and the column-vectors of the matrices in $\mathcal{M} \setminus \mathcal{P}$. All the vectors delimiting C' are distinct from $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and Proposition 1.6 implies that $P_n(\omega)V \in C'$ for any $\omega \in \mathcal{S}^{\mathbb{N}}$. Hence m and M that is, the bounds of $\mathfrak{n}(P_n(\omega)V)$, are positive.

Suppose first that M_{ω_n} is lower-triangular for any $n \in \mathbb{N}$ and let $\begin{pmatrix} \alpha_n & 0 \\ \gamma_n & \delta_n \end{pmatrix} = P_n(\omega)$. The hypotheses of the case 3 imply $\lim_{n \rightarrow \infty} \frac{\delta_n}{\alpha_n} = 0$. A simple computation gives $d_{\text{columns}}(P_n(\omega)M_V) \leq \frac{\delta_n}{\alpha_n} \cdot \frac{M-m}{Mm}$ hence $\lim_{n \rightarrow \infty} d_{\text{columns}}(P_n(\omega)M_V) = 0$. This conclusion remains valid if M_{ω_n} is eventually lower-triangular.

Suppose now M_{ω_n} is not lower-triangular for infinitely many n . The hypotheses mentioned in the case 3 and (H) imply that M_{ω_n} is upper-triangular for any $n > N$ (because for each n such that M_{ω_n} is lower-triangular or has the form $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$, (H) implies that $M_{\omega_{n+1}}$ is lower-triangular). Proposition 1.5 (iii) and (iv) implies that $\lim_{n \rightarrow \infty} d_{\text{columns}}(P_n(\omega)M_V) = 0$.

Proof in the case 4: Let \mathcal{M}' be the set of matrices $M'_k = \Delta^{-1}M_k\Delta$ for $k = 0, \dots, \mathfrak{s}-1$, and let $V' = \Delta^{-1}V$ (here we can choose $\Delta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$). The set \mathcal{M}' satisfies the hypotheses of the case 3 hence $\lim_{n \rightarrow \infty} d_{\text{columns}}(P_n(\omega)M_V) = \lim_{n \rightarrow \infty} d_{\text{columns}}(\Delta M'_{\omega_1} \dots M'_{\omega_n} V') = 0$.

1.4 Proof of the converse assertion in Theorem 1.1

Now we suppose the existence of the uniform limit $V(\cdot) := \lim_{n \rightarrow \infty} \mathbf{N}(P_n(\cdot)V)$ and we want to check the conditions contained in one of the five cases involved in Theorem 1.1. Let \mathcal{M}^2 be the set of matrices MM' for $M, M' \in \mathcal{M}$, and let \mathcal{U} (resp. \mathcal{L}) be the set of upper-triangular (resp. lower-triangular) matrices $M \in \mathcal{M} \cup \mathcal{M}^2$.

We first prove that \mathcal{U} cannot contain a couple of matrices $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and $A' = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}$ such that $a \geq d$ and $a' < d'$: suppose that \mathcal{U} contain such matrices let, for simplicity, $M_0 = A$ and $M_1 = A'$. One has $V(\bar{0}) = \lim_{n \rightarrow \infty} \mathbf{N}(A^n V)$, and this limit is also the normalized nonnegative right-eigenvector of A associated to its maximal eigenvalue, hence $V(\bar{0}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Similarly, $V(\bar{1})$ is colinear to $\begin{pmatrix} b' \\ d' - a' \end{pmatrix}$ (eigenvector of A') hence distinct from $V(\bar{0})$. Moreover, for fixed $N \in \mathbb{N}$

$$\begin{aligned} V(1^N \bar{0}) &= \lim_{n \rightarrow \infty} \mathbf{N}(A'^N A^n V) = \lim_{n \rightarrow \infty} \mathbf{N}(A'^N \mathbf{N}(A^n V)) \\ &= \mathbf{N}(A'^N V(\bar{0})) \\ &= V(\bar{0}). \end{aligned} \tag{8}$$

Since $1^N \bar{0}$ tends to $\bar{1}$ when $N \rightarrow \infty$, the inequality $V(\bar{0}) \neq V(\bar{1})$ contradicts the continuity of the map V . This proves that the couple of matrices $A, A' \in \mathcal{U}$ such that $a \geq d$ and $a' < d'$ do not exist. Similarly, the couple of matrices $A, A' \in \mathcal{L}$ such that $a \leq d$ and $a' > d'$ do not exist.

- Suppose that all the matrices in \mathcal{U} satisfy $a \geq d$ and all the ones in \mathcal{L} satisfy $a \leq d$. If \mathcal{M} contains a matrix of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, it is necessarily an homothetic matrix. If it contains a matrix of the form $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$, the square of this matrix is homothetic. So in both cases $\mathcal{M} \cup \mathcal{M}^2$ contains an homothetic matrix, let H . We use the same method as above: since the map V is continuous, the vector $\lim_{n \rightarrow \infty} \mathbf{N}(H^n V)$ must be equal to $\lim_{N \rightarrow \infty} \left(\lim_{n \rightarrow \infty} \mathbf{N}(H^N M H^n V) \right)$ for any $M \in \mathcal{M}$. But the first is $\mathbf{N}(V)$ and the second $\mathbf{N}(MV)$, hence V is an eigenvector of all the matrices in \mathcal{M} . Suppose now that \mathcal{M} do not contain matrices of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ nor $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$: all the conditions of the case 1 are satisfied.
- Suppose that all the matrices in \mathcal{U} satisfy $a < d$ and all the ones in \mathcal{L} satisfy $a > d$; then the conditions of the case 2 are satisfied.

- Suppose that all the matrices $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{U}$ satisfy $a \geq d$ and all the matrices $A' = \begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix} \in \mathcal{L}$ satisfy $a' > d'$. If there exists $A \in \mathcal{U}$, $A' \in \mathcal{L}$ and $M = \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix} \in \mathcal{M}$, the map V is discontinuous because

$$\lim_{n \rightarrow \infty} \mathbb{N} \left(A'^n M A^n \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) = \lim_{n \rightarrow \infty} \mathbb{N} \left(\begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix}^n \begin{pmatrix} \beta & 0 \\ \delta & \gamma \end{pmatrix} \begin{pmatrix} d & 0 \\ b & a \end{pmatrix}^n \begin{pmatrix} v_2 \\ v_1 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

differs from $\lim_{n \rightarrow \infty} \mathbb{N} \left(A'^n \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)$ which is colinear to $\begin{pmatrix} a' - d' \\ c' \end{pmatrix}$. Hence, either \mathcal{M} do not contain a matrix of the form $\begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}$ and we are in the case 3, or $\mathcal{U} = \emptyset$ and we are in the case 2, or $\mathcal{L} = \emptyset$ and we are in the case 1.

- The case when all the matrices $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{U}$ satisfy $a < d$ and all the matrices $A' = \begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix} \in \mathcal{L}$ satisfy $a' \leq d'$ is symmetrical to the previous, by using the set of matrices $\mathcal{M}' := \{\Delta^{-1} M \Delta ; M \in \mathcal{M}\}$ and the vector $V' = \Delta^{-1} V$, where $\Delta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

2 Some properties of the Bernoulli convolutions in base $\beta > 1$

Given a real $\beta > 1$, an integer $d > \beta$ and a d -dimensional probability vector $\mathbf{p} := (p_i)_{i=0}^{d-1}$, the \mathbf{p} -distributed (β, d) -Bernoulli convolution is by definition the probability distribution $\mu_{\mathbf{p}}$ of the random variable X defined by

$$\forall \omega \in \mathcal{D}^{\mathbb{N}} := \{0, \dots, d-1\}^{\mathbb{N}}, \quad X(\omega) = \sum_{k=1}^{\infty} \frac{\omega_k}{\beta^k},$$

where $\omega \mapsto \omega_k$ ($k = 1, 2, \dots$) is a sequence of i.i.d. random variables assuming the values $i = 0, 1, \dots, d-1$ with probability p_i .

Denoting by $\bar{\omega}$ the sequence such that $\bar{\omega}_k = d-1 - \omega_k$ for any k , one has the relation $X(\omega) + X(\bar{\omega}) = \alpha := \frac{d-1}{\beta-1}$. Hence, setting $\bar{p}_i = p_{d-1-i}$ for any $i = 0, 1, \dots, d-1$, the following symmetry relation holds for any Borel set $B \subset \mathbb{R}$:

$$\mu_{\mathbf{p}}(B) = \mu_{\bar{\mathbf{p}}}(\alpha - B) \tag{9}$$

(notice that the support of $\mu_{\mathbf{p}}$ is a subset of $[0, \alpha]$).

The measure $\mu_{\mathbf{p}}$ also satisfy the following selfsimilarity relation: denoting by σ the shift on $\mathcal{D}^{\mathbb{N}}$ one has – for any Borel set $B \subset \mathbb{R}$

$$X(\omega) \in B \Leftrightarrow X(\sigma\omega) \in (\beta B - \omega_0)$$

hence, using the independance of the random variables $\omega \mapsto \omega_k$,

$$\mu_{\mathbf{p}}(B) = \sum_{k=0}^{\mathfrak{d}-1} \mathbf{p}_k \cdot \mu_{\mathbf{p}}(\beta B - k) \quad \text{for any Borel set } B \subset \mathbb{R} \quad (10)$$

and in particular

$$\mu_{\mathbf{p}}(B) = \mathbf{p}_0 \cdot \mu_{\mathbf{p}}(\beta B) \quad \text{if } \beta B \subset [0, 1]. \quad (11)$$

The following proposition is proved in [3, Theorem 2.1 and Proposition 5.4] in case the probability vector \mathbf{p} is uniform:

PROPOSITION 2.1 *The 1-periodic map $H :] - \infty, 0] \rightarrow \mathbb{R}$ defined by*

$$H(x) = (\mathbf{p}_0)^x \cdot \mu_{\mathbf{p}}([0, \beta^x])$$

is continuous and a.e. differentiable. Moreover H is not differentiable on a certain continuum of points if β is an irrational Pisot number or an integer and – in this latter case – if β do not divide \mathfrak{d} .

Let us give also the matricial form of the relation (10) (from [7, §2.1]). We define the (finite or countable) set $\mathcal{I}_{(\beta, \mathfrak{d})} = \{0 = \mathfrak{i}_0, \mathfrak{i}_1, \dots\}$ as follows (where \mathcal{B} is the alphabet $\{0, 1, \dots, \mathfrak{b} - 1\}$ such that $\mathfrak{b} - 1 < \beta \leq \mathfrak{b}$):

DEFINITION 2.2 $\mathcal{I}_{(\beta, \mathfrak{d})}$ *is the set of $i \in] - 1, \alpha[$ for which there exists $-1 < i_1, \dots, i_n < \alpha$ with $0 \triangleright i_1 \triangleright \dots \triangleright i_n \triangleright i$, where $x \triangleright y$ means that exists $(\varepsilon, k) \in \mathcal{B} \times \mathcal{D}$ such that $y = \beta x + (\varepsilon - k)$.*

Let $\varepsilon \in \mathcal{B}$; the entries of the matrix M_{ε} are – for the row index i and the column index j , with $\mathfrak{i}_i, \mathfrak{i}_j \in \mathcal{I}_{(\beta, \mathfrak{d})}$,

$$M_{\varepsilon}(i, j) = \begin{cases} \mathbf{p}_k & \text{if } k = \varepsilon + \beta \mathfrak{i}_i - \mathfrak{i}_j \in \mathcal{D} \\ 0 & \text{otherwise.} \end{cases}$$

Setting $\mathbb{R}_{\varepsilon}(x) = \frac{x + \varepsilon}{\beta}$ for any $\varepsilon \in \mathcal{B}$ and $x \in \mathbb{R}$, we have the following

PROPOSITION 2.3 ([7, Lemma 2.2]) If $\mathcal{I}_{(\beta, \mathbf{d})} = \{\mathbf{i}_0, \dots, \mathbf{i}_{r-1}\}$ then, for any Borel set $B \subset [0, 1]$ and any $\varepsilon \in \mathcal{B}$ such that $\mathbb{R}_\varepsilon^{-1}(B) \subset [0, 1]$,

$$\begin{pmatrix} \mu_{\mathbf{p}}(B + \mathbf{i}_0) \\ \vdots \\ \mu_{\mathbf{p}}(B + \mathbf{i}_{r-1}) \end{pmatrix} = M_\varepsilon \begin{pmatrix} \mu_{\mathbf{p}}(\mathbb{R}_\varepsilon^{-1}(B) + \mathbf{i}_0) \\ \vdots \\ \mu_{\mathbf{p}}(\mathbb{R}_\varepsilon^{-1}(B) + \mathbf{i}_{r-1}) \end{pmatrix}.$$

REMARK 2.4 The finiteness of $\mathcal{I}_{(\beta, \mathbf{d})}$ is assured, according to [7, §2.2], if β is an irrational Pisot number or an integer.

We shall use also the probability distribution of the fractionnal part of the random variable X , that we denote by $\mu_{\mathbf{p}}^*$. Suppose that α belongs to $]1, 2[$, or equivalently that $\beta < \mathbf{d} < 2\beta - 1$. Then $\mu_{\mathbf{p}}^*$ – which has support $[0, 1]$ – satisfy the following relation for any Borel set $B \subset [0, 1]$:

$$\mu_{\mathbf{p}}^*(B) = \mu_{\mathbf{p}}(B) + \mu_{\mathbf{p}}(B + 1)$$

and, if $B \subset [\alpha - 1, 1]$,

$$\mu_{\mathbf{p}}^*(B) = \mu_{\mathbf{p}}(B). \quad (12)$$

The following proposition points out that in certain cases, the restriction of $\mu_{\mathbf{p}}$ (or $\mu_{\mathbf{p}}^*$) to the interval $[\alpha - 1, 1]$ is "representative" of $\mu_{\mathbf{p}}$ itself.

PROPOSITION 2.5 Suppose $\beta < \mathbf{d} \leq \beta + 1 - \frac{1}{\beta}$.

(i) The interval $]0, \alpha[$ is the reunion of $I_k := \left[\frac{1}{\beta^{k+1}}, \frac{1}{\beta^k} \right]$ and $I'_k := \left[\alpha - \frac{1}{\beta^k}, \alpha - \frac{1}{\beta^{k+1}} \right]$ for $k \in \mathbb{N} \cup \{0\}$

(ii) Let $B \subset \mathbb{R}$ be a Borel set . If $B \subset I_k$ (or equivalently if $\alpha - B \subset I'_k$), then $\beta^k B$ and $\alpha - \beta^k B$ are two subsets of $[\alpha - 1, 1]$ such that

$$\begin{aligned} \mu_{\mathbf{p}}(B) &= \mathbf{p}_0^k \cdot \mu_{\mathbf{p}}^*(\beta^k B) \\ \mu_{\mathbf{p}}(\alpha - B) &= \mathbf{p}_{\mathbf{d}-1}^k \cdot \mu_{\mathbf{p}}^*(\alpha - \beta^k B). \end{aligned}$$

Proof. (i) The hypothesis on \mathbf{d} implies $\alpha < 2$ hence $]0, \alpha[$ is the reunion of $]0, 1]$ and $[\alpha - 1, \alpha[$.

(ii) $B \subset I_k \Rightarrow \beta^k B \subset \left[\frac{1}{\beta}, 1 \right] \subset [\alpha - 1, 1]$. The equality $\mu_{\mathbf{p}}(B) = \mathbf{p}_0^k \cdot \mu_{\mathbf{p}}^*(\beta^k B)$ results from (11) and (12).

Since $\beta^k B \subset [\alpha - 1, 1]$ one has $\alpha - \beta^k B \subset [\alpha - 1, 1]$. The equality $\mu_{\mathbf{p}}(\alpha - B) = \mathbf{p}_{a-1}^k \cdot \mu_{\mathbf{p}}(\alpha - \beta^k B)$ follows from (9), (11) and (12). ■

3 Bernoulli convolution in Pisot quadratic bases

In this section $\beta > 1$ is solution of the equation $x^2 = ax + b$ (with integral a and b), and we suppose that the other solution belongs to $] -1, 0[$. This implies $1 \leq b \leq a \leq \beta - \frac{1}{\beta} < \beta < a + 1$. Let $\mathbf{p} = (\mathbf{p}_0, \dots, \mathbf{p}_a)$ be a positive probability vector; the Bernoulli convolution $\mu_{\mathbf{p}}$ has support $[0, \alpha]$, where $\alpha = \frac{a}{\beta - 1}$ belongs to $]1, 2[$. The condition in Proposition 2.5 is satisfied hence it is sufficient to study the Gibbs properties of $\mu_{\mathbf{p}}^*$ on its support $[0, 1]$, to get the local properties of $\mu_{\mathbf{p}}$ on $[0, \alpha]$ (see [6] for the multifractal analysis of the weak Gibbs measures).

With the notations of the previous subsection one has $\mathcal{B} = \mathcal{D} = \{0, \dots, a\}$,

$\mathcal{I}_{(\beta, a+1)} = \{0, 1, \beta - a\}$ and – for any $\varepsilon \in \mathcal{B}$

$$M_{\varepsilon} = \begin{pmatrix} \mathbf{p}_{\varepsilon} & \mathbf{p}_{\varepsilon-1} & 0 \\ 0 & 0 & \mathbf{p}_{a+\varepsilon} \\ \mathbf{p}_{b+\varepsilon} & \mathbf{p}_{b+\varepsilon-1} & 0 \end{pmatrix},$$

where, by convention, $\mathbf{p}_i = 0$ if $i \notin \mathcal{D}$.

Notice that the intervals $\mathbb{R}_{\varepsilon}([0, 1])$ do not make a partition of $[0, 1]$ for $\varepsilon \in \mathcal{B}$ but, setting

$$\mathbb{S}_{\varepsilon} := \begin{cases} \mathbb{R}_{\varepsilon} & \text{for } 0 \leq \varepsilon \leq a - 1 \\ \mathbb{R}_a \circ \mathbb{R}_{\varepsilon-a} & \text{for } a \leq \varepsilon \leq a + b - 1 \end{cases}$$

the intervals $\mathbb{S}_{\varepsilon}([0, 1])$ make such a partition for $\varepsilon \in \mathcal{A} := \{0, \dots, a + b - 1\}$.

THEOREM 3.1 *The measure $\mu_{\mathbf{p}}^*$ is weak Gibbs w.r.t. $\{\mathbb{S}_{\varepsilon}\}_{\varepsilon=0}^{a+b-1}$ if and only if $\mathbf{p}_0^2 \geq \mathbf{p}_a \mathbf{p}_{b-1}$ and $\mathbf{p}_0 \mathbf{p}_{a-b+1} \geq \mathbf{p}_a^2$.*

Proof. The n -step potential of $\mu_{\mathbf{p}}^*$ can be computed by means of the matrices

$$M_{\varepsilon}^* := \begin{cases} M_{\varepsilon} & \text{for } 0 \leq \varepsilon \leq a - 1 \\ M_a M_{\varepsilon-a} & \text{for } a \leq \varepsilon \leq a + b - 1. \end{cases}$$

Indeed by applying Proposition 2.3 to the sets $B = [\xi_1 \dots \xi_n]$ and $B' = [\xi_2 \dots \xi_n]$, one has

$$\exp(\phi_n(\xi)) = \log \left(\frac{(1 \ 1 \ 0) M_{\xi_1}^* \dots M_{\xi_n}^* V}{(1 \ 1 \ 0) M_{\xi_2}^* \dots M_{\xi_n}^* V} \right), \quad \text{where } V := \begin{pmatrix} \mu_{\mathbf{p}}([0, 1]) \\ \mu_{\mathbf{p}}([0, 1] + 1) \\ \mu_{\mathbf{p}}([0, 1] + \beta - a) \end{pmatrix}. \quad (13)$$

Now the matrices M_ε^* are 3×3 and we shall use 2×2 ones. The matrices defined – for any $\varepsilon \in \mathcal{A}' := \{0, \dots, 2a\}$ – by

$$M_\varepsilon^* := \begin{cases} M_0 M_\varepsilon & \text{if } \varepsilon \leq a \\ M_{\varepsilon-a} & \text{if } \varepsilon > a \end{cases}$$

satisfy the commutation relation $Y M_\varepsilon^* = P_\varepsilon Y$, where

$$Y := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad P_\varepsilon := \begin{cases} \begin{pmatrix} \mathbf{p}_0 \mathbf{p}_\varepsilon & \mathbf{p}_0 \mathbf{p}_{\varepsilon-1} \\ \mathbf{p}_a \mathbf{p}_{b+\varepsilon} & \mathbf{p}_a \mathbf{p}_{b+\varepsilon-1} \end{pmatrix} & \text{if } \varepsilon \leq a \\ \begin{pmatrix} \mathbf{p}_{\varepsilon-a} & \mathbf{p}_{\varepsilon-a-1} \\ 0 & 0 \end{pmatrix} & \text{if } \varepsilon > a. \end{cases}$$

Let $\xi \in \mathcal{A}^\mathbb{N}$ such that $\sigma\xi \neq \bar{0}$. There exists an integer $k \geq 0$ and $\varepsilon \in \mathcal{A} \setminus \{0\}$ such that

$$M_{\xi_2} \dots M_{\xi_{k+2}} = M_0^k M_\varepsilon.$$

One can associate to the sequence ξ , the sequence $\zeta \in \mathcal{A}^\mathbb{N}$ such that

$$\forall n \geq k+4, \exists k(n) \in \mathbb{N}, \quad M_{\xi_{k+3}}^* \dots M_{\xi_n}^* = M_{\zeta_1}^* \dots M_{\zeta_{k(n)}}^* \quad \text{or} \quad M_{\zeta_1}^* \dots M_{\zeta_{k(n)}}^* M_0.$$

Now – according to (13) and the commutation relation

$$n \geq k+4 \Rightarrow \exp(\phi_n(\xi)) = \frac{(1 \ 1 \ 0) M_{\xi_1}^* M_0^k Q_\varepsilon \mathbf{N}(P_{\zeta_1} \dots P_{\zeta_{k(n)}} W)}{(1 \ 1 \ 0) M_0^k Q_\varepsilon \mathbf{N}(P_{\zeta_1} \dots P_{\zeta_{k(n)}} W)} \quad (14)$$

where $Q_\varepsilon := \begin{pmatrix} \mathbf{p}_\varepsilon & \mathbf{p}_{\varepsilon-1} \\ 0 & 0 \\ \mathbf{p}_{b+\varepsilon} & \mathbf{p}_{b+\varepsilon-1} \end{pmatrix}$ and $W = YV$ or $Y M_0 V$.

If $\mathbf{p}_0 \mathbf{p}_{a-b+1} \geq \mathbf{p}_a^2$, the uniform convergence – on $\mathcal{A}'^\mathbb{N}$ – of the sequence $\mathbf{N}(P_{\zeta_1} \dots P_{\zeta_k} YV)$ and $\mathbf{N}(P_{\zeta_1} \dots P_{\zeta_k} Y M_0 V)$ to the same vector $V(\zeta)$ is insured by Theorem 1.1 and Corollary 1.2. When $n \rightarrow \infty$ the numerator in (14) converges to $V_1(\xi) := (1 \ 1 \ 0) M_{\xi_1}^* M_0^k Q_\varepsilon V(\zeta)$, and the denominator to $V_2(\xi) := (1 \ 1 \ 0) M_0^k Q_\varepsilon V(\zeta)$; this convergence is uniform on each cylinder $[[\varepsilon' 0^k \varepsilon]]$. Since the first entrie in $Q_\varepsilon V(\zeta)$ is at least $\min\{\mathbf{p}_\varepsilon, \mathbf{p}_{\varepsilon-1}\} > 0$, $V_1(\xi)$ and $V_2(\xi)$ are positive and consequently $\phi_n(\xi)$ converges uniformly to $\log \frac{V_1(\xi)}{V_2(\xi)}$. This is also true on any finite reunion of such cylinders; let us denote by $X(k_0)$ the reunion of $[[\varepsilon' 0^k \varepsilon]]$ for $k < k_0, \varepsilon \in \mathcal{A} \setminus \{0\}$ and $\varepsilon' \in \mathcal{A}$; then

$$\forall \eta > 0, \exists n_0 \in \mathbb{N}, n \geq n_0 \text{ and } \xi \in X(k_0) \Rightarrow \left| \phi_n(\xi) - \log \frac{V_1(\xi)}{V_2(\xi)} \right| \leq \eta. \quad (15)$$

We consider now a sequence $\xi \in \mathcal{A}^{\mathbb{N}} \setminus X(k_0)$. By using the left and right eigenvectors of M_0 – for the eigenvalue \mathbf{p}_0 – we obtain

$$\lim_{k \rightarrow \infty} A_k = \lambda_0 \begin{pmatrix} \mathbf{p}_0^2 - \mathbf{p}_a \mathbf{p}_{b-1} & 0 & 0 \\ \mathbf{p}_a \mathbf{p}_b & 0 & 0 \\ \mathbf{p}_0 \mathbf{p}_b & 0 & 0 \end{pmatrix} \quad \text{where } \lambda_0 > 0, A_k := \begin{cases} \mathbf{p}_0^{-k} M_0^k & \text{if } \mathbf{p}_a \mathbf{p}_{b-1} < \mathbf{p}_0^2 \\ k^{-1} \mathbf{p}_0^{-k} M_0^k & \text{if } \mathbf{p}_a \mathbf{p}_{b-1} = \mathbf{p}_0^2. \end{cases}$$

The entries $\mathbf{p}_a \mathbf{p}_b$ and $\mathbf{p}_0 \mathbf{p}_b$ being positive, there exists $\lambda(\varepsilon') > 0$ such that

$$\lim_{k \rightarrow \infty} (1 \quad 1 \quad 0) M_{\varepsilon'}^* A_k Q_{\varepsilon} = \lambda(\varepsilon') (\mathbf{p}_{\varepsilon} \quad \mathbf{p}_{\varepsilon-1}).$$

Moreover the convergence of $(1 \quad 1 \quad 0) M_{\varepsilon'}^* A_k Q_{\varepsilon} \begin{pmatrix} x \\ y \end{pmatrix}$ to $\lambda(\varepsilon')(\mathbf{p}_{\varepsilon} x + \mathbf{p}_{\varepsilon-1} y)$ is uniform on the set of normalized nonnegative column-vectors $\begin{pmatrix} x \\ y \end{pmatrix}$. Similarly, there exists $\lambda_1 > 0$ such that $(1 \quad 1 \quad 0) A_k Q_{\varepsilon} \begin{pmatrix} x \\ y \end{pmatrix}$ converges uniformly to $\lambda_1(\mathbf{p}_{\varepsilon} x + \mathbf{p}_{\varepsilon-1} y)$. Both limits are positive if $\varepsilon \neq 0$, implying that the ratio converges uniformly to $\frac{\lambda(\varepsilon')}{\lambda_1}$. Hence, using (14) one can choose k_0 such that – if we assume $\xi \in \mathcal{A}^{\mathbb{N}} \setminus X(k_0)$ and $\sigma \xi \neq \bar{0}$

$$n \geq k_0 + 4 \Rightarrow \left| \phi_n(\xi) - \log \frac{\lambda(\varepsilon')}{\lambda_1} \right| \leq \eta. \quad (16)$$

The uniform convergence of $\phi_n(\xi)$ on $\mathcal{A}^{\mathbb{N}}$ follows from (15) and (16) since, in the remaining case $\sigma \xi = \bar{0}$ one has $\lim_{n \rightarrow \infty} \phi_n(\xi) = \log \frac{\lambda(\xi_1)}{\lambda_1}$.

Conversely, suppose $\mathbf{p}_a \mathbf{p}_{b-1} > \mathbf{p}_0^2$. If $\mu_{\mathbf{p}}^*$ is weak Gibbs w.r.t. $\{\mathbb{S}_{\varepsilon}\}_{\varepsilon=0}^{s-1}$ then, from (1) and (2) one has $\phi_n(\xi) = o(n)$ for any $\xi \in \mathcal{A}^{\mathbb{N}}$. But this is not true: $\phi_{2n+1}(1\bar{0}) \sim n \log \frac{\mathbf{p}_a \mathbf{p}_{b-1}}{\mathbf{p}_0^2}$.

Suppose now $\mathbf{p}_0 \mathbf{p}_{a-b+1} < \mathbf{p}_a^2$. If $b = 1$ we have $\mathbf{p}_0 < \mathbf{p}_a$ hence $\mathbf{p}_a \mathbf{p}_{b-1} > \mathbf{p}_0^2$; that is, we are in the previous case. If $b \neq 1$, $\mu_{\mathbf{p}}^*$ is no more weak Gibbs w.r.t. $\{\mathbb{S}_{\varepsilon}\}_{\varepsilon=0}^{s-1}$ because there exists a contradiction between the limit in (1) and the following:

$$\lim_{n \rightarrow \infty} \left(\frac{\mu_{\mathbf{p}}^*[(0(a-b+1))^n 1^n]}{\mu_{\mathbf{p}}^*[(0(a-b+1))^n] \cdot \mu_{\mathbf{p}}^*[1^n]} \right)^{1/n} = \frac{\mathbf{p}_0 \mathbf{p}_{a-b+1}}{\mathbf{p}_a^2} < 1.$$

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