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# Infinite products of $2 \times 2$ matrices and the Gibbs properties of Bernoulli convolutions

by ERIC OLIVIER & ALAIN THOMAS

**Abstract.**— We consider the infinite sequences  $(A_n)_{n \in \mathbb{N}}$  of  $2 \times 2$  matrices with nonnegative entries, where the  $A_n$  are taken in a finite set of matrices. Given a vector  $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  with  $v_1, v_2 > 0$ , we give a necessary and sufficient condition for  $\frac{A_1 \dots A_n V}{\|A_1 \dots A_n V\|}$  to converge uniformly. In application we prove that the Bernoulli convolutions related to the numeration in Pisot quadratic bases are weak Gibbs.

**Key-words:** Infinite products of matrices, weak Gibbs measures, Bernoulli convolutions, Pisot numbers,  $\beta$ -numeration.

**2000 Mathematics Subject Classification:** 28A12, 11A67, 15A48.

## Introduction

Let  $\mathcal{M} = \{M_0, \dots, M_{s-1}\}$  be a finite subset of the set – stable by matrix multiplication – of nonnegative and column-allowable  $d \times d$  matrices (i.e., the matrices with nonnegative entries and without null column). We associate to any sequence  $(\omega_n)_{n \in \mathbb{N}}$  with terms in  $\mathcal{S} := \{0, 1, \dots, s-1\}$ , the sequence of product matrices

$$P_n(\omega) = M_{\omega_1} M_{\omega_2} \dots M_{\omega_n}.$$

Experimentally, in most cases each normalized column of  $P_n(\omega)$  converges when  $n \rightarrow \infty$  to a limit-vector, which depends on  $\omega \in \mathcal{S}^{\mathbb{N}}$  and may depend on the index of the column. Nevertheless the normalized rows of  $P_n(\omega)$  in general do not converge: suppose for instance that all the matrices in  $\mathcal{M}$  are positive but do not have the same positive normalized left-eigenvector, let  $L_k$  such that  $L_k M_k = \rho_k L_k$ . For any positive matrix  $M$ , the normalized rows of  $MM_0^n$  converge to  $L_0$  and the ones of  $MM_1^n$  to  $L_1$ . Consequently we can choose the sequence  $(n_k)_{k \in \mathbb{N}}$  sufficiently increasing such that the normalized rows of  $M_0^{n_1} M_1^{n_2} \dots M_0^{n_{2k-1}}$  converge to  $L_0$  while the ones of  $M_0^{n_1} M_1^{n_2} \dots M_0^{n_{2k-1}} M_1^{n_{2k}}$  converge to  $L_1$ . This proves – if  $L_0 \neq L_1$  – that the normalized rows in  $P_n(\omega)$  do not converge when  $\omega = 0^{n_1} 1^{n_2} 0^{n_3} 1^{n_4} \dots$

Now in case  $\mathcal{M}$  is a set of positive matrices it is clear that, if both normalized columns and normalized rows in  $P_n(\omega)$  converge then – after replacing each matrix  $M_k$  by  $\frac{1}{\rho_k}M_k$  – the matrix  $P_n(\omega)$  itself converges: the previous counterexample proves that the matrices  $P_n(\omega)$  have a common left-eigenvector for any  $n$ , and a straightforward computation (using the limits of the normalized columns in  $P_n(\omega)$ ) proves the existence of  $\lim_{n \rightarrow \infty} P_n(\omega)$ .

The existence of a common left-eigenvector is settled in a more general context by L. Elsner and S. Friedland ([5, Theorem 1]), in case  $\mathcal{M}$  is a finite set of matrices with entries in  $\mathbb{C}$ . This theorem means (after transposition of the matrices) that if  $P_n(\omega)$  converges to a non-null limit, then there exists  $N \in \mathbb{N}$  such that the matrices  $M_{\omega_n}$  for  $n \geq N$  have a common left-eigenvector for the eigenvalue 1. Now, L. Elsner & S. Friedland (in [5, Main Theorem]) and I. Daubechies & J. C. Lagarias (in [2, Theorem 5.1] (resp. [1, Theorem 4.2])) give necessary and sufficient conditions for  $P_n(\omega)$  to converge for any  $\omega \in \mathcal{S}^{\mathbb{N}}$  (resp., to converge to a continuous map).

By these theorems we see that the problem of the convergence of the normalized columns in  $P_n(\omega)$  is very different from the problem of the convergence of  $P_n(\omega)$  itself. Let for instance  $M_0 = \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$  and  $M_1 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ ; then the normalized columns in

$$P_n(\omega) = \begin{cases} \frac{1}{10} \cdot \begin{pmatrix} 4 + 6 \cdot 6^{-n} & 6 - 6 \cdot 6^{-n} \\ 4 - 4 \cdot 6^{-n} & 6 + 4 \cdot 6^{-n} \end{pmatrix} & \text{if } \omega_1 \dots \omega_n = 0 \dots 0 \\ \frac{1}{10} \cdot \begin{pmatrix} 4 + 6^{-h} & 6 - 6^{-h} \\ 4 + 6^{-h} & 6 - 6^{-h} \end{pmatrix} & \text{if } \omega_1 \dots \omega_n = \omega_1 \dots \omega_{n-h-1} 10 \dots 0 \end{cases}$$

converge to  $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$  for any  $\omega \in \{0, 1\}^{\mathbb{N}}$ , but  $P_n(\omega)$  diverges (although it is bounded) if  $\omega$  is not eventually constant.

In Section 1 we study the uniform convergence – in direction – of  $P_n(\omega)V$  in case the  $M_k$  are  $2 \times 2$  nonnegative column-allowable matrices and  $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  a positive vector (Theorem 1.1). Notice that the convergence in direction of the columns of  $P_n(\omega)$ , to a same vector, implies the ones of  $P_n(\omega)V$ , but the converse is not true: see for instance the case  $\mathcal{M} = \left\{ \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ .

The second section is devoted to the *Bernoulli convolutions* [4], which have been studied since the early 1930's (see [8] for the other references). We give a matricial relation for such measures.

In the third section we apply more precisely Theorem 1.1 to prove that certain Bernoulli convolutions are weak Gibbs in the following sense (see [10]): given a system of affine contractions  $\mathbb{S}_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$  such that the intervals  $\mathbb{S}_\varepsilon([0, 1])$  make a partition of  $[0, 1]$  for  $\varepsilon \in \mathcal{S} = \{0, 1, \dots, s-1\}$ , a measure  $\eta$  supported by  $[0, 1]$  is weak Gibbs w.r.t.  $\{\mathbb{S}_\varepsilon\}_{\varepsilon=0}^{s-1}$  if there exists a map  $\Phi : \mathcal{S}^{\mathbb{N}} \rightarrow \mathbb{R}$ , continuous for the product topology, such that

$$\lim_{n \rightarrow \infty} \left( \frac{\eta[\xi_1 \dots \xi_n]}{\exp \left( \sum_{k=0}^{n-1} \Phi(\sigma^k \xi) \right)} \right)^{1/n} = 1 \quad \text{uniformly on } \xi \in \mathcal{S}^{\mathbb{N}}, \quad (1)$$

where  $[\xi_1 \dots \xi_n] := \mathbb{S}_{\xi_1} \circ \dots \circ \mathbb{S}_{\xi_n}([0, 1])$  and  $\sigma$  is the shift on  $\mathcal{S}^{\mathbb{N}}$ . Let us give a sufficient condition for  $\eta$  to be weak Gibbs. For each  $\xi \in \mathcal{S}^{\mathbb{N}}$  we put  $\phi_1(\xi) = \log \eta[\xi_1]$  and for  $n \geq 2$ ,

$$\phi_n(\xi) = \log \left( \frac{\eta[\xi_1 \dots \xi_n]}{\eta[\xi_2 \dots \xi_n]} \right). \quad (2)$$

The continuous map  $\phi_n : \mathcal{S}^{\mathbb{N}} \rightarrow \mathbb{R}$  ( $n \geq 1$ ) is the  $n$ -step potential of  $\eta$ . Assume the existence of the uniform limit  $\Phi = \lim_{n \rightarrow \infty} \phi_n$ ; it is then straightforward that for  $n \geq 1$ ,

$$\frac{1}{K_n} \leq \frac{\eta[\xi_1 \dots \xi_n]}{\exp \left( \sum_{k=0}^{n-1} \Phi(\sigma^k \xi) \right)} \leq K_n \quad \text{with} \quad K_n = \exp \left( \sum_{k=1}^n \|\Phi - \phi_k\|_\infty \right). \quad (3)$$

By a well known lemma on the Cesàro sums,  $K_1, K_2, \dots$  form a subexponential sequence of positive real numbers, that is  $\lim_{n \rightarrow \infty} (K_n)^{1/n} = 1$  and thus, (3) means  $\eta$  is weak Gibbs w.r.t.  $\{\mathbb{S}_\varepsilon\}_{\varepsilon=0}^{s-1}$ .

Now the weak Gibbs property can be proved for certain Bernoulli convolutions by computing the  $n$ -step potential by means of products of matrices (see [6] for the Bernoulli convolution associated with the golden ratio  $\beta = \frac{1 + \sqrt{5}}{2}$  – called the *Erdős measure* – and the application to the multifractal analysis). In Theorem 3.1 we generalize this result in case  $\beta > 1$  is a quadratic number with conjugate  $\beta' \in ]-1, 0[$ .

## 1 Infinite product of $2 \times 2$ matrices

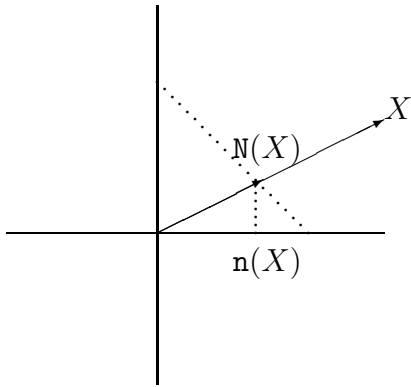
From now the vectors  $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$  and the matrices  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  we consider are supposed to be nonnegative and column-allowable that is,  $x_1, x_2, a, b, c, d$  are nonnegative

and  $x_1 + x_2, a + c, b + d$  are positive. In particular we suppose that the matrices in  $\mathcal{M} = \{M_0, \dots, M_{s-1}\}$  satisfy these conditions. We associate to  $X$  the normalized vector:

$$\mathbf{N}(X) := \begin{pmatrix} \frac{x_1}{x_1+x_2} \\ \frac{x_2}{x_1+x_2} \end{pmatrix} = \begin{pmatrix} \mathbf{n}(X) \\ 1 - \mathbf{n}(X) \end{pmatrix} \quad \text{where} \quad \mathbf{n}(X) := \frac{x_1}{x_1 + x_2}$$

and define the distance between the column of  $A$  (or the rows of  ${}^tA$ ):

$$d_{\text{columns}}(A) := \left| \mathbf{n} \left( \begin{pmatrix} a \\ c \end{pmatrix} \right) - \mathbf{n} \left( \begin{pmatrix} b \\ d \end{pmatrix} \right) \right| = \frac{|\det A|}{(a+c)(b+d)} =: d_{\text{rows}}({}^tA).$$



**THEOREM 1.1** Given  $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  with  $v_1, v_2 > 0$ , the sequence of vectors  $\mathbf{N}(P_n(\omega)V)$  converges uniformly for  $\omega \in \mathcal{S}^{\mathbb{N}}$  only in the five following cases:

Case 1:  $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{M} \Rightarrow a \geq d$ ;  $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \mathcal{M} \Rightarrow a \leq d$ ;

$\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}, \begin{pmatrix} 0 & b' \\ c' & d' \end{pmatrix} \in \mathcal{M} \Rightarrow bc' \geq b'c$ ; no matrix in  $\mathcal{M}$  has the form  $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$  or  $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ .

Case 2:  $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{M} \Rightarrow a < d$ ;  $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \mathcal{M} \Rightarrow a > d$ ;

$\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}, \begin{pmatrix} 0 & b' \\ c' & d' \end{pmatrix} \in \mathcal{M} \Rightarrow bc' < b'c$ .

Case 3:  $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{M} \Rightarrow a \geq d$ ;  $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \mathcal{M} \Rightarrow a > d$ ;

no matrix in  $\mathcal{M}$  has the form  $\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$ .

Case 4:  $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{M} \Rightarrow a < d$ ;  $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \mathcal{M} \Rightarrow a \leq d$ ;

no matrix in  $\mathcal{M}$  has the form  $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ ;

Case 5:  $V$  is an eigenvector of all the matrices in  $\mathcal{M}$ .

**COROLLARY 1.2** *If  $\mathbb{N}(P_n(\cdot)V)$  converges uniformly on  $\mathcal{S}^{\mathbb{N}}$ , the limit do not depend on the positive vector  $V$ , except in the fifth case of Theorem 1.1.*

*Proof.* Suppose that  $\mathcal{M}$  satisfies the conditions of the case 1,2,3 or 4 in Theorem 1.1 and let  $V, W$  be two positive vectors. Then the following set  $\mathcal{M}'$  also do:

$\mathcal{M}' := \mathcal{M} \cup \{M_s\}$ , where  $M_s$  is the matrix whose both columns are  $W$ .

Denoting by  $\omega' = \omega_1 \dots \omega_n \bar{s}$  the sequence defined by  $\omega'_i = \begin{cases} \omega_i & \text{if } i \leq n \\ s & \text{if } i > n \end{cases}$  one has for any  $\omega \in \mathcal{S}^{\mathbb{N}}$

$$\mathbb{N}(P_n(\omega)V) - \mathbb{N}(P_n(\omega)W) = \mathbb{N}(P_n(\omega')V) - \mathbb{N}(P_{n+1}(\omega')V)$$

and this tends to 0, according to the uniform Cauchy property of the sequence  $\mathbb{N}(P_n(\cdot)V)$ . ■

Nevertheless, this limit may depend of  $V$  if one assume only that  $V$  is nonnegative. For instance, if  $\mathcal{M} = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$  then  $\lim_{n \rightarrow \infty} \mathbb{N} \left( P_n(\omega) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$  differs from  $\lim_{n \rightarrow \infty} \mathbb{N} \left( P_n(\omega) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$  iff  $\omega = \bar{0}$  (implying the second limit is not uniform on  $\mathcal{S}^{\mathbb{N}}$ ).

## 1.1 Geometric considerations

We follow the ideas of E. Seneta about products of nonnegative matrices in Section 3 of [9], or stochastic matrices in Section 4. In what follows we denote the matrices by  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,  $A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$  or  $A_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$  for  $n \in \mathbb{N}$ , and we suppose they are nonnegative and column-allowable. We define the coefficient

$$\tau(A) := \sup_{d_{\text{columns}}(A') \neq 0} \frac{d_{\text{columns}}(A'A)}{d_{\text{columns}}(A')}.$$

The straightforward formula

$$d_{\text{columns}} \left( \prod_{k=1}^n (A_k) \right) \leq d_{\text{columns}}(A_1) \prod_{k=2}^n \tau(A_k) \quad (4)$$

is of use to prove Theorem 1.1 because, according to the following proposition one has  $\tau(A) < 1$  if  $A$  is positive.

PROPOSITION 1.3

$$\tau(A) = \begin{cases} \frac{|\sqrt{ad} - \sqrt{bc}|}{\sqrt{ad} + \sqrt{bc}} & \text{if } A \text{ do not have any null row} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.*  $\frac{d_{\text{columns}}(A'A)}{d_{\text{columns}}(A')} = \frac{|\det A|}{(a + c/x)(bx + d)}$  (where  $x = \frac{a' + c'}{b' + d'}$ ) is maximal for  $x = \sqrt{\frac{cd}{ab}}$ . ■

REMARK 1.4 One can consider – instead of  $d_{\text{columns}}$  – the angle between the columns of  $A$ :

$$\alpha(A) := \left| \arctan \frac{a}{c} - \arctan \frac{b}{d} \right|,$$

or the Hilbert distance between the columns of a positive matrix  $A$ :

$$d_{\text{Hilbert}}(A) := \left| \log \frac{a}{c} - \log \frac{b}{d} \right|.$$

This last can be interpreted either as the distance between the columns or the rows of  $A$ , because  $d_{\text{Hilbert}}(A) = d_{\text{Hilbert}}({}^t A)$ . The Birkhoff coefficient  $\tau_{\text{Birkhoff}}(A) := \sup_{d_{\text{Hilbert}}(A') \neq 0} \frac{d_{\text{Hilbert}}(A'A)}{d_{\text{Hilbert}}(A')}$  has – from [9, Theorem (3.12)] – the same value as  $\tau(A)$  in Proposition 1.3, and probably as a large class of coefficients defined in this way.

In the following proposition we list the properties of  $d_{\text{columns}}$  that are required for proving Theorem 1.1.

PROPOSITION 1.5 (i)  $\sup_{d_{\text{columns}}(A') \neq 0} \frac{d_{\text{columns}}(AA')}{d_{\text{columns}}(A')} = \frac{|\det A|}{\min((a+c)^2, (b+d)^2)} =: \tau_1(A)$ .

(ii) If  $A$  is positive then  $\sup_{d_{\text{columns}}(A') \neq 0} \frac{d_{\text{rows}}(AA')}{d_{\text{columns}}(A')} \leq \frac{|\det A|}{\min(a, b) \cdot \min(c, d)} =: \tau_2(A)$ .

(iii) If  $\lim_{n \rightarrow \infty} d_{\text{columns}}(A_n) = 0$  then  $\lim_{n \rightarrow \infty} d_{\text{columns}}(AA_n A') = 0$  and, assuming that  $A$  is positive,  $\lim_{n \rightarrow \infty} d_{\text{rows}}(AA_n A') = 0$ .

(iv) Suppose the matrices  $A_n$  are upper-triangular. If  $\inf_{k \in \mathbb{N}} \frac{a_k}{d_k} \geq 1$  and  $\sum_{k \in \mathbb{N}} \frac{b_k}{d_k} = \infty$  then

$$\lim_{n \rightarrow \infty} d_{\text{columns}}(A_1 \dots A_n) = \lim_{n \rightarrow \infty} d_{\text{columns}}(A_n \dots A_1) = 0.$$

(v) Suppose the matrices  $A_n$  are lower-triangular. If  $\inf_{k \in \mathbb{N}} \frac{d_k}{a_k} \geq 1$  and  $\sum_{k \in \mathbb{N}} \frac{c_k}{a_k} = \infty$  then

$$\lim_{n \rightarrow \infty} d_{\text{columns}}(A_1 \dots A_n) = \lim_{n \rightarrow \infty} d_{\text{columns}}(A_n \dots A_1) = 0.$$

*Proof.* (i) and (ii) are obtained from the formula

$$d_{\text{columns}}(AA') = \frac{\det A \cdot \det A'}{((a+c)a' + (b+d)c') \cdot ((a+c)b' + (b+d)d')},$$

and the relation  $d_{\text{rows}}(AA') = d_{\text{columns}}({}^t A' {}^t A)$ .

(iii) is due to the fact that the inequalities of items (i), (ii) and (4) imply  $d_{\text{columns}}(AA_n A') \leq \tau_1(A)d_{\text{columns}}(A_n)\tau(A')$  and – if  $A$  is positive –  $d_{\text{rows}}(AA_n A') \leq \tau_2(A)d_{\text{columns}}(A_n)\tau(A')$ .

(iv) follows from the formula

$$A_1 \dots A_n = \begin{pmatrix} a_1 \dots a_n & s_n \\ 0 & d_1 \dots d_n \end{pmatrix},$$

where  $s_n = \sum_{k=1}^n a_1 \dots a_{k-1} b_k d_{k+1} \dots d_n \geq d_1 \dots d_n \sum_{k=1}^n \frac{b_k}{d_k}$ .

(v) can be deduced from (iv) by using the relation  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ 0 & a \end{pmatrix}$ . ■

We need also the following:

**PROPOSITION 1.6** *Let  $V_A$  be a nonnegative eigenvector associated to the maximal eigenvalue of  $A$ , and  $C$  a cone of nonnegative vectors containing  $V_A$ . If  $\det A \geq 0$  then  $C$  is stable by left-multiplication by  $A$ .*

*Proof.* The discriminant of the characteristic polynomial of  $A$  is  $(a-d)^2 + 4bc$ . In case this discriminant is null the proof is obtained by direct computation, because  $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$  or  $\begin{pmatrix} a & 0 \\ c & a \end{pmatrix}$ . Otherwise  $A$  has two eigenvalues  $\lambda > \lambda'$  and, given a nonnegative vector  $X$ , there exists a real  $\alpha$  and an eigenvector  $W_A$  (associated to  $\lambda'$ ) such that

$$X = \alpha V_A + W_A \quad \text{and} \quad AX = \lambda \alpha V_A + \lambda' W_A = \lambda' X + (\lambda - \lambda') \alpha V_A.$$

Notice that  $\alpha \geq 0$  (because the nonnegative vector  $A^n X = \lambda^n \alpha V_A + \lambda'^n W_A$  converges in direction to  $\alpha V_A$ ) and  $\lambda' \geq 0$  (from the hypothesis  $\det A \geq 0$ ). Hence  $AX$  is a nonnegative linear combination of  $X$  and  $V_A$ ; if  $X$  belongs to  $C$  then  $AX$  also do. ■

## 1.2 How pointwise convergence implies uniform convergence

Let  $m$  and  $M$  be the bounds of  $\mathbf{n}(P_n(\omega)V)$  for  $n \in \mathbb{N}$  and  $\omega \in \mathcal{S}^{\mathbb{N}}$ , and let  $M_V := \begin{pmatrix} m & M \\ 1-m & 1-M \end{pmatrix}$ . Each real  $x \in [m, M]$  can be written  $x = mx_1 + Mx_2$  with  $x_1, x_2 \geq 0$

and  $x_1 + x_2 = 1$ ; in particular the real  $x = \mathbf{n}(P_n(\omega)V)$  can be written in this form, hence

$$\forall \omega \in \mathcal{S}^{\mathbb{N}}, \exists t_1, t_2 \geq 0, P_n(\omega)V = M_V \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}. \quad (5)$$

**PROPOSITION 1.7** *If  $d_{\text{columns}}(P_n(\cdot)M_V)$  converges pointwise to 0 when  $n \rightarrow \infty$ , then  $\mathbb{N}(P_n(\cdot)V)$  converges uniformly on  $\mathcal{S}^{\mathbb{N}}$ .*

*Proof.* Suppose the pointwise convergence holds. Given  $\omega \in \mathcal{S}^{\mathbb{N}}$  and  $\varepsilon > 0$ , there exists the integer  $n = n(\omega, \varepsilon)$  such that  $d_{\text{columns}}(P_n(\omega)M_V) \leq \varepsilon$ . The family of cylinders  $C(\omega, \varepsilon) := \llbracket \omega_1 \dots \omega_{n(\omega, \varepsilon)} \rrbracket$ , for  $\omega$  running over  $\mathcal{S}^{\mathbb{N}}$ , is a covering of the compact  $\mathcal{S}^{\mathbb{N}}$ ; hence there exists a finite subset  $X \subset \mathcal{S}^{\mathbb{N}}$  such that  $\mathcal{S}^{\mathbb{N}} = \bigcup_{\omega \in X} C(\omega, \varepsilon)$ . Let  $p > q \geq n_\varepsilon := \max\{n(\omega, \varepsilon) ; \omega \in X\}$ . For any  $\xi \in \mathcal{S}^{\mathbb{N}}$ , there exists  $\zeta \in X$  such that  $\xi \in C(\zeta, \varepsilon)$  that is,  $\xi_k = \zeta_k$  for any  $k \leq n = n(\zeta, \varepsilon)$ . From (5) there exists two nonnegative vectors  $V_p$  and  $V_q$  such that  $P_p(\xi)V = P_n(\zeta)M_V V_p$  and  $P_q(\xi)V = P_n(\zeta)M_V V_q$ . Denoting by  $M(p, q)$  the column-allowable matrix whose columns are  $V_p$  and  $V_q$  we have – in view of (4)

$$\begin{aligned} |\mathbf{n}(P_p(\xi)V) - \mathbf{n}(P_q(\xi)V)| &= d_{\text{columns}}(P_n(\zeta)M_V M(p, q)) \\ &\leq d_{\text{columns}}(P_n(\zeta)M_V) \\ &\leq \varepsilon, \end{aligned}$$

implying the uniform Cauchy property for  $\mathbb{N}(P_n(\cdot)V)$ . ■

### 1.3 Proof of the uniform convergence of $\mathbb{N}(P_n(\cdot)V)$

According to Proposition 1.7 it is sufficient to prove that  $\lim_{n \rightarrow \infty} d_{\text{columns}}(P_n(\omega)M_V) = 0$  for each  $\omega \in \mathcal{S}^{\mathbb{N}}$ . This convergence is obvious in the following cases:

- If there exists  $N$  such that  $M_{\omega_N}$  has rank 1, then  $P_n(\omega)M_V$  has rank 1 for  $n \geq N$  and

$$\forall n \geq N, d_{\text{columns}}(P_n(\omega)M_V) = 0.$$

- If there exists infinitely many integers  $n$  such that  $M_{\omega_n}$  is a positive matrix, one has  $\tau(M_{\omega_n}) \leq \rho := \max_{M \in \mathcal{M}, M > 0} \tau(M) < 1$ , and the formula (4) implies

$$\lim_{n \rightarrow \infty} d_{\text{columns}}(P_n(\omega)M_V) = 0.$$

- Similarly, this limit is null also in case there exists infinitely many integers  $n$  such that  $M_{\omega_n} M_{\omega_{n+1}}$  is a positive matrix.

So we can make from now the following hypotheses on the sequence  $\omega$  under consideration:

(H):  $\det M_{\omega_n} \neq 0$  for any  $n \in \mathbb{N}$ , and there exists  $N$  such that the matrix  $M_{\omega_n} M_{\omega_{n+1}}$  has at least one null entrie for any  $n > N$ .

Proof in the case 1: Since the couples of matrices  $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}, \begin{pmatrix} 0 & b' \\ c' & d' \end{pmatrix} \in \mathcal{M}$  satisfy  $\frac{b}{c} \geq \frac{b'}{c'}$ , there exists a real  $\alpha$  such that

$$\forall \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}, \begin{pmatrix} 0 & b' \\ c' & d' \end{pmatrix} \in \mathcal{M}, \quad \frac{b}{c} \geq \alpha \geq \frac{b'}{c'}.$$

Let  $\Delta = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$ . We denote by  $\mathcal{P}$  the set of  $2 \times 2$  matrices with nonnegative determinant and by  $\tilde{\mathcal{M}}$  the subset of  $\mathcal{P}$  defined as follows:

$$\tilde{\mathcal{M}} := \{\Delta^{-1}M, M\Delta ; M \in \mathcal{M} \setminus \mathcal{P}\} \cup \{M, \Delta^{-1}M\Delta ; M \in \mathcal{M} \cap \mathcal{P}\}.$$

This set of matrices also satisfies the conditions mentionned in the case 1: for instance if  $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \in \mathcal{M}$ , the matrix  $\Delta^{-1} \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} = \begin{pmatrix} c & 0 \\ a/\alpha & b/\alpha \end{pmatrix}$  satisfies  $c \leq b/\alpha$ , and so one. For any sequence  $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots$  of elements of  $\{0, 1\}$  such that  $\varepsilon_0 = 0$  we can write

$$\begin{aligned} P_n(\omega) &= M_{\omega_1} M_{\omega_2} \dots M_{\omega_n} \\ &= (\Delta^{-\varepsilon_0} M_{\omega_1} \Delta^{\varepsilon_1}) \cdot (\Delta^{-\varepsilon_1} M_{\omega_2} \Delta^{\varepsilon_2}) \dots \cdot (\Delta^{-\varepsilon_{n-1}} M_{\omega_n} \Delta^{\varepsilon_n}) \cdot \Delta^{-\varepsilon_n} \\ &= A_1 A_2 \dots A_n \Delta^{-\varepsilon_n} \end{aligned} \quad (6)$$

where  $A_n := \Delta^{-\varepsilon_{n-1}} M_{\omega_n} \Delta^{\varepsilon_n}$  for any  $n \in \mathbb{N}$ . By the following choice of the sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$ , the matrices  $A_n$  belong to  $\tilde{\mathcal{M}}$ :

$$\varepsilon_n = \begin{cases} \varepsilon_{n-1} & \text{if } \det M_{\omega_n} > 0 \\ 1 - \varepsilon_{n-1} & \text{otherwise.} \end{cases}$$

The hypotheses (H) imply that either all the matrices  $A_n$  for  $n > N$  are upper-triangular, or all of them are lower-triangular (otherwise  $M_{\omega_n} M_{\omega_{n+1}} = \Delta^{\varepsilon_{n-1}} A_n A_{n+1} \Delta^{-\varepsilon_{n+1}}$  is positive for some  $n > N$ ). By Proposition 1.5 (iv) and (v),

$$\lim_{n \rightarrow \infty} d_{\text{columns}}(A_{N+1} \dots A_n) = 0.$$

From (6) and Proposition 1.5 (iii),  $\lim_{n \rightarrow \infty} d_{\text{columns}}(P_n(\omega) M_V) = 0$ .

Proof in the case 2: We use the matrix  $\Delta$  and the set of matrices  $\tilde{\mathcal{M}}$  defined in the previous case; here the real  $\alpha$  is supposed such that  $\frac{b}{c} \leq \alpha \leq \frac{b'}{c'}$  for any  $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}, \begin{pmatrix} 0 & b' \\ c' & d' \end{pmatrix} \in \mathcal{M}$ ,

and consequently  $\tilde{\mathcal{M}}$  satisfies the hypotheses of the case 2. This imply that each matrix in  $\tilde{\mathcal{M}}$  has a positive eigenvector. Let  $C$  be the (minimal) cone containing  $V$ ,  $\Delta^{-1}V$  and the positive eigenvectors of the matrices in  $\tilde{\mathcal{M}}$ . From (6) and Proposition 1.6,  $P_n(\omega)V$  belongs to this cone for any  $\omega \in \mathcal{S}^{\mathbb{N}}$  hence  $M_V$  is positive.

Using again the relation (6) we have

$$d_{\text{columns}}(P_n(\omega)M_V) = d_{\text{rows}}({}^tM_V {}^t\Delta^{-\varepsilon_n} {}^tA_n \dots {}^tA_1). \quad (7)$$

Each matrix  ${}^tA_n$  for  $n > N$  satisfy  $a > d$  if  ${}^tA_n$  is upper-triangular, and  $a < d$  if it is lower-triangular. By Proposition 1.5 (iv) and (v),  $\lim_{n \rightarrow \infty} d_{\text{columns}}({}^tA_n \dots {}^tA_{N+1}) = 0$ . This implies  $\lim_{n \rightarrow \infty} d_{\text{columns}}(P_n(\omega)M_V) = 0$  by applying Proposition 1.5 (iii) to the r.h.s. in (7).

Proof in the case 3: Let  $C'$  be the (minimal) cone containing  $V$ , the nonnegative eigenvectors (associated to the maximal eigenvalues) of the matrices in  $\mathcal{M} \cap \mathcal{P}$ , and the column-vectors of the matrices in  $\mathcal{M} \setminus \mathcal{P}$ . All the vectors delimiting  $C'$  are distinct from  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ , and Proposition 1.6 implies that  $P_n(\omega)V \in C'$  for any  $\omega \in \mathcal{S}^{\mathbb{N}}$ . Hence  $m$  and  $M$  that is, the bounds of  $\mathfrak{n}(P_n(\omega)V)$ , are positive.

Suppose first that  $M_{\omega_n}$  is lower-triangular for any  $n \in \mathbb{N}$  and let  $\begin{pmatrix} \alpha_n & 0 \\ \gamma_n & \delta_n \end{pmatrix} = P_n(\omega)$ . The hypotheses of the case 3 imply  $\lim_{n \rightarrow \infty} \frac{\delta_n}{\alpha_n} = 0$ . A simple computation gives  $d_{\text{columns}}(P_n(\omega)M_V) \leq \frac{\delta_n}{\alpha_n} \cdot \frac{M-m}{Mm}$  hence  $\lim_{n \rightarrow \infty} d_{\text{columns}}(P_n(\omega)M_V) = 0$ . This conclusion remains valid if  $M_{\omega_n}$  is eventually lower-triangular.

Suppose now  $M_{\omega_n}$  is not lower-triangular for infinitely many  $n$ . The hypotheses mentioned in the case 3 and (H) imply that  $M_{\omega_n}$  is upper-triangular for any  $n > N$  (because for each  $n$  such that  $M_{\omega_n}$  is lower-triangular or has the form  $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ , (H) implies that  $M_{\omega_{n+1}}$  is lower-triangular). Proposition 1.5 (iii) and (iv) implies that  $\lim_{n \rightarrow \infty} d_{\text{columns}}(P_n(\omega)M_V) = 0$ .

Proof in the case 4: Let  $\mathcal{M}'$  be the set of matrices  $M'_k = \Delta^{-1}M_k\Delta$  for  $k = 0, \dots, \mathfrak{s}-1$ , and let  $V' = \Delta^{-1}V$  (here we can choose  $\Delta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ). The set  $\mathcal{M}'$  satisfies the hypotheses of the case 3 hence  $\lim_{n \rightarrow \infty} d_{\text{columns}}(P_n(\omega)M_V) = \lim_{n \rightarrow \infty} d_{\text{columns}}(\Delta M'_{\omega_1} \dots M'_{\omega_n} V') = 0$ .

## 1.4 Proof of the converse assertion in Theorem 1.1

Now we suppose the existence of the uniform limit  $V(\cdot) := \lim_{n \rightarrow \infty} \mathbf{N}(P_n(\cdot)V)$  and we want to check the conditions contained in one of the five cases involved in Theorem 1.1. Let  $\mathcal{M}^2$  be the set of matrices  $MM'$  for  $M, M' \in \mathcal{M}$ , and let  $\mathcal{U}$  (resp.  $\mathcal{L}$ ) be the set of upper-triangular (resp. lower-triangular) matrices  $M \in \mathcal{M} \cup \mathcal{M}^2$ .

We first prove that  $\mathcal{U}$  cannot contain a couple of matrices  $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$  and  $A' = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}$  such that  $a \geq d$  and  $a' < d'$ : suppose that  $\mathcal{U}$  contain such matrices let, for simplicity,  $M_0 = A$  and  $M_1 = A'$ . One has  $V(\bar{0}) = \lim_{n \rightarrow \infty} \mathbf{N}(A^n V)$ , and this limit is also the normalized nonnegative right-eigenvector of  $A$  associated to its maximal eigenvalue, hence  $V(\bar{0}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ . Similarly,  $V(\bar{1})$  is colinear to  $\begin{pmatrix} b' \\ d' - a' \end{pmatrix}$  (eigenvector of  $A'$ ) hence distinct from  $V(\bar{0})$ . Moreover, for fixed  $N \in \mathbb{N}$

$$\begin{aligned} V(1^N \bar{0}) &= \lim_{n \rightarrow \infty} \mathbf{N}(A'^N A^n V) = \lim_{n \rightarrow \infty} \mathbf{N}(A'^N \mathbf{N}(A^n V)) \\ &= \mathbf{N}(A'^N V(\bar{0})) \\ &= V(\bar{0}). \end{aligned} \tag{8}$$

Since  $1^N \bar{0}$  tends to  $\bar{1}$  when  $N \rightarrow \infty$ , the inequality  $V(\bar{0}) \neq V(\bar{1})$  contradicts the continuity of the map  $V$ . This proves that the couple of matrices  $A, A' \in \mathcal{U}$  such that  $a \geq d$  and  $a' < d'$  do not exist. Similarly, the couple of matrices  $A, A' \in \mathcal{L}$  such that  $a \leq d$  and  $a' > d'$  do not exist.

- Suppose that all the matrices in  $\mathcal{U}$  satisfy  $a \geq d$  and all the ones in  $\mathcal{L}$  satisfy  $a \leq d$ . If  $\mathcal{M}$  contains a matrix of the form  $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ , it is necessarily an homothetic matrix. If it contains a matrix of the form  $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ , the square of this matrix is homothetic. So in both cases  $\mathcal{M} \cup \mathcal{M}^2$  contains an homothetic matrix, let  $H$ . We use the same method as above: since the map  $V$  is continuous, the vector  $\lim_{n \rightarrow \infty} \mathbf{N}(H^n V)$  must be equal to  $\lim_{N \rightarrow \infty} \left( \lim_{n \rightarrow \infty} \mathbf{N}(H^N M H^n V) \right)$  for any  $M \in \mathcal{M}$ . But the first is  $\mathbf{N}(V)$  and the second  $\mathbf{N}(MV)$ , hence  $V$  is an eigenvector of all the matrices in  $\mathcal{M}$ . Suppose now that  $\mathcal{M}$  do not contain matrices of the form  $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$  nor  $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ : all the conditions of the case 1 are satisfied.
- Suppose that all the matrices in  $\mathcal{U}$  satisfy  $a < d$  and all the ones in  $\mathcal{L}$  satisfy  $a > d$ ; then the conditions of the case 2 are satisfied.

- Suppose that all the matrices  $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{U}$  satisfy  $a \geq d$  and all the matrices  $A' = \begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix} \in \mathcal{L}$  satisfy  $a' > d'$ . If there exists  $A \in \mathcal{U}$ ,  $A' \in \mathcal{L}$  and  $M = \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix} \in \mathcal{M}$ , the map  $V$  is discontinuous because

$$\lim_{n \rightarrow \infty} \mathbb{N} \left( A'^N M A^n \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) = \lim_{n \rightarrow \infty} \mathbb{N} \left( \begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix}^N \begin{pmatrix} \beta & 0 \\ \delta & \gamma \end{pmatrix} \begin{pmatrix} d & 0 \\ b & a \end{pmatrix}^n \begin{pmatrix} v_2 \\ v_1 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

differs from  $\lim_{n \rightarrow \infty} \mathbb{N} \left( A'^n \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)$  which is colinear to  $\begin{pmatrix} a' - d' \\ c' \end{pmatrix}$ . Hence, either  $\mathcal{M}$  do not contain a matrix of the form  $\begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}$  and we are in the case 3, or  $\mathcal{U} = \emptyset$  and we are in the case 2, or  $\mathcal{L} = \emptyset$  and we are in the case 1.

- The case when all the matrices  $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{U}$  satisfy  $a < d$  and all the matrices  $\begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix} \in \mathcal{L}$  satisfy  $a' \leq d'$  is symmetrical to the previous, by using the set of matrices  $\mathcal{M}' := \{\Delta^{-1} M \Delta ; M \in \mathcal{M}\}$  and the vector  $V' = \Delta^{-1} V$ , where  $\Delta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

## 2 Some properties of the Bernoulli convolutions in base $\beta > 1$

Given a real  $\beta > 1$ , an integer  $d > \beta$  and a  $d$ -dimensional probability vector  $\mathbf{p} := (\mathbf{p}_i)_{i=0}^{d-1}$ , the  $\mathbf{p}$ -distributed  $(\beta, d)$ -Bernoulli convolution is by definition the probability distribution  $\mu_{\mathbf{p}}$  of the random variable  $X$  defined by

$$\forall \omega \in \mathcal{D}^{\mathbb{N}} := \{0, \dots, d-1\}^{\mathbb{N}}, \quad X(\omega) = \sum_{k=1}^{\infty} \frac{\omega_k}{\beta^k},$$

where  $\omega \mapsto \omega_k$  ( $k = 1, 2, \dots$ ) is a sequence of i.i.d. random variables assuming the values  $i = 0, 1, \dots, d-1$  with probability  $\mathbf{p}_i$ .

Denoting by  $\bar{\omega}$  the sequence such that  $\bar{\omega}_k = d-1 - \omega_k$  for any  $k$ , one has the relation  $X(\omega) + X(\bar{\omega}) = \alpha := \frac{d-1}{\beta-1}$ . Hence, setting  $\bar{\mathbf{p}}_i = \mathbf{p}_{d-1-i}$  for any  $i = 0, 1, \dots, d-1$ , the following symmetry relation holds for any Borel set  $B \subset \mathbb{R}$ :

$$\mu_{\mathbf{p}}(B) = \mu_{\bar{\mathbf{p}}}(\alpha - B) \tag{9}$$

(notice that the support of  $\mu_{\mathbf{p}}$  is a subset of  $[0, \alpha]$ ).

The measure  $\mu_{\mathbf{p}}$  also satisfy the following selfsimilarity relation: denoting by  $\sigma$  the shift on  $\mathcal{D}^{\mathbb{N}}$  one has – for any Borel set  $B \subset \mathbb{R}$

$$X(\omega) \in B \Leftrightarrow X(\sigma\omega) \in (\beta B - \omega_0)$$

hence, using the independance of the random variables  $\omega \mapsto \omega_k$ ,

$$\mu_{\mathbf{p}}(B) = \sum_{k=0}^{\mathfrak{d}-1} \mathbf{p}_k \cdot \mu_{\mathbf{p}}(\beta B - k) \quad \text{for any Borel set } B \subset \mathbb{R} \quad (10)$$

and in particular

$$\mu_{\mathbf{p}}(B) = \mathbf{p}_0 \cdot \mu_{\mathbf{p}}(\beta B) \quad \text{if } \beta B \subset [0, 1]. \quad (11)$$

The following proposition is proved in [3, Theorem 2.1 and Proposition 5.4] in case the probability vector  $\mathbf{p}$  is uniform:

**PROPOSITION 2.1** *The 1-periodic map  $H : ] - \infty, 0] \rightarrow \mathbb{R}$  defined by*

$$H(x) = (\mathbf{p}_0)^x \cdot \mu_{\mathbf{p}}([0, \beta^x])$$

*is continuous and a.e. differentiable. Moreover  $H$  is not differentiable on a certain continuum of points if  $\beta$  is an irrational Pisot number or an integer and – in this latter case – if  $\beta$  do not divide  $\mathfrak{d}$ .*

Let us give also the matricial form of the relation (10) (from [7, §2.1]). We define the (finite or countable) set  $\mathcal{I}_{(\beta, \mathfrak{d})} = \{0 = \mathbf{i}_0, \mathbf{i}_1, \dots\}$  as follows (where  $\mathcal{B}$  is the alphabet  $\{0, 1, \dots, \mathfrak{b} - 1\}$  such that  $\mathfrak{b} - 1 < \beta \leq \mathfrak{b}$ ):

**DEFINITION 2.2**  $\mathcal{I}_{(\beta, \mathfrak{d})}$  is the set of  $i \in ] - 1, \alpha[$  for which there exists  $-1 < i_1, \dots, i_n < \alpha$  with  $0 \triangleright i_1 \triangleright \dots \triangleright i_n \triangleright i$ , where  $x \triangleright y$  means that exists  $(\varepsilon, k) \in \mathcal{B} \times \mathcal{D}$  such that  $y = \beta x + (\varepsilon - k)$ .

Let  $\varepsilon \in \mathcal{B}$ ; the entries of the matrix  $M_{\varepsilon}$  are – for the row index  $i$  and the column index  $j$ , with  $\mathbf{i}_i, \mathbf{i}_j \in \mathcal{I}_{(\beta, \mathfrak{d})}$ ,

$$M_{\varepsilon}(i, j) = \begin{cases} \mathbf{p}_k & \text{if } k = \varepsilon + \beta \mathbf{i}_i - \mathbf{i}_j \in \mathcal{D} \\ 0 & \text{otherwise.} \end{cases}$$

Setting  $\mathbb{R}_{\varepsilon}(x) = \frac{x + \varepsilon}{\beta}$  for any  $\varepsilon \in \mathcal{B}$  and  $x \in \mathbb{R}$ , we have the following

PROPOSITION 2.3 ([7, Lemma 2.2]) If  $\mathcal{I}_{(\beta, \mathbf{d})} = \{\mathbf{i}_0, \dots, \mathbf{i}_{r-1}\}$  then, for any Borel set  $B \subset [0, 1]$  and any  $\varepsilon \in \mathcal{B}$  such that  $\mathbb{R}_\varepsilon^{-1}(B) \subset [0, 1]$ ,

$$\begin{pmatrix} \mu_{\mathbf{p}}(B + \mathbf{i}_0) \\ \vdots \\ \mu_{\mathbf{p}}(B + \mathbf{i}_{r-1}) \end{pmatrix} = M_\varepsilon \begin{pmatrix} \mu_{\mathbf{p}}(\mathbb{R}_\varepsilon^{-1}(B) + \mathbf{i}_0) \\ \vdots \\ \mu_{\mathbf{p}}(\mathbb{R}_\varepsilon^{-1}(B) + \mathbf{i}_{r-1}) \end{pmatrix}.$$

REMARK 2.4 The finiteness of  $\mathcal{I}_{(\beta, \mathbf{d})}$  is assured, according to [7, §2.2], if  $\beta$  is an irrational Pisot number or an integer.

We shall use also the probability distribution of the fractionnal part of the random variable  $X$ , that we denote by  $\mu_{\mathbf{p}}^*$ . Suppose that  $\alpha$  belongs to  $]1, 2[$ , or equivalently that  $\beta < \mathbf{d} < 2\beta - 1$ . Then  $\mu_{\mathbf{p}}^*$  – which has support  $[0, 1]$  – satisfy the following relation for any Borel set  $B \subset [0, 1]$ :

$$\mu_{\mathbf{p}}^*(B) = \mu_{\mathbf{p}}(B) + \mu_{\mathbf{p}}(B + 1)$$

and, if  $B \subset [\alpha - 1, 1]$ ,

$$\mu_{\mathbf{p}}^*(B) = \mu_{\mathbf{p}}(B). \quad (12)$$

The following proposition points out that in certain cases, the restriction of  $\mu_{\mathbf{p}}$  (or  $\mu_{\mathbf{p}}^*$ ) to the interval  $[\alpha - 1, 1]$  is "representative" of  $\mu_{\mathbf{p}}$  itself.

PROPOSITION 2.5 Suppose  $\beta < \mathbf{d} \leq \beta + 1 - \frac{1}{\beta}$ .

(i) The interval  $]0, \alpha[$  is the reunion of  $I_k := \left[ \frac{1}{\beta^{k+1}}, \frac{1}{\beta^k} \right]$  and  $I'_k := \left[ \alpha - \frac{1}{\beta^k}, \alpha - \frac{1}{\beta^{k+1}} \right]$  for  $k \in \mathbb{N} \cup \{0\}$

(ii) Let  $B \subset \mathbb{R}$  be a Borel set . If  $B \subset I_k$  (or equivalently if  $\alpha - B \subset I'_k$ ), then  $\beta^k B$  and  $\alpha - \beta^k B$  are two subsets of  $[\alpha - 1, 1]$  such that

$$\begin{aligned} \mu_{\mathbf{p}}(B) &= \mathbf{p}_0^k \cdot \mu_{\mathbf{p}}^*(\beta^k B) \\ \mu_{\mathbf{p}}(\alpha - B) &= \mathbf{p}_{\mathbf{d}-1}^k \cdot \mu_{\mathbf{p}}^*(\alpha - \beta^k B). \end{aligned}$$

*Proof.* (i) The hypothesis on  $\mathbf{d}$  implies  $\alpha < 2$  hence  $]0, \alpha[$  is the reunion of  $]0, 1]$  and  $[\alpha - 1, \alpha[$ .

(ii)  $B \subset I_k \Rightarrow \beta^k B \subset \left[ \frac{1}{\beta}, 1 \right] \subset [\alpha - 1, 1]$ . The equality  $\mu_{\mathbf{p}}(B) = \mathbf{p}_0^k \cdot \mu_{\mathbf{p}}^*(\beta^k B)$  results from (11) and (12).

Since  $\beta^k B \subset [\alpha - 1, 1]$  one has  $\alpha - \beta^k B \subset [\alpha - 1, 1]$ . The equality  $\mu_{\mathbf{p}}(\alpha - B) = \mathbf{p}_{a-1}^k \cdot \mu_{\mathbf{p}}(\alpha - \beta^k B)$  follows from (9), (11) and (12). ■

### 3 Bernoulli convolution in Pisot quadratic bases

In this section  $\beta > 1$  is solution of the equation  $x^2 = ax + b$  (with integral  $a$  and  $b$ ), and we suppose that the other solution belongs to  $] -1, 0[$ . This implies  $1 \leq b \leq a \leq \beta - \frac{1}{\beta} < \beta < a + 1$ . Let  $\mathbf{p} = (\mathbf{p}_0, \dots, \mathbf{p}_a)$  be a positive probability vector; the Bernoulli convolution  $\mu_{\mathbf{p}}$  has support  $[0, \alpha]$ , where  $\alpha = \frac{a}{\beta - 1}$  belongs to  $]1, 2[$ . The condition in Proposition 2.5 is satisfied hence it is sufficient to study the Gibbs properties of  $\mu_{\mathbf{p}}^*$  on its support  $[0, 1]$ , to get the local properties of  $\mu_{\mathbf{p}}$  on  $[0, \alpha]$  (see [6] for the multifractal analysis of the weak Gibbs measures).

With the notations of the previous subsection one has  $\mathcal{B} = \mathcal{D} = \{0, \dots, a\}$ ,

$\mathcal{I}_{(\beta, a+1)} = \{0, 1, \beta - a\}$  and – for any  $\varepsilon \in \mathcal{B}$

$$M_{\varepsilon} = \begin{pmatrix} \mathbf{p}_{\varepsilon} & \mathbf{p}_{\varepsilon-1} & 0 \\ 0 & 0 & \mathbf{p}_{a+\varepsilon} \\ \mathbf{p}_{b+\varepsilon} & \mathbf{p}_{b+\varepsilon-1} & 0 \end{pmatrix},$$

where, by convention,  $\mathbf{p}_i = 0$  if  $i \notin \mathcal{D}$ .

Notice that the intervals  $\mathbb{R}_{\varepsilon}([0, 1])$  do not make a partition of  $[0, 1]$  for  $\varepsilon \in \mathcal{B}$  but, setting

$$\mathbb{S}_{\varepsilon} := \begin{cases} \mathbb{R}_{\varepsilon} & \text{for } 0 \leq \varepsilon \leq a - 1 \\ \mathbb{R}_a \circ \mathbb{R}_{\varepsilon-a} & \text{for } a \leq \varepsilon \leq a + b - 1 \end{cases}$$

the intervals  $\mathbb{S}_{\varepsilon}([0, 1])$  make such a partition for  $\varepsilon \in \mathcal{A} := \{0, \dots, a + b - 1\}$ .

**THEOREM 3.1** *The measure  $\mu_{\mathbf{p}}^*$  is weak Gibbs w.r.t.  $\{\mathbb{S}_{\varepsilon}\}_{\varepsilon=0}^{a+b-1}$  if and only if  $\mathbf{p}_0^2 \geq \mathbf{p}_a \mathbf{p}_{b-1}$  and  $\mathbf{p}_0 \mathbf{p}_{a-b+1} \geq \mathbf{p}_a^2$ .*

*Proof.* The  $n$ -step potential of  $\mu_{\mathbf{p}}^*$  can be computed by means of the matrices

$$M_{\varepsilon}^* := \begin{cases} M_{\varepsilon} & \text{for } 0 \leq \varepsilon \leq a - 1 \\ M_a M_{\varepsilon-a} & \text{for } a \leq \varepsilon \leq a + b - 1. \end{cases}$$

Indeed by applying Proposition 2.3 to the sets  $B = [\xi_1 \dots \xi_n]$  and  $B' = [\xi_2 \dots \xi_n]$ , one has

$$\exp(\phi_n(\xi)) = \log \left( \frac{(1 \ 1 \ 0) M_{\xi_1}^* \dots M_{\xi_n}^* V}{(1 \ 1 \ 0) M_{\xi_2}^* \dots M_{\xi_n}^* V} \right), \quad \text{where } V := \begin{pmatrix} \mu_{\mathbf{p}}([0, 1]) \\ \mu_{\mathbf{p}}([0, 1] + 1) \\ \mu_{\mathbf{p}}([0, 1] + \beta - a) \end{pmatrix}. \quad (13)$$

Now the matrices  $M_\varepsilon^*$  are  $3 \times 3$  and we shall use  $2 \times 2$  ones. The matrices defined – for any  $\varepsilon \in \mathcal{A}' := \{0, \dots, 2a\}$  – by

$$M_\varepsilon^* := \begin{cases} M_0 M_\varepsilon & \text{if } \varepsilon \leq a \\ M_{\varepsilon-a} & \text{if } \varepsilon > a \end{cases}$$

satisfy the commutation relation  $Y M_\varepsilon^* = P_\varepsilon Y$ , where

$$Y := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad P_\varepsilon := \begin{cases} \begin{pmatrix} \mathbf{p}_0 \mathbf{p}_\varepsilon & \mathbf{p}_0 \mathbf{p}_{\varepsilon-1} \\ \mathbf{p}_a \mathbf{p}_{b+\varepsilon} & \mathbf{p}_a \mathbf{p}_{b+\varepsilon-1} \end{pmatrix} & \text{if } \varepsilon \leq a \\ \begin{pmatrix} \mathbf{p}_{\varepsilon-a} & \mathbf{p}_{\varepsilon-a-1} \\ 0 & 0 \end{pmatrix} & \text{if } \varepsilon > a. \end{cases}$$

Let  $\xi \in \mathcal{A}^\mathbb{N}$  such that  $\sigma\xi \neq \bar{0}$ . There exists an integer  $k \geq 0$  and  $\varepsilon \in \mathcal{A} \setminus \{0\}$  such that

$$M_{\xi_2} \dots M_{\xi_{k+2}} = M_0^k M_\varepsilon.$$

One can associate to the sequence  $\xi$ , the sequence  $\zeta \in \mathcal{A}^\mathbb{N}$  such that

$$\forall n \geq k+4, \exists k(n) \in \mathbb{N}, \quad M_{\xi_{k+3}}^* \dots M_{\xi_n}^* = M_{\zeta_1}^* \dots M_{\zeta_{k(n)}}^* \quad \text{or} \quad M_{\zeta_1}^* \dots M_{\zeta_{k(n)}}^* M_0.$$

Now – according to (13) and the commutation relation

$$n \geq k+4 \Rightarrow \exp(\phi_n(\xi)) = \frac{(1 \ 1 \ 0) M_{\xi_1}^* M_0^k Q_\varepsilon \mathbf{N}(P_{\zeta_1} \dots P_{\zeta_{k(n)}} W)}{(1 \ 1 \ 0) M_0^k Q_\varepsilon \mathbf{N}(P_{\zeta_1} \dots P_{\zeta_{k(n)}} W)} \quad (14)$$

where  $Q_\varepsilon := \begin{pmatrix} \mathbf{p}_\varepsilon & \mathbf{p}_{\varepsilon-1} \\ 0 & 0 \\ \mathbf{p}_{b+\varepsilon} & \mathbf{p}_{b+\varepsilon-1} \end{pmatrix}$  and  $W = YV$  or  $Y M_0 V$ .

If  $\mathbf{p}_0 \mathbf{p}_{a-b+1} \geq \mathbf{p}_a^2$ , the uniform convergence – on  $\mathcal{A}'^\mathbb{N}$  – of the sequence  $\mathbf{N}(P_{\zeta_1} \dots P_{\zeta_k} YV)$  and  $\mathbf{N}(P_{\zeta_1} \dots P_{\zeta_k} Y M_0 V)$  to the same vector  $V(\zeta)$  is insured by Theorem 1.1 and Corollary 1.2. When  $n \rightarrow \infty$  the numerator in (14) converges to  $V_1(\xi) := (1 \ 1 \ 0) M_{\xi_1}^* M_0^k Q_\varepsilon V(\zeta)$ , and the denominator to  $V_2(\xi) := (1 \ 1 \ 0) M_0^k Q_\varepsilon V(\zeta)$ ; this convergence is uniform on each cylinder  $[[\varepsilon' 0^k \varepsilon]]$ . Since the first entrie in  $Q_\varepsilon V(\zeta)$  is at least  $\min\{\mathbf{p}_\varepsilon, \mathbf{p}_{\varepsilon-1}\} > 0$ ,  $V_1(\xi)$  and  $V_2(\xi)$  are positive and consequently  $\phi_n(\xi)$  converges uniformly to  $\log \frac{V_1(\xi)}{V_2(\xi)}$ . This is also true on any finite reunion of such cylinders; let us denote by  $X(k_0)$  the reunion of  $[[\varepsilon' 0^k \varepsilon]]$  for  $k < k_0, \varepsilon \in \mathcal{A} \setminus \{0\}$  and  $\varepsilon' \in \mathcal{A}$ ; then

$$\forall \eta > 0, \exists n_0 \in \mathbb{N}, n \geq n_0 \text{ and } \xi \in X(k_0) \Rightarrow \left| \phi_n(\xi) - \log \frac{V_1(\xi)}{V_2(\xi)} \right| \leq \eta. \quad (15)$$

We consider now a sequence  $\xi \in \mathcal{A}^{\mathbb{N}} \setminus X(k_0)$ . By using the left and right eigenvectors of  $M_0$  – for the eigenvalue  $\mathbf{p}_0$  – we obtain

$$\lim_{k \rightarrow \infty} A_k = \lambda_0 \begin{pmatrix} \mathbf{p}_0^2 - \mathbf{p}_a \mathbf{p}_{b-1} & 0 & 0 \\ \mathbf{p}_a \mathbf{p}_b & 0 & 0 \\ \mathbf{p}_0 \mathbf{p}_b & 0 & 0 \end{pmatrix} \quad \text{where } \lambda_0 > 0, A_k := \begin{cases} \mathbf{p}_0^{-k} M_0^k & \text{if } \mathbf{p}_a \mathbf{p}_{b-1} < \mathbf{p}_0^2 \\ k^{-1} \mathbf{p}_0^{-k} M_0^k & \text{if } \mathbf{p}_a \mathbf{p}_{b-1} = \mathbf{p}_0^2. \end{cases}$$

The entries  $\mathbf{p}_a \mathbf{p}_b$  and  $\mathbf{p}_0 \mathbf{p}_b$  being positive, there exists  $\lambda(\varepsilon') > 0$  such that

$$\lim_{k \rightarrow \infty} (1 \ 1 \ 0) M_{\varepsilon'}^* A_k Q_{\varepsilon} = \lambda(\varepsilon') (\mathbf{p}_{\varepsilon} \ \mathbf{p}_{\varepsilon-1}).$$

Moreover the convergence of  $(1 \ 1 \ 0) M_{\varepsilon'}^* A_k Q_{\varepsilon} \begin{pmatrix} x \\ y \end{pmatrix}$  to  $\lambda(\varepsilon')(\mathbf{p}_{\varepsilon} x + \mathbf{p}_{\varepsilon-1} y)$  is uniform on the set of normalized nonnegative column-vectors  $\begin{pmatrix} x \\ y \end{pmatrix}$ . Similarly, there exists  $\lambda_1 > 0$  such that  $(1 \ 1 \ 0) A_k Q_{\varepsilon} \begin{pmatrix} x \\ y \end{pmatrix}$  converges uniformly to  $\lambda_1(\mathbf{p}_{\varepsilon} x + \mathbf{p}_{\varepsilon-1} y)$ . Both limits are positive if  $\varepsilon \neq 0$ , implying that the ratio converges uniformly to  $\frac{\lambda(\varepsilon')}{\lambda_1}$ . Hence, using (14) one can choose  $k_0$  such that – if we assume  $\xi \in \mathcal{A}^{\mathbb{N}} \setminus X(k_0)$  and  $\sigma \xi \neq \bar{0}$

$$n \geq k_0 + 4 \Rightarrow \left| \phi_n(\xi) - \log \frac{\lambda(\varepsilon')}{\lambda_1} \right| \leq \eta. \quad (16)$$

The uniform convergence of  $\phi_n(\xi)$  on  $\mathcal{A}^{\mathbb{N}}$  follows from (15) and (16) since, in the remaining case  $\sigma \xi = \bar{0}$  one has  $\lim_{n \rightarrow \infty} \phi_n(\xi) = \log \frac{\lambda(\xi_1)}{\lambda_1}$ .

Conversely, suppose  $\mathbf{p}_a \mathbf{p}_{b-1} > \mathbf{p}_0^2$ . If  $\mu_{\mathbf{p}}^*$  is weak Gibbs w.r.t.  $\{\mathbb{S}_{\varepsilon}\}_{\varepsilon=0}^{s-1}$  then, from (1) and (2) one has  $\phi_n(\xi) = o(n)$  for any  $\xi \in \mathcal{A}^{\mathbb{N}}$ . But this is not true:  $\phi_{2n+1}(1\bar{0}) \sim n \log \frac{\mathbf{p}_a \mathbf{p}_{b-1}}{\mathbf{p}_0^2}$ .

Suppose now  $\mathbf{p}_0 \mathbf{p}_{a-b+1} < \mathbf{p}_a^2$ . If  $b = 1$  we have  $\mathbf{p}_0 < \mathbf{p}_a$  hence  $\mathbf{p}_a \mathbf{p}_{b-1} > \mathbf{p}_0^2$ ; that is, we are in the previous case. If  $b \neq 1$ ,  $\mu_{\mathbf{p}}^*$  is no more weak Gibbs w.r.t.  $\{\mathbb{S}_{\varepsilon}\}_{\varepsilon=0}^{s-1}$  because there exists a contradiction between the limit in (1) and the following:

$$\lim_{n \rightarrow \infty} \left( \frac{\mu_{\mathbf{p}}^*[(0(a-b+1))^n 1^n]}{\mu_{\mathbf{p}}^*[(0(a-b+1))^n] \cdot \mu_{\mathbf{p}}^*[1^n]} \right)^{1/n} = \frac{\mathbf{p}_0 \mathbf{p}_{a-b+1}}{\mathbf{p}_a^2} < 1.$$

■

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